Critical comparison of microscopic calculations for a model neutron liquid*

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The calculated energy-density relation of a well-defined superdense neutron liquid model is compared with other microscopic calculations. All techniques except one give consistent results. This finding has an important bearing on the existence of solid cores in neutron stars.

The equation of state of a dense neutron liquid can be calculated microscopically. A number of such calculations have appeared in recent literature. Even though a wide variety of nuclear models and theoretical techniques are employed, the results obtained are not too different, in the sense that when the corresponding Tolman-Oppenheimer-Volkoff equations are solved for the structure of a neutron star the resulting stability conditions do not vary appreciably. The ostensibly small discrepancies, however, become significant when one considers the question of the solidification of neutron-star matter. It is not surprising that the solidification density should depend critically on the form of the nuclear potential chosen. But since 1973 it has become increasingly clear that the difference in the calculational methods may well play a major role in the disagreement. This discrepancy is alarming and unacceptable, since most of the methods employed in microscopic neutron-star work have heretofore weathered the rigorous tests posed by the theory of liquid helium. In the effort to clarify this intolerable situation, a number of the groups involved have agreed to apply their methods to a common model—one that is unabashedly hypothetical, but simple and well defined. We wish to report in this paper our observation that all who participated except one have obtained consistent results.

The model calls for the adoption of the spin-independent, state-independent, repulsive part of the Reid $^1S_0$ potential. (Some confusion has arisen concerning the assumed values of the parameters, but as will be seen shortly this does not prevent a satisfactory comparison of results.) Boltzmann statistics is assumed for further simplification. The search is then on for the unsymmetrized ground-state solution of the many-particle Schrödinger equation:

$$H\Psi_0(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = E_0\Psi_0(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N), \quad (1)$$

where

$$H = \frac{-\hbar^2}{2m}\sum_{i=1}^{N}\nabla_i^2 + \sum_{1\le i<j\le N} v(r_{ij}) \quad (2)$$

and

$$v(r) = 6484.2\,\frac{e^{-7X}}{X} = 9263.1\,\frac{e^{-4.9r}}{r} = 46.78\hbar c\,\frac{e^{-4.9r}}{r}, \quad (3)$$

with r measured in fm and $X = 0.7r$, for a range of densities $0.2 \lesssim \rho \lesssim 2.0$ fm^{-3}.

The case of the solid phase has been stated in an earlier paper by Chakravarty, Miller, and Woo[1] (CMW). Reasonably good agreement exists between the results obtained by Canuto and Chitre[2] (CC) and CMW. Both disagree significantly with Pandharipande[3] (P). In this note we wish to concentrate on the case of the *liquid* phase, which undoubtedly provides the simplest and therefore the most illuminating test that one can devise.

We have on hand for comparison the results obtained by P, CMW, Cochran and Chester[4] (CoC), and the present work (SW). All are variational calculations using correlated (or Jastrow) trial wave functions:

$$\Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = \prod_{1\le i<j\le N} f(r_{ij}) \equiv \prod_{1\le i<j\le N} \exp[\tfrac{1}{2}u(r_{ij})]. \quad (4)$$

P evaluated the energy expectation value $E \equiv \langle\Psi|H|\Psi\rangle/\langle\Psi|\Psi\rangle$ by means of a straightforward cluster expansion. To permit truncation at a very low order, retaining in fact just the first cluster in the series, he resorted to the application of a healing-length constraint on $f(r)$. CMW chose two parameterized forms for $f(r)$, and solved in each case the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) equation for the radial distribution function, whose appearance in the energy expression

corresponds to partially summing selected cluster terms to all orders. An upper bound to E_0 resulted from minimizing E with respect to the variational parameters. CoC carried out a similar calculation, the difference being that E was evaluated by means of a Monte Carlo procedure using up to 128 particles in a periodically extended box. It was a happy coincidence that the two classes of variational wave functions used by CoC turned out to be of precisely the same forms as those chosen by CMW. In the present work another time-honored procedure (in liquid-helium theory) is employed.

E is written in the form

$$E = \tfrac{1}{2} N\rho \int \left[v(r) - \frac{\hbar^2}{4m} \nabla^2 u(r) \right] g(r) d\vec{r} , \tag{5}$$

where the radial distribution function $g(r)$ is related to the wave function through the definition

$$g(r_{12}) = \frac{N(N-1)}{\rho^2} \frac{\int \Psi^2 d\vec{r}_3 \cdots d\vec{r}_N}{\int \Psi^2 d\vec{r}_1 \cdots d\vec{r}_N} , \tag{6}$$

or through the following approximate integral equations[5]:

PY (Percus-Yevick):

$$u(r) = \ln g(r) - \ln[1 + P(r)] , \tag{7}$$

HNC (hypernetted chain):

$$u(r) = \ln g(r) - P(r) , \tag{8}$$

where

$$P(r) = \frac{1}{(2\pi)^3 \rho} \int \frac{[S(k)-1]^2}{S(k)} e^{-i\vec{k}\cdot\vec{r}} d\vec{k} \tag{9}$$

and

$$S(k) = 1 + \rho \int [g(r) - 1] e^{i\vec{k}\cdot\vec{r}} d\vec{r} . \tag{10}$$

Both of these equations are easy to solve if one begins with $g(r)$ rather than $u(r)$. Restrictions on the form of $g(r)$ and the range of variational parameters must then be imposed,[5] to ensure the compatibility of $g(r)$ with its definition and a unique correspondence between $g(r)$ and $u(r)$. The form that we choose is given by[5]

$$g(r) = (1 + C) \exp[-(d/r)^m] - C \exp[-(1+\alpha)(d/r)^n] . \tag{11}$$

Every choice of the set of parameters (C, m, n, d, α) is subjected to testing against the above-mentioned restrictions so that all the undesirable members of the set are screened out. Still a large class of variational functions have remained to afford us a great deal of flexibility. The equation that we use in this calculation is the HNC. Using PY at random points shows that the differences are negligible, amounting, e.g., to only 1% at $\rho = 1\,\mathrm{fm}^{-3}$.

A detailed analysis of our results will be presented elsewhere, along with elaborate, perturbative improvements on theoretical calculations of the present generation. In Fig. 1 we show the discrepancy among various calculations. The SW results differ from the CMW results by absolute amounts never more than 25 MeV throughout the range of densities considered. Note that all quantities used in the curve marked $(E'_{\mathrm{CoC}} - E'_{\mathrm{SW}})$ have been obtained with a potential chosen by CoC,

$$v'(r) = 5742 \frac{e^{-4.0r}}{r} = 29\hbar c \frac{e^{-4.0r}}{r} , \tag{12}$$

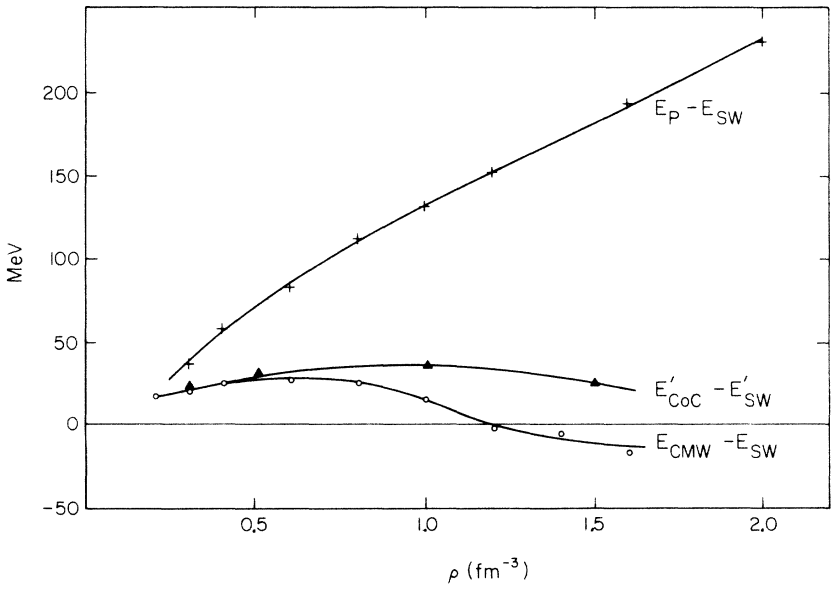

FIG. 1. Discrepancies among results of variational calculations.

which is believed to be the repulsive part of the triplet ($^3P_2 - {}^3F_2$) Reid potential. E'_{CoC} is slightly higher than E'_{SW}. This is not unexpected, since CoC's variational functions consist of no more than two parameters and are not as flexible as Eq. (11). The agreement between these calculations is remarkable, especially when viewed from direct plots of the energy-density data, as shown in Fig. 2. However, the discrepancy between P and SW (and therefore between P and other calculations) is unacceptably large. E_{P} is higher by as much as 130 MeV around the important region of $\rho \sim 1$ fm^{-3}, and more than 200 MeV at $\rho \gtrsim 1.5$ fm^{-3}. This may well be a direct consequence of the strong healing-length constraint placed on $f(r)$ in P, which drastically cuts down the extent of the variational wave-function space and results in upper bounds which are much too high. The situation, as expected, deteriorates with increasing density. The isolation of P from all other independently obtained results leads to the conclusion that Pandharipande's calculation must be less accurate.[6] The other procedures are viable alternatives; the particular choice reduces to a matter of familiarity and convenience.

Let us briefly recall certain earlier work on neutron matter using more realistic nuclear potentials. Canuto and Chitre[7] found that neutron matter solidifies at 0.9 fm^{-3}, while Nosanow and Parish,[8] by means of a Monte Carlo calculation, placed the solidification density at an even lower value (~0.3 fm^{-3}). Pandharipande,[9] using the healing-length constraint, obtained the only result showing no solidification up to 2.1 fm^{-3}. Even though the controversy cannot yet be resolved, our present finding seems to have cast Pandharipande's conclusion into some doubt.[3]

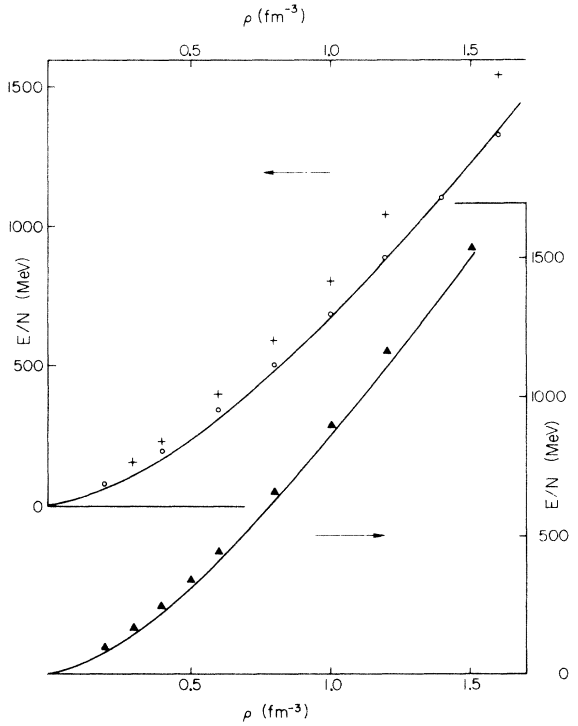

FIG. 2. Energy-density relations. Upper set (left scale): $v(r)$ used; +: P; ○: CMW; —— : present work (SW). Lower set (right scale): $v'(r)$ used; ▲: CoC; —— : present work (SW).

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[1]S. Chakravarty, M. D. Miller, and C.-W. Woo, Nucl. Phys. A209, 170 (1974).

[2]V. Canuto and S. M. Chitre, private communication.

[3]V. R. Pandharipande, private communication. In private communication, G. Baym indicated to us that better results are obtained when the full Bethe-Pandharipande procedure is followed.

[4]S. Cochran and G. V. Chester, report (unpublished).

[5]E. Feenberg, *Theory of Quantum Fluids* (Academic, New York, 1969), Chap. 1 and 6.

[6]At a glance E_{P} and E'_{CoC} appear to be in reasonably good agreement. This is, however, coincidental, since very different potentials are used in the two calculations.

[7]V. Canuto and S. M. Chitre, in Proceedings of the Sixth Texas Symposium on Relativistic Astrophysics, New York, 1972 (unpublished); Phys. Rev. Lett. 30, 999 (1973).

[8]L. H. Nosanow and L. J. Parish, report (unpublished).

[9]V. R. Pandharipande, in Proceedings of the Sixth Texas Symposium on Relativistic Astrophysics, New York, 1972 (unpublished); report (unpublished).