Regge-calculus model for the Tolman universe

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A model for the Tolman universe is constructed from 600 equilateral tetrahedrons, each containing blackbody radiation, and connected so as to form a closed space. The dynamics of this universe is studied using the techniques of Regge calculus.

In a previous paper[1] we have shown how the Friedmann universe[2] may be approximated by a collection of equilateral tetrahedrons joined together to form a closed space and how the time development of this space can be obtained using Regge calculus.[3,4] Like the Friedmann universe, the Tolman universe[5] is an isotropic, homogeneous space having the geometry of a 3-sphere. The only difference is that whereas the Friedmann universe is filled with dust the Tolman universe contains radiation. In this addendum we show how the models considered in Ref. 1 may be used for the Tolman universe.

The models are constructed by taking a number of equilateral tetrahedrons and joining them together so that an equal number of tetrahedrons meet at each edge and vertex, thus ensuring homogeneity and isotropy. Where there are 3, 4, or 5 tetrahedrons meeting at each edge closure is obtained using a total of 5, 16, or 600 tetrahedrons. By assuming that each block contains the same amount of dust or radiation we obtain a model for the Friedmann or Tolman universe. At any instant of time the geometry of the model may be specified by giving the edge length, l, of the tetrahedrons. If we divide the time axis into intervals i starting at time t_i and ending at t_{i+1} and construct the corresponding 3-space with edge length l_i at each t_i, then by joining corresponding vertices of the tetrahedrons at t_i and t_{i+1} we may approximate the space-time between t_i and t_{i+1} by a set of 4-dimensional blocks as illustrated in Fig. 1. The space-time within each of these blocks is assumed to be Minkowskian. Regge calculus tells us that such a collection of 4-simplexes will provide an approximate solution to Einstein's equations if the equation[1,3]

$$\delta\left(\sum_{\text{all hinges } i} 2L_i\delta_i - \frac{16\pi G}{c^2}\sum_{\text{all blocks } j}\int_{\text{block } j}\rho\, d^4x\right) = 0 \tag{1}$$

is satisfied. The first term in this equation corresponds to the gravitational action. The hinges are the 2-dimensional surfaces (e.g., ABC or $AA'B'B$ in Fig. 1) on which the 3-dimensional faces of the blocks meet. L_i is the area of hinge i and δ_i is the deficit angle defined as 2π minus the sum of the dihedral angles between the faces of the 4-dimensional blocks meeting on hinge i. ρ is the total invariant mass density which includes a contribution from any internal energy resulting from random kinetic motion as well as the rest mass of the particles which comprise the matter. The variations are with respect to the edge lengths l_i and the invariant distances m_i between corresponding vertices in the 3-dimensional hypersurfaces at t_i and t_{i+1} (see Fig. 1).

If we restrict ourselves to the 600-tetrahedron model, which, as one would expect, provides the best approximation, we find that letting $t_i \to t_{i+1}$ we can write Eq. (1) as[1]

$$\rho l^2\left[1-\frac{1}{8c^2}\left(\frac{dl}{dt}\right)^2\right]^{1/2} = \frac{9\sqrt{2}\,c^2}{10\pi G}(2\pi - 5\theta)\,, \tag{2}$$

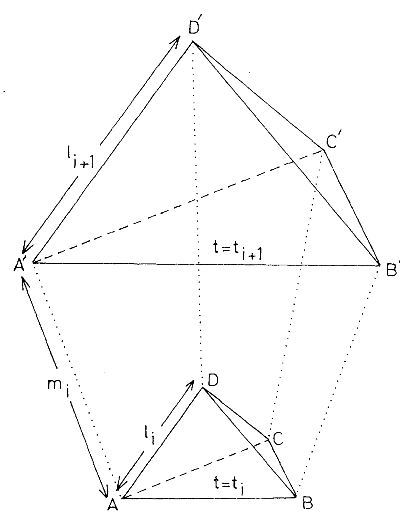

FIG. 1. Diagram illustrating a 4-dimensional block.

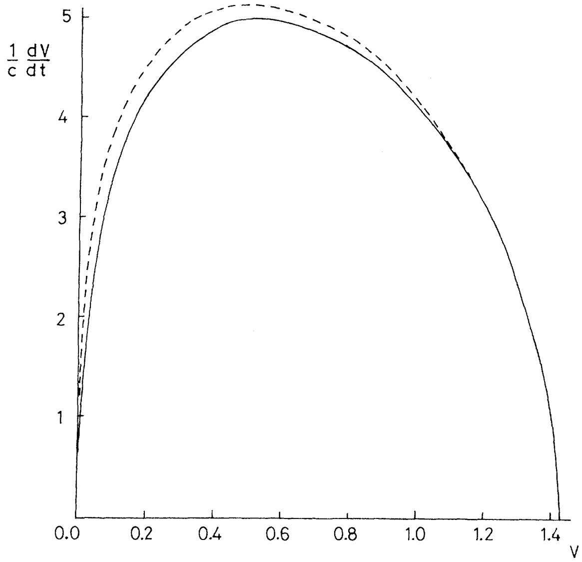

FIG. 2. Rate of change of the volume of the universe plotted against the volume for $d=1$. The Regge-calculus solution is given by the solid line, the analytic solution by the dashed line.

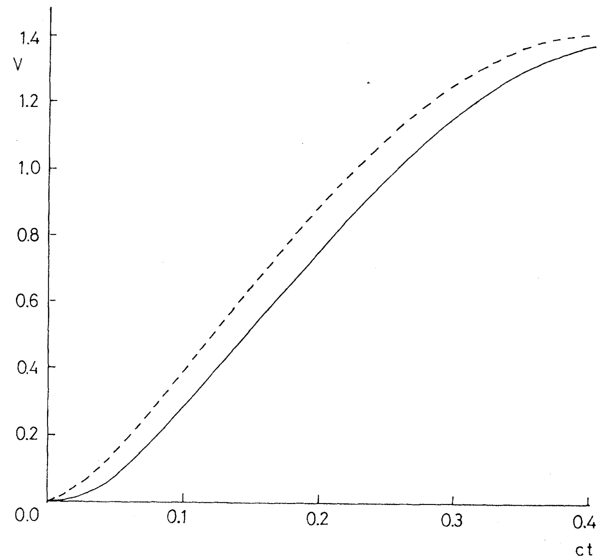

FIG. 3. Time dependence of the volume of the universe for $d=1$. The Regge-calculus solution is given by the solid line, the analytic solution by the dashed line.

where

$$\cos\theta = \left[8c^2 + \left(\frac{dl}{dt}\right)^2\right] \Big/ \left[24c^2 - \left(\frac{dl}{dt}\right)^2\right].$$

If we now use the fact that the energy density, u, of blackbody radiation confined in a volume V satisfies the equation

$$u \propto V^{-4/3} \tag{3}$$

during adiabatic expansion, we may write

$$\rho = \frac{b}{l^4}, \tag{4}$$

where b is some constant. Substituting in Eq. (2) we obtain a parametric solution for l and dl/dt:

$$l = \left(\frac{d\pi \tan(\frac{1}{2}\theta)}{2\pi - 5\theta}\right)^{1/2}, \quad d = \frac{10bG}{9c^2} \tag{5}$$

$$\frac{dl}{dt} = 2c\,[2 - 4\tan^2(\tfrac{1}{2}\theta)]^{1/2}. \tag{6}$$

Comparison of this solution with the exact solution is most conveniently effected by considering the volume of the universe, which is given by

$$V = 100\,l^3/\sqrt{2}. \tag{7}$$

The exact solution takes the form[4,5]

$$V = 2\pi^2 a_0^3 \sin^3\mu, \tag{8}$$

$$t = \frac{a_0}{c}(1 - \cos\mu). \tag{9}$$

a_0 is an arbitrary constant equaling the maximum radius of the 3-sphere.

In Figs. 2 and 3 we show graphs of dV/dt against V and of V against t [obtained by numerical integration of Eqs. (5) and (6)] for $d=1$. We also show the exact solution with a_0 chosen so that both solutions have the same maximum volume. From Fig. 2 we see that agreement with the analytic solution is good, especially for larger values of the volume, but it must be remembered that the maximum values have been constrained to be equal. As with the Friedmann universe, agreement at smaller values of V is poorer. The inaccuracy at small V results from a qualitative difference in the solutions. The analytic solution shows that the rate of change of the radius, a, of the 3-sphere becomes infinite as $a \to 0$, whereas in the Regge-calculus solution dl/dt has a finite maximum value as $l \to 0$. It would appear (not unexpectedly) that Regge calculus will not produce accurate results for the time development of 3-spaces in situations where the extrinsic curvature of the 3-dimensional spacelike hypersurface is very large, even though the intrinsic geometry of the hypersurface itself may be accurately described by the subdivision into blocks.

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