In the limit in which quarks $\mathcal{P}'$, q^0, q'^0, q^-, and q'^- are regarded as heavy, the above reproduces the Cabibbo coupling

$$e \sin\beta[-W_\mu^+(\overline{\mathcal{P}}\gamma_\mu \mathfrak{N}_L') - W_\mu^-(\overline{\mathfrak{N}}_L'\gamma_\mu \mathcal{P})] + eA_\mu(\overline{\mathcal{P}}\gamma_\mu\mathcal{P}) .$$

In addition, the triplets of leptons and triplets and singlets of hadrons are coupled invariantly to a Higgs scalar-meson triplet ϕ which is introduced in order to facilitate spontaneous breaking of the gauge symmetry. The neutral member of ϕ develops a vacuum expectation value, the charged members disappear, and the $W^\pm$ bosons become massive when the gauge symmetry is broken.

[12]Because the model is "vectorlike" the leptons and hadrons may have masses before the gauge symmetry is broken. See Ref. 2.
[13]J. R. Primack and H. R. Quinn, Phys. Rev. D 6, 3171 (1972); K. Fujikawa, B. W. Lee, and A. I. Sanda, *ibid.* 6, 2923 (1972).
[14]W. A. Bardeen, R. Gastmans, and B. Lautrup, Nucl. Phys. B46, 319 (1972).
[15]S. Matsuda, survey of Experimental Proposals at NAL, CERN report, 1972 (unpublished).
[16]T. D. Lee and C. N. Yang, Phys. Rev. 128, 885 (1962).
[17]J. Bernstein and T. D. Lee, Phys. Rev. Lett. 11, 512 (1963); P. H. Meyer and D. Schiff, Phys. Lett. 8, 217 (1964).
[18]See for example, J. Schechter and Y. Ueda, Phys. Rev. D 2, 736 (1970) and K. Tanaka, *ibid.* 5, 3243 (1972).

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Comment on "New approach to the renormalization group"

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Following previous discussions concerning the field-theoretical derivation of Kadanoff's scaling laws, we apply the method of "soft quantization" to the derivation of a homogeneous renormalization-group equation. This equation is similar to the one proposed recently by S. Weinberg. In addition to our attempt to close the "communication gap" between physicists working on critical phenomena and high-energy physics, we discuss some new applications of such homogeneous differential equations to perturbations of scale-invariant models.

In a recent paper S. Weinberg[1] derived a homogeneous parametric differential equation which for certain problems in high-energy physics seems to have a larger range of applicability than the Callan-Symanzik[2,3] equation. A similar equation for the scalar A^4 coupling has been known to physicists working on applications of field-theoretical methods to critical phenomena. In fact it is the infinitesimal version of the Kadanoff scaling law[4] for correlation functions at noncritical temperature. In Ref. 5 this equation was derived on the basis of "normal product" properties. Subsequently its validity was argued on the basis of loopwise summations.[6] Using methods similar to those of Coleman and E. Weinberg,[7] the authors in Ref. 8 gave a third argument in favor of its validity and also showed how results of Kadanoff,[9] Wilson,[10] and Wegner and Riedel[11] can be obtained in a very economical way by using methods of renormalized quantum field theory. In this note we want to give first a finite (i.e., without using cutoffs or regulators) derivation of the homogeneous scaling equation in $D=4$ dimensions and then point out some interesting applications to perturbations of exactly soluble models. We also derive a similar, slightly more complicated homogeneous scaling equation, which stays infrared-finite for $D<4$. Our derivation is an elaboration of the remarks made after formula (7.13) of Ref. 5. In the Bogoliubov-Parasiuk-Hepp (BPH) renormalization approach, in the version of Zimmermann,[12] one obtains the renormalized Green's functions by application of the finite-part prescription to the Gell-Mann–Low formula for the time-ordered functions (for brevity we argue with an A^4 self-coupling):

$$\langle TX\rangle = \text{finite part of } \left\langle TX_0 \exp\left[i\int :\mathcal{L}_{\text{int}}(A_0):dx\right]\right\rangle_\otimes^{(0)} ,$$

$$X=\prod_{i=1}^{N} A(x_i), \quad \otimes = \text{omission of vacuum bubbles.} \tag{1}$$

With the help of Feynman rules in momentum space and by the application of Taylor operators on each renormalization part[12] one obtains absolutely convergent Feynman integrands, i.e., any subintegration leads to a convergent expression. By adding finite counterterms to the Lagrangian, i.e.,

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2}\partial_\mu A \partial^\mu A - \tfrac{1}{2}m^2 A^2 + \tfrac{1}{2}aA^2 + \tfrac{1}{2}b\partial_\mu A\partial^\mu A - \frac{\lambda - c}{4!}A^4, \tag{2}$$

one obtains through formula (1) the Green's functions (i.e., vertex functions) with prescribed normalization conditions[13] at fixed spots in momentum space. The desired homogeneous equation (1) is, however, only consistent with normalization at a fixed value of the mass parameter. Hence one needs a Taylor subtraction scheme in which the Taylor operators act not only on the external momenta of the renormalization subgraphs but also on their masses. Such a scheme was proposed by Gomes, Lowenstein, and Zimmermann[14] in connection with the treatment of symmetry breaking.[15] Adapted to our situation, we define the following "Taylor" operators on renormalization subgraphs: a zero-degree Taylor operator

$$\tau^{(0)}F(\mathfrak{p}, m) = F(0,\mu), \quad \mathfrak{p} = (p_1, \ldots, p_N) \tag{3}$$

and a second-degree "Taylor" operator

$$\tau^{(2)}F(\mathfrak{p}, m) = F(0,0) + \sum_{i=1}^{N} p_i^\mu \left(\frac{\partial F}{\partial p_i^\mu}\right)_{\mathfrak{p}=0, m=\mu} + \tfrac{1}{2}\sum_{i<k} p_i^\mu p_k^\nu \left(\frac{\partial}{\partial p_i^\mu}\frac{\partial F}{\partial p_k^\nu}\right)_{\mathfrak{p}=0, m=\mu} + m^2\left(\frac{\partial F}{\partial m^2}\right)_{\mathfrak{p}=0, m=\mu}. \tag{4}$$

The $F(\mathfrak{p}, m)$ are either the self-energy or the vertex-normalization parts. The renormalized Feynman integrand associated with a graph Γ is given by the forest formula[12] which just solves the problem of overlapping Taylor subtractions. Note that the Taylor subtraction scheme (3), (4) does not create infrared divergencies. The first subtraction of the two-point function is done at $m=0$, but the higher subtractions, which if done at $m=0$ would lead to infrared divergencies, are actually done at $m=\mu$.

It is now easy to see that the chosen subtraction scheme gives the following normalization conditions for the vertex functions:

$$\Gamma^{(4)}(p_i = 0, m=\mu) = -i\lambda, \tag{5a}$$

$$\left.\frac{\partial\Gamma^{(2)}}{\partial p^2}\right|_{p=0, m=\mu} = i, \tag{5b}$$

$$\left.\frac{\partial\Gamma^{(2)}}{\partial m^2}\right|_{p=0, m=\mu} = -i, \tag{5c}$$

and

$$\Gamma^{(2)}(p=0, m=0) = 0. \tag{5d}$$

As the usual BPHZ Taylor subtraction would correspond to "intermediate" normalizations of $\Gamma^{(N)}$ at $p=0$ (and m arbitrary), the Lagrangian (2) with the subtraction scheme (3), (4) and $a=b=c=0$ leads to the normalization (5) for the vertex functions. If one wants to change (5) one has to add finite a, b, and c counterterms. For the derivation of the parametric differential equations we follow the usual procedure of the normal-product formalism. We define integrated composite fields ("differential vertex operations")

$$\Delta_0 = \frac{i}{2}\int d^4x N_2[A^2], \tag{6a}$$

$$\Delta_1 = \frac{i}{2}\int d^4x N_4[m^2A^2], \tag{6b}$$

$$\Delta_2 = \frac{i}{2}\int d^4x N_4[\partial_\mu A\partial^\mu A], \tag{6c}$$

$$\Delta_3 = \frac{i}{4!}\int d^4x N_4[A^4]. \tag{6d}$$

With the help of the renormalized Gell-Mann–Low formula (the subscript of N is related to the degree of the Taylor operator for graphs containing the composite vertex), we first note that there is an algebraic identity between Δ_0 and the Δ_i:

$$m^2\Delta_0\Gamma^{(N)} = (\lambda_1\Delta_1 + \lambda_2\Delta_2 + \lambda_3\Delta_3)\Gamma^{(N)}, \tag{7}$$

$$\lambda_1 = 1 - i\mu^2 \frac{\partial}{\partial m^2}\Delta_0\Gamma^{(2)}\Big|_{p=0, m=\mu}, \tag{8a}$$

$$\lambda_2 = -\tfrac{1}{8}i\mu^2\partial_\mu^p\partial_p^\mu\Delta_0\Gamma^{(2)}(p,-p)|_{p=0, m=\mu}, \tag{8b}$$

$$\lambda_3 = -i\mu^2\Delta_0\Gamma^{(4)}(p=0, m=\mu). \tag{8c}$$

The parametric changes for the vertex functions may be expressed in terms of the differential vertex operations[16] ("renormalized Schwinger action formula")

$$\frac{\partial\Gamma^{(N)}}{\partial m^2} = -\Delta_0\Gamma^{(N)}, \tag{9a}$$

$$\frac{\partial\Gamma^{(N)}}{\partial\lambda} = -\Delta_3\Gamma^{(N)}, \tag{9b}$$

$$\frac{\partial\Gamma^{(N)}}{\partial\mu^2} = (\alpha_1\Delta_1 + \alpha_2\Delta_2 + \alpha_3\Delta_3)\Gamma^{(N)}, \tag{9c}$$

$$\alpha_1 = -i \frac{\partial}{\partial m^2} \frac{\partial \Gamma^{(2)}}{\partial \mu^2}\bigg|_{p=0,m=\mu}, \tag{10a}$$

$$\alpha_2 = -\frac{i}{8} \frac{\partial}{\partial p^\mu} \frac{\partial}{\partial p_\mu} \frac{\partial \Gamma^{(2)}}{\partial \mu^2}\bigg|_{p=0,\, m=\mu}, \tag{10b}$$

$$\alpha_3 = -i \frac{\partial \Gamma^{(4)}}{\partial \mu^2}\bigg|_{p=0,m=\mu}. \tag{10c}$$

Note that it is the validity of these rules which allows us to reinterpret the original Lagrangian (which was just a "bookkeeper" to manufacture the renormalized Gell-Mann–Low perturbation theory) as a composite field:

$$\mathcal{L} = \tfrac{1}{2} N_4[\partial_\mu A \partial^\mu A] - \tfrac{1}{2} m^2 N_2[A^2] - (\lambda/4!) N_4[A^4]. \tag{11}$$

The integrated bilinear field equation[17]

$$\langle TN_4[A\partial^2 A](x)X\rangle^{\text{prop}} = \langle TN_4[m^2A^2](x)X\rangle^{\text{prop}} + \frac{\lambda}{3!}\langle TN_4[A^4](x)X\rangle^{\text{prop}} + \sum_{i=1}^{N} \delta(x-x_i)\langle TX\rangle^{\text{prop}} \tag{12}$$

gives the counting identity[16]

$$N\Gamma^{(N)} = (-4\lambda\Delta_3 + 2\Delta_2 - 2\Delta_1)\Gamma^{(N)}. \tag{13}$$

We now have five operations $\partial/\partial m^2$, $\partial/\partial\lambda$, $\partial/\partial\mu^2$, N, and the mass insertion Δ_0 expressed in terms of three (linearly independent) Δ_i, $i = 1, 2, 3$. Hence there must be two linear relations between the five operations. In other words, in addition to the already established relation

$$\frac{\partial \Gamma^{(N)}}{\partial m^2} = -\Delta_0 \Gamma^{(N)} \tag{14}$$

there is a homogeneous parametric differential equation

$$\left(2\mu^2 \frac{\partial}{\partial \mu^2} + 2\delta m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} - N\gamma_A\right)\Gamma^{(N)} = 0, \tag{15}$$

with

$$2\mu^2\alpha_1 - 2\delta\lambda_1 + 2\gamma_A = 0, \tag{16a}$$

$$2\mu^2\alpha_2 - 2\delta\lambda_2 - 2\gamma_A = 0, \tag{16b}$$

$$2\mu^2\alpha_3 - 2\delta\lambda_3 - \beta - \lambda\gamma_A = 0, \tag{16c}$$

where the λ_i's and $\mu^2\alpha_i$'s depend only on g.

Since the determinant is nonvanishing in lowest order this system is soluble for δ, β, and γ_A in perturbation theory. However, for the determination of these coefficients it is more convenient to use the normalization conditions (5) directly. The "mass" m^2 is according to (14) a parameter "conjugate" to the composite operator $N[A^2]$. In the field-theoretical treatment[18] of critical phenomena this operator represents the energy fluctuations and therefore m^2 is the same as the temperature t (more precisely the deviation from the critical temperature). Once one is aware of this physical interpretation, the statement that 2δ is the "would-be" anomalous dimension of the energy fluctuation [i.e., it is the anomalous dimension at a scale-invariant point λ_0 where $\beta(\lambda_0) = 0$] is to be expected. In order to see this formally, we derive the parametric differential equation for

$$\Gamma^{(N)}_{A^2} = \langle TN_2[A^2](x)X\rangle^{\text{prop}}. \tag{17}$$

Going through the standard arguments,[19,20] we obtain

$$\left(2\mu^2 \frac{\partial}{\partial \mu^2} + 2\delta m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} - N\gamma_A + \gamma_{A^2}\right)\Gamma^{(N)}_{A^2} = 0, \tag{18}$$

where γ_{A^2} is given in terms of "cat graphs."[5] The normalization condition

$$\Gamma^{(2)}_{A^2}\big|_{p=0,m=\mu} = 2 \tag{19}$$

yields

$$\frac{\delta-1}{2}\mu^2\, 2\, \frac{\partial \Gamma^{(2)}_{A^2}}{\partial m^2}\bigg|_{p=0,m=\mu} - 2\gamma_A + \gamma_{A^2} = 0. \tag{20}$$

On the other hand, from (14) one has

$$\frac{\partial \Gamma^{(2)}}{\partial m^2}\bigg|_{p=0} = -\frac{i}{2}\Gamma^{(2)}_{A^2}\big|_{p=0}, \tag{21}$$

the normalization condition (5c) reads

$$2(\delta-1)\mu^2 \frac{\partial}{\partial m^2}\frac{\partial \Gamma^{(2)}}{\partial m^2}\bigg|_{p=0,m=\mu} + (\delta - 2\gamma_A)(-i) = 0, \tag{22}$$

and hence together with (21) and (20) gives

$$2\delta = \gamma_{A^2}. \tag{23}$$

Anybody who is familiar with the theory of critical phenomena will now realize that the homogeneous parametric differential equation (18) at a zero of β is nothing but the infinitesimal version of the Kadanoff[4] scaling law ($m^2 = t$) at zero magnetic field

$$\Gamma^{(N)}(p_1\cdots p_N; t, \mu) = t^{-(D-Nd_A)/(\delta_0-1)} F(p_i t^{1/2(\delta_0-1)}; \mu), \tag{24}$$

$$D = \text{dimensional of model},\quad d_A = \text{dimension of the field} = \tfrac{1}{2}(D-2) + \gamma_A(\lambda_0),\quad \delta_0 = \delta(\lambda_0). \tag{25}$$

The integration of (18) for $\beta \neq 0$ leads to a generalized scaling law which in the case of the existence of a long-distance zero λ_0 of β with $\beta'(\lambda_0) > 0$ corrects the Kadanoff scaling law. The generalizations to correlation functions with a higher number of energy fluctuations and other composite fields are straightforward and entirely analogous to the derivation in the case of the Callan-Symanzik equation.[19,20]

The inclusion of broken symmetries also does not present any difficulties. Since the renormalization theory of broken discrete symmetries as the linear breaking of the A^4 model is a bit tricky,[21] we will only comment on a continuous broken symmetry, say in a two-component model, when Ward-Takahashi identities simplify the renormalization procedure. It has been demonstrated elsewhere[22] that the loopwise resummation procedure of Lee[23] can be performed in the Lagrangian by using "soft quantization" around the pion mass. Suitable normalization conditions consistent with this quantization lead to three parametric differential equations, an inhomogeneous "Goldstone-limit" equation involving only the mass insertion operator, a Callan-Symanzik equation having the bilinear mass insertion and in addition a trilinear insertion (which also can be neglected at high spacelike momenta), and a homogeneous Gell-Mann–Low type renormalization-group equation. However, by changing the quantization in such a way that the symmetric mass is also quantized softly, i.e., by using Taylor operators which act on the symmetric mass in the same way as (3) and (4), we obtain a subtraction scheme which is compatible with the normalization conditions [t= (symmetric mass)^2]:

$$\left.\frac{\partial\Gamma_\pi^{(2)}}{\partial t}\right|_{p=0,t=\mu^2,F=0}=-i\,,\quad \left.\frac{\partial\Gamma_\pi^{(2)}}{\partial p^2}\right|_{p=0,t=\mu^2,F=0}=i\,,$$

$$\Gamma_\pi^{(4)}|_{p=0,t=\mu^2,F=0}=-i\lambda,\quad \Gamma_\pi^{(2)}|_{p=0,t=0,F=0}=0,$$

$$F=\langle A\rangle=\text{magnetization}.$$

We obtain two inhomogeneous differential equations expression $\partial\Gamma^{(N)}/\partial t$ and $\partial\Gamma^{(N)}/\partial F$ in terms of bilinear mass insertion as well as the homogeneous equation

$$\left[2\mu^2\frac{\partial}{\partial\mu^2}+2\delta(\lambda)t\frac{\partial}{\partial t}+\beta(\lambda)\frac{\partial}{\partial\lambda}-\gamma_A\left(N+F\frac{\partial}{\partial F}\right)\right]\Gamma^{(N)}=0\,. \quad (26)$$

The integration of this equation with the methods of characteristics leads for $\beta(\lambda_0)=0$ and $\beta'(\lambda_0)>0$ to the Kadanoff scaling law with the built-in corrections.[5] Starting from such homogeneous equations in a situation with several coupling terms, Di Castro, Jona-Lasinio, and Peliti[8] showed that all the critical-phenomena problems which had been discussed previously in the Kadanoff-Wilson-Wegner framework (including tricritical behavior and crossover indices) may also be very elegantly described in standard local quantum field theory language.

The normalization condition and the related Taylor subtraction scheme (3), (4), (5) on which we have based our consideration lead to the infrared divergencies for supernormalizable couplings. Thus our model in $D=4-\epsilon$ dimension develops the well-known poles at rational ϵ (Ref. 24) due to the normalization (5d). This shortcoming can be repaired by replacing (5d) by

$$\Gamma^{(2)}|_{p^2=0,m^2=\mu^2}=i\mu^2\,. \quad (27)$$

The corresponding Taylor operators are slightly modified. Instead of (4) we have

$$\begin{aligned}\tau^{(2)}F(\mathfrak{p},m)=F(0,\mu)&+\sum_{i=1}^{N}p_i^\mu\left(\frac{\partial F}{\partial p_i^\mu}\right)_{\mathfrak{p}=0,m=\mu}\\&+\tfrac{1}{2}\sum_{i<k}p_i^\mu p_k^\nu\left(\frac{\partial^2F}{\partial p_i^\mu\,\partial p_k^\nu}\right)_{\mathfrak{p}=0,m=\mu}\\&+(m^2-\mu^2)\left(\frac{\partial F}{\partial m^2}\right)_{\mathfrak{p}=0,m=\mu}\,.\end{aligned} \quad (28)$$

The Lagrangian in normal-product notation now has the form

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}N_4[\partial_\mu A\partial^\mu A]-\tfrac{1}{2}(m^2-\mu^2)N_2[A^2]\\&-\tfrac{1}{2}\mu^2N_4[A^2]-(\lambda/4!)N_4[A^4]\,.\end{aligned} \quad (29)$$

Note that part of the mass term is quantized soft, i.e., with N_2.

The inhomogeneous equation (9a) as well as the relations (9b) follows as before. The mass term in the counting identity (13) consists now of two parts,

$$N\Gamma^{(N)}=[-4\lambda\Delta_3+2\Delta_2-2(m^2-\mu^2)\Delta_0-2\Delta_1']\Gamma^{(N)}$$

with

$$\Delta_1'=\frac{i}{2}\int N_4[\mu^2A^2]\,d^Dx\,. \quad (30)$$

Finally, $2\mu^2\partial/\partial\mu^2$ is (as can be checked directly by use of the forest formula) a linear combination of the linearly independent operators Δ_1', Δ_2, Δ_3, and $(m^2-\mu^2)\Delta_0=\Delta_4$:

$$2\mu^2\frac{\partial\Gamma^{(N)}}{\partial\mu^2}=(\alpha_1\Delta_1'+\alpha_2\Delta_2+\alpha_3\Delta_3+\alpha_4\Delta_4)\Gamma^{(N)}. \quad (31)$$

The Zimmermann identity reads

$$\mu^2\Delta_0\Gamma^{(N)}=\left(\Delta_1'+\sum_{i=2}^{4}\lambda_i\Delta_i\right)\Gamma^{(N)}\,. \quad (32)$$

The α's and λ's can be computed from the normalization conditions (27) and (5b), (5c), (5d). They are numbers which just depend on g, in particular because with condition (27) $\alpha_1=0$. Hence again using (9a) we see that $\partial/\partial g$, $N-2(m^2-\mu^2)\partial/\partial m^2$, $2\mu^2\partial/\partial\mu^2+\alpha_4(m^2-\mu^2)\partial/\partial m^2$, and $-\mu^2\partial/\partial m^2 + \lambda_4(m^2-\mu^2)\partial/\partial m^2$ are linear combinations of Δ_1', Δ_2, and Δ_3. The linear relation must be of the form

$$\left[2\mu^2\frac{\partial}{\partial\mu^2}+\left(\delta_1+\frac{\mu^2}{m^2}\delta_2\right)2m^2\frac{\partial}{\partial m^2}+\beta\frac{\partial}{\partial g}-N\gamma_A\right]\Gamma^{(N)}=0,$$

where δ_1, δ_2, β, and γ_A are only functions of g. Again one shows from the normalization conditions that

$$\delta_1+\delta_2=\gamma_A$$

and

$$2\delta_1=\gamma_{A^2}.$$

By using the methods of characteristics one obtains a global scaling law of the form

$$\Gamma^{(N)}(p_1\cdots p_N;m,\mu,g)=\kappa^{D-N(D-2)/2}a^{-N}\Gamma^{(N)}(p_1/\kappa\cdots p_N/\kappa;\bar m,\mu,\bar g),$$

with $\bar g$ defined by

$$\ln\kappa=\int_g^{\bar g}\frac{1}{\beta(g')}dg',$$

$$a(g,\kappa)=\exp\left(\int_g^{\bar g}\frac{\gamma}{\beta}dg'\right),$$

and

$$\frac{d\bar m^2}{d\kappa}=2[\delta_1(\bar g)-1]\bar m^2+2\mu^2\delta_2(\bar g),$$

i.e.,

$$\bar m^2=2\mu^2\int_g^{\bar g}\frac{\delta_2}{\beta}\exp\left(2\int_{g'}^{\bar g}\frac{\delta_1-1}{\beta}dg''\right)dg'+m^2\exp\left(2\int_g^{\bar g}\frac{\delta_2-1}{\beta}dg'\right).$$

For $\kappa\to 0$ the assumption of the existence of a long-distance eigenvalue λ_0 still leads to the scaling law (24). The reason is that asymptotically the $\bar m^2$ still behaves as

$$\bar m^2\to\kappa^{-2(1-\delta_{10})}m^2,$$

where δ_{10} is the value of δ_1 at λ_0. The new normalization leads to a more complicated "effective scaling mass," but asymptotically anything looks as it did in the old framework.

The only additional problem is to show that $m\to 0$ really means zero mass, i.e.,

$$\Gamma^{(2)}|_{p=0}\underset{m\to 0}{\longrightarrow}0.$$

For this we have to use the existence of a zero λ_0 of β with $\beta'(\lambda_0)>0$.[24] Fortunately the existence of such a zero can be argued on much more solid grounds than in the case of a nontrivial short-distance (Gell-Mann–Low) zero. Namely, in two dimensions we know that the soluble Lenz-Ising model leads to critical powers for correlation functions at large distances. On the other hand, according to Wilson[10] the Lenz-Ising model can be approximated to arbitrary accuracy by 4th-degree polynomials. The evidence for scale-invariant power behavior at criticality for three-dimensional systems comes from high-temperature expansions as well as from Wilson's "approximate renormalization group" discussion.[10]

For a detailed treatment of Kadanoff scaling laws in D-dimensional A^4 theories based on our new normalization conditions (in particular for the proof of existence of the $m\to 0$ correlation function for nonexceptional momenta) we refer to a forthcoming publication.[31]

We finally would like to mention another interesting application of homogeneous parametric differential equations involving the "temperature."

Consider mass perturbations in the Thirring model:

$$\mathcal{L}=\mathcal{L}_{\rm Th}+tN_1[\bar\Phi\Phi],\tag{33}$$

where Φ is the two-component Thirring field. In this case the "temperature" normalization conditions (5) together with soft quantization via "Taylor" operators (3), (4) lead again to (18) with

$$\Gamma^{(2N)}=\langle T\Phi(x_1)\cdots\Phi(x_N)\bar\Phi(y_1)\cdots\bar\Phi(y_N)\rangle.$$

But an adaptation of an argument[25] to the case of soft quantization leads immediately to $\beta(\lambda)=0$. By a simple reparametrization of the coupling constant in the massive theory, one can arrange things in such a way that the anomalous dimensions of Φ and $N_1[\bar\Phi\Phi]$ are identical to those in the massless Thirring model,[26] namely

$$\gamma_\Phi=\frac{\lambda^2}{4\pi^2}$$

and

$$\gamma_{\bar\Phi\Phi}=\frac{b(\lambda)}{\pi}=\frac{1}{\pi}\left\{\left[\frac{\lambda}{2\pi}+\left(\frac{\lambda^2}{4\pi}+\pi\right)^{1/2}\right]^2-\pi\right\}.\tag{34}$$

Note that $\gamma_{\bar\Phi\Phi}$ runs through the range of all values allowed by general principles of positive-definite-metric quantum field theory: For $-\infty<\lambda<\infty$,

$$0<\frac{b(\lambda)}{\pi}+1=\dim\bar\Phi\Phi<\infty.$$

In order to construct the (nontrivial) massive theory from the massless one, one may think of two different methods:

(a) Use the standard Gell-Mann–Low perturbation theory for time-ordered functions (1) where instead of free-field products the X_0 is replaced by products of operators in the Thirring model. In such an approach the perturbation by $\int N[\bar\Phi\Phi]\,d^2x$

would become infinitely strong either at long distances if $\dim\bar{\Phi}\Phi < 2$ or at short distances for $\dim\bar{\Phi}\Phi > 2$.

In the first case one has to add renormalization counterterms of dimensionality smaller than 2, whereas the second possibility leads to a nonrenormalizable situation with increasing perturbation order. It is obvious that for the first case the counterterm is again of the $\bar{\Phi}\Phi$ form, since the mass operator is the only symmetry-preserving operator of dimension smaller than 2. In the nonrenormalizable case $\dim\bar{\Phi}\Phi > 2$, it seems that the scaling equation (18) restricts the structure of possible counterterms. In fact this "nonrenormalizable" interaction may be the first example of a case where the usual infinity ambiguity of counterterms is eliminated by the requirement that scaling equations holds in every order of the perturbation parameter t. These remarks are at the moment somewhat speculative because we have not carried out any detailed investigation of this perturbation theory.

(b) Using techniques which were recently developed by Symanzik,[27] one may construct asymptotic expansions for small t. The use of differential equations (18) instead of the Callan-Symanzik equation turns out to be somewhat more convenient. In the case of the Thirring model this asymptotic expansion is an expansion of $F^{(N)}$ [see Eq. (24)] for $\delta < 1$ into fractional powers of $\hat{t} = t^{1-\delta}$. The coefficient functions of this expansion are functions of the momenta (of the coordinates, since these computations for the Thirring model are somewhat simpler in x space) which can be computed solely within the massless Thirring model with the help of Wilson's operator-product expansion. A detailed discussion of the application of Symanzik's methods to the massive Thirring model will be given elsewhere. The connection of this approach with the conventional perturbation theory discussed previously is at the moment not completely clear. In our opinion investigations on the massive Thirring model as we proposed will be important for the further development of "constructive quantum field theory," which up to now has been mainly concerned with a particular class of superrenormalizable theories.[28]

Finally we want to point out that the Thirring model provides a nice illustration for the concepts of "thermodynamic relevance" introduced by Kadanoff, Wilson, and Wegner. We remind the reader that this model has two dimensionless parameters: the anomalous dimension of the field γ_Φ and the "continuous spin" s. The appearance of this s is related to the fact that in two dimensions the usual concept of spin loses its meaning. There are two "relevant" fields of dimension smaller than 2 (in a certain range of the coupling constant λ), the symmetry- (phase symmetry) conserving term $N[\bar{\Phi}\Phi]$ and the symmetry-breaking term $N[\Phi\gamma_0\Phi] + \mathrm{H.c.}$, where $\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If we put $s = 0$ we also may introduce the linear symmetry-breaking term $\Phi + \Phi^\dagger$. Because of the lack of spontaneous symmetry breaking in two dimensions this last interaction cannot lead to first-order phase transitions; however, it nevertheless plays an important role as a perturbation of the scale-invariant theory. For

$$\mathcal{L} = \mathcal{L}_{\mathrm{Th}} + tN[\bar{\Phi}\Phi] + sN[\Phi\gamma_0\Phi + \mathrm{H.c.}] + h(\Phi + \Phi^\dagger) \quad (35)$$

we obtain with the normalization conditions (F = Legendre conjugate variable to h)

$$\left.\frac{\partial\Gamma^{(2)}}{\partial t}\right|_{\substack{p=0,s=0\\ t=\mu^2,F=0}}, \quad \left.\frac{\partial\Gamma^{(2)}}{\partial p^2}\right|_{\substack{p=0,s=0\\ t=\mu^2,F=0}}, \quad \left.\Gamma^{(4)}\right|_{\substack{p=0,s=0\\ t=\mu^2,F=0}}, \quad (36)$$

which are equal to their zero-order values, and with the help of soft quantization the homogeneous equation

$$\left[\mu^2\frac{\partial}{\partial\mu^2} + \delta_t(\lambda)t\frac{\partial}{\partial t} + \delta_s(\lambda)s\frac{\partial}{\partial s} - \left(N + F\frac{\partial}{\partial F}\right)\gamma_\Phi(\lambda)\right]\Gamma^{(N)} = 0 \quad (37)$$

and three inhomogeneous equations which we will not write down we obtain a Kadanoff scaling law for three "relevant" variables.

The operator $j_\mu j^\mu$ with $j_\mu = N[\bar{\Phi}\gamma_\mu\Phi]$ is marginal, i.e., has dimension 2. If we introduce it as an additional perturbation on $\mathcal{L}_{\mathrm{Th}}$, it *remains marginal* because of the asymptotic conservation laws of j_μ and $j_{\mu 5}$. Conservation laws of this type, which *maintain the scale invariance of the marginal perturbation* $j_\mu j^\mu$ under its own action, are in our view the necessary prerequisites for obtaining critical indices (anomalous dimension) which depend continuously on a dimensionless coupling strength.[29] From this viewpoint one should expect a deep connection between the continuous version of the lattice Baxter model[30] and the Thirring model.

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[1]S. Weinberg, Phys. Rev. D 8, 3497 (1973).

[2]C. G. Callan, Jr., Phys. Rev. D 2, 1541 (1970).

[3]K. Symanzik, Commun. Math. Phys. 16, 48 (1970).

[4]L. P. Kadanoff *et al.*, Rev. Mod. Phys. 39, 395 (1967).

[5]F. Jegerlehner and B. Schroer, Acta Phys. Austriaca Suppl. XI, 389 (1973).

[6]E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D 8, 434 (1973); 8, 2418 (1973).

[7]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

[8]C. Di Castro, G. Jona-Lasinio, and L. Peliti, University of Rome report, 1973 (unpublished).

[9]L. P. Kadanoff, in *Proceedings of the International School of Physics "Enrico Fermi" Course 51*, edited by M. S. Green (Academic, New York, 1971).

[10]K. G. Wilson, Phys. Rev. B 4, 3174 (1971); 4, 3184 (1971).

[11]F. Wegner and E. K. Riedel, Phys. Rev. B 7, 248 (1973).

[12]W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory*, 1970 Brandeis Summer Institute in Theoretical Physics, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1970), Vol. 1, p. 397.

[13]The normal-product approach may seem more complicated than the better-known multiplicative renormalization theory using cutoffs or regulators. However, if one wants to make the arguments of the latter approach rigorous and convincing, one has to go through rather complicated iterative arguments based on Schwinger-Dyson equations and Bethe-Salpeter equations. The formal elegance of the multiplicative renormalization theory is lost in the mathematical proofs. For a comparison of the two methods see Ref. 15.

[14]M. Gomes, J. H. Lowenstein, and W. Zimmermann, NYU report, 1974 (unpublished).

[15]B. Schroer, in proceedings of the 1973 Cargèse Summer Institute (to be published). In these lectures notes the methods proposed in Ref. 14 were applied to the σ model.

[16]J. H. Lowenstein, Commun. Math. Phys. 24, 1 (1971).

[17]J. H. Lowenstein, Phys. Rev. D 4, 2281 (1971).

[18]B. Schroer, Phys. Rev. B 8, 4200 (1973).

[19]K. Symanzik, Commun. Math. Phys. 23, 49 (1971).

[20]K. Mitter, Phys. Rev. D 8, 2927 (1973).

[21]K. Symanzik, Commun. Math. Phys. 16, 48 (1970). See also last section of Ref. 15.

[22]F. Jegerlehner and B. Schroer, Nucl. Phys. B68, 461 (1974).

[23]B. W. Lee, Nucl. Phys. B9, 649 (1969).

[24]Compare G. Parisi and K. Symanzik, in proceedings of the 1973 Cargèse Summer Institute (to be published).

[25]M. Gomes and J. H. Lowenstein, Nucl. Phys. B45, 252 (1972).

[26]We use normalizations identical to those of J. H. Lowenstein, Commun. Math. Phys. 16, 265 (1970).

[27]K. Symanzik, Commun. Math. Phys. 23, 49 (1971); Capri lectures, 1973 (to be published).

[28]J. Glimm and A. Jaffe, in *Mathematics of Contemporary Physics*, edited by R. F. Streater (Academic, New York, 1972).

[29]This view is similar, but not identical, to that of L. P. Kadanoff and F. Wegner, Phys. Rev. B 4, 3989 (1971).

[30]R. J. Baxter, Phys. Rev. Lett. 26, 832 (1971).

[31]B. Schroer, Rev. Bras. Fiz. (to be published).