Operator formulation of the multi–Regge-pole hypothesis

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The conventional multi-Regge-pole behaviors of all many-body scattering amplitudes are shown to be consequences of two assumptions: (i) the validity of large-distance operator-product expansions (LDOPE's) in a strong sense for all local field products, and (ii) the validity of the conventional single-Regge-pole behavior for all 2-to-2 amplitudes. The LDOPE's (as weak limits) were previously deduced from the assumption of single-Regge-pole dominance of many-body amplitudes in the appropriate high-energy limits, but are here abstracted (as strong limits) to form the basis of an operator formulation of the multi-Regge hypothesis. This hypothesis was previously based on O(3) or O(2,1) expansions and strong but unproven analyticity and boundedness assumptions. The proposed operator framework avoids such assumptions and proceeds by repeatedly using the LDOPE's to reduce a multiparticle amplitude to a sum over 2-to-2 amplitudes. In this way, the correct multi-Regge behavior with 2-body Regge trajectories is obtained. Furthermore, the deduced Reggeon-particle vertices are expressed in terms of particle matrix elements of operator structures and simpler vertices. This leads to consistency conditions and restrictions on the vertices, and further restrictions can be obtained by performing the above reductions in different ways.

I. INTRODUCTION

The simplicity and generality of the Regge-pole description of the high-energy behavior of the 2-to-2 scattering amplitude in potential theory led some years ago to the introduction of the Regge-pole hypothesis in elementary-particle physics.[1] For the 2-to-2 process, there are at present two precise formulations of this hypothesis, one based on the O(3) (partial wave) expansion and the other based on the O(2, 1) (Toller[2]) expansion of the scattering amplitude. In this case, the necessary analyticity and boundedness properties have been proven (at least in perturbation theory) and precise definitions of the analytically continued partial-wave amplitudes have been given.[1] If the leading singularity in the complex J plane is a pole,[3] then the high-energy behavior will have the factorized Regge form. Specifically, if $T^{AB}(s, t)$ is the amplitude for the reaction $a(k)+b(p)\to a'(k')+b'(p')$ [A stands for the particles a and a', B stands for the particles b and b', $s=(k+p)^2$, and $t=(p'-p)^2\equiv\Delta^2$], one has

$$T^{AB}(s, t)\sim\beta^A(t)\,\beta^B(t)\,s^{\alpha(t)} \tag{1.1}$$

in the Regge limit $s\to\infty$ with t fixed. This is illustrated in Fig. 1. One obtains in this way an elegant and experimentally verified relation between s-channel high-energy behavior and t-channel particle poles [at $\alpha(t)=$ integer].[4]

The real power and usefulness of the Regge-pole concept lies, however, in the realm of multiparticle processes.[5] One wants to obtain two generalizations of the 2-to-2 Regge behavior. First, one assumes that multiparticle reactions $a+b\to a'+b'$, where now a, a', b, b' represent clusters of particles, have the behavior given by the exchange of a Regge pole in the limit where the clusters a and b have a large relative energy with cluster subenergies and a-a' cluster momentum transfers fixed. This is illustrated in Fig. 2. The assumed factorization of this pole leads to the existence of Reggeon-particle amplitudes. The second generalization is that such Reggeon-particle amplitudes themselves have Regge asymptotic behavior in the high-energy limits of the previous type with the Reggeon treated as an off-mass-and-spin-shell external particle. This is illustrated in Fig. 3. Factorization and continuation of this procedure leads to multi-Reggeon-particle amplitudes.

There have been a number of serious problems encountered in such attempts to generalize the original Regge-pole hypothesis:

(i) It is not *a priori* clear what the appropriate variables are.

(ii) It has not been possible to prove or even precisely formulate either an appropriate energy-plane analyticity structure or an appropriate analytically continued (in the complex J plane) multiple partial-wave amplitude.

(iii) No justification has been given for the assumption that only the two-body Regge trajectories are encountered in the multiparticle amplitudes.

The first problem has been solved by the use of group-theoretic methods, both in the O(2, 1) (Refs. 2, 6, and 7) and O(3) (Refs. 8 and 9) formalisms. The second problem is avoided in the O(2, 1) framework but must be replaced there by presumably equally strong assumptions of analyticity and boundedness. Furthermore, it has not been

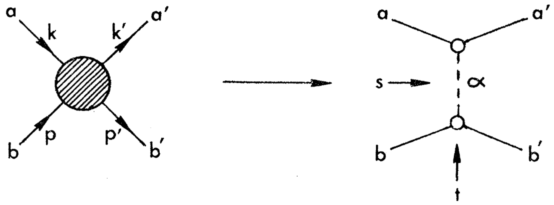

FIG. 1. The 2-to-2 scattering amplitude and its behavior in the Regge limit. $\alpha(t)$ is the leading Regge trajectory that couples to $A=(a,a')$ and $B=(b,b')$.

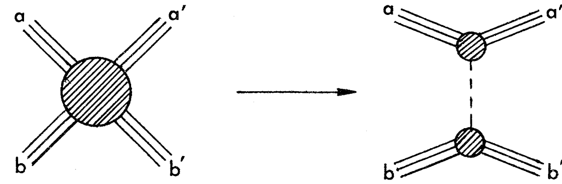

FIG. 2. A general scattering amplitude and its behavior in a Regge limit. a, a', b, and b' represent clusters of particles, and the high-energy limit is that in which all the subenergies $k\cdot p$ (with k in the a cluster and p in the b cluster) become large and all the masses and momentum transfers $k\cdot k'$ (with k in the a cluster and k' in the a' cluster, etc.) are held fixed. α_1 is the dominant Regge trajectory.

possible to formulate unitarity and discontinuity relations in this formalism. t-channel unitarity implies that the two-body Regge trajectories occur in the multiparticle amplitudes, but not that these are the only, or the leading, singularities, and so neither approach really resolves problem (iii).

In this paper we shall present an alternative formulation of the multi-Regge hypothesis which avoids all of the above problems and has a number of other virtues. It is an operator formulation in which the desired multi-Regge behaviors of all multiparticle amplitudes follow simply and precisely from a single strong operator-product expansion (OPE) plus the usual 2-to-2 Regge behavior. No energy-plane or J-plane analyticity assumptions are necessary and only the two-body Regge trajectories are encountered. So all of the usual assumptions about all of the N-to-N' amplitudes for $N+N'=5,6,\ldots$ are here replaced by a single strong operator statement.[10]

This operator statement was previously deduced[11,12] as a weak[13] OPE from the assumption that the amplitudes for all of the reactions $a+b\to a'+b'$, for a and a' single particles and b and b' arbitrary clusters of particles, have the usual single-Regge-pole behavior in the high-energy limit illustrated in Fig. 2.[14] In the present work, we abstract this operator statement, but as a strong[15] OPE, and take it, along with 2-to-2 Regge behavior, as our basic hypothesis. Put differently, we assume the 2-to-2 Regge behavior and the consequent validity of the OPE between single-particle states, and we assume the same OPE is valid as a strong limit between multiparticle states.

The OPE reduces an N-point function to an infinite sum over $(N-1)$-point functions. This procedure can be repeated until a 4-point function is reached. The assumed 4-point Regge behavior, together with the consistency requirements arising from the fact that the reductions can be accomplished in different ways, then leads to the usual multi-Regge behavior with two-body Regge trajectories. This is all accomplished without ever invoking analyticity or boundedness assumptions.

Other advantages of this approach are its simplicity (no group theory is needed, and invariant, as opposed to group-theoretic, variables are used throughout), its economy (the unphysical complex J plane is avoided), and its explicitness (the high-energy behavior is directly seen to be built out of summing the exchange of higher spin states).

In spite of these virtues, it would be difficult to pretend that another formulation of the multi-Regge hypothesis constitutes real progress. Our final result is, after all, just the usual multi-Regge behavior. We believe, however, that our formulation may lead to further results which do not follow from the previous frameworks. The expressions we obtain for the various Reggeon-particle couplings involve lower-point couplings and this leads to possibly interesting consistency conditions among the couplings and restrictions on the couplings. Other such restrictions are obtainable by exploiting the freedom of performing the particle reduction in different ways. (We plan to study these relations in a future paper.) Also, a new formulation might lead to new ideas and extensions of the theory and might suggest conceptual generalizations. At the very least it should improve our understanding of the theory and its limitations.

It remains to be seen whether the Reggeon unitarity and discontinuity relations, which have been much discussed of late,[9,16] are more simply studied

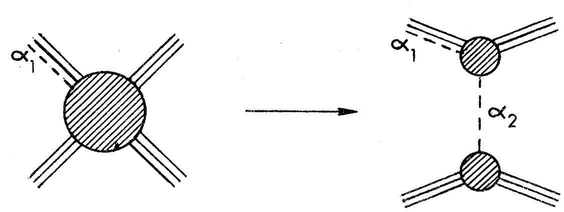

FIG. 3. A Reggeon (α_1)-multiparticle vertex and its assumed behavior in the high-energy limit of the type defined in Fig. 2. α_2 is the dominant Regge trajectory.

in this framework. In any case, we are not proposing a replacement for the other approaches. The various formalisms can be considered as complimentary, each having its own advantages. One advantage of our approach is that it enables all of the consequences of multi-Regge behavior to be studied and summarized in terms of the consequences of a single operator statement. This should prove to be highly economical from a logical point of view.

In this paper, we shall illustrate our procedures in detail for the scalar five-point functions (2-to-3 amplitude). The generalizations are immediate and indicated at the end of the paper. Generalizations to include certain classes of Regge cuts can be carried out as in RI.

In Sec. II we state our notation, summarize the conventional single- and double-Regge limits of the 2-to-3 amplitude, and note how these limits are related. We also recall the effect of spin for the 2-to-2 Reggeization. The large-distance operator-product expansion (LDOPE) and some of its properties are given in Sec. III. In Sec. IV we deduce the double-Regge behavior of the 2-to-3 amplitude from our stated assumptions. In Sec. V we further assume the existence of an analytic complex n-plane structure and deduce that the leading singularity of the 2-body Regge residue is a pole with a factorized residue. We also note connections with results in RI and RII. The final Sec. VI contains a general discussion which summarizes our results, indicates some generalizations, and suggests further related work.

II. THE REGGE LIMITS

We consider the 2-to-3 reaction

$$a(p_1)+b(p_2)\to a'(p_1')+b'(p_2')+c(k) \tag{2.1}$$

among five scalar particles or currents[17] labeled as indicated. We divide the five particles into three distinct sets as follows:

$$A=(a,a'),\quad B=(b,b'),\quad C=c\,. \tag{2.2}$$

The variables we will use are indicated in the tree diagram of Fig. 4. These are the "energy" variables

$$\nu_1=p_1'\cdot k,\quad \nu_2=p_2'\cdot k,\quad \nu=p_1\cdot p_2\,, \tag{2.3}$$

the "momentum transfer" variables

$$t_1=\Delta_1^{\,2},\quad t_2=\Delta_2^{\,2}\,, \tag{2.4}$$

defined from the four-momentum transfers

$$\begin{aligned}\Delta_1&=p_1'-p_1=p_2-p_2'-k\,,\\ \Delta_2&=p_2'-p_2=-k-p_1'+p_1\,,\end{aligned} \tag{2.5}$$

and the "mass" variables

$$\begin{aligned}\kappa_1&=p_1^{\,2},\quad \delta_1=p_1\cdot\Delta_1,\\ \kappa_2&=p_2^{\,2},\quad \delta_2=p_2\cdot\Delta_2\,.\end{aligned} \tag{2.6}$$

It is convenient to use the Toller-type variable

$$\eta=\nu_1\nu_2/\nu \tag{2.7}$$

in place of ν and to represent the variables associated purely with the particle sets A and B by

$$K^A=(\kappa_1,\delta_1,t_1),\quad K^B=(\kappa_2,\delta_2,t_2)\,, \tag{2.8}$$

respectively. The amplitude for (2.1) will thus be written

$$T^{ABC}(\nu_1,\nu_2,\eta;K^A;K^B)\,. \tag{2.9}$$

Particle c will always be kept on-shell and so its mass k^2 need not be indicated.

There are three well-known Regge limits corresponding to the tree diagram of Fig. 4.[5-9] These are the "target fragmentation" limit

R_1: $\nu_1\to\infty$ with all else in (2.9) fixed,

the "particle fragmentation" limit

R_2: $\nu_2\to\infty$ with all else in (2.9) fixed,

and the "pionization" (or double-Regge) limit

R_{12}: $\nu_1,\nu_2\to\infty$ with all else in (2.9) fixed.

The Regge-pole behaviors of (2.9) in these limits are the following:

$$T^{ABC}\overset{R_1}{\sim}\beta^{A1}(K^A)(2\nu_1)^{\alpha_1(t_1)}\beta^{BC1}(\nu_2,\eta,t_1;K^B)\,, \tag{2.10}$$

$$T^{ABC}\overset{R_2}{\sim}\beta^{AC2}(\nu_1,\eta,t_2;K^A)(2\nu_2)^{\alpha_2(t_2)}\beta^{B2}(K^B)\,, \tag{2.11}$$

$$\begin{aligned}T^{ABC}\overset{R_{12}}{\sim}\,&\beta^{A1}(K^A)(2\nu_1)^{\alpha_1(t_1)}\beta^{C12}(t_1,\eta,t_2)\\ &\times(2\nu_2)^{\alpha_2(t_2)}\beta^{B2}(K^B).\end{aligned} \tag{2.12}$$

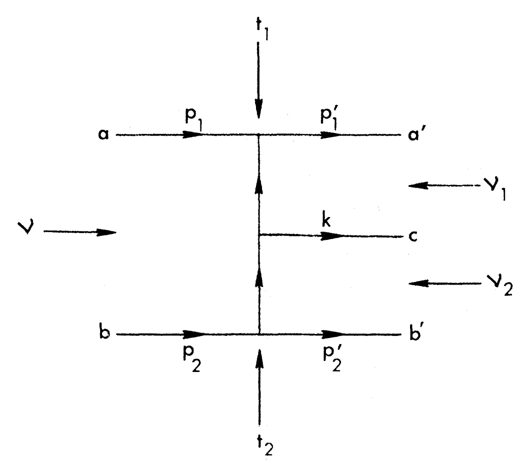

FIG. 4. A tree diagram illustrating the kinematics of the 2-to-3 scattering amplitude.

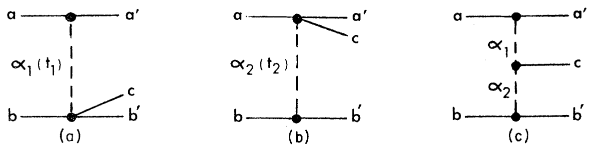

FIG. 5. The three Regge limits of the 2-to-3 amplitudes of Fig. 4. (a) and (b) illustrate the single-Regge (fragmentation) limits and (c) illustrates the double-Regge (pionization) limit.

Here $\alpha_1(t_1)$ is the leading trajectory which couples to particles A, and $\alpha_2(t_2)$ is the leading trajectory which couples to particles B. Superscripts 1 and 2 will always refer to these respective trajectories. Thus β^{A1} and β^{B2} are the 2-body Regge residues, β^{BC1} and β^{AC2} are the 3-body Regge residues, and β^{C12} is the (Reggeon α_1)-(particle c)-(Reggeon α_2) vertex. The three Regge limits are pictured in Fig. 5.

We would like to note here that the double-Regge behavior (2.12) follows from the single-Regge behaviors (2.10) and (2.11) if a certain amount of smoothness is granted. Namely, suppose that the R_{12} limit can be obtained by first taking the R_1 limit and then letting $\nu_2 \to \infty$, or by first taking the R_2 limit and then letting $\nu_1 \to \infty$. In symbols,

$$\lim_{R_{12}} = \lim_{\nu_2\to\infty} \lim_{R_1} = \lim_{\nu_1\to\infty} \lim_{R_2} . \tag{2.13}$$

The requirement that the large-ν_2 limit of (2.10) agrees with the large-ν_1 limit of (2.11) implies that these limits have the factorized forms

$$\beta^{BC1} \underset{\nu_2\to\infty}{\sim} \gamma^{C12}(t_1, \eta, t_2)(2\nu_2)^{\alpha_2(t_2)}\beta^{B2}(K^B) \tag{2.14}$$

and

$$\beta^{AC2} \underset{\nu_1\to\infty}{\sim} \beta^{A1}(K^A)(2\nu_1)^{\alpha_1(t_1)}\gamma^{C12}(t_1, \eta, t_2) \tag{2.15}$$

for some function γ of the indicated labels and arguments. Thus the second equality in (2.13) implies that the equated limits each have the form (2.12) of the R_{12} limit. The first equality implies the identification of γ^{C12} with β^{C12}. This argument and its generalizations shows that the first generalizations, shown in Fig. 2, of 2-to-2 Regge behavior together with smoothness (commutativity) relations of the type (2.13), are sufficient to imply the second generalization shown in Fig. 3, namely, that Reggeon-particle amplitudes themselves have Regge behavior, as we have seen in (2.14) and (2.15). In other words, the multi-Regge-pole hypothesis is implied by the single-Regge-pole hypothesis for multiparticle states and enough smoothness. We will use similar arguments in a different framework in Sec. IV.

The crucial commutativity relation (2.13) and its generalizations can themselves be deduced from Sommerfeld-Watson integral representations.[18,19] Our approach in the following will, however, be an opposite one. We will deduce (2.12) from (1.1) and a strong OPE which embodies (2.13).

The Regge behaviors (1.1), (2.10)–(2.12), etc. can be unambiguously extended to the case when the particles have nonzero spins.[1] For our purposes, we need only consider the case of the 2-to-2 reaction $a+b \to a'+b'$ illustrated in Fig. 1 when $a'(k')$ is a maximum spin-n object[20] $a'_{\alpha_1 \cdots \alpha_n}(k')$ and the other particles are spinless. The amplitude $T^{AB}_{\alpha_1 \cdots \alpha_n}$ for the process has the general decomposition

$$T^{AB}_{\alpha_1 \cdots \alpha_n} = \sum_{\substack{ikl\\ \text{perm}}} G^{AB}_{nikl}(s,t)\, k'_{\alpha_1} \cdots k'_{\alpha_i} \Delta_{\alpha_{i+1}} \cdots \Delta_{\alpha_{i+k}} p_{\alpha_{i+k+1}} \cdots p_{\alpha_{i+k+l}} g_{\alpha_{i+k+l+1}\alpha_{i+k+l+2}} \cdots g_{\alpha_{n-1}\alpha_n} , \tag{2.16}$$

the permutations being taken over the various ways of choosing i, k, and l from $n \geq i+k+l$. The behavior of the scalar amplitudes G^{AB}_{nikl} in the Regge limit is

$$G^{AB}_{nikl}(s,t) \underset{s\to\infty}{\sim} \beta^{A}_{nikl}(t)\,\beta^{B}(t)\, s^{\alpha(t)-l} , \tag{2.17}$$

which should be compared with (1.1). Here l is the helicity flip involved in the (ikl) term in (2.16), which in this case is just the number of p's which occur.

III. THE LARGE-DISTANCE OPERATOR-PRODUCT EXPANSION

Consider the amplitude T for the process shown in Fig. 2 with $a(p_1)$ and $a'(p_1+\Delta_1)$ single particles and b and b' arbitrary clusters of particles. If $j^a(x)$ and $j^{a'}(y)$ are sources for particles a and a' (or, more generally, aribitrary local fields[17]), then T can be written as

$$T = \int d^4x\, e^{ip_1\cdot x} \langle b| T[j^a(x) j^{a'}(0)] |b'\rangle . \tag{3.1}$$

Assuming that all such amplitudes have the Regge-pole-dominated high-energy behavior indicated in Fig. 2, one arrives at the large distance (LD)[21] OPE[11,12,14]

$$j^a(x)j^{a'}(0) \overset{\text{LD}}{\to} B^{A1}(\Box_x, \partial_x \cdot \partial_0, \Box_0) \times \sum_{n=0}^{\infty} F_n^A(x^2) x^{\alpha_1} \cdots x^{\alpha_n} O_{\alpha_1 \cdots \alpha_n}(0) . \tag{3.2}$$

Here B^{A1} is a c-number function of the indicated differential operators[22] determined by the currents A and the Regge trajectory $\alpha_1(t_1)$, F_n^A is a c-number function of x^2 depending on A, and $O_{\alpha_1 \cdots \alpha_n}(0)$ is a maximal spin-n local field operator.[23] The sum over n in (3.2) is the leading light-cone operator-product expansion[24] about which no assumptions are made concerning the singularity structure of $F_n^A(x^2)$. The differential operator converts this light-cone OPE into a LDOPE. The Regge behavior of (3.1) implies that (3.2) is valid between the states $\langle b|$ and $|b'\rangle$:

$$\langle b|j^a(x)j^{a'}(0)|b'\rangle \overset{\text{LD}}{\to} B^{A1}(\Box_x, i\partial_x \cdot \Delta_1, -t_1)\sum_n F_n^A(x^2)x^{\alpha_1} \cdots x^{\alpha_n}\langle b|O_{\alpha_1 \cdots \alpha_n}(0)|b'\rangle . \tag{3.3}$$

Regge behavior for all such states implies the validity of (3.2) as a weak operator statement.[25] Equation (3.2) thus provides an exact operator description of the construction of the exchanged Regge pole out of the exchange of a sequence of increasing-integral-spin objects.

The differential operator has the form[11]

$$B^{A1}(-K^A) = \beta^{A1}(K^A)/b^{A1}(K^A) , \tag{3.4}$$

where β^{A1} is the 2-body Regge residue. The function b^{A1} can be expressed as

$$b^{A1}(K^A) = \sum_i D^A_{\alpha_1(t_1)+i,i}(K^A) I^1_i(t_1) , \tag{3.5}$$

in terms of the functions

$$D^A_{ni}(K^A) \equiv \sum_{j=0}^{[i/2]} d_{ij}(2\delta_1)^{i-2j}(2t_1)^j \tilde{F}_n^{A(n-j)}(\kappa_1) \tag{3.6}$$

defined from the Fourier transforms

$$\tilde{F}_n^{A(m)}(\kappa_1) \equiv \left(2i\frac{\partial}{\partial \kappa_1}\right)^m \int d^4x\, e^{ip_1\cdot x} F_n^A(x^2) \tag{3.7}$$

and the numbers

$$d_{ij} \equiv i!/[(i-2j)!j!2^j] . \tag{3.8}$$

The function $I^1_i(t_1)$ in (3.5) is most simply defined from the factorization property of the residue at the Regge pole $\alpha_1(t_1) = n - i$ of the coefficient $C^A_{ni}(t_1)$ of $\Delta_1^{\alpha_1} \cdots \Delta_1^{\alpha_i} p_1^{\alpha_{i+1}} \cdots p_1^{\alpha_n}$ in the decomposition of $\langle a(p_1)|O_{\alpha_1 \cdots \alpha_n}(0)|a'(p_1')\rangle\!\rangle$, as displayed in[11]

$$C^A_{ni}(t_1) \sim [n - i - \alpha_1(t_1)]^{-1} I^1_i(t_1)\beta^{A1}(K^A) \times(\sin\pi\alpha_1/\pi) . \tag{3.9}$$

It is not important for our present purposes to understand the derivation of (3.2). This derivation assumes the existence of unproved (and even unformulated) J-plane analyticity. Even given (3.2), it is not straightforward to rederive the Regge behavior of (3.1). One must still actually use this assumed Regge behavior to conclude, via Bethe-Salpeter equations, that the matrix elements $\langle b|O_{\alpha_1 \cdots \alpha_n}|b'\rangle$ can be continued into the complex-n-plane and have the leading scattering-amplitude Regge pole as their leading singularity.[11] It is just such continuations we wish to avoid in the present paper. Rather, we shall *assume* the validity of (3.2) as a strong operator statement. This, together with the 2-to-2 Regge behavior (1.1), will be seen in Sec. IV to imply the full multi-Regge hypothesis.

IV. OPERATOR DERIVATION OF DOUBLE-REGGE BEHAVIOR

In this section we consider the 2-to-3 amplitude (2.9). We shall show that the double-Regge behavior (2.12) of (2.9) simply follows from two assumptions: (i) the validity of the LDOPE (3.2) as a strong expansion, and (ii) the validity of the Regge behaviors (1.1) and (2.17) for all 2-to-2 amplitudes. One representation for (2.9) is

$$T^{ABC}(\nu_1, \nu_2, \eta; K^A, K^B) = \int d^4x\, e^{ip_1\cdot x}\langle b(p_2)|T[j^a(x)j^{a'}(0)]|b'(p_2'), c(k)\rangle\!\rangle . \tag{4.1}$$

Another representation similarly expresses (2.9) in terms of $\langle a|T[j^b j^{b'}]|a', t\rangle$, where $j^b, j^{b'}$ are sources for b and b', respectively (or, more generally, any local fields).

If we wanted to evaluate just the limit of (4.1) for $\nu_1 \to \infty$, the validity of (3.2) as a weak expansion would enable us to substitute (3.2) into (4.1), to obtain

$$\int d^4x\, e^{ip_1\cdot x} \left\langle b\left|\sum_n F_n^A(x^2)x^{\alpha_1}\cdots x^{\alpha_n} B^{A1}(\vec{\Box}_x, \vec{\partial}_x \cdot \vec{\partial}_0, \vec{\Box}_0) O_{\alpha_1\cdots\alpha_n}(0)\right| b', c\right\rangle , \tag{4.2}$$

and to evaluate the $\nu_1 \to \infty$ limit of (4.2). The validity of (3.2) as a strong expansion means that its convergence is uniform in the matrix elements so that the limit $\nu_2 \to \infty$ on the matrix element[26] can be interchanged with the limit $\nu_1 \to \infty$, and that furthermore the limit $\nu_2 \to \infty$ can be taken inside the n-sum in (4.2).[15] In symbols,

$$\begin{aligned}\lim_{R_{12}} \langle b|T[j^a j^{a'}]|b'c\rangle &= \lim_{\nu_2\to\infty}\lim_{\nu_1\to\infty} \langle b|T[j^a j^{a'}]|b'c\rangle \\ &= \lim_{\nu_2\to\infty}\lim_{\nu_1\to\infty} \left\langle b \left| B^A \sum_n \tilde{F}_n^A O_n \right| b'c \right\rangle \\ &= \lim_{\nu_1\to\infty}\lim_{\nu_2\to\infty} \left[B^A \sum_n \tilde{F}_n^A \langle b|O_n|b'c\rangle \right] \\ &= \lim_{\nu_1\to\infty} \left[B^A \sum_n \tilde{F}_n^A \lim_{\nu_2\to\infty} \langle b|O_n|b'c\rangle \right] . \end{aligned} \tag{4.3}$$

To evaluate (4.2), we decompose the matrix element as in (2.16):

$$\langle b|x^{\alpha_1}\cdots x^{\alpha_n} O_{\alpha_1\cdots\alpha_n}(0)|b'c\rangle = \sum_{ik} G_{nik}^{BC}(\nu_2, t_1; K^B)(x\cdot\Delta_1)^i (x\cdot\Delta_2)^k (x\cdot p_2)^{n-i-k} . \tag{4.4}$$

There are no $g_{\alpha\beta}$ terms here because $x^2=0$.[27] The resulting x integration in (4.2) can be performed as in RI to give

$$\int d^4x\, e^{ip_1\cdot x} F_n^A(x^2)(x\cdot\Delta_1)^i(x\cdot\Delta_2)^k(x\cdot p_2)^{n-i-k} = D_{ni}^A(K^A)(2\nu_1)^k(2\nu)^{n-i-k} + \text{NLR} , \tag{4.5}$$

where we have used (3.6)–(3.8) and where "NLR" stands for terms which are nonleading in the double-Regge limit R_{12}. We can thus write

$$T^{ABC} = \sum_{nik} D_{ni}^A(K^A) B^{A1}(-K^A) G_{nik}^{BC}(\nu_2, t_1; K^B) \times (2\nu_1)^{n-i}(2\nu_2)^{n-i-k}\eta^{k+i-n} + \text{NLR} , \tag{4.6}$$

where we have used (2.7).

According to (4.3), we can now evaluate the $\nu_2 \to \infty$ limit of (4.6) termwise. The functions $G_{nik}^{BC}(\nu_2, t_1; K^B)$ are 2-to-2 amplitudes so that their large-ν_2 behavior can be determined from our assumption that all such amplitudes have Regge-pole behavior. The appropriate Regge behavior is given by (2.17):

$$G_{nik}^{BC}(\nu_2, t_1; K^B) \underset{\nu_2\to\infty}{\sim} \beta_{nik}^{C2}(t_1, t_2)\beta^{B2}(K^B) \times (2\nu_2)^{\alpha_2(t_2)-(n-i-k)} . \tag{4.7}$$

Equation (4.6) therefore becomes[28]

$$T^{ABC} = B^{A1}(-K^A)(2\nu_2)^{\alpha_2(t_2)}\beta^{B2}(K^B) \times \sum_{nik} D_{ni}^A(K^A)\beta_{nik}^{C2}(t_1, t_2)(2\nu_1)^{n-i}\eta^{k+i-n} + \text{NLR} . \tag{4.8a}$$

The expression (4.8a) has been derived starting from the representation (4.1). If instead we start from the representation in terms of $T[j^b j^{b'}]$ and proceed in the same way, reversing the roles of A and B and of ν_1 and ν_2, we would obtain

$$T^{ABC} = B^{B2}(-K^B)(2\nu_1)^{\alpha_1(t_1)}\beta^{A1}(K^A) \times \sum_{nik} D_{ni}^B(K^B)\beta_{nik}^{C1}(t_2, t_1)(2\nu_2)^{n-i}\eta^{k+i-n} + \text{NLR} . \tag{4.8b}$$

Here $B^{B2}(-K^B)$ is the differential operator occurring in the $j^b(x)j^{b'}(0)$ LDOPE [cf. Eq. (3.2)], $D_{ni}^B(K^B)$ is given by the expression (3.6) with A replaced by B, where $F_n^B(x^2)$ are the c-number functions occurring in the $j^b j^{b'}$ LDOPE, and the β_{nik}^{C1} are the (O_n)-(α_1)-C Regge residues occurring in the factorized large-ν_1 behavior of the 2-to-2 amplitudes G_{nik}^{AC} defined analogously to (4.4) from the matrix elements $\langle a|O_n|a'c\rangle$.

Using (3.4) and the analogous relation for B^{B2},[29] Eqs. (4.8a) and (4.8b) take on the forms

$$T^{ABC} = \beta^{A1}(K^A)(2\nu_2)^{\alpha_2(t_2)}\beta^{B2}(K^B) \times R^{AC12}(\nu_1, \eta, t_2, K^A) + \text{NLR} \tag{4.9a}$$

and

$$T^{ABC} = \beta^{A1}(K^A)(2\nu_1)^{\alpha_1(t_1)}\beta^{B2}(K^B) \times R^{BC21}(\nu_2, \eta, t_1, K^B) + \text{NLR} , \tag{4.9b}$$

respectively. Consistency between these relations requires that

$$R^{AC12}(\nu_1, \eta, t_2, K^A) \underset{\nu_1\to\infty}{\sim} (2\nu_1)^{\alpha_1(t_1)}\gamma^{AC12}(\eta, t_2, K^A) \tag{4.10a}$$

and

$$R^{BC21}(\nu_2, \eta, t_1, K^B) \underset{\nu_2\to\infty}{\sim} (2\nu_2)^{\alpha_2(t_2)}\gamma^{BC21}(\eta, t_1, K^B) , \tag{4.10b}$$

and that furthermore

$$\gamma^{AC12}(\eta, t_2, K^A) = \gamma^{BC21}(\eta, t_1, K^B) \equiv \gamma^{C12}(t_1, \eta, t_2) . \tag{4.11}$$

The first equality in (4.11) has arisen from the requirement that (4.9a) and (4.9b) agree. This equality implies that the equated functions are actually independent of the labels A and B and the variables κ_1, δ_1, κ_2, and δ_2 not appearing on both sides. Each function therefore depends only on the labels and variables exhibited in the function γ defined in (4.11).

With (4.10) and (4.11), the Eqs. (4.9) each give the correct double-Regge form (2.12) with $\beta^{C12} = \gamma^{C12}$. This is the advertized result that the double-Regge behavior follows from (1.1) and a suitably strong version of (3.2). The vertex β^{C12} is here not an arbitrary function, but is expressed in terms of the quantities appearing in Eqs. (4.8).

V. COMPLEX n-PLANE STRUCTURE

In our derivation of the double-Regge behavior (2.12) in Sec. IV from our assumptions (1.1) and (3.2), we made no commitments about whether or not the summations in Eq. (4.8) could be expressed as Sommerfeld-Watson-type integrals in the complex n plane. In this section, we shall assume that these summations can be so expressed and that the deduced behaviors (4.10) are determined by the leading singularities in this complex plane. This (relatively weak) assumption will lead to some interesting structures and will enable us to compare our results here with those in our previous work.

Assuming then that D_{ni} and β_{nik} in (4.8) can be continued into the complex n plane[30] and that these continuations are Carlsonian,[1] the consistency between (4.8a) and (4.8b) implies that the leading singularities are the poles

$$\beta^{C2}_{nik}(t_1, t_2) \sim \frac{\beta^{C21}_{ik}(t_1, t_2)}{n - i - \alpha_1(t_1)} \; \frac{\sin\pi\alpha_1(t_1)}{\pi} \,, \tag{5.1a}$$

$$\beta^{C1}_{nik}(t_2, t_1) \sim \frac{\beta^{C12}_{ik}(t_2, t_1)}{n - i - \alpha_2(t_2)} \; \frac{\sin\pi\alpha_2(t_2)}{\pi} \,, \tag{5.1b}$$

where the trigonometric functions have been factored out for later convenience. Introducing (5.1) and (3.4) into (4.8), we obtain

$$T^{ABC} = \beta^{A1}(K^A)\,(2\nu_1)^{\alpha_1}(2\nu_2)^{\alpha_2}\beta^{B2}(K^B)\,[b^{A1}(K^A)]^{-1} \times \sum_{ik} D^A_{i+\alpha_1,\,i}(K^A)\,\beta^{C21}_{ik}(t_1, t_2)\,\eta^{k-\alpha_1} + \text{NLR} \tag{5.2a}$$

and

$$T^{ABC} = \beta^{A1}(K^A)\,(2\nu_1)^{\alpha_1}(2\nu_2)^{\alpha_2}\beta^{B2}(K^B)\,[b^{B2}(K^B)]^{-1} \times \sum_{ik} D^B_{i+\alpha_2,\,i}(K^B)\,\beta^{C12}_{ik}(t_2, t_1)\,\eta^{k-\alpha_2} + \text{NLR}'. \tag{5.2b}$$

The requirement that (5.2a) and (5.2b) are consistent now gives the equality

$$[b^{A1}(K^A)]^{-1}\sum_i D^A_{i+\alpha_1,\,i}(K^A)\beta^{C21}_{i,k+\alpha_1}(t_1, t_2) = [b^{B2}(K^B)]^{-1}\sum_i D^B_{i+\alpha_2,\,i}(K^B)\,\beta^{C12}_{i,k+\alpha_2}(t_2, t_1) \tag{5.3}$$

$$\equiv \Gamma^{C12}_k(t_1, t_2)\,. \tag{5.4}$$

The equality (5.3) implies that each side is independent of the labels A and B and the variables κ_1, δ_1, κ_2, and δ_2 and so is given by a function Γ of the arguments indicated in (5.4). This function clearly satisfies the symmetry property

$$\Gamma^{C12}_k(t_1, t_2) = \Gamma^{C21}_k(t_2, t_1)\,. \tag{5.5}$$

Equations (5.2)–(5.4) thus exhibit the correct double-Regge behavior (2.12), with the identification

$$\beta^{C12}(t_1, \eta, t_2) = \sum_k \Gamma^{C12}_{k-\alpha_1}(t_1, t_2)\,\eta^{k-\alpha_1} = \sum_k \Gamma^{C12}_k(t_1, t_2)\,\eta^k\,. \tag{5.6}$$

Such a power-series representation of the Reggeon-particle-Reggeon vertex also arises in the conventional formalisms[6-9] as well as in model calculations.

In the above analysis we have not only reproduced the correct double-Regge behavior but have also obtained the expressions (5.3) for the vertex function (5.4) in terms of quantities determined solely from 2-to-2 amplitudes. Using (3.5), we have

$$\Gamma^{C12}_k(t_1, t_2) = \Big[\sum_i D^A_{i+\alpha_1,\,i}(K^A)\,\beta^{C21}_{i,k+\alpha_1}(t_1, t_2)\Big] \times \Big[\sum_i D^A_{i+\alpha_1,\,i}(K^A)\,I^1_i(t_1)\Big]^{-1}. \tag{5.7}$$

This equality implies that β^{C21} is given by

$$\beta^{C21}_{i,k+\alpha_1}(t_1, t_2) = I^1_i(t_1)\,\Gamma^{C21}_k(t_1, t_2)\,. \tag{5.8}$$

We have thus deduced that the residue of the pole of β^{C2}_{nik} at $n - i = \alpha_1(t_1)$ [cf. Eq. (5.1a)] factorizes as indicated. If we assume the validity of the multi-Regge hypothesis to begin with, we could, by the methods of RI and RII, deduce the existence of the pole (5.1a) and the factorization property (5.8) of the residue from the Bethe-Salpeter equation for β^{C2}_{nik} which follows from the corresponding equation for the vertex function G^{BC}_{nik}. This is illustrated in Fig. 6. We have here derived (5.1a) and (5.8), assuming sufficient analyticity, from (1.1) and (3.2).

Our procedure in the above was to first take the large-ν_2 limit (4.7) of G^{BC}_{nik} and then go to the pole (5.1a) at $n - i = \alpha_1(t_1)$ with the factorized residue (5.8):

$$G^{BC}_{nik}(\nu_2, t_1; K^B) \underset{\nu_2\to\infty}{\sim} \beta^{C2}_{nik}(t_1, t_2)\,\beta^{B2}(K^B) \times (2\nu_2)^{\alpha_2-(n-i-k)} \sim (n - i - \alpha_1)^{-1} I^1_i(t_1)\,\Gamma^{C12}_k(t_1, t_2) \times \beta^{B2}(K^B)\,(2\nu_2)^{\alpha_2-\alpha_1+k}(\sin\pi\alpha_1/\pi)\,. \tag{5.9}$$

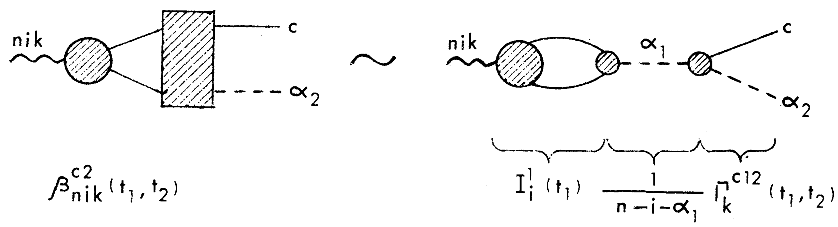

FIG. 6. The Bethe-Salpeter structure of the (field O_n)-(Reggeon α_2)-(particle c) vertex β^{C2}_{nik}. This equation feeds the assumed Regge pole α_1 of the (3-particle)-(Reggeon α_2) amplitude into β^{C2}_{nik} as a pole in n with a residue factorized as indicated.

In our previous paper RI, we went directly to the pole of G^{BC}_{nik} (Refs. 31 and 32):

$$G^{BC}_{nik}(\nu_2, t_1; K^B) \sim (n-i-\alpha_1)^{-1} I^1_i(t_1) \times \Lambda^{BC1}_k(\nu_2, t_1; K^B)(\sin\pi\alpha_1/\pi) . \qquad (5.10)$$

If the limits $\nu_2 \to \infty$ and $n-i \to \alpha_1(t_1)$ commute, then Λ^{BC1}_k must have the large-ν_2 behavior

$$\Lambda^{BC1}_k(\nu_2, t_1; K^B) \underset{\nu_2\to\infty}{\sim} \beta^{B2}(K^B)\Gamma^{C12}_k(t_1, t_2) \times (2\nu_2)^{\alpha_2-\alpha_1+k} . \qquad (5.11)$$

Exactly this behavior was deduced in our previous paper RII from the conventional double-Regge assumption.[33] The combined assumptions thus imply that these limits do in fact commute.

VI. DISCUSSION

We have seen that the conventional double-Regge behavior (2.12) of the general 2-to-3 amplitude (2.9) follows from our two basic assumptions: (i) the validity of LDOPE's of the form (3.2) in a strong sense for all current products, and (ii) the validity of the conventional single-Regge behavior (2.17) for all 2-to-2 amplitudes (2.16). Our procedure can obviously be generalized to show that the conventional multi-Regge behavior of the general N-to-N' amplitude ($N, N' \geq 2$) follows from these same two assumptions.[34] These assumptions therefore provide an operator formulation of the multi-Regge hypothesis. Because the 2-to-2 Reggeizations imply the validity of the LDOPE between single-particle states, our approach can alternatively be said to be based on a strong operator generalization of 2-to-2 Regge behavior.

Our procedure was to use the LDOPE to reduce the 2-to-3 amplitude (4.1) to a sum (4.6) over the 2-to-2 amplitudes (4.4). The assumed Regge behaviors (4.7) of these 2-to-2 amplitudes then yielded the expression (4.8a). If at this point we were to *assume* sufficient analyticity and the conventional 2-to-3 Regge behavior, we would deduce the leading pole behavior (5.1a) with the factorized residue (5.8) from Bethe-Salpeter equations (Fig. 6), from which would follow the same 2-to-3 Regge behavior for (4.1). We were able to *deduce* this 2-to-3 Reggeization without invoking these unproven strong assumptions. The operator nature of the LDOPE's provided us with the possibility of reducing the 2-to-3 amplitude (4.1) in another way (in terms of A-matrix elements of B currents instead of B-matrix elements of A currents) to also obtain the relation (4.8b). The requirement that (4.8a) and (4.8b) be consistent then directly yielded the 2-to-3 Regge behavior. *If* a suitable continuation into the complex n plane is possible, our consistency conditions also imply the leading pole structure (5.1a) with the factorized residue (5.8). It is to be emphasized, however, that even this simple intrusion into the complex n plane at the 2-to-2 level, which requires far weaker analyticity assumptions than those invoked in the conventional approach (at the 2-to-3 level), is unnecessary in our framework.

We should note here the role played by factorization in our analysis. Both the assumed 2-to-2 Regge behavior (2.17) and the LDOPE (3.2) and (3.3) have factorization properties. It was an interplay between these factorizations that yielded the correctly factorized double-Regge behavior (2.12).

Our use of consistency between (4.9a) and (4.9b) is very analogous to our use in Sec. II of consistency between the single-Regge fragmentation limits (2.10) and (2.11) to obtain the double-Regge behavior (2.12). The smoothness (2.13) used in this analysis[18,19] is of the same form (4.3) as that embodied in our assumption of strong convergence of the LDOPE. The qualitative difference between the frameworks of Sec. II and Sec. IV is that in Sec. II it was by assuming single-Regge behavior for multiparticle states and smoothness that the double-Regge behavior was obtained, whereas in Sec. IV only single-Regge behavior for single-particle states (and smoothness) was assumed. Logically, however, in view of Sec. II, it should be no surprise that the LDOPE's plus 2-to-2 Regge behavior imply multi-Regge behavior. In Sec. II we saw how single-Regge behavior for multiparticle states (plus smoothness) implies multi-Regge behavior, and in RI we saw how these same assumptions implied the LDOPE's, from which multi-Regge behavior was deduced in Sec. IV. The main point of this paper was, of course, not to repeat the argument of Sec. II in an operator framework but was to show that the LDOPE's abstracted from the multi-Regge formalism are sufficient in themselves to imply the physical consequences of the formalism.

A major virtue of our operator approach is its simplicity. The simple final multi-Regge behavior, with the two-body trajectories, directly emerges without the need to use group theory or venture into the unphysical complex J plane with complicated signatured amplitudes, where unproven analyticity assumptions must be made. Consequences of assuming that all multiparticle amplitudes have multi-Regge behavior are here simply consequences of a single[10] operator expansion.

Another virtue of this approach is its explicitness. We obtain an exact picture of the Regge pole being built out of an infinite sum of integral spin operators. This circumstance is more than just picturesque. It provides exact expressions for Reggeon-particle vertices in terms of purely particle matrix elements of operator structures. These expressions constitute consistency conditions which must be satisfied by any field theory possessing multi-Regge behavior. Equation (5.4), for example, expresses the Reggeon-particle-Reggeon amplitude[35] Γ_k^{C12} in terms of quantities determined from 2-to-2 amplitudes. One consequence of this expression is the factorization property (5.8) and the implied more direct expression (5.9) of Γ_k^{C12} in terms of the 2-to-2 vertex functions G_{nik}^{BC}. There are further consequences of such expressions which we plan to explore elsewhere.

One might doubt that our consistency conditions are useful because of the fact that our LDOPE (3.2) is a *consequence* of the general Regge behavior illustrated in Fig. 2 and so cannot lead to anything new. The point is that the light-cone OPE,[24] on which the LDOPE is based, incorporates a number of field-theoretic constraints (locality, asymptotic completeness, etc.) and the conversion of the light-cone OPE into the LDOPE requires unitaritylike equations to be satisfied.[11] Furthermore, as we have already stressed, the LDOPE is derived from single-Regge-pole exchange as a *weak* OPE, whereas it is used as a *strong* OPE to deduce the multi-Regge-pole behavior. The consistency conditions are, to some extent at least, consequences of this new assumption. The fact that the use of the LDOPE's as strong limits implies the correct double-Regge form strongly suggests the validity of this assumption in theories which exhibit multi-Regge behavior. The validity in such theories would, however, not be strictly proven because of the possibility that the function γ^{C12} of Eq. (4.11) might not be the correct vertex β^{C12} occurring in (2.12). We consider this to be very unlikely and mention it only because it is a logical possibility.

As Eq. (5.9) demonstrates, the approach in Sec. V was to obtain the large-ν_2 behavior of (4.1) directly and to obtain the large-ν_1 behavior from the pole arising from the continuation in n of $\langle b|O_n|b'c\rangle$. As in RI, the large-ν_2 behavior can itself be obtained from the LDOPE of the current product $j^b j^{b'}$. Assuming sufficient analyticity, the double-Regge behavior of (4.1) then would be determined by the poles in n and m of the vertex functions $\langle 0|T[O_n O_m]|c\rangle$. The residue of the leading simultaneous pole is essentially the Reggeon-particle-Reggeon vertex. This is illustrated in Fig. 7.

The analysis in this paper treated only the spinless 2-to-3 amplitude in the absence of Regge cuts. The generalization to the general N-to-N' amplitude is simple. Consider, for example, the 3-to-3 amplitude. The LDOPE reduces it to a sum over 2-to-3 amplitudes. Using the now established (from *our* assumptions) double-Regge behavior of these amplitudes, and performing the reduction in different ways, one easily arrives at the correct multi-Regge behavior. The generalization to include spin is also simple along the lines discussed in RI.[36] We also expect to be able to include the presence of Regge cuts in the formalism—at least those of the type discussed in RI. The LDOPE's remain valid in this case and we can proceed as in Sec. IV except that the pure-pole behaviors of the 2-to-2 amplitude would be replaced by Regge-cut behaviors.

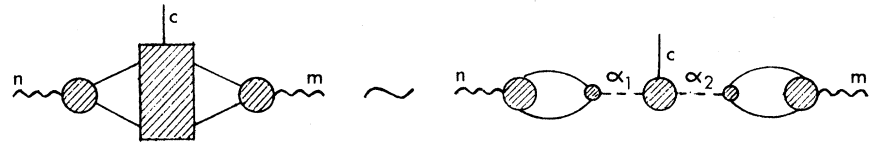

FIG. 7. The Bethe-Salpeter structure of the (field O_n)-(particle c)-(field O_m) vertex. This equation feeds the assumed Regge poles α_1 and α_2 of the 2-to-3 particle amplitude into the vertex.

Further work to be done in the direction pursued here should include the investigation of models of Regge behavior,[37] the exploration of our consistency conditions, and the study of Reggeon unitarity and discontinuity relations[16] and helicity limits.[5,9,38] The results of such studies should indicate whether or not the ideas developed here are useful tools for particle physics.

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[1]See, for example, P. Collins and E. Squires, *Springer Tracts in Modern Physics* (Springer, Berlin, 1968), Vol. 45, for an account of 2-body Regge-pole theory. References to the original articles can be found here.

[2]M. Toller, Nuovo Cimento 37, 631 (1965).

[3]Regge cuts arising from Regge poles are not meant to be excluded. In this paper we will for simplicity deal exclusively with poles.

[4]Cuts are also needed experimentally. See Ref. 1.

[5]See, for example, *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vols. 1 and 4; D. Horn and F. Zachariasen, *Hadron Physics at Very High Energies* (Benjamin, Reading, Mass., 1973); G. C. Fox and C. Quigg, Annu. Rev. Nucl. Sci. 23, 219 (1973); G. C. Fox, in *High Energy Collisions—1973,* proceedings of the fifth international conference on high energy collisions, Stony Brook, 1973, edited by C. Quigg (A.I.P., New York, 1973).

[6]N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. 163, 1572 (1967), and references therein.

[7]P. Goddard and A. R. White, Nucl. Phys. B17, 45 (1970); B17, 88 (1970); C. E. Jones, F. E. Low, and J. E. Young, Ann. Phys. (N.Y.) 63, 476 (1971); 70, 286 (1972); C. Cronstrom and W. H. Klink, *ibid.* 69, 218 (1972).

[8]J. B. Hartle and C. E. Jones, Phys. Rev. 184, 1564 (1969), and references therein; A. R. White, Nucl. Phys. B39, 432 (1972).

[9]A. R. White, Nucl. Phys. B67, 189 (1973).

[10]More precisely, it is assumed that each field product has a strongly convergent OPE.

[11]R. A. Brandt, Nucl. Phys. B72, 125 (1974). We shall refer to this reference as RI.

[12]R. A. Brandt, Nucl. Phys. B (to be published). We shall refer to this reference as RII.

[13]Weak convergence means that each matrix element converges, but at a rate which can depend on the matrix element.

[14]The validity of light-cone OPE's was assumed and some further technical (analyticity) assumptions were also made.

[15]Strong convergence means for us that the rate of convergence is independent of the matrix element taken so that operator limits and state limits can be freely interchanged and so that state limits can be taken termwise.

[16]V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, Phys. Rev. 139, B184 (1965); A. R. White, Nucl. Phys. B50, 93 (1972); B50, 130 (1972); H. D. I. Abarbanel, Phys. Rev. D 6, 2788 (1972); I. J. Muzinich, F. E. Paige, T. L. Trueman, and L. L. Wang, *ibid.* 6, 1048 (1972).

[17]In physically realistic cases, these would be the weak and electromagnetic currents of hadrons measured by lepton-hadron scattering.

[18]R. A. Brandt, Nuovo Cimento 22A, 461 (1974).

[19]If (2.13) were not true, then it would not be immediately clear what the R_{12} limit meant and further specification of how the limit is to be taken must be given. It would also not be clear that the Sommerfeld-Watson representation is sensible.

[20]The α's are Lorentz indices running from 0 to 3.

[21]The LD limit is $x_0 \to \infty$ with x^2 fixed. This is the configuration-space limit which corresponds to the R_1 limit in momentum space. See the Introduction in Ref. 11 and references therein.

[22]$\partial_0^\mu f(0) \equiv [(\partial/\partial y)^\mu f(y)]_{y=0}$.

[23]In general there will be $r_n > 1$ such fields which occur in (3.2). A sum over these r_n fields for each n, with possibly different coefficient functions $F_{nr}^A(x^2)$, should be understood. For notational simplicity, we suppress this r sum in this paper.

[24]R. A. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971).

[25]A separate argument is necessary for the only case ($\langle b|$ the vacuum and $|b'\rangle$ a single-particle state) where Regge behavior is not relevant. Equation (3.2) turns out to be identically valid in this case. See RII, Sec. 2.5.

[26]All of the $\nu_2 = p_2' \cdot k$ dependence of (4.1) is in the state $|b'(p_2'),\ c(k)\rangle$.

[27]An equally valid LDOPE could be derived without keeping $x^2=0$ here. This would only change the differential operator in (4.2). It would, however, make the calculations more cumbersome.

[28]In spite of the fact that (4.8a) has the correct R_2-limit form (2.11) it will not in general give the correct 3-body residue β^{AC2} because it was evaluated from (4.2), which is the dominant part of the full amplitude (4.1) only for $\nu_1 \to \infty$.

[29]We insert (3.4) at this point only for convenience. Since we will not assume anything about b^{A1} in this section, Eq. (3.4) contains no new information.

[30]Continuations into complex i and k planes can also be understood.

[31]Ref. 11, Eq. (3.19). The notation here is slightly different.

[32]Equation (3.9) of this paper is the analogous relation for the 3-point vertex.

[33]Ref. 12, Eq. (4.48). The notation here is slightly different. The Ω function of RII(4.48) is a part of the absorptive part of Λ. Compare Eqs. RII(4.41) and RII(4.43).

[34]More on this will be said below.

[35]As a matter of terminology, it is better to refer to such helicity coefficients as the Reggeon amplitudes rather than the full vertex β^{C12}. See Refs. 9 and 12.

[36]See RI, Sec. 5.4.

[37]For example, dual resonance models, Gribov calculus, gauge models in perturbation theory.

[38]J. H. Weis, Phys. Rev. D 6, 2823 (1972), and references therein.