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Multiparticle scattering amplitudes with spin-independent Poincaré-transformation properties

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For systems of massive particles with arbitrary spins, we define new classes of invariant scattering amplitudes. No auxiliary spin group is used, and the amplitudes are parametrized by the eigenvalues of Poincaré-invariant operator-valued functions of single-particle observables alone. We construct frame-independent partial-wave decomposition formulas for these amplitudes, and give a detailed derivation of the constraint equations which they must satisfy if scattering processes are to be invariant under space-time translations and proper homogeneous Lorentz transformations. In appendixes we collect together the definitions and transformation properties of many multiparticle states and scattering amplitudes, and derive some useful kinematical transformation formulas.

I. INTRODUCTION

The kinematical structure of a relativistic quantum-mechanical S matrix is determined by the physical principle of special relativity. However, for the sake of mathematical simplicity, we frequently construct hadronic scattering amplitudes which resemble the scattering amplitudes of quantum electrodynamics. Such amplitudes have an apparent extended kinematical structure which is characterized by the appearance in the mathematical formalism of nonunitary finite-dimensional representations of a homogeneous Lorentz group. Many widely differing dynamical theories and models have this kinematical form. For example, the M functions of Stapp,[1] the Regge amplitudes in some Lorentz-symmetry models,[2,3] and the projected amplitudes of Feldman and Matthews[4] all have this conventional field-theoretic structure. S-matrix elements are then implicitly parametrized by unphysical eigenvalues of non-Hermitian auxiliary spin-group operators. We suggest that a deeper understanding of the physical nature of field theories and S-matrix theories can be obtained if we avoid the introduction of auxiliary spin groups, since they have no intrinsic physical significance.

In a separate paper[5] we shall take this point of view and examine the structure of perturbation field theory. In this paper we shall concern ourselves with the definitions and Poincaré-transformation properties of complete sets of invariant amplitudes and partial-wave amplitudes which are parametrized by the eigenvalues of observables alone. The new amplitudes will then have an intrinsic physical significance, and the physical implications of any functional similarity between them will be immediately apparent.

In Sec. II we construct multiparticle states with simple Poincaré-transformation properties. They closely resemble those which we have described elsewhere,[6] but we now take into account phase factors of the form $(-1)^{2\sigma}$ which we neglected in our earlier work. In order to follow our notation, which differs from that used before,[6] one should refer to Appendix A where, for convenience, we have listed the definitions and transformation properties of many multiparticle states. Our new states which characterize r-particle systems are all parametrized by the eigenvalues of frame-independent single-particle spin-component operators $\hat{S}_{(k)}{}^{q(k)}$ [Eq. (A23)]. They are of three types. States of type I are labeled by single-particle three-momenta $\vec{p}_{(k)}$. States of type II are labeled by six frame-dependent momentum-component variables and a number of Poincaré-scalar momentum products. States of type III, which have single-particle-state-type transformation properties, are parametrized by the total three-momentum $\vec{p}$, the total effective squared mass s and spin σ, the third component of spin λ, a scalar spin component μ, and $3r-7$ scalar momentum products.

In Sec. III we sandwich a scattering operator $\hat{S}$ between these states of types I, II, and III to obtain three complete sets of relativistic scattering amplitudes. For relativistically invariant scattering theories, they prove to be functions of Poincaré scalars alone. We give an expansion of the standard frame-dependent scattering amplitudes of field theory in terms of invariant type-I amplitudes which are parametrized by the eigenvalues of Poincaré-invariant observables. One may then relate[5] the type-I amplitudes to the M functions of Stapp[1] or to the scalar amplitudes of conventional field theories. One may also derive a partial-wave expansion of type-I amplitudes in terms of type-III amplitudes which is, by construction, frame-independent.

In Appendix A we give the definitions and Poincaré-transformation properties of many multiparticle states. We also give an equation which relates different types of multiparticle states to each other. In Appendix B we define some multiparticle scattering amplitudes and give the constraint equations which they must satisfy in Poincaré-invariant theories. In Appendix C we give formulas expressing the components of any momentum $p_{(k)}$ measured in a special Lorentz frame determined by a momentum triplet p, q, f, in terms of scalar momentum products $p \cdot p_{(k)}$, $q \cdot p_{(k)}$, and $f \cdot p_{(k)}$. These formulas enable one to determine Poincaré-scalar Wigner-rotation angles directly, without using the geometrical constructions of Wick.[7]

II. SCALAR SPIN-COMPONENT MULTIPARTICLE STATES

We have examined elsewhere several classes of multiparticle states with simple Poincaré-transformation properties.[6] The states, which characterized systems of at least three particles, were parametrized by the eigenvalues $\lambda_{(k)}$ of scalar spin-component operators $\hat{S}_{(k)}{}^{q(k)}$ which are define by Eq. (A23). The essential difference between such states and the conventional direct-product states was the measurement of spin components relative to Lorentz frames which moved with an observer.

In our earlier discussions we neglected for simplicity the effect of sign factors $(-1)^{2\sigma}$ which become important when systems of particles with half-integral spins are considered. We shall now determine the complications which arise when such spin-dependent phases are taken into account, and shall show that new states may be defined for which these spin factors become relatively unimportant.

A. States of type I

Multiparticle scalar spin-component states of type I, $|\vec{p}_{(k)} : \lambda_{(k)}]$, have been defined in terms of standard rest states $|\vec{0} : \lambda_{(k)}\rangle$ by Eq. (A25),

$$|\vec{p}_{(k)} : \lambda_{(k)}] = \prod_{k=1}^{r} [\hat{L}(p_{(k)}; q_{(k)}, f_{(k)})] |\vec{0} : \lambda_{(k)}\rangle . \quad (1)$$

Homogeneous Lorentz transformations $L^{-1}(p_{(k)}; q_{(k)}, f_{(k)})$ take vectors $\vec{p}_{(k)}$ to zero, vectors $\vec{q}_{(k)}$ into the negative three-direction, and vectors $\vec{f}_{(k)}$ into such a direction that the second component is zero and the first component is positive. Conversely, these properties determine the 4×4 self-representation matrices $L(p_{(k)}; q_{(k)}, f_{(k)})$ uniquely, provided that the four-momenta $p_{(k)}$, $q_{(k)}$, and $f_{(k)}$ are not linearly dependent[8]:

$$\Delta(p_{(k)}, q_{(k)}, f_{(k)}) \neq 0 . \quad (2)$$

Unlike these matrices, the operators $\hat{L}(p_{(k)}; q_{(k)}, f_{(k)})$ are not uniquely determined by momenta $p_{(k)}$, $q_{(k)}$, and $f_{(k)}$. They may differ by factors of the form $\exp(2\pi i \hat{J}_3)$. Indeed, in Appendix A we have defined the operators $\hat{H}_+(p_{(k)}; q_{(k)}, f_{(k)})$ and $\hat{L}(p_{(k)}; q_{(k)}, f_{(k)})$, which differ by factors $\exp(2\pi i \hat{J}_3)$ for some values of the momenta $p_{(k)}$, $q_{(k)}$, and $f_{(k)}$, but have the same self-representation matrix $L(p_{(k)}; q_{(k)}, f_{(k)})$.

We have chosen our phase convention in such a way that multiparticle states (1) are defined in terms of operators $\hat{L}(p_{(k)}; q_{(k)}, f_{(k)})$, where

$$\hat{L}(p_{(k)}; q_{(k)}, f_{(k)}) = \hat{L}(p_{(k)})\hat{R}_-(q_{(k)}) \times \hat{R}_3(R^{-1}(\dot{q}_{(k)})L^{-1}(p_{(k)})f_{(k)}) \quad (3)$$

and

$$\dot{q}_{(k)} = L^{-1}(p_{(k)})q_{(k)} . \quad (4)$$

The homogeneous Lorentz-transformation property of the states (1) then follows directly:

$$\hat{\Lambda}_{[r]} |\vec{p}_{(k)} : \lambda_{(k)}] = \prod_{k=1}^{r} \{[l(\hat{\Lambda} : p_{(k)}; q_{(k)}, f_{(k)})]^{2\sigma(k)}\} |\vec{p}^{\dagger}_{(k)} : \lambda_{(k)}] , \quad (5)$$

where the transformed momenta are given by

$$p^{\dagger}_{(k)} = \Lambda p_{(k)} . \quad (6)$$

The signs $l(\hat{\Lambda} : p_{(k)}; q_{(k)}, f_{(k)})$ are simply related to generalized Wigner rotation functions:

$$[l(\hat{\Lambda} : p_{(k)}; q_{(k)}, f_{(k)})]^{2\hat{J}_3} = \hat{L}(\hat{\Lambda} : p_{(k)}; q_{(k)}, f_{(k)}) = \hat{L}^{-1}(p^{\dagger}_{(k)}; q^{\dagger}_{(k)}, f^{\dagger}_{(k)}) \times \hat{\Lambda}\hat{L}(p_{(k)}; q_{(k)}, f_{(k)}) , \quad (7)$$

and may be neglected unless spins $\sigma_{(k)}$ are half-integral.

We see that in general the multiparticle states $|\vec{p}_{(k)} : \lambda_{(k)}]$ will not have spin-independent Poincaré-transformation properties. For half-integral spin systems they may undergo a change of sign on Lorentz transformation which is determined in a complicated way by the transformation $\hat{\Lambda}_{[r]}$ and by the eigenvalues $\vec{p}_{(k)}$ of the r single-particle momentum observables $\hat{\underline{p}}_{(k)}$. We now intend to construct a set of states $|\vec{p}_{(k)} : \lambda_{(k)}]_{qf}$ for which the condition for a sign change depends only on the Lorentz transformation $\hat{\Lambda}$, the total momentum p, and two linearly independent momenta q and f which are functions of the single-particle momenta $p_{(k)}$.

We define such states $|\vec{p}_{(k)} : \lambda_{(k)}]_{qf}$ in terms of states $|\vec{p}_{(k)} : \lambda_{(k)}]$ by

$$|\vec{p}_{(k)} : \lambda_{(k)}]_{qf} = \prod_{k=1}^{r} \{[\, l(\hat{L}^{-1}(p;q,f) : p_{(k)}; q_{(k)}, f_{(k)})]^{2\sigma_{(k)}}\} \, |\vec{p}_{(k)} : \lambda_{(k)}] . \tag{8}$$

These states have the simple space-time translation property

$$\hat{a}_{[r]} |\vec{p}_{(k)} : \lambda_{(k)}]_{qf} = e^{ip\cdot a} \, |\vec{p}_{(k)} : \lambda_{(k)}]_{qf} . \tag{9}$$

It follows from Eqs. (5), (7), and (8) that they behave in the following way under homogeneous Lorentz transformations $\hat{\Lambda}_{[r]}$:

$$\begin{aligned}\hat{\Lambda}_{[r]} |\vec{p}_{(k)} : \lambda_{(k)}]_{qf} &= \prod_{k=1}^{r} [\, \hat{L}_{(k)}(L^{-1}(\hat{\Lambda} : p; q,f) : p_{(k)}; q_{(k)}, f_{(k)})] \, |\vec{p}^{\dagger}_{(k)} : \lambda_{(k)}]_{qf} \\ &= \hat{L}_{[r]}(\hat{\Lambda} : p; q,f) \, |\vec{p}^{\dagger}_{(k)} : \lambda_{(k)}]_{qf} .\end{aligned} \tag{10}$$

If we now define a spin parameter $\sigma_{\{r\}}$ by

$$\sigma_{\{r\}} = \sum_{k=1}^{r} \sigma_{(k)} , \tag{11}$$

we may rewrite Eq. (10) in the simplified form

$$\hat{\Lambda}_{[r]} |\vec{p}_{(k)} : \lambda_{(k)}]_{qf} = [\, l(\hat{\Lambda} : p; q,f)]^{2\sigma_{\{r\}}} \, |\vec{p}^{\dagger}_{(k)} : \lambda_{(k)}]_{qf} . \tag{12}$$

Like the standard states $|\vec{p}_{(k)} : \lambda_{(k)}\rangle$ our scalar spin-component states $|\vec{p}_{(k)} : \lambda_{(k)}]_{qf}$ have the normalization

$$_{qf}[\vec{p}'_{(k)} : \lambda'_{(k)} \, |\vec{p}_{(k)} : \lambda_{(k)}]_{qf} = \prod_{k=1}^{r} [\, 2p_{(k)0}\delta(\vec{p}'_{(k)} - \vec{p}_{(k)})\delta_{\lambda'_{(k)}\lambda_{(k)}}] . \tag{13}$$

B. States of types II and III

We define angles ϕ, θ, and ψ in terms of momenta p, $p_{(2)}$, and $p_{(3)}$ by

$$\hat{R}(\phi, \theta, \psi - \phi) = \hat{L}^{-1}(p)\hat{L}(p; p_{(2)}, p_{(3)}) . \tag{14}$$

We then introduce type-II q-spin states

$$|s : \vec{p}; \phi, \theta, \psi : s_{(l)(m)}, \lambda_{(k)}]_{qf}$$

parametrized by the square s of the total four-momentum p, the total three-momentum $\vec{p}$, angles ϕ, θ, and ψ, scalar spin components $\lambda_{(k)}$, and $3r-7$ independent momentum variables $s_{(l)(m)}$ defined by

$$s_{(l)(m)} = (p_{(l)} + p_{(m)})^2 , \quad m, l \neq 0 \tag{15}$$

and

$$s_{(0)(m)} = (p - p_{(m)})^2 , \quad m \neq 0 \tag{16}$$

with the range of integers l and m depending on the number of particles r,

$$l=0,\ m=2,3,\ r=3 \qquad \begin{cases} l=0,\ m=2,3,4,\ r=4 \\ l=2,\ m=3,4. \end{cases} \qquad \begin{cases} l=0,\ m=2,3,\ldots,r, & r\geq 5 \\ l=2,\ m=3,4,\ldots,r \\ l=3,\ m=5,6,\ldots,r \end{cases} \tag{17}$$

These scalar products $s_{(l)(m)}$ are closely related to momentum components $p_{(m)l}$ measured in a special Lorentz frame determined by the total momentum p_μ and angles ϕ, θ, and ψ,

$$p_{(m)l} \to p^{\star}_{(m)l} = [\, L^{-1}(p; p_{(2)}, p_{(3)}) \, p_{(m)}]_l . \tag{18}$$

Type-II states are related to type-I states by

$$|s : \vec{p}; \phi, \theta, \psi : s_{(l)(m)}, \lambda_{(k)}]_{qf} = J_{\{r\}}^{-1/2} \, |\vec{p}_{(k)} : \lambda_{(k)}]_{qf} . \tag{19}$$

The Jacobians $J_{\{r\}}$, which have been defined in Appendix A, are functions of Poincaré-scalar momentum products, and have been chosen to give type-II states the simple normalization

$$\begin{aligned}&_{qf}[s' : \vec{p}'; \phi', \theta', \psi' : s'_{(l)(m)}, \lambda'_{(k)} \, |s : \vec{p}; \phi, \theta, \psi : s_{(l)(m)}, \lambda_{(k)}]_{qf} \\ &\quad = \delta(s'-s)2p_0\delta(\vec{p}' - \vec{p})8\pi^2\delta(\phi' - \phi)\delta(\cos\theta' - \cos\theta)\delta(\psi' - \psi) \prod_{klm} [\, \delta(s'_{(l)(m)} - s_{(l)(m)})\delta_{\lambda'_{(k)}\lambda_{(k)}}] .\end{aligned} \tag{20}$$

Under space-time translations $\hat{a}_{[r]}$ and homogeneous Lorentz transformations $\hat{\Lambda}_{[r]}$, type-II states behave in the following way:

$$\hat{a}_{[r]} |s : \vec{p}; \phi, \theta, \psi : s_{(l)(m)}, \lambda_{(k)}]_{qf} = e^{ip\cdot a} \, |s : \vec{p}; \phi, \theta, \psi : s_{(l)(m)}, \lambda_{(k)}]_{qf} \tag{21}$$

and

$$\hat{\Lambda}_{[r]} \, |s : \vec{p}; \phi, \theta, \psi : s_{(l)(m)}, \lambda_{(k)}]_{qf} = [\, l(\hat{\Lambda} : p; q,f)]^{2\sigma_{\{r\}}} \, |s : \vec{p}^{\dagger}; \phi^{\dagger}, \theta^{\dagger}, \psi^{\dagger} : s_{(l)(m)}, \lambda_{(k)}]_{qf} , \tag{22}$$

where the angles $\phi^\dagger$, $\theta^\dagger$, and $\psi^\dagger$ are given in terms of transformed momenta $p^\dagger$, $p^\dagger_{(2)}$, and $p^\dagger_{(3)}$ by

$$\hat{R}(\phi^\dagger, \theta^\dagger, \psi^\dagger - \phi^\dagger) = \hat{L}^{-1}(p^\dagger)\hat{L}(p^\dagger; p_{(2)}^\dagger, p_{(3)}^\dagger). \tag{23}$$

One may use Eqs. (22) and (14) to show that states of type II satisfy the relation

$$|s:\vec{p}; \phi, \theta, \psi: s_{(l)(m)}, \lambda_{(k)}]_{qf} = [\,l(\hat{L}^{-1}(p; p_{(2)}, p_{(3)}): p; q, f)]^{2\sigma\{r\}}\,\hat{L}_{[r]}(p; p_{(2)}, p_{(3)})|s:\vec{0}; 0, 0, 0: s_{(l)(m)}, \lambda_{(k)}]_{qf}. \tag{24}$$

We make use of this equation to define states of type III, $|s, \sigma:\vec{p}, \lambda:\mu; s_{(l)(m)}, \lambda_{(k)}]_{qf}$, which are eigenstates of the spin operators $\underline{\hat{S}}_{[r]}{}^2$, $\hat{S}_{[r]3}$, and $\hat{S}_{[r]}{}^{p(2)}$ with the corresponding eigenvalues $\sigma(\sigma+1)$, λ, and μ,

$$|s, \sigma:\vec{p}, \lambda:\mu; s_{(l)(m)}, \lambda_{(k)}]_{qf}$$
$$= \frac{(2\sigma+1)^{1/2}}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} D^{\sigma *}_{\lambda\mu}(\hat{R}(\phi, \theta, \psi-\phi))[\,l(\hat{L}^{-1}(p; p_{(2)}, p_{(3)}): p; q, f)]^{2\sigma}|s:\vec{p}; \phi, \theta, \psi: s_{(l)(m)}, \lambda_{(k)}]_{qf} \times \sin\theta\, d\phi\, d\theta\, d\psi \tag{25}$$

$$= (2\sigma+1)^{1/2} L_{[r]}(p)\int D^{\sigma *}_{\lambda\mu}(\hat{R}(\alpha, \beta, \gamma))\hat{R}_{[r]}(\alpha, \beta, \gamma)|s:\vec{0}; 0, 0, 0: s_{(l)(m)}, \lambda_{(k)}]_{qf}\, d\mu(\hat{R}(\alpha, \beta, \gamma)), \tag{26}$$

where the measure on the covering group of the rotation group is such that

$$d\mu(\hat{R}(\alpha, \beta, \gamma)) = \frac{\sin\beta\, d\alpha\, d\beta\, d\gamma}{16\pi^2}, \quad \alpha \in [0, 2\pi), \quad \beta \in [0, \pi], \quad \gamma \in [0, 4\pi). \tag{27}$$

One may use the properties of the group-representation function to invert Eq. (25) and express type-II states as a sum of states of type III. We find

$$|s:\vec{p}; \phi, \theta, \psi: s_{(l)(m)}, \lambda_{(k)}]_{qf} = \sum_{\sigma\lambda\mu}(2\sigma+1)^{1/2} D^{\sigma}_{\lambda\mu}(\hat{R}(\phi, \theta, \psi-\phi)) \times [\,l(R^{-1}(\phi, \theta, \psi-\phi): P, Q, F)]^{2\sigma}|s, \sigma:\vec{p}, \lambda:\mu; s_{(l)(m)}, \lambda_{(k)}]_{qf}, \tag{28}$$

where capital letters denote c.m. momenta defined by equations of the form (A15). The relation (28) will prove useful when we come to consider partial-wave decompositions in Sec. III.

Under space-time translations $\hat{a}_{[r]}$ and homogeneous Lorentz transformations $\hat{\Lambda}_{[r]}$, our multiparticle q-spin states of type III behave like single-particle states $|\vec{p}:\lambda\rangle$,

$$\hat{a}_{[r]}|s, \sigma:\vec{p}, \lambda:\mu; s_{(l)(m)}, \lambda_{(k)}]_{qf} = e^{ip\cdot a}|s, \sigma:\vec{p}, \lambda:\mu; s_{(l)(m)}, \lambda_{(k)}]_{qf} \tag{29}$$

and

$$\hat{\Lambda}_{[r]}|s, \sigma:\vec{p}, \lambda:\mu; s_{(l)(m)}, \lambda_{(k)}]_{qf} = \sum_{\lambda'} D^{\sigma}_{\lambda'\lambda}(\hat{L}(\hat{\Lambda}:p))|s, \sigma:\vec{p}^{\dagger}, \lambda':\mu: s_{(l)(m)}, \lambda_{(k)}]_{qf}, \tag{30}$$

where the Wigner rotation $\hat{L}(\hat{\Lambda}:p)$ is defined by Eq. (A33). Moreover, the form of Eq. (25) has been chosen to give type-III states the normalization

$${}_{qf}[s', \sigma':\vec{p}', \lambda':\mu'; s'_{(l)(m)}, \lambda'_{(k)}|s, \sigma:\vec{p}, \lambda:\mu; s_{(l)(m)}, \lambda_{(k)}]_{qf} = \delta(s'-s)\delta_{\sigma'\sigma}2p_0\delta(\vec{p}'-\vec{p})\delta_{\lambda'\lambda}\delta_{\mu'\mu}\prod_{klm}[\,\delta(s'_{(l)(m)}-s_{(l)(m)})\delta_{\lambda'_{(k)}\lambda_{(k)}}]. \tag{31}$$

III. PARTIAL-WAVE AND INVARIANT AMPLITUDE EXPANSIONS OF THE MULTIPARTICLE S MATRIX

We shall investigate the properties of various S matrices obtained by sandwiching a scattering operator $\hat{S}$ between complete sets of normalized on-mass-shell multiparticle states. Consider, for example, the standard multiparticle states $|\vec{p}_{(k)}:\lambda_{(k)}\rangle$ defined formally in terms of rest states $|\vec{0}:\lambda_{(k)}\rangle$ by Eq. (A25). They are eigenstates of single-particle momentum operators $\hat{\underline{p}}_{(k)}$ and spin-component operators $\hat{S}_{(k)3}$ with eigenvalues $\vec{p}_{(k)}$ and $\lambda_{(k)}$, respectively. The corresponding S matrix is defined by Eq. (B1):

$$S\langle\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)}\rangle = \langle\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\hat{S}|\vec{p}_{(k)}:\lambda_{(k)}\rangle, \tag{32}$$

where the parameters $\vec{p}_{(k)}$, $\lambda_{(k)}$ and $\vec{p}_{(\bar{k})}$, $\lambda_{(\bar{k})}$ are associated with r initial particles (k) and $\bar{r}$ final particles $(\bar{k})$, respectively.

The generators of space-time translations $\hat{a}$ and homogeneous Lorentz transformations $\hat{\Lambda}$ of the scattering system, as a whole, are of the form

$$\hat{p}_\mu = \hat{p}_{[\bar{r}]\mu} + \hat{p}_{[r]\mu} \tag{33}$$

and

$$\hat{J}_{\mu\nu} = \hat{J}_{[\bar{r}]\mu\nu} + \hat{J}_{[r]\mu\nu}, \tag{34}$$

where $\hat{p}_{[\bar{r}]\mu}, \hat{J}_{[\bar{r}]\mu\nu}$ and $\hat{p}_{[r]\mu}, \hat{J}_{[r]\mu\nu}$ are the generators of Poincaré transformations of the set of final $\bar{r}$-particle states and the set of initial r-particle states, respectively. The Poincaré invariance of a scattering operator $\hat{S}$ is then implied by the equations

$$\hat{a}\hat{S}\hat{a}^{-1} = \hat{S} \tag{35}$$

and

$$\hat{\Lambda}\hat{S}\hat{\Lambda}^{-1} = \hat{S}. \tag{36}$$

We may use the translational invariance (35) of the scattering operator $\hat{S}$ and the transformation property (A34) of standard states $|\vec{p}_{(k)} : \lambda_{(k)}\rangle$ to show that amplitudes $S\langle \vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}\rangle$ satisfy the spin-independent constraint equation

$$(1 - e^{ia\cdot(\bar{p}-p)}) S\langle \vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}\rangle = 0, \tag{37}$$

where $\bar{p}$ is the eigenvalue of the total final-particle momentum operator $\hat{p}_{[\bar{r}]}$, defined by

$$\hat{p}_{[\bar{r}]} = \sum_{k=1}^{\bar{r}} \hat{p}_{(\bar{k})}. \tag{38}$$

Equation (37) implies that S-matrix elements vanish unless energy and momentum are conserved,

$$\bar{p} = p. \tag{39}$$

For this reason we may define reduced amplitudes $S'\langle \vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}\rangle$, which are functions of only $3(\bar{r}+r)-4$ independent momentum components

$$S\langle \vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}\rangle = \delta^4(\bar{p}-p) S'\langle \vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}\rangle. \tag{40}$$

We now use the Lorentz-invariant nature (36) of the scattering operator $\hat{S}$ and the homogeneous Lorentz-transformation properties (A35) of standard states $|\vec{p}_{(k)} : \lambda_{(k)}\rangle$ to show that amplitudes $S'\langle \vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}\rangle$ satisfy the constraint equation

$$S'\langle \vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}\rangle = \prod_{\bar{k}=1}^{\bar{r}} \left[\sum_{\lambda'_{(\bar{k})}} D^{\sigma(\bar{k})\,*}_{\lambda'_{(\bar{k})}\lambda_{(\bar{k})}}(\hat{L}(\hat{\Lambda} : p_{(\bar{k})})) \right] \times \prod_{k=1}^{r} \left[\sum_{\lambda'_{(k)}} D^{\sigma(k)}_{\lambda'_{(k)}\lambda_{(k)}}(\hat{L}(\hat{\Lambda} : p_{(k)})) \right] S'\langle \vec{p}^{\dagger}_{(\bar{k})} : \lambda'_{(\bar{k})} | \vec{p}^{\dagger}_{(k)} : \lambda'_{(k)}\rangle, \tag{41}$$

where $p^{\dagger}_{(\bar{k})} = \Lambda p_{(\bar{k})}$ and $p^{\dagger}_{(k)} = \Lambda p_{(k)}$. This equation is very complicated. However, we shall see later that, for amplitudes which are defined in terms of our new multiparticle states $|\vec{p}_{(k)} : \lambda_{(k)}]_{qf}$, the corresponding constraint equation will be trivial.

A. Invariant amplitudes of type I

We use Eq. (8) to define sets of initial-particle states $|\vec{p}_{(k)} : \lambda_{(k)}]_{qf}$ and final-particle states $|\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})}]_{qf}$. We then sandwich a scattering operator $\hat{S}$ between these states to obtain a complete set of S-matrix elements $S^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}]$ of type I,

$$S^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}] = {}_{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \hat{S} | \vec{p}_{(k)} : \lambda_{(k)}]_{qf}. \tag{42}$$

Such scattering amplitudes are evidently parametrized by single-particle momenta $\vec{p}_{(\bar{k})}$ and $\vec{p}_{(k)}$ and the eigenvalues $\lambda_{(\bar{k})}$ and $\lambda_{(k)}$ of the Poincaré-invariant q-spin operators $\hat{S}_{(\bar{k})}{}^{q(\bar{k})}$ and $\hat{S}_{(k)}{}^{q(k)}$ defined by Eq. (A23).

Since the multiparticle states $|\vec{p}_{(k)} : \lambda_{(k)}]_{qf}$ have the same translation property (9) as the standard states $|\vec{p}_{(k)} : \lambda_{(k)}\rangle$, Eq. (A34), we may show once more that S-matrix elements vanish unless energy and momentum are conserved. For this reason we may define reduced S-matrix elements $S'^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}]$ which are functions of only $3(\bar{r}+r)-4$ independent-momentum component parameters,

$$S^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}] = \delta^4(\bar{p}-p) S'^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}]. \tag{43}$$

Under homogeneous Lorentz transformations $\hat{\Lambda}$, initial states $|\vec{p}_{(k)} : \lambda_{(k)}]_{qf}$ and final states $|\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})}]_{qf}$ transform according to Eq. (12), and the scattering operator $\hat{S}$ remains unchanged [Eq. (36)]. Our reduced S-matrix elements must thus satisfy the constraint equation

$$S'^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}] = [l(\Lambda : p; q, f)]^{2(\sigma\{r\} + \sigma\{\bar{r}\})} \times S'^{qf}[\vec{p}^{\dagger}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}^{\dagger}_{(k)} : \lambda_{(k)}], \tag{44}$$

where $\sigma_{\{\bar{r}\}}$ is the sum of the spins of the final particles $(\bar{k})$,

$$\sigma_{\{\bar{r}\}} = \sum_{\bar{k}=1}^{\bar{r}} \sigma_{(\bar{k})} \,. \tag{45}$$

Consider first of all the effect of a transformation $\hat{\Lambda}$ of the form

$$\hat{\Lambda} = \exp(2\pi i \hat{J}_3) \,. \tag{46}$$

Equation (44) then takes the form

$$S'^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}] = (-1)^{2(\sigma\{r\} + \sigma\{\bar{r}\})} S'^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}] \,, \tag{47}$$

and, if S-matrix elements are not to vanish, the sum of all individual particle spins must be an integer. In this case, for an arbitrary Lorentz transformation $\hat{\Lambda}$, (44) becomes a spin-independent constraint equation

$$S'^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}] = S'^{qf}[\vec{p}^{\dagger}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}^{\dagger}_{(k)} : \lambda_{(k)}] \,, \tag{48}$$

where $p^{\dagger}_{(\bar{k})} = \Lambda p_{(\bar{k})}$ and $p^{\dagger}_{(k)} = \Lambda p_{(k)}$. Our S-matrix elements are thus frame-independent invariant amplitudes. In order to see this explicitly, consider the effect of introducing a Lorentz transformation $\hat{\Lambda}$ of the form $\hat{L}^{-1}(p; q, f)$ into Eq. (48). Evidently, the momenta $\vec{p}_{(k)}$ and $\vec{p}_{(\bar{k})}$ will be replaced by $P^{\star}_{(k)}$ and $P^{\star}_{(\bar{k})}$, where

$$P^{\star}_{(k)} = L^{-1}(p : q, f) p_{(k)}$$

and

$$P^{\star}_{(\bar{k})} = L^{-1}(p; q, f) p_{(\bar{k})} \,. \tag{49}$$

TABLE I. Multiparticle scattering amplitudes.

Amplitude	$S(\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} \| \vec{p}_{(k)} : \lambda_{(k)})$	$\|\vec{p}_{(k)} : \lambda_{(k)})$
Standard	$S\langle \vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} \| \vec{p}_{(k)} : \lambda_{(k)}\rangle$	$\|\vec{p}_{(k)} : \lambda_{(k)}\rangle$
Helicity	$S^{+}\{\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} \| \vec{p}_{(k)} : \lambda_{(k)}\}$	$\|\vec{p}_{(k)} : \lambda_{(k)}\}_{+}$
	$S\{\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} \| \vec{p}_{(k)} : \lambda_{(k)}\}$	$\|\vec{p}_{(k)} : \lambda_{(k)}\}$
q spin	$S^{+}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} \| \vec{p}_{(k)} : \lambda_{(k)}]$	$\|\vec{p}_{(k)} : \lambda_{(k)}]_{+}$
	$S[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} \| \vec{p}_{(k)} : \lambda_{(k)}]$	$\|\vec{p}_{(k)} : \lambda_{(k)}]$
	$S^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} \| \vec{p}_{(k)} : \lambda_{(k)}]$	$\|\vec{p}_{(k)} : \lambda_{(k)}]_{qf}$

Then

$$S'^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}] = S'^{qf}[P^{\star}_{(\bar{k})} : \lambda_{(\bar{k})} | P^{\star}_{(k)} : \lambda_{(k)}] \,, \tag{50}$$

and we see from Eqs. (C10)–(C14) that each momentum component $P^{\star}_{(\bar{k})\mu}$ and $P^{\star}_{(k)\mu}$ is a function of scalar momentum products alone.

Our invariant amplitudes $S'^{qf}[\vec{P}^{\star}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{P}^{\star}_{(k)} : \lambda_{(k)}]$ are labeled by the eigenvalues of simple operator-valued functions of single-particle observables. In this respect they are quite different from many of the invariant amplitudes of conventional field theories and S-matrix theories which are implicitly parametrized by "unphysical" eigenvalues of non-Hermitian auxiliary spin-group operators.

We may use the defining equation (A25) of states $|\vec{p}_{(k)} : \lambda_{(k)}]_{qf}$ to expand any of the scattering amplitudes listed in Table I [see (B6)] in terms of a complete set of invariant amplitudes $S'^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}]$. In particular, in the case of the standard frame-dependent amplitudes $S'\langle \vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}\rangle$ of field theory, the general equation (B6) takes the form

$$S'\langle \vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}\rangle = \prod_{\bar{k}=1}^{\bar{r}} \left[\sum_{\lambda'_{(\bar{k})}} D^{\sigma(\bar{k})\,*}_{\lambda'_{(\bar{k})}\lambda_{(\bar{k})}} (\hat{L}^{-1}(p; q, f : p_{(\bar{k})}; q_{(\bar{k})}, f_{(\bar{k})}) \hat{L}(p_{(\bar{k})})) \right] \times \prod_{k=1}^{r} \left[\sum_{\lambda'_{(k)}} D^{\sigma(k)}_{\lambda'_{(k)}\lambda_{(k)}} (\hat{L}^{-1}(p; q, f : p_{(k)}; q_{(k)}, f_{(k)}) \hat{L}(p_{(k)})) \right] \times S'^{qf}[\vec{p}_{(\bar{k})} : \lambda'_{(\bar{k})} | \vec{p}_{(k)} : \lambda'_{(k)}] \,. \tag{51}$$

Moreover, since the transformation matrix relating the amplitudes is unitary, we may invert this equation directly and express invariant amplitudes as sums of standard frame-dependent amplitudes:

$$S'^{qf}[\vec{p}_{(\bar{k})} : \lambda_{(\bar{k})} | \vec{p}_{(k)} : \lambda_{(k)}] = \prod_{\bar{k}=1}^{\bar{r}} \left[\sum_{\lambda'_{(\bar{k})}} D^{\sigma(\bar{k})\,*}_{\lambda'_{(\bar{k})}\lambda_{(\bar{k})}} (\hat{L}^{-1}(p_{(\bar{k})}) \hat{L}(p; q, f : p_{(\bar{k})}; q_{(\bar{k})}, f_{(\bar{k})})) \right] \times \prod_{k=1}^{r} \left[\sum_{\lambda'_{(k)}} D^{\sigma(k)}_{\lambda'_{(k)}\lambda_{(k)}} (\hat{L}^{-1}(p_{(k)}) \hat{L}(p; q, f : p_{(k)}; q_{(k)}, f_{(k)})) \right] \times S'\langle \vec{p}_{(\bar{k})} : \lambda'_{(\bar{k})} | \vec{p}_{(k)} : \lambda'_{(k)}\rangle \,. \tag{52}$$

One may, of course, expand standard amplitudes $S\langle \vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)}\rangle$ in terms of invariant amplitudes in many ways. We obtain several different decompositions on changing the momenta $q_{(\bar{k})}$, $q_{(k)}$, $f_{(\bar{k})}$, and $f_{(k)}$ which enter into the definitions of multiparticle states $|\vec{p}_{(k)}:\lambda_{(k)}]_{qf}$ and $|\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}]_{qf}$. However, unlike the position in conventional field theories, each decomposition in terms of new amplitudes $S'^{qf}[\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)}]$ has an immediate physical interpretation, and the different sets of invariant amplitudes constructed in this way are unitarily equivalent.

In conclusion, we note that the type-I invariant amplitudes $S'^{qf}[\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)}]$ will only be well defined if all matrices $L(p_{(\bar{k})};q_{(\bar{k})},f_{(\bar{k})})$, $L(p_{(k)};q_{(k)},f_{(k)})$, and $L(p;q,f)$ are uniquely defined as functions of components of four-momenta. For this to be the case, various momentum triplets must be linearly independent:

$$\Delta(p_{(\bar{k})},q_{(\bar{k})},f_{(\bar{k})})\neq 0,\quad \Delta(p_{(k)},q_{(k)},f_{(k)})\neq 0,\quad \text{and}\quad \Delta(p,q,f)\neq 0. \tag{53}$$

Moreover, if we are to avoid sign ambiguities when spins $\sigma_{(\bar{k})}$, $\sigma_{(k)}$, or σ are half-integral, we must ensure that the three-momenta associated with four-momenta $L^{-1}(p_{(\bar{k})})q_{(\bar{k})}$, $L^{-1}(p_{(k)})q_{(k)}$, and $L^{-1}(p)q$ do not lie in the positive three-direction.

B. Invariant amplitudes of type II

We have shown that scattering amplitudes associated with scalar spin-component states of type I $|\vec{p}_{(k)}:\lambda_{(k)}]_{qf}$ are invariant amplitudes. We should now like to decompose them into amplitudes parametrized by the eigenvalues $\sigma(\sigma+1)$ of the total effective spin operator $\underline{\hat{S}}^2$, where

$$\hat{S}_i=\hat{s}^{-1}L^{-1}(\hat{p})_i{}^{\mu}\epsilon_{\mu}{}^{\nu\rho\sigma}\hat{p}_\nu\hat{J}_{\rho\sigma}. \tag{54}$$

Before doing this we shall examine the properties of related scattering amplitudes of type II. Such amplitudes are obtained by sandwiching the scattering operator $\hat{S}$ between type-II multiparticle states (19),

$$S^{qf}[\bar{s}:\bar{p};\bar{\phi},\bar{\theta},\bar{\psi}:\bar{s}_{(l)(m)},\lambda_{(\bar{k})}|s:p;\phi,\theta,\psi:s_{(l)(m)},\lambda_{(k)}]={}_{qf}[\bar{s}:\bar{p};\bar{\phi},\bar{\theta},\bar{\psi}:\bar{s}_{(l)(m)},\lambda_{(\bar{k})}|\hat{S}|s:p;\phi,\theta,\psi:s_{(l)(m)},\lambda_{(k)}]_{qf}. \tag{55}$$

Since type-II states satisfy the space-time translation equation (21), we also define a set of reduced amplitudes:

$$S'^{qf}[\bar{\phi},\bar{\theta},\bar{\psi}:\bar{s}_{(l)(m)},\lambda_{(\bar{k})}|s,\vec{p}|\phi,\theta,\psi:s_{(l)(m)},\lambda_{(k)}],$$

$$S^{qf}[\bar{s}:\bar{p};\bar{\phi},\bar{\theta},\bar{\psi}:\bar{s}_{(l)(m)},\lambda_{(\bar{k})}|s:p;\phi,\theta,\psi:s_{(l)(m)},\lambda_{(k)}] =\delta^4(\bar{p}-p)S'^{qf}[\bar{\phi},\bar{\theta},\bar{\psi}:\bar{s}_{(l)(m)},\lambda_{(\bar{k})}|s,\vec{p}|\phi,\theta,\psi:s_{(l)(m)},\lambda_{(k)}]. \tag{56}$$

We now use the homogeneous Lorentz-transformation properties (22) and (36) of type-II states and of the scattering operator $\hat{S}$ to show that these reduced amplitudes satisfy the constraint equation

$$S'^{qf}[\bar{\phi},\bar{\theta},\bar{\psi}:\bar{s}_{(l)(m)},\lambda_{(\bar{k})}|s,\vec{p}|\phi,\theta,\psi:s_{(l)(m)},\lambda_{(k)}] =S'^{qf}[\bar{\phi}^\dagger,\bar{\theta}^\dagger,\bar{\psi}^\dagger:\bar{s}_{(l)(m)},\lambda_{(\bar{k})}|s,\vec{p}^\dagger|\phi^\dagger,\theta^\dagger,\psi^\dagger:s_{(l)(m)},\lambda_{(k)}], \tag{57}$$

where $p^\dagger=\Lambda p$, and the transformed angles are defined by

$$R(\bar{\phi}^\dagger,\bar{\theta}^\dagger,\bar{\psi}^\dagger-\bar{\phi}^\dagger)=L^{-1}(p^\dagger)L(p^\dagger;p^\dagger_{(\bar{2})},p^\dagger_{(\bar{3})}) \tag{58}$$

and

$$R(\phi^\dagger,\theta^\dagger,\psi^\dagger-\phi^\dagger)=L^{-1}(p^\dagger)L(p^\dagger;p^\dagger_{(2)},p^\dagger_{(3)}). \tag{59}$$

Let us take the Lorentz transformation $\hat{\Lambda}$ to be of the form $\hat{L}^{-1}(p;p_{(2)},p_{(3)})$. This leads to the relation

$$S'^{qf}[\bar{\phi},\bar{\theta},\bar{\psi}:\bar{s}_{(l)(m)},\lambda_{(\bar{k})}|s,\vec{p}|\phi,\theta,\psi:s_{(l)(m)},\lambda_{(k)}]=S'^{qf}[\Phi,\Theta,\Psi:\bar{s}_{(l)(m)},\lambda_{(\bar{k})}|s,\vec{0}|0,0,0:s_{(l)(m)},\lambda_{(k)}] \equiv S'^{qf}[\bar{s}_{(l)(m)},\lambda_{(\bar{k})}|s:\Phi,\Theta,\Psi|s_{(l)(m)},\lambda_{(k)}], \tag{60}$$

where angles Φ, Θ, and Ψ are given by

$$R(\Phi,\Theta,\Psi-\Phi)=L^{-1}(p;p_{(2)},p_{(3)})L(p;p_{(\bar{2})},p_{(\bar{3})}), \tag{61}$$

with

$$\Phi\in[0,2\pi),\quad \Theta\in[0,\pi],\quad \text{and}\ \Psi\in[0,2\pi). \tag{62}$$

These angles are Poincaré scalars. Indeed, one may use Eqs. (C17) and (C18) to show explicitly that these angles are functions of scalar momentum products,[8]

$$\cos\Theta = \frac{\Omega(p;p_{(2)},p_{(\bar{2})})}{\Delta(p,p_{(2)})\Delta(p,p_{(\bar{2})})}, \quad \sin\Theta = \frac{\Delta(p)\Delta(p,p_{(2)},p_{(\bar{2})})}{\Delta(p,p_{(2)})\Delta(p,p_{(\bar{2})})},$$

$$\cos\Phi = \frac{\Omega(p;p_{(\bar{2})},p_{(3)})\Delta^2(p,p_{(2)})+\Omega(p;p_{(2)},p_{(\bar{2})})\Omega(p;p_{(2)},p_{(3)})}{p^2\Delta(p,p_{(2)},p_{(3)})\Delta(p,p_{(2)},p_{(\bar{2})})},$$

$$\sin\Phi = \frac{\Delta(p,p_{(2)})\Delta(p,p_{(2)},p_{(3)},p_{(\bar{2})})}{\Delta(p,p_{(2)},p_{(3)})\Delta(p,p_{(2)},p_{(\bar{2})})}, \tag{63}$$

$$\cos(\Psi-\Phi) = -\frac{\Omega(p;p_{(2)},p_{(\bar{3})})\Delta^2(p,p_{(\bar{2})})+\Omega(p;p_{(2)},p_{(\bar{2})})\Omega(p;p_{(\bar{2})},p_{(\bar{3})})}{p^2\Delta(p,p_{(\bar{2})},p_{(\bar{3})})\Delta(p,p_{(\bar{2})},p_{(2)})},$$

and

$$\sin(\Psi-\Phi) = \frac{\Delta(p,p_{(\bar{2})})\Delta(p,p_{(\bar{2})},p_{(\bar{3})},p_{(2)})}{\Delta(p,p_{(\bar{2})},p_{(\bar{3})})\Delta(p,p_{(\bar{2})},p_{(2)})}.$$

We may use Eq. (19) to show that the reduced amplitudes of type II [Eq. (60)] are simply related to invariant amplitudes of type I:

$$S'^{qf}[\bar{s}_{(l)(m)},\lambda_{(\bar{k})}\,|\,s;\Phi,\Theta,\Psi\,|\,s_{(l)(m)},\lambda_{(k)}] = J_{\{\bar{r}\}}{}^{-1/2}J_{\{r\}}{}^{-1/2}S'^{qf}[\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}\,|\,\vec{p}_{(k)}:\lambda_{(k)}], \tag{64}$$

where the Jacobians $J_{\{\bar{r}\}}$ and $J_{\{r\}}$ are defined by Eqs. (A38)–(A40).

C. Partial-wave amplitudes

We now construct type-III amplitudes by sandwiching the scattering operator $\hat{S}$ between type-III multiparticle states,

$$S^{qf}[\bar{s},\bar{\sigma}:\bar{p},\bar{\lambda}:\bar{\mu};\bar{s}_{(l)(m)},\lambda_{(\bar{k})}\,|\,s,\sigma:p,\lambda:\mu;s_{(l)(m)},\lambda_{(k)}]$$
$$={}_{qf}[\bar{s},\bar{\sigma}:\bar{p},\bar{\lambda}:\bar{\mu};\bar{s}_{(l)(m)},\lambda_{(\bar{k})}\,|\hat{S}|\,s,\sigma:p,\lambda:\mu;s_{(l)(m)},\lambda_{(k)}]_{qf}. \tag{65}$$

These amplitudes are parametrized by the eigenvalues $\bar{\sigma}(\bar{\sigma}+1)$, $\sigma(\sigma+1)$, $\bar{\mu}$, and μ of spin operators $\underline{\hat{S}}_{[\bar{r}]}{}^2$, $\underline{\hat{S}}_{[r]}{}^2$, $\hat{S}_{[\bar{r}]}{}^{p(\bar{2})}$, and $\hat{S}_{[r]}{}^{p(2)}$, respectively.

Since type-III states have the space-time translation property (29), the total four-momentum is conserved, and we may define a set of reduced amplitudes:

$$S'^{qf}[\bar{\sigma}:\bar{\lambda}:\bar{\mu};\bar{s}_{(l)(m)},\lambda_{(\bar{k})}\,|\,s,\vec{p}\,|\,\sigma:\lambda:\mu;s_{(l)(m)},\lambda_{(k)}],$$
$$S^{qf}[\bar{s},\bar{\sigma}:\bar{p},\bar{\lambda}:\bar{\mu};\bar{s}_{(l)(m)},\lambda_{(\bar{k})}\,|\,s,\sigma:p,\lambda:\mu;s_{(l)(m)},\lambda_{(k)}] \tag{66}$$
$$=\delta^4(\bar{p}-p)S'^{qf}[\bar{\sigma}:\bar{\lambda}:\bar{\mu};\bar{s}_{(l)(m)},\lambda_{(\bar{k})}\,|\,s,\vec{p}\,|\,\sigma:\lambda:\mu;s_{(l)(m)},\lambda_{(k)}].$$

We now derive a reduced amplitude constraint equation from the homogeneous Lorentz-transformation properties of type-III states [Eq. (30)], and of the scattering operator $\hat{S}$ [Eq. (36)]. If we suppress for the moment the Poincaré-scalar labels $\bar{\mu}$, $\bar{s}_{(l)(m)}$, μ, $s_{(l)(m)}$, $\lambda_{(\bar{k})}$, and $\lambda_{(k)}$ we find

$$S'^{qf}[\bar{\sigma},\bar{\lambda}\,|\,s,\vec{p}\,|\,\sigma,\lambda]$$
$$=\sum_{\bar{\lambda}'\lambda'} D^{\bar{\sigma}*}_{\bar{\lambda}'\bar{\lambda}}(\hat{L}(\hat{\Lambda}:p))D^{\sigma}_{\lambda'\lambda}(\hat{L}(\hat{\Lambda}:p))$$
$$\times S'^{qf}[\bar{\sigma},\bar{\lambda}'\,|\,s,\vec{p}^{\dagger}\,|\,\sigma,\lambda']. \tag{67}$$

When the Lorentz transformation $\hat{\Lambda}$ is of the form $\exp(2\pi i\hat{J}_3)$, we have the relation

$$S'^{qf}[\bar{\sigma},\bar{\lambda}\,|\,s,\vec{p}\,|\,\sigma,\lambda] = (-1)^{2(\bar{\sigma}+\sigma)}S'^{qf}[\bar{\sigma},\bar{\lambda}\,|\,s,\vec{p}\,|\,\sigma,\lambda], \tag{68}$$

and the amplitudes vanish unless the sum of initial- and final-particle spins is an integer. If we now take $\hat{\Lambda}$ to be of the form $\hat{L}(p')\hat{L}^{-1}(p)$, for any momentum p' we find

$$S'^{qf}[\bar{\sigma},\bar{\lambda}\,|\,s,\vec{p}\,|\,\sigma,\lambda] = S'^{qf}[\bar{\sigma},\bar{\lambda}\,|\,s,\vec{p}'\,|\,\sigma,\lambda], \tag{69}$$

and the amplitudes must be independent of the total three-momentum $\vec{p}$. Finally, if we take the Lorentz transformation $\hat{\Lambda}$ to be $\hat{L}(p)\hat{R}(\alpha,\beta,\gamma)\hat{L}^{-1}(p)$ and integrate over angles α, β, and γ, we obtain the spin-dependent constraint equation:

$$S'^{qf}[\bar{\sigma},\bar{\lambda}\,|\,s,\vec{p}\,|\,\sigma,\lambda] = \delta_{\bar{\lambda}\lambda}\delta_{\bar{\sigma}\sigma}\sum_{\lambda'}S'^{qf}[\sigma,\lambda'\,|\,s,\vec{p}\,|\,\sigma,\lambda']. \tag{70}$$

This implies that the amplitudes are diagonal in parameters σ and λ, and independent of the value of the third component of spin λ. This leads us to define a new reduced amplitude of type III, $S'^{qf}_{\bar{\mu}\mu}[\bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s, \sigma | s_{(l)(m)}, \lambda_{(k)}]$, by

$$S'^{qf}[\bar{\sigma}:\bar{\lambda}:\bar{\mu}; \bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s, \vec{p} | \sigma:\lambda:\mu; s_{(l)(m)}, \lambda_{(k)}] = \delta_{\bar{\sigma}\sigma}\delta_{\bar{\lambda}\lambda} S'^{qf}_{\bar{\mu}\mu}[\bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s, \sigma | s_{(l)(m)}, \lambda_{(k)}]. \tag{71}$$

We have now shown explicitly that type-III amplitudes are functions of Poincaré scalars alone. We should like to relate them to the type-II invariant amplitudes defined in Sec. III B.

First of all, we use Eq. (28), which relates type-II states to type-III states, to obtain the expansion

$$\begin{aligned} &S^{qf}[\bar{s}:\bar{p}; \bar{\phi}, \bar{\theta}, \bar{\psi}: \bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s:p; \phi, \theta, \psi: s_{(l)(m)}, \lambda_{(k)}] \\ &= \sum_{\bar{\sigma}\sigma\bar{\lambda}\lambda\bar{\mu}\mu} [(2\sigma+1)(2\bar{\sigma}+1)]^{1/2} D^{\bar{\sigma}*}_{\bar{\lambda}\bar{\mu}}(\hat{R}(\bar{\phi}, \bar{\theta}, \bar{\psi}-\bar{\phi})) D^{\sigma}_{\lambda\mu}(\hat{R}(\phi, \theta, \psi-\phi)) [l(\hat{L}^{-1}(p, p_{(2)}, p_{(3)}):p; q, f)]^{2\sigma} \\ &\quad \times [l(L^{-1}(p; p_{(\bar{2})}, p_{(\bar{3})}):p; q, f)]^{2\bar{\sigma}} S^{qf}[\bar{s}, \bar{\sigma}:\bar{p}, \bar{\lambda}:\bar{\mu}; \bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s, \sigma:p, \lambda:\mu; s_{(l)(m)}, \lambda_{(k)}]. \end{aligned} \tag{72}$$

If we now substitute into this equation our expressions (60) and (71) for the reduced amplitudes, we obtain the relation

$$\begin{aligned} &S'^{qf}[\bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s:\Phi, \Theta, \Psi | s_{(l)(m)}, \lambda_{(k)}] \\ &= \sum_{\sigma\bar{\mu}\mu} [2\sigma+1] D^{\sigma}_{\bar{\mu}\mu}(R(\Phi, \Theta, \Psi-\Phi)) [l(\hat{R}^{-1}(\Phi, \Theta, \Psi-\Phi):p^{\star}; q^{\star}, f^{\star})]^{2\sigma} S'^{qf}_{\bar{\mu}\mu}[\bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s, \sigma | s_{(l)(m)}, \lambda_{(k)}], \end{aligned} \tag{73}$$

where

$$p^{\star} = L^{-1}(p; p_{(2)}, p_{(3)})p, \quad q^{\star} = L^{-1}(p; p_{(2)}, p_{(3)})q,$$

and (74)

$$f^{\star} = L^{-1}(p; p_{(2)}, p_{(3)})f.$$

We see that our type-III amplitudes $S'^{qf}_{\bar{\mu}\mu}[\bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s, \sigma | s_{(l)(m)}, \lambda_{(k)}]$ are partial-wave amplitudes associated with type-II invariant amplitudes $S'^{qf}[\bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s:\Phi, \Theta, \Psi | s_{(l)(m)}, \lambda_{(k)}]$. By construction, parameters $\bar{\mu}$ and μ are the eigenvalues of the commuting operators $\hat{S}_{[\bar{r}]}{}^{p(\bar{2})}$ and $\hat{S}_{[r]}{}^{p(2)}$, respectively, where

$$\hat{S}_{[\bar{r}]}{}^{p(\bar{2})} = [\Delta(\hat{p}_{(\bar{2})}, \hat{p}_{[\bar{r}]})]^{-1} \epsilon^{\mu\nu\rho\sigma} \hat{p}_{(\bar{2})\mu} \hat{p}_{[\bar{r}]\nu} \hat{J}_{[\bar{r}]\rho\sigma} \tag{75}$$

and

$$\hat{S}_{[r]}{}^{p(2)} = [\Delta(\hat{p}_{(2)}, \hat{p}_{[r]})]^{-1} \epsilon^{\mu\nu\rho\sigma} \hat{p}_{(2)\mu} \hat{p}_{[r]\nu} \hat{J}_{[r]\rho\sigma}. \tag{76}$$

One may wish to interpret them as eigenvalues of operators $\hat{S}^{p(\bar{2})}$ and $\hat{S}^{p(2)}$ defined by

$$\hat{S}^{p(\bar{2})} = [\Delta(\hat{p}_{(\bar{2})}, \hat{p})]^{-1} \epsilon^{\mu\nu\rho\sigma} \hat{p}_{(\bar{2})\mu} \hat{p}_{\nu} \hat{J}_{\rho\sigma} \tag{77}$$

and

$$\hat{S}^{p(2)} = [\Delta(\hat{p}_{(2)}, \hat{p})]^{-1} \epsilon^{\mu\nu\rho\sigma} \hat{p}_{(2)\mu} \hat{p}_{\nu} \hat{J}_{\rho\sigma}. \tag{78}$$

However, these operators do not commute. The parameter $\sigma(\sigma+1)$ may of course be regarded as the eigenvalue of any of the commuting operators $\underline{\hat{S}}_{[r]}{}^2$, $\underline{\hat{S}}_{[\bar{r}]}{}^2$, and $\underline{\hat{S}}^2$.

We may now use Eqs. (73), (64), and (B6) to expand any of the frame-dependent amplitudes listed in Table I of Appendix B in terms of the partial-wave amplitudes $S'^{qf}_{\bar{\mu}\mu}[\bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s, \sigma | s_{(l)(m)}, \lambda_{(k)}]$. We may also use the orthogonality and completeness properties of representations of the rotation group SU(2), or use Eq. (25) relating states of types II and III to express partial-wave amplitudes as integrals over the angle parameters of type-II invariant amplitudes,

$$\begin{aligned} &S'^{qf}_{\bar{\mu}\mu}[\bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s, \sigma | s_{(l)(m)}, \lambda_{(k)}] \\ &= \frac{(2\pi+1)^{1/2}}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} D^{\sigma*}_{\bar{\mu}\mu}(\hat{R}(\Phi, \Theta, \Psi-\Phi)) [l(\hat{R}^{-1}(\Phi, \Theta, \Psi-\Phi):p^{\star}; q^{\star}, f^{\star})]^{2\sigma} \\ &\quad \times S'^{qf}[\bar{s}_{(l)(m)}, \lambda_{(\bar{k})} | s:\Phi, \Theta, \Psi | s_{(l)(m)}, \lambda_{(k)}] \sin\Theta \, d\Phi \, d\Theta \, d\Psi. \end{aligned} \tag{79}$$

If we take momenta q and f to coincide with $p_{(2)}$ and $p_{(3)}$, respectively, we may omit the frame-independent phases $[l(R^{-1}(\Phi, \Theta, \Psi-\Phi):p^{\star}; q^{\star}, f^{\star})]^{2\sigma}$ from Eqs. (73) and (79), provided we take angles Φ, Θ, and Ψ in the range (62).

In conclusion we note that amplitudes of types II and III are well defined provided constraints (53) are satisfied, and in addition

$$\Delta(p, p_{(2)}, p_{(3)}) \neq 0$$

and (80)

$$\Delta(p, p_{(\overline{2})}, p_{(\overline{3})}) \neq 0 .$$

If we are also to avoid ambiguities of sign when the spin $\sigma_{\{r\}}$ is half-integral, the angle parameters $\overline{\theta}$, θ, and Θ of type-II amplitudes must not be equal to π. These conditions reflect the extent to which Lorentz frames can be uniquely specified by four-momentum triplets.

IV. SUMMARY AND CONCLUSIONS

In Sec. II we defined complete sets of multiparticle states $|\vec{p}_{(k)}; \lambda_{(k)}]_{qf}$ [Eq. (8)] which had particularly simple Poincaré transformation properties Eqs. (9) and (12). These states were formally defined in terms of the standard rest states $|\vec{0}:\lambda_{(k)}\rangle$ which we encounter in conventional field theories. They could easily be related to helicity states with the aid of Eq. (A34). The parameters $\lambda_{(k)}$ labeling the new states were eigenvalues of q-spin operators $\hat{S}_{(k)}{}^{q(k)}$ [(A23)] which are Poincaré-invariant observables. Had we taken the momentum operators $\hat{q}_{(k)}$ to coincide with the total-momentum observable $\hat{p}_{[r]}$ for all (k), the parameters $\lambda_{(k)}$ would have been the eigenvalues of c.m. helicity operators $\hat{S}_{(k)}{}^{p[r]}$. These operators $\hat{S}_{(k)}{}^{p[r]}$ were first introduced by Feldman and Matthews[9] in their discussion of some analyticity properties of two-particle scattering amplitudes.

We proceeded to define type-II states $|s:\vec{p}; \phi, \theta, \psi: s_{(l)(m)}, \lambda_{(k)}]_{qf}$ [Eq, (19)] parametrized by a maximum number of frame-independent scalar momentum product variables s and $s_{(l)(m)}$ [Eq. (15)], and a set of six frame-dependent functions of momentum components $\vec{p}$, ϕ, θ, and ψ [Eq. (14)]. These states were simply related [Eq. (25)] to type-III multiparticle states $|s, \sigma:\vec{p}, \lambda:\mu: s_{(l)(m)}, \lambda_{(k)}]_{qf}$, eigenstates of spin operators $\underline{\hat{S}}^2$, $\hat{S}_3$, and $\hat{S}^{p(2)}$, which had single-particle-state-type Poincaré-transformation properties [Eqs. (29) and (30)].

In Sec. III we defined Eqs. (42), (55), and (65), complete sets of scattering amplitudes in terms of our q-spin states of types I, II, and III, and showed that they were functions of Poincaré scalars alone. We defined sets of reduced amplitudes

$S'^{qf}[\vec{p}_{(\overline{k})}:\lambda_{(\overline{k})} | \vec{p}_{(k)}:\lambda_{(k)}]$ [Eq. (43)],

$S'^{qf}[\overline{s}_{(l)(m)}, \lambda_{(\overline{k})} | s:\Phi, \Theta, \Psi | s_{(l)(m)}, \lambda_{(k)}]$ [Eq. (60)],

and

$S'^{qf}_{\overline{\mu}\mu}[\overline{s}_{(l)(m)}, \lambda_{(\overline{k})} | s:\sigma | s_{(l)(m)}, \lambda_{(k)}]$ [Eq. (71)],

and constructed a frame-independent expansion (73) of invariant multiparticle amplitudes of type II in terms of multiparticle amplitudes of type III,[10]

$$S'^{qf}[\overline{s}_{(l)(m)}, \lambda_{(\overline{k})} | s:\Phi, \Theta, \Psi | s_{(l)(m)}, \lambda_{(k)}] = \sum_{\sigma\overline{\mu}\mu} (2\sigma+1) D^{\sigma}_{\overline{\mu}\mu}(\Phi, \Theta, \Psi-\Phi) S'^{qf}_{\overline{\mu}\mu}[\overline{s}_{(l)(m)}, \lambda_{(\overline{k})} | s, \sigma | s_{(l)(m)}, \lambda_{(k)}] . \tag{81}$$

Moreover, we pointed out that any frame-dependent scattering amplitude could be expanded in terms of our invariant amplitudes with the aid of Eq. (B6).

Throughout this paper we have indicated the extent to which our scattering amplitudes are well defined. We have made no reference to analyticity, crossing, or unitarity properties of our amplitudes, as we hope to investigate them in a separate paper.

Our aim has not been merely to give a formalism which may prove useful for multiparticle scattering-amplitude analysis. We wished to give a physical interpretation of the invariant-amplitude decomposition of an S matrix and clarify the physical significance of partial-wave decomposition formulas. With our "observable" approach we now propose to examine in detail[5] the kinematical structure of several dynamical theories with which we are all familiar.

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APPENDIX A: MULTIPARTICLE STATES

1. Momentum-dependent Lorentz transformations

With each spacelike or timelike momentum p_μ of the form[8]

$$p_\mu \to \Delta(p)(\cosh\delta; \sinh\delta \sin\theta \cos\phi, \sinh\delta \sin\theta \sin\phi, \sinh\delta \cos\theta), \quad p^2 > 0 \tag{A1}$$

or

$$p_\mu \to |\Delta(p)|(\sinh\delta, \cosh\delta \sin\theta \cos\phi, \cosh\delta \sin\theta \sin\phi, \cosh\delta \cos\theta), \quad p^2 < 0 \tag{A2}$$

we associate a triplet of parameters δ, θ, and ϕ lying in the intervals

$$\phi \in [0, 2\pi), \quad \theta \in [0, \pi],$$

and (A3)

$$\delta \in [0, \infty).$$

When the boost parameter δ is zero and the momentum p is timelike, the parameters ϕ and θ become indeterminate as functions of components of the momentum p_μ. Similarly, when the angle θ is zero or π, the angle ϕ is indeterminate for both timelike and spacelike momenta p_μ. We list below various Lorentz-transformation operators which are functions of parameters ϕ, θ, and δ, and indicate the points within the intervals (A3) at which these operators are not well defined as functions of momentum components p_μ:

$$\hat{R}_+(p) = \hat{R}_+(\vec{p}) = \hat{R}(\phi, \theta, -\phi) = e^{-i\hat{J}_3\phi} e^{-i\hat{J}_2\theta} e^{i\hat{J}_3\phi}, \quad \delta \neq 0, \ \theta \neq \pi \tag{A4}$$

$$\hat{R}_-(p) = \hat{R}_-(\vec{p}) = \hat{R}_+(-\vec{p}) = \hat{R}(\phi, \theta - \pi, -\phi), \quad \delta \neq 0, \ \theta \neq 0 \tag{A5}$$

$$\hat{R}(p) = \hat{R}(\vec{p}) = R(\phi, \theta, 0), \quad \delta \neq 0, \ \theta \neq 0, \ \pi \tag{A6}$$

$$\hat{R}_3(p) = \hat{R}_3(\vec{p}) = \hat{R}(\phi, 0, 0), \quad \delta \neq 0, \ \theta \neq 0, \ \pi \tag{A7}$$

$$\hat{Z}(p) = e^{-i\hat{K}_3\delta}, \tag{A8}$$

$$\hat{H}(p) = \hat{R}(p)\hat{Z}(p), \quad \delta \neq 0, \ \theta \neq 0, \ \pi \tag{A9}$$

$$\hat{H}_+(p) = \hat{R}_+(p)\hat{Z}(p), \quad \delta \neq 0, \ \theta \neq \pi \tag{A10}$$

and

$$L(p) = R(p)Z(p)R^{-1}(p). \tag{A11}$$

It is to be noted that we shall always use symbols $\hat{\Lambda}$, $\hat{R}$, $\hat{H}$, etc., to denote Lorentz-transformation operators or the corresponding 2×2 representation matrices with unit determinant [SL(2, C)]. We shall denote the associated 4×4 self-representation matrices by the symbols Λ, R, H, etc.

In terms of a timelike momentum p and two spacelike or timelike momenta q and f, we define Lorentz-transformation operators $\hat{H}_+(q;p)$, $\hat{H}_+(q;p,f)$, and $\hat{L}(p;q,f)$ by

$$\hat{H}_+(q;p) = \hat{L}(p)\hat{H}_+(Q), \quad \Delta(p,q) \neq 0, \ \theta(Q) \neq \pi \tag{A12}$$

$$\hat{H}_+(q;p,f) = \hat{L}(p)\hat{H}_+(Q)\hat{R}_3(R_+^{-1}(Q)F), \quad \Delta(p,q,f) \neq 0, \ \theta(Q) \neq \pi \tag{A13}$$

$$\hat{L}(p;q,f) = \hat{L}(p)\hat{R}_-(Q)\hat{R}_3(R_-^{-1}(Q)F), \quad \Delta(p,q,f) \neq 0, \ \theta(Q) \neq 0 \tag{A14}$$

where we always use capital letters Q, F, etc., to denote momenta of the form

$$Q = L^{-1}(p)q \tag{A15}$$

and

$$F = L^{-1}(p)f, \ \text{etc.} \tag{A16}$$

It should be noted that for the indicated values of $\theta(Q)$ the operators $\hat{H}_+(q;p,f)$ and $\hat{L}(p;q,f)$ are well defined up to a factor $\exp(2\pi i\hat{J}_3)$, provided that $\Delta(p,q,f) \neq 0$.

In terms of two timelike momenta p and $p_{(k)}$, and four-momenta q, $q_{(k)}$, f, and $f_{(k)}$ satisfying the equations

$$\Delta(p,q,f) \neq 0$$

and (A17)

$$\Delta(p_{(k)}, q_{(k)}, f_{(k)}) \neq 0,$$

we define a Lorentz transformation $\hat{L}(p;q,f:p_{(k)};q_{(k)},f_{(k)})$ by

$$\hat{L}(p;q,f:p_{(k)};q_{(k)},f_{(k)}) = \hat{L}(p;q,f)\hat{L}(P^\star_{(k)};Q^\star_{(k)},F^\star_{(k)}), \tag{A18}$$

with momenta $P^\star_{(k)}$, $Q^\star_{(k)}$, and $F^\star_{(k)}$ determined by Eqs. (C10)–(C13).

Finally, for any Lorentz transformation $\hat{\Lambda}$, and four-momentum triplet p, q, and f, we define the signs $l(\hat{\Lambda}:p;q,f)$ and a generalized Wigner rotation $\hat{L}(\hat{\Lambda}:p;q,f)$ by

$$[l(\hat{\Lambda}:p;q,f)]^{2\hat{J}_3} = \hat{L}(\hat{\Lambda}:p;q,f) = \hat{L}^{-1}(p^\dagger;q^\dagger,f^\dagger)\hat{\Lambda}\hat{L}(p;q,f), \tag{A19}$$

where we always use the dagger to denote transformed momenta

$$p^\dagger = \Lambda p, \quad q^\dagger = \Lambda q, \quad f^\dagger = \Lambda f, \ \text{etc.} \tag{A20}$$

2. Multiparticle states

We define r-particle states $|\vec{p}_{(k)}:\lambda_{(k)})$ to be eigenstates of r three-momentum operators $\hat{\underline{p}}_{(k)}$ and r spin-component operators $\hat{\Sigma}_{(k)}$ with eigenvalues $\vec{p}_{(k)}$ and $\lambda_{(k)}$, respectively. The various spin-component operators $\hat{\Sigma}_{(k)}$ are given in terms of single-particle observables by[8]

$$\hat{S}_{(k)3} = m_{(k)}{}^{-1}L^{-1}(\hat{p}_{(k)})_3{}^\mu \hat{W}_{(k)\mu}, \tag{A21}$$

$$\hat{S}_{(k)} = |\hat{\underline{p}}_{(k)}|^{-1}\hat{W}_{(k)0}, \tag{A22}$$

and

$$\hat{S}_{(k)}{}^{q(k)} = [\Delta(\hat{p}_{(k)}, \hat{q}_{(k)})]^{-1}\hat{q}_{(k)}\cdot\hat{W}_{(k)}, \tag{A23}$$

where the Pauli-Lubanski spin $\hat{W}_{(k)}$ is defined by

$$\hat{W}_{(k)\mu} = -\tfrac{1}{2}\epsilon_\mu{}^{\nu\rho\sigma}\hat{p}_{(k)\nu}\hat{J}_{(k)\rho\sigma}. \tag{A24}$$

We formally define states $|\vec{p}_{(k)}:\lambda_{(k)})$ in terms of standard rest states $|\vec{0}:\lambda_{(k)}\rangle$ which are eigenstates of spin-component operators $\hat{S}_{(k)3}$ and three-momentum operators $\hat{\underline{p}}_{(k)}$ with eigenvalues $\lambda_{(k)}$ and $\vec{0}$, respectively,

$$|\vec{p}_{(k)}:\lambda_{(k)}) = \prod_{k=1}^{r} [\hat{B}_{(k)}(p_{(k)})]|\vec{0}:\lambda_{(k)}\rangle . \tag{A25}$$

Each Lorentz transformation $\hat{B}_{(k)}(p_{(k)})$, which is generated by operators $\hat{J}_{(k)\mu\nu}$, may be a function of all momentum eigenvalues $\vec{p}_{(k)}$ for $k=1,2,\ldots,r$. The states $|\vec{p}_{(k)}:\lambda_{(k)})$ and the associated operators $\hat{\Sigma}_{(k)}$ and $\hat{B}(p_{(k)})$ are listed in Table II.

The overlap of any two states $|\vec{p}_{(k)}:\lambda_{(k)})$ and $|\vec{p}_{(k)}:\lambda_{(k)})^{\star}$ is given by

$$^{\star}(\vec{p}'_{(k)}:\lambda'_{(k)}|\vec{p}_{(k)}:\lambda_{(k)}) = \prod_{k=1}^{r} [2p_{(k)0}\delta(\vec{p}'_{(k)}-\vec{p}_{(k)})D^{\sigma(k)}_{\lambda'_{(k)}\lambda_{(k)}}(\hat{B}^{\star -1}(p_{(k)})\hat{B}(p_{(k)}))], \tag{A26}$$

where the functions $D^{\sigma}_{\lambda'\lambda}(\hat{R})$ are irreducible unitary rotation group [SU(2)] representations of rotations $\hat{R}$.

We now define Poincaré-group generators $\hat{J}_{[r]\mu\nu}$ and $\hat{p}_{[r]\mu}$ by

$$\hat{J}_{[r]\mu\nu} = \sum_{k=1}^{r} \hat{J}_{(k)\mu\nu} \tag{A27}$$

and

$$\hat{p}_{[r]\mu} = \sum_{k=1}^{r} \hat{p}_{(k)\mu} . \tag{A28}$$

The corresponding Lorentz transformations $\hat{\Lambda}_{[r]}$ and translations $\hat{a}_{[r]}$ of the form

$$\hat{\Lambda}_{[r]} = \hat{R}_{[r]}(\alpha,\beta,\gamma)\hat{Z}_{[r]}(\delta)\hat{R}_{[r]}(0,\beta',\gamma') \tag{A29}$$

and

$$\hat{a}_{[r]} = \exp(ia\cdot\hat{p}_{[r]}) \tag{A30}$$

have the following matrix elements between states $|\vec{p}_{(k)}:\lambda_{(k)})$:

$$(\vec{p}'_{(k)}:\lambda'_{(k)}|\hat{a}_{[r]}|\vec{p}_{(k)}:\lambda_{(k)}) = e^{ip\cdot a}\prod_{k=1}^{r}[2p_{(k)0}\delta(\vec{p}'_{(k)}-\vec{p}_{(k)})\delta_{\lambda'_{(k)}\lambda_{(k)}}] \tag{A31}$$

and

$$(\vec{p}'_{(k)}:\lambda'_{(k)}|\hat{\Lambda}_{[r]}|\vec{p}_{(k)},\lambda_{(k)}) = \prod_{k=1}^{r}[2p_{(k)0}\delta(\vec{p}'_{(k)}-\vec{p}_{(k)})D^{\sigma(k)}_{\lambda'_{(k)}\lambda_{(k)}}(\hat{B}(\hat{\Lambda}:p_{(k)}))], \tag{A32}$$

where the momentum p is the eigenvalue of the operator (A28) and the generalized Wigner rotations are defined by

$$\hat{B}(\hat{\Lambda}:p_{(k)}) = \hat{B}^{-1}(p^{\dagger}_{(k)})\hat{\Lambda}\hat{B}(p_{(k)}). \tag{A33}$$

One may use Eqs. (A31) and (A32) to determine the Poincaré-transformation properties of multiparticle states. For the states $|\vec{p}_{(k)}:\lambda_{(k)})$ we have

$$\hat{a}_{[r]}|\vec{p}_{(k)}:\lambda_{(k)}) = \prod_{k=1}^{r}\left[\sum_{\lambda'_{(k)}}\int\frac{d\vec{p}'_{(k)}}{2p'_{(k)0}}\right](\vec{p}'_{(k)}:\lambda'_{(k)}|\hat{a}_{[r]}|\vec{p}_{(k)}:\lambda_{(k)})|\vec{p}'_{(k)}:\lambda'_{(k)})$$
$$= e^{ip\cdot a}|\vec{p}_{(k)}:\lambda_{(k)}) \tag{A34}$$

TABLE II. Multiparticle states.

State	$\vert\vec{p}_{(k)}:\lambda_{(k)})$	$\hat{B}(p_{(k)})$		$\hat{\Sigma}_{(k)}$	$\hat{B}(\hat{\Lambda}:p_{(k)})$ (A21)	Reference
Standard	$\vert\vec{p}_{(k)}:\lambda_{(k)}\rangle$	$\hat{L}(p_{(k)})$	(A11)	$\hat{S}_{(k)3}$	$\hat{L}(\hat{\Lambda}:p_{(k)})$	11
Helicity	$\vert\vec{p}_{(k)}:\lambda_{(k)}\}_{+}$	$\hat{H}_{+}(p_{(k)})$	(A10)	$\hat{S}_{(k)}$	$\hat{H}_{+}(\hat{\Lambda}:p_{(k)})$	7,12
	$\vert\vec{p}_{(k)}:\lambda_{(k)}\}$	$\hat{H}(p_{(k)})$	(A9)	$\hat{S}_{(k)}$	$\hat{H}(\hat{\Lambda}:p_{(k)})$	7
q spin	$\vert\vec{p}_{(k)}:\lambda_{(k)}]_{+}$	$\hat{H}_{+}(p_{(k)};q_{(k)})$	(A12)	$\hat{S}_{(k)}{}^{q(k)}$	$\hat{H}_{+}(\hat{\Lambda}:p_{(k)};q_{(k)})$	6
	$\vert\vec{p}_{(k)}:\lambda_{(k)}]$	$\hat{L}(p_{(k)};q_{(k)},f_{(k)})$	(A14)	$\hat{S}_{(k)}{}^{q(k)}$	$\hat{L}(\hat{\Lambda}:p_{(k)};q_{(k)},f_{(k)})$	6
	$\vert\vec{p}_{(k)}:\lambda_{(k)}]_{qf}$	$\hat{L}(p;q,f:p_{(k)};q_{(k)},f_{(k)})$	(A18)	$\hat{S}_{(k)}{}^{q(k)}$	$\hat{L}(\hat{\Lambda}:p;q,f)$	Sec. II

and

$$\Lambda_{[r]}|\vec{p}_{(k)}:\lambda_{(k)}) = \prod_{k=1}^{r}\left[\sum_{\lambda'_{(k)}}\int\frac{d\vec{p}'_{(k)}}{2p'_{(k)0}}\right](\vec{p}'_{(k)}:\lambda'_{(k)}|\hat{\Lambda}_{[r]}|\vec{p}_{(k)}:\lambda_{(k)})|\vec{p}'_{(k)}:\lambda'_{(k)})$$

$$= \prod_{k=1}^{r}\left[\sum_{\lambda'_{(k)}} D^{\sigma(k)}_{\lambda'_{(k)}\lambda_{(k)}}(\hat{B}(\hat{\Lambda}:p_{(k)}))\right]|\vec{p}^{\dagger}_{(k)}:\lambda'_{(k)}). \tag{A35}$$

3. States of types II and III

States of type II are parametrized by a maximum number of scalar momentum product variables s and $s_{(l)(m)}$ which are defined by equations of Sec. II B:

$$|s:\vec{p};\phi,\theta,\psi,s_{(l)(m)}:\lambda_{(k)}) = J_{\{r\}}^{-1/2}|\vec{p}_{(k)}:\lambda_{(k)}). \tag{A36}$$

The angles ϕ, θ, and ψ are defined by

$$\hat{R}(\phi,\theta,\psi-\phi) = \hat{L}^{-1}(p)\hat{L}(p;p_{(2)},p_{(3)}), \tag{A37}$$

and the Poincaré-scalar Jacobians $J_{\{r\}}$ are given by

$$J_{\{3\}} = 4s\pi^{-2}, \tag{A38}$$

$$J_{\{4\}} = 64s\pi^{-2}|n'\cdot p_{(4)}| = 64\pi^{-2}s|\Delta(p_{(1)},p_{(2)},p_{(3)},p_{(4)})|, \tag{A39}$$

and

$$J_{\{r\}} = \frac{-64s}{\pi^2 n'^2}\left|\prod_{k=5}^{r}[16n'\cdot p_{(k)}]\right|\left|\sum_{k=5}^{r}[m'\cdot p_{(4)}n'\cdot p_{(k)} - n'\cdot p_{(4)}m'\cdot p_{(k)}] + n'^2 n'\cdot p_{(4)}\right|, \quad r\geq 5 \tag{A40}$$

where the four-vectors m' and n' are defined by

$$n'_\mu = \epsilon_\mu{}^{\nu\rho\sigma}p_\nu p_{(2)\rho}p_{(3)\sigma} \tag{A41}$$

and

$$m'_\mu = \epsilon_\mu{}^{\nu\rho\sigma}p_\nu p_{(2)\rho}n'_\sigma\ . \tag{A42}$$

Type-II states have the following normalizations:

$$(s':\vec{p}';\phi',\theta',\psi',s'_{(l)(m)}:\lambda'_{(k)}|s:\vec{p};\phi,\theta,\psi,s_{(l)(m)}:\lambda_{(k)}) \tag{A43}$$

$$= \delta(s'-s)2p_0\delta(\vec{p}'-\vec{p})8\pi^2\delta(\phi'-\phi)\delta(\cos\theta'-\cos\theta)\delta(\psi'-\psi)\prod_{klm}[\delta(s'_{(l)(m)}-s_{(l)(m)})\delta_{\lambda'_{(k)}\lambda_{(k)}}], \tag{A44}$$

and transform in the following way under space-time translations $\hat{a}_{[r]}$ and homogeneous Lorentz transformations $\hat{\Lambda}_{[r]}$:

$$\hat{a}_{[r]}|s:\vec{p};\phi,\theta,\psi,s_{(l)(m)}:\lambda_{(k)}) = e^{ip\cdot a}|s:\vec{p};\phi,\theta,\psi,s_{(l)(m)}:\lambda_{(k)}), \tag{A45}$$

$$\hat{\Lambda}_{[r]}|s:\vec{p};\phi,\theta,\psi,s_{(l)(m)}:\lambda_{(k)}) = \prod_{k=1}^{r}\left[\sum_{\lambda'_{(k)}}\right]D^{\sigma(k)}_{\lambda'_{(k)}\lambda_{(k)}}(\hat{B}(\hat{\Lambda}:p_{(k)}))|s:\vec{p}^{\dagger};\phi^{\dagger},\theta^{\dagger},\psi^{\dagger},s_{(l)(m)}:\lambda'_{(k)}), \tag{A46}$$

where the angles $\phi^\dagger$, $\theta^\dagger$, and $\psi^\dagger$ are defined in terms of transformed momenta by Eq. (A16).

We define states of type III by

$$|s,\sigma:\vec{p},\lambda:\mu:s_{(l)(m)},\lambda_{(k)}]_{qf}$$

$$= \frac{(2\sigma+1)^{1/2}}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}D^{\sigma*}_{\lambda\mu}(\hat{R}(\phi,\theta,\psi-\phi))[l(\hat{L}^{-1}(p;p_{(2)},p_{(3)}):p;q,f)]^{2\sigma}|s:\vec{p};\phi,\theta,\psi:s_{(l)(m)},\lambda_{(k)}]_{qf}$$

$$\times\sin\theta\,d\phi\,d\theta\,d\psi. \tag{A47}$$

These states transform like single-particle states under space-time translations $\hat{a}_{[r]}$ and homogeneous Lorentz transformations $\hat{\Lambda}_{[r]}$,

$$\hat{a}_{[r]}|s,\sigma:\vec{p},\lambda:\mu:s_{(l)(m)},\lambda_{(k)}]_{qf}=e^{ip\cdot a}|s,\sigma:\vec{p},\lambda:\mu:s_{(l)(m)},\lambda_{(k)}]_{qf}, \tag{A48}$$

$$\hat{\Lambda}_{[r]}|s,\sigma:\vec{p},\lambda:\mu:s_{(l)(m)},\lambda_{(k)}]_{qf}=\sum_{\lambda'}D^{\sigma}_{\lambda'\lambda}(\hat{L}(\hat{\Lambda}:p))|s,\sigma:\vec{p}^{\dagger},\lambda':\mu:s_{(l)(m)},\lambda_{(k)}]_{qf}, \tag{A49}$$

and have the simple normalization

$$_{qf}[s',\sigma':\vec{p}',\lambda':\mu':s'_{(l)(m)},\lambda'_{(k)}|s,\sigma:\vec{p},\lambda:\mu:s_{(l)(m)},\lambda_{(k)}]_{qf}$$
$$=\delta(s'-s)\delta_{\sigma'\sigma}2p_0\delta(\vec{p}'-\vec{p})\delta_{\lambda'\lambda}\delta_{\mu'\mu}\prod_{klm}[\delta(s'_{(l)(m)}-s_{(l)(m)})\delta_{\lambda'_{(k)}\lambda_{(k)}}]. \tag{A50}$$

APPENDIX B: SOME MULTIPARTICLE SCATTERING AMPLITUDES

We define scattering amplitudes to be matrix elements of a scattering operator $\hat{S}$ between the multiparticle states of the type listed in Table II of Appendix A:

$$S(\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)})=(\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\hat{S}|\vec{p}_{(k)}:\lambda_{(k)}). \tag{B1}$$

In order to follow our notation one should refer to Table I in which we list various amplitudes together with the corresponding multiparticle states.

Each amplitude satisfies a space-time-translation constraint equation of the form

$$S(\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)})$$
$$=e^{-i(\bar{p}-p)\cdot a}S(\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)}), \tag{B2}$$

where $\bar{p}$ denotes the eigenvalue of the operator $\hat{p}_{[\bar{r}]}$ defined by

$$\hat{p}_{[\bar{r}]}=\sum_{\bar{k}=1}^{\bar{r}}\hat{p}_{(\bar{k})}. \tag{B3}$$

We define the closely related reduced amplitudes $S'(\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)})$ by

$$S(\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)})$$
$$=\delta^4(\bar{p}-p)S'(\vec{p}_{(\bar{k})},\lambda_{(\bar{k})}|\vec{p}_{(k)},\lambda_{(k)}). \tag{B4}$$

These amplitudes satisfy homogeneous-Lorentz-transformation constraint equations of the form

$$S'(\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)})=\prod_{\bar{k}=1}^{\bar{r}}\Big[\sum_{\lambda'_{(\bar{k})}}D^{\sigma_{(\bar{k})}*}_{\lambda'_{(\bar{k})}\lambda_{(\bar{k})}}(\hat{B}(\hat{\Lambda}:p_{(\bar{k})}))\Big]$$
$$\times\prod_{k=1}^{r}\Big[\sum_{\lambda'_{(k)}}D^{\sigma_{(k)}}_{\lambda'_{(k)}\lambda_{(k)}}(\hat{B}(\hat{\Lambda}:p_{(k)}))\Big]S'(\vec{p}_{(\bar{k})}:\lambda'_{(\bar{k})}|\vec{p}_{(k)};\lambda'_{(k)}), \tag{B5}$$

where the transformed momenta $p^{\dagger}_{(k)}$ and generalized Wigner rotations $\hat{B}_+(\hat{\Lambda}:p_{(k)})$, and $\hat{B}(\hat{\Lambda}:p_{(k)})$ are defined by Eqs. (A20) and (A33).

Since each set of scattering amplitudes is complete, we may use Eq. (A26) connecting multiparticle states to relate them to each other,

$$S'^{\star}(\vec{p}_{(\bar{k})}:\lambda_{(\bar{k})}|\vec{p}_{(k)}:\lambda_{(k)})=\prod_{\bar{k}=1}^{\bar{r}}\Big[\sum_{\lambda'_{(\bar{k})}}D^{\sigma_{(\bar{k})}*}_{\lambda'_{(\bar{k})}\lambda_{(\bar{k})}}(\hat{B}^{\star-1}(p_{(\bar{k})})\hat{B}(p_{(\bar{k})}))\Big]$$
$$\times\prod_{k=1}^{r}\Big[\sum_{\lambda'_{(k)}}D^{\sigma_{(k)}}_{\lambda'_{(k)}\lambda_{(k)}}(\hat{B}^{\star-1}(p_{(k)})\hat{B}(p_{(k)}))\Big]S'(\vec{p}_{(\bar{k})}:\lambda'_{(\bar{k})}|\vec{p}_{(k)};\lambda'_{(k)}). \tag{B6}$$

In Sec. III we define the corresponding amplitudes of types II and III and derive some of their properties.

APPENDIX C: MOMENTA IN A SPECIAL LORENTZ FRAME

1. Kinematical Δ functions

We define four kinematical Δ functions of momenta $p_{(1)}$, $p_{(2)}$, $p_{(\bar{1})}$, and $p_{(\bar{2})}$:

$$\Delta(p_{(1)})=(p_{(1)}{}^2)^{1/2}, \tag{C1}$$

$$\Delta(p_{(1)},p_{(2)})=[(p_{(1)}\cdot p_{(2)})^2-p_{(1)}{}^2p_{(2)}{}^2]^{1/2}, \tag{C2}$$

$$\Delta(p_{(1)},p_{(2)},p_{(\bar{1})})=[p_{(1)}{}^2p_{(2)}{}^2p_{(\bar{1})}{}^2+2p_{(1)}\cdot p_{(2)}\,p_{(2)}\cdot p_{(\bar{1})}\,p_{(\bar{1})}\cdot p_{(1)}$$
$$-p_{(1)}{}^2(p_{(2)}\cdot p_{(\bar{1})})^2-p_{(2)}{}^2(p_{(\bar{1})}\cdot p_{(1)})^2-p_{(\bar{1})}{}^2(p_{(1)}\cdot p_{(2)})^2]^{1/2}, \tag{C3}$$

and

$$\Delta(p_{(1)}, p_{(2)}, p_{(\bar{1})}, p_{(\bar{2})}) = \epsilon^{\mu\nu\rho\sigma} p_{(1)\mu} p_{(2)\nu} p_{(\bar{1})\rho} p_{(\bar{2})\sigma}\,. \tag{C4}$$

The square roots in Eqs. (C1)–(C3) are taken to be positive when momentum $p_{(1)}$ is timelike. We may then identify the Δ function (C1) with the mass $m_{(1)}$:

$$\Delta(p_{(1)}) = m_{(1)},\; p_{(1)}{}^2 > 0\,. \tag{C5}$$

In general, the squares of Δ functions may be expressed as determinants of matrices with scalar elements:

$$\Delta^2(p_{(1)}, \ldots, p_{(r)}) = (-1)^{r+1} \det(p_{(l)} \cdot p_{(m)}), \quad l, m = 1, 2, \ldots, r\,. \tag{C6}$$

The momentum products $p_{(l)} \cdot p_{(m)}$ are then simply related to the scalar variables $s_{(l)(m)}$,

$$p_{(l)} \cdot p_{(m)} = -\tfrac{1}{2}(s_{(l)(m)} - m_{(l)}{}^2 - m_{(m)}{}^2)\,. \tag{C7}$$

For future reference we also define a kinematical function $\Omega(p; p_{(2)}, p_{(\bar{2})})$ which is symmetrical under the interchange of momenta $p_{(2)}$ and $p_{(\bar{2})}$,

$$\Omega(p; p_{(2)}, p_{(\bar{2})}) = p^2 p_{(2)} \cdot p_{(\bar{2})} - p \cdot p_{(\bar{2})}\, p \cdot p_{(2)}\,. \tag{C8}$$

2. Momenta in a special frame

In a Lorentz frame in which momentum p is zero, momentum $\vec{q}$ lies in the negative three-direction and momentum $\vec{f}$ has positive 1st component and zero 2nd component, the components of a momentum $p_{(k)}$ are given by

$$p_{(k)} \to P^{\star}_{(k)} = L^{-1}(p; q, f)\, p_{(k)}\,. \tag{C9}$$

Each momentum component $P^{\star}_{(k)\mu}$ may be expressed in a manifestly Poincaré-scalar form[6]:

$$P^{\star}_{(k)0} = \frac{p \cdot p_{(k)}}{\Delta(p)}\,, \tag{C10}$$

$$P^{\star}_{(k)1} = \frac{m'' \cdot p_{(k)}}{\Delta(p, q)\Delta(p, q, f)} = \frac{\Omega(p; q, p_{(k)})\Omega(p; q, f) + \Delta^2(p, q)\Omega(p; f, p_{(k)})}{p^2 \Delta(p, q, f)\Delta(p, q)}\,, \tag{C11}$$

$$P^{\star}_{(k)2} = \frac{n'' \cdot p_{(k)}}{\Delta(p, q, f)} = -\frac{\Delta(p, q, f, p_{(k)})}{\Delta(p, q, f)}\,, \tag{C12}$$

and

$$P^{\star}_{(k)3} = \frac{\Omega(p; q, p_{(k)})}{\Delta(p)\Delta(p, q)}\,, \tag{C13}$$

where the vectors n'' and m'' are defined by

$$n''_\mu = \epsilon_\mu{}^{\nu\rho\sigma} p_\nu q_\rho f_\sigma \tag{C14}$$

and

$$m''_\mu = \epsilon_\mu{}^{\nu\rho\sigma} p_\nu q_\rho n''_\sigma\,. \tag{C15}$$

When momentum $P^{\star}_{(k)}$ is timelike and of the form (A1), we may obtain explicit expressions for the associated Poincaré-scalar parameters ϕ, θ, and δ in terms of products of four-momenta:

$$\cosh\delta = \frac{p \cdot p_{(k)}}{\Delta(p)\Delta(p_{(k)})}\,, \qquad \sinh\delta = \frac{\Delta(p, p_{(k)})}{\Delta(p)\Delta(p_{(k)})}\,, \tag{C16}$$

$$\cos\theta = \frac{\Omega(p; q, p_{(k)})}{\Delta(p, q)\Delta(p, p_{(k)})}\,, \qquad \sin\theta = \frac{\Delta(p)\Delta(p, q, p_{(k)})}{\Delta(p, q)\Delta(p, p_{(k)})}\,, \tag{C17}$$

$$\cos\phi = \frac{m'' \cdot p_{(k)}}{\Delta(p, q, f)\Delta(p, q, p_{(k)})}\,, \qquad \sin\phi = \frac{-\Delta(p, q)\Delta(p, q, f, p_{(k)})}{\Delta(p, q, f)\Delta(p, q, p_{(k)})}\,. \tag{C18}$$

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[8]Kinematical Δ and Ω functions are defined in Appendix C.

[9]We use the term "c.m. helicity" to denote the eigenvalue of a c.m. helicity or $p_{[r]}$-spin operator $\hat{S}_{(k)}{}^{p[r]}$ [Eq. (A23)]. Such operators were first introduced by G. Feldman and P. T. Matthews [Phys. Rev. 168, 1587 (1968)], and examined in greater detail by M. King and G. Feldman [Nuovo Cimento 60A, 86 (1969)].

[10]For simplicity we take momenta q and f to coincide with momenta $p_{(2)}$ and $p_{(3)}$, respectively.

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