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# Quark confinement and the puzzle of the ninth axial-vector current

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A solution to the U(3) × U(3) problem—the absence of a ninth pseudoscalar Goldstone boson—is proposed. The mechanism for eliminating the unwanted Goldstone boson involves the same infrared instabilities of Yang-Mills theory which confine quarks and eliminate physical "color" states. A formal argument is presented, and two models of quark confinement illustrate the mechanism involved in eliminating the spurious massless state.

## I. THE PUZZLE

There are a number of reasons to believe in a Yang-Mills theory of strong interactions. These include the possibilities of quark confinement[1,2] and asymptotic freedom.[3] The most reasonable candidate for such a theory employs an additional "hidden" SU(3) symmetry of quarks which is called "color."[4] The color quantum number is coupled to an octet of Yang-Mills gauge fields. This model is potentially capable of explaining the absence of free quarks and the approximate free-particle behavior of quarks at small distances.

One serious difficulty of this model is that it appears to have too much symmetry. Let us begin with the quark-gluon Lagrangian,

$$\mathcal{L}=i\overline{\psi}\gamma_\mu\partial^\mu\psi-\overline{\psi}\mathfrak{M}\psi+g\overline{\psi}\gamma_\mu C^\alpha\psi A^\alpha_\mu-\tfrac{1}{4}F^{\ \alpha}_{\mu\nu}F^{\mu\nu}_\alpha, \tag{1.1}$$

where the quark field $\psi$ carries both color and ordinary SU(3) indices. The $C^\alpha$ are the eight color matrices in the adjoint representation. $\mathfrak{M}$ is the fermion mass matrix which must be color-invariant but need not be SU(3)-invariant. In particular, $\mathfrak{M}$ splits the masses of the nonstrange quarks from the strange quark.

There are many reasons for believing that the strong interactions are approximately SU(3) × SU(3)-chiral-invariant. In particular, the almost-massless character of pions, the success of the Goldberger-Treiman relation, the Adler-Weisberger relation and other soft-pion theorems, and the approximate validity of SU(3) all suggest that SU(3)×SU(3) chiral is realized as an approximate Goldstone symmetry.

In order that the Lagrangian in Eq. (1.1) be SU(3)×SU(3)-chiral-invariant, the mass matrix $\mathfrak{M}$ must vanish. Indeed, the octet of axial-vector currents satisfies

$$\partial_\mu\overline{\psi}\gamma^\mu\gamma^5\lambda^i\psi=2\overline{\psi}\gamma_5\{\lambda^i,\mathfrak{M}\}\psi, \tag{1.2}$$

where $\lambda^i$ are the eight SU(3) matrices. However, the vanishing of $\mathfrak{M}$ also implies the additional symmetry associated with the *ninth* axial-vector current $\overline{\psi}\gamma^\mu\gamma_5\psi$.

How might this apparent additional symmetry be realized? Its realization cannot be through degenerate multiplets because that would require massless nucleons. Realization as a Goldstone symmetry would require the existence of a massless pseudoscalar SU(3) singlet. The only available candidate for such a particle is the $\eta'$. However, the squared mass of the $\eta'$ is $\approx 1\ \text{GeV}^2$ which seems much too big to be a symmetry-breaking effect. In fact, the success of the Gell-Mann–Okubo mass formula requires the $\eta$ and $\eta'$ to be almost completely unmixed which in turn suggests that the $\eta'$ would be much more massive than the octet even when the $\lambda$-quark mass is switched off.

To summarize, we find that SU(3)×SU(3) approximate symmetry implies U(3)×U(3) approximate symmetry in the quark-gluon model. However, neither the ordinary nor the Goldstone realizations of the ninth axial symmetry appear consistent with the empirical spectrum.

There are two ways to avoid this dilemma. The first is to try to add terms to $\mathcal{L}$ which break U(3)×U(3) without breaking SU(3)×SU(3). Such terms exist but are nonrenormalizable. The second way introduces a multiplet of fundamental pseudoscalar- and scalar-meson fields which transform according to the $(3,\overline{3})+(\overline{3},3)$ representation of U(3)×U(3).[5] These fields may be coupled to the quarks in a way which preserves U(3)×U(3). However, a cubic renormalizable SU(3)×SU(3)-invariant interaction between these fields allows a violation of U(3)×U(3).

The difficulty with this last suggestion is that it would introduce fundamental boson fields which contribute to the weak and electromagnetic currents of hadrons. Unless the effects of these fields are particularly small in the nucleon, they would ruin various successful quark-model predictions for deep-inelastic lepton scattering.

In this paper we suggest a solution to the puzzle which is fully consistent with asymptotic freedom, approximate SU(3)×SU(3) symmetry, lack of fundamental charged boson fields, and renormalizability. The elimination of the massless ninth pseudoscalar (henceforth called the $\eta''$) is one of the several effects associated with the peculiar large-distance properties of Yang-Mills theory. This phenomenon, which was first discovered by Schwinger[6] in two-dimensional quantum electrodynamics, involves the elimination of the long-range gauge field and the attendant massless gauge bosons. In addition to the elimination of the massless gauge bosons, the Schwinger phenomenon also removes all color nonsinglet states, thus insuring the absence of free quarks. We shall also argue that the same phenomenon eliminates the $\eta''$ Goldstone boson. In much that follows it will not be essential to carry along the color quantum number. For simplicity we will use the notations of an abelian vector-gluon model.

## II. GAUGE INVARIANCE AND THE AXIAL-VECTOR ANOMALY

The proper definitions of the axial charge densities require careful point separation,[7]

$$\rho_5^i(x) = \operatorname*{sym\,lim}_{\epsilon\to 0} \psi^\dagger(x)\gamma_5\lambda^i\psi(x+\epsilon)\,,$$
$$\rho_5(x) = \operatorname*{sym\,lim}_{\epsilon\to 0} \psi^\dagger(x)\gamma_5\psi(x+\epsilon)\,. \tag{2.1}$$

The notation sym lim as $\epsilon\to 0$ indicates a limit in which the spacelike separation $\epsilon$ tends to zero and a symmetrization over the directions of $\epsilon$ is taken. We also introduce the notation

$$\rho_5^i(x,\epsilon) = \psi^\dagger(x)\gamma_5\lambda^i\psi(x+\epsilon)\,,$$
$$\rho_5(x,\epsilon) = \psi^\dagger(x)\gamma_5\psi(x+\epsilon)\,. \tag{2.2}$$

The operators $\rho_5^i$ and $\rho_5(x)$ are the local generators of chiral transformations. Their spatial integrals generate the global transformations

$$\psi\to e^{i\alpha\gamma_5\lambda^i}\psi,\quad \psi\to e^{i\alpha\gamma_5}\psi\,, \tag{2.3}$$

which are exact symmetries of $\mathcal{L}$ when $\mathfrak{M}=0$. Accordingly the densities $\rho_5^i$ and $\rho_5(x)$ satisfy local continuity equations,

$$\partial_t\rho_5^i + \vec\nabla\cdot\vec{\mathfrak{J}}_5^i = 0\,,\quad \partial_t\rho_5 + \vec\nabla\cdot\vec{\mathfrak{J}}_5 = 0\,, \tag{2.4}$$

where $\vec{\mathfrak{J}}_5^i$ and $\vec{\mathfrak{J}}_5$ are the appropriate axial-vector fluxes. Assuming that the symmetries associated with the operators $\rho_5^i$ and $\rho_5$ are not realized algebraically, the densities should generate soft Goldstone bosons when applied to the vacuum. Thus for $k\approx 0$,

$$|\varphi^i\rangle = \int e^{ik\cdot x}\rho_5^i(x)d^3x\,|0\rangle\,,$$
$$|\eta''\rangle = \int e^{ik\cdot x}\rho_5(x)d^3x\,|0\rangle \tag{2.5}$$

are identified as the pseudoscalar octet and singlet of massless Goldstone bosons.

It is important to note that the operators $\rho_5^i(x,\epsilon)$ and $\rho_5(x,\epsilon)$ are *not* gauge-invariant since $\psi(x)$ and $\psi(x+\epsilon)$ transform differently under *local* gauge transformations. In order to compensate this noninvariance, one usually follows Schwinger[7] and multiplies $\rho_5^i(x,\epsilon)$ and $\rho_5(x,\epsilon)$ by the factor $\exp(ig\int dx_\mu A^\mu)$, where the line integral extends from $x$ to $x+\epsilon$. It is sufficient to use simply $\exp[ig\vec{A}(x)\cdot\vec\epsilon\,]$. Thus we define the gauge-invariant (hatted) densities,

$$\hat\rho_5^i(x,\epsilon) = \psi^\dagger(x)\gamma_5\lambda^i\psi(x+\epsilon)e^{ig\vec{A}(x)\cdot\vec\epsilon}\,,$$
$$\hat\rho_5(x,\epsilon) = \psi^\dagger(x)\gamma_5\psi(x+\epsilon)e^{ig\vec{A}(x)\cdot\vec\epsilon}\,. \tag{2.6}$$

When taking the symmetric limit of Eq. (2.6) we cannot generally ignore the factors $\exp(ig\vec{A}\cdot\vec\epsilon)$. This is so because the operator products $\psi^\dagger(x)\gamma_5\psi(x+\epsilon)$ have short-distance singularities which diverge as $\epsilon^{-1}$. The behavior of this singularity was computed originally by Schwinger,[7]

$$\psi^\dagger(x)\gamma_5\psi(x+\epsilon)\underset{\epsilon\to 0}{\sim}\frac{3g}{4\pi^2}\,\frac{\vec{B}\cdot\vec\epsilon}{i\epsilon^2} + \text{regular terms}\,, \tag{2.7}$$

where $B_i$ is the "magnetic" gluon field $\epsilon_{ijk}F_{jk}$. [The octet of operators $\rho_5^i(x,\epsilon)$ does not contain such singularities.] Using Eq. (2.7) we may expand the factor $\exp(ig\vec{A}\cdot\vec\epsilon)$ in Eq. (2.6) and take the symmetric limit as $\epsilon\to 0$ of the axial densities,

$$\hat\rho_5^i(x) = \rho_5^i(x)\,,$$
$$\hat\rho_5(x) = \rho_5(x) + \operatorname*{sym\,lim}_{\epsilon\to 0}\frac{g}{8\pi^2}\,\frac{\vec{B}\cdot\vec\epsilon}{i\epsilon^2}\,ig\vec{A}\cdot\vec\epsilon$$
$$= \rho_5(x) + \frac{g^2}{4\pi^2}\,\vec{B}(x)\cdot\vec{A}(x)\,. \tag{2.8}$$

Therefore, the ninth current differs from the other eight in a significant way.

The gauge-invariant $\hat\rho_5$ is not conserved.[8] Instead it satisfies

$$\partial_t\hat\rho_5 + \vec\nabla\cdot\hat{\vec{\mathfrak{J}}}_5 = \frac{g^2}{2\pi^2}\,\vec{E}\cdot\vec{B}\,, \tag{2.9}$$

where

$$\hat{\mathfrak{J}}_5^j = \mathfrak{J}_5^j + \frac{g^2}{8\pi^2}\epsilon_{j\mu\sigma\tau}A_\mu F_{\sigma\tau} \quad (j=1,2,3).$$

Since the gauge-invariant current $\hat{\rho}_5$ is not conserved, it cannot be used to prove the existence of a massless particle.

## III. GAUGE-INVARIANT DESCRIPTION OF THE MASSLESS $\eta''$

Since $\rho_5(x)$ is not gauge-invariant, we must consider the possibility that a Goldstone boson might be generated by a *gauge-variant* current. We shall carry out our discussion of gauge invariance in a class of gauges satisfying $A_0=0$. This restricts us to gauge transformations of the form

$$\vec{A}(x) \to \vec{A}(x) + g^{-1}\vec{\nabla}\Lambda(x), \quad \psi(x) \to e^{i\Lambda(x)}\psi(x), \tag{3.1}$$

with

$$\frac{\partial}{\partial t}\Lambda(x) = 0. \tag{3.2}$$

The Lagrangian in this gauge is

$$\begin{aligned}\mathcal{L} &= i\bar{\psi}\nabla\!\!\!/\psi + \tfrac{1}{2}\vec{E}^2 - \tfrac{1}{2}\vec{B}^2 + g\bar{\psi}\vec{\gamma}\psi\cdot\vec{A} \\ &= i\bar{\psi}\nabla\!\!\!/\psi + \tfrac{1}{2}\dot{\vec{A}}^2 - \tfrac{1}{2}(\vec{\nabla}\times\vec{A})^2 + g\vec{j}(x)\cdot\vec{A}(x).\end{aligned} \tag{3.3}$$

In the $A_0=0$ gauge the canonical momentum of $\vec{A}$ is $-\dot{\vec{A}}$, the electric field. Therefore,

$$[E_i(x), A_j(y)] = i\delta_{ij}\delta^3(x-y) \tag{3.4}$$

at equal times.

An infinitesimal gauge transformation is generated by

$$\int [\vec{\nabla}\cdot\vec{E} - \rho]\Lambda(x)d^3x, \tag{3.5}$$

where $\rho(x) = j^0(x)$ is the vector charge density. By using Eq. (3.4) and the canonical commutation relation $[\rho(\vec{x},t), \psi(\vec{x}',t)] = -\psi(\vec{x},t)\delta^3(\vec{x}-\vec{x}')$ one can verify that Eq. (3.5) is the generator of gauge transformations Eq. (3.1). The condition for a state $|\psi\rangle$ to be physical is that it be gauge-invariant, i.e., it must satisfy

$$(\vec{\nabla}\cdot\vec{E} - \rho)|\psi\rangle = 0. \tag{3.6}$$

Let us consider the value of the electric field in a state

$$|U\rangle = \exp\left[-i\int \vec{A}(x)\cdot\vec{U}(x)d^3x\right]|0\rangle, \tag{3.7}$$

where $\vec{U}(x)$ is a $c$-number vector field. From Eq. (3.4) it follows that

$$\langle U|\vec{E}(x)|U\rangle = \vec{U}(x). \tag{3.8}$$

Therefore, it is clear that the factor $\exp(ig\vec{A}\cdot\vec{\epsilon})$ in $\hat{\rho}$ creates an electric field between the positive and negative charges created by the fields $\psi^\dagger$ and $\psi$. Without this factor the state $\rho_5|0\rangle$ fails to satisfy $\vec{\nabla}\cdot\vec{E}=\rho$, and is therefore not gauge-invariant. This fact, however, does not prove that the $\eta''$ does not exist. In fact, creating a state with a gauge-variant operator is not unusual. For example, both the electron and the photon are described by gauge-variant fields $\psi$ and $A$ in electrodynamics. However, the projection $Z_2^{1/2}$ of the state $\psi|0\rangle$ on the physical gauge-invariant electron state is gauge-dependent.

Since the presence of a single electron is gauge-invariant, there must be a gauge-invariant way to excite it from the vacuum. The clue is to provide the Coulomb field of the electron by employing an operator similar to that in Eq. (3.7). If $\vec{U}(x)$ satisfies

$$\vec{\nabla}\cdot\vec{U}(x) = g\delta^3(x) \tag{3.9}$$

then

$$\exp\left[-i\int \dot{\vec{A}}(x)\cdot\vec{U}(x)d^3x\right]\psi^\dagger(0)|0\rangle \tag{3.10}$$

is gauge-invariant. We note that since the electron has a nonvanishing charge, Gauss's theorem requires $\vec{U}(x)$ to have a long-range component.

We shall demonstrate that the operator $\rho_5$ may be provided with a similar long-range photon cloud which renders it gauge-invariant. Furthermore, unlike $\hat{\rho}_5$, the new operator $\tilde{\rho}_5$ will be conserved as is $\rho_5$.

Consider the operator

$$\tilde{\rho}_5(x,\epsilon) = \psi^\dagger(x)\gamma_5\psi(x+\epsilon)\exp\left[-i\int \vec{A}(r)\cdot\vec{V}(r-x)d^3r\right], \tag{3.11}$$

where $\vec{V}$ is a $c$-number vector field satisfying

$$\vec{\nabla}\cdot\vec{V}(r) = g\delta^3(r) - g\delta^3(r+\epsilon) \underset{\epsilon\to 0}{\sim} -g\vec{\epsilon}\cdot\vec{\nabla}\delta^3(r). \tag{3.12}$$

$\tilde{\rho}_5(x,\epsilon)$ is gauge-invariant by construction. It is convenient to write

$$V(r) = -\vec{\epsilon}\cdot\vec{\nabla}U(r), \tag{3.13}$$

where $U(r)$ satisfies Eq. (3.9). We may now use Eq. (2.7) to write

$$\tilde{\rho}_5(x,\epsilon) = \psi^\dagger(x)\gamma_5\psi(x+\epsilon) + \frac{g^2}{4\pi^2}\frac{\vec{B}(x)\cdot\vec{\epsilon}}{\epsilon^2}\int \vec{\epsilon}\cdot\vec{\nabla}_r[\vec{U}(r-x)\cdot\vec{A}(x)]\,d^3x. \tag{3.14}$$

Taking the symmetric limit ($\epsilon\to 0$) of Eq. (3.14) gives

$$\tilde{\rho}_5(x) = \rho_5(x) + \frac{g^2}{4\pi^2}B_j(x)\int \partial_j U_k(r-x)A_k(r)d^3r. \tag{3.15}$$

Integrating the second term in Eq. (3.15) by parts gives

$$B_j(x)\int \partial_j U_k(r-x)A_k(r)d^3r = -B_j(x)\int U_k(r-x)\partial_j A_k\, d^3r$$
$$= -B_j(x)\int [\vec{U}(r-x)\times\vec{B}(r)]_j\, d^3r - B_j(x)\int U_k(r-x)\partial_k A_j\, d^3r$$
$$= -\vec{B}(x)\cdot\int \vec{U}(r-x)\times\vec{B}(r)d^3r + \vec{B}(x)\cdot\int [\vec{\nabla}\cdot\vec{U}(r-x)]\vec{A}(r)d^3r$$
$$= \vec{B}(x)\cdot\vec{A}(x) - \vec{B}(x)\cdot\int \vec{U}(r-x)\times\vec{B}(r)d^3r\,. \tag{3.16}$$

Thus $\tilde{\rho}_5$ becomes

$$\tilde{\rho}_5 = \rho_5 + \frac{g^2}{4\pi^2}\vec{B}\cdot\vec{A} - \frac{g^2}{4\pi^2}\vec{B}\cdot\int \vec{U}(r-x)\times\vec{B}(r)d^3r$$
$$= \hat{\rho}_5 - \frac{g^2}{4\pi^2}\vec{B}\cdot\int \vec{U}(r-x)\times\vec{B}(r)d^3r\,. \tag{3.17}$$

We shall now show that $\tilde{\rho}_5$ is the density associated with a conserved current and can therefore be used in discussing the existence of the Goldstone boson $\eta''$. Consider the time derivative $(d/dt)\tilde{\rho}_5$. Since we know that $\rho_5$ is conserved, we need only prove that

$$\frac{d}{dt}\left[\vec{B}(x)\cdot\vec{A}(x) - \vec{B}(x)\cdot\int \vec{U}(r-x)\times\vec{B}(r)d^3r\right]$$

is a pure spatial divergence. The first term, $(d/dt)(\vec{B}\cdot\vec{A})$, gives

$$\frac{d}{dt}(\vec{B}\cdot\vec{A}) = 2\vec{B}\cdot\vec{E} + \vec{\nabla}\cdot(\vec{A}\times\vec{E})\,. \tag{3.18}$$

The second term gives

$$\frac{d}{dt}\left[\vec{B}(x)\cdot\int \vec{U}(r-x)\times\vec{B}(r)d^3r\right] = \dot{\vec{B}}(x)\cdot\int \vec{U}(r-x)\times\vec{B}(r)d^3r + \vec{B}(x)\cdot\int \vec{U}(r-x)\times\dot{\vec{B}}(r)d^3r\,. \tag{3.19}$$

Using $\dot{\vec{B}} = \vec{\nabla}\times\vec{E}$ we obtain

$$\frac{d}{dt}\left[\vec{B}(x)\cdot\int \vec{U}(r-x)\times\vec{B}(r)d^3r\right] = \vec{\nabla}_x\times\vec{E}(x)\cdot\int \vec{U}(r-x)\times\vec{B}(r)d^3r + \vec{B}(x)\cdot\int \vec{U}(r-x)\times\vec{\nabla}_r\times\vec{E}(r)d^3r$$
$$= \partial_i E_j(x)\int U_i(r-x)B_j(r)d^3r - \partial_i E_j(x)\int U_j(r-x)B_i(r)d^3r$$
$$+ B_i(x)\int U_j(r-x)\partial_i E_j(r)d^3r - B_i(x)\int U_j(r-x)\partial_j E_i(r)d^3r$$
$$= \partial_{x_i}\left[E_j(x)\int U_i(r-x)B_j(r)d^3r\right] - \vec{E}\cdot\vec{B} - \partial_{x_i}\left[E_j(x)\int U_j(r-x)B_i(r)d^3r\right]$$
$$+ \partial_{x_i}\left[B_i(x)\int U_j(r-x)E_j(r)d^3r\right] - \vec{E}\cdot\vec{B}$$
$$= -2\vec{E}\cdot\vec{B} + \partial_{x_i}\left[\int U_i(r-x)\vec{E}(x)\cdot\vec{B}(r)d^3r\right.$$
$$\left. - \int \vec{E}(x)\cdot\vec{U}(r-x)B_i(r)d^3r + B_i(x)\int \vec{E}(r)\cdot\vec{U}(r-x)d^3r\right]\,. \tag{3.20}$$

Collecting everything we can combine Eq. (3.20) with Eq. (3.18) to obtain

$$\frac{d}{dt}\tilde{\rho}_5 + \vec{\nabla}\cdot\tilde{\vec{\mathfrak{J}}}_5 = 0\,, \tag{3.21}$$

where

$$\tilde{\vec{\mathfrak{J}}}_5(x) = \vec{\mathfrak{J}}_5(x) + \frac{g^2}{4\pi^2}\vec{A}(x)\times\vec{E}(x) + \frac{g^2}{4\pi^2}\int \vec{U}(r-x)\vec{E}(x)\cdot\vec{B}(r)d^3r$$
$$- \frac{g^2}{4\pi^2}\int \vec{E}(x)\cdot\vec{U}(r-x)\vec{B}(r)d^3r + \frac{g^2}{4\pi^2}\vec{B}(x)\int \vec{U}(r-x)\cdot\vec{E}(r)d^3r\,. \tag{3.22}$$

Therefore, the hatted current is locally conserved and may be used to generate a symmetry which might be realized algebraically or through Goldstone bosons. The relevant global symmetry is just the original chiral symmetry generated by $\int\rho_5(x)d^3x$. To see this integrate $\tilde{\rho}_5(x)$ over space. After several integrations by parts, we obtain

$$\int\tilde{\rho}_5(x)d^3x=\int\rho_5(x)d^3x\,. \tag{3.23}$$

The soft Goldstone boson is given by

$$|\eta''\rangle=\int e^{ik\cdot x}\tilde{\rho}_5(x)d^3x|0\rangle\,,\quad k\approx 0\,. \tag{3.24}$$

We interpret Eq. (3.24) to mean that gauge invariance constrains the $\eta''$ to have a long-range photon field given by

$$\exp\left[i\int\vec{A}(r)\cdot\vec{U}(r-x)d^3r\right]|0\rangle\,.$$

Recently it has been argued that if the Schwinger phenomenon occurs in a gauge theory, then all color-carrying objects will be removed from the physical spectrum of the theory.[1] We remind the reader that the Schwinger phenomenon is an effect in which the long-range gauge field and attendant massless gauge bosons are eliminated from the spectrum.[9] The physical space is composed only of states in which the electric field falls rapidly (exponentially) to zero at large distances. Since states with long-range electric fields are absent, it follows that operators of the form

$$O(x)\exp\left[i\int\vec{A}(r)\cdot\vec{U}(r-x)d^3r\right]\,,$$

where $O(x)$ is a smeared local operator, do not create physical states. Depending on the gauge used, these states may formally be identified as having infinite energy or as violating subsidiary conditions.[10] In particular, the Schwinger phenomenon eliminates the charged particles from the spectrum. In its non-Abelian version it eliminates quarks and other colored objects. Since gauge invariance also requires a long-range field for the $\eta''$, we conclude that *it too is absent from the physical spectrum*. This concludes the formal argument.

## IV. TWO EXAMPLES

### A. A four-dimensional theory of confinement

In this section we discuss two quark-confining theories in which the elimination of the $\eta''$ is explicit. The first example uses theories considered recently by Blaha,[11] Kaufman,[12] and others. In these theories the bare gauge field propagator, $g_{\mu\nu}/(q^2+i\epsilon)$, is modified to

$$g_{\mu\nu}\,\lambda^2\mathrm{P}\frac{1}{q^4}\,, \tag{4.1}$$

where P denotes the principal part. The resulting force law between charges is given by a potential

$$V(r)\sim r \tag{4.2}$$

and is sufficient to confine quarks—it would require an infinite amount of energy to separate quarks to infinity. Blaha's model[11] follows from a covariant Lagrangian

$$\mathcal{L}=i\bar{\psi}\gamma_\mu\partial^\mu\psi-\tfrac{1}{2}F_{\mu\nu}D^{\mu\nu}-\tfrac{1}{2}\lambda^2B_\mu B^\mu-gj_\mu A^\mu\,, \tag{4.3}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,,\;D_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu\,,\qquad j_\mu=\bar{\psi}\gamma_\mu\psi\,. \tag{4.4}$$

The theory is invariant under the gauge transformation

$$A_\mu\to A_\mu+g^{-1}\partial_\mu\Lambda,\;B_\mu\to B_\mu\,,\qquad \psi\to e^{-ig\Lambda}\psi\,. \tag{4.5}$$

The equations of motion in the Lorentz gauge $(\partial_\mu A^\mu=0)$ read

$$\Box A_\mu=-\lambda^2B_\mu\,, \tag{4.6a}$$

$$\Box B_\mu=gj_\mu\,, \tag{4.6b}$$

$$(i\partial\!\!\!/-gA\!\!\!/)\psi=0\,. \tag{4.6c}$$

Quark-confining properties of this theory can be read off from Eq. (4.6b). The charge density satisfies

$$\Box B_0=gj_0\,. \tag{4.7}$$

Integrating this over all space and dropping boundary terms at infinity give

$$Q=\int j_0(x)d^3x=\frac{d^2}{dt^2}\int B_0(x)d^3x\,. \tag{4.8}$$

However, from Eq. (4.6a) it follows that $\partial_\mu B^\mu=0$. Therefore

$$Q=-\frac{d}{dt}\int\vec{\nabla}\cdot\vec{B}\,d^3x=-\frac{d}{dt}\int_\infty\vec{B}\cdot d\vec{\sigma}\,, \tag{4.9}$$

which vanishes by appropriate boundary conditions.[11] Perturbation-theory rules for this theory have been derived by Blaha. They are rather straightforward modifications of the usual electrodynamics rules: Replace photon propagators with $g_{\mu\nu}\lambda^2\mathrm{P}/q^4$.

In order to have a theory in which a Goldstone boson appears perturbatively, we add to Eq. (4.3) the usual degrees of freedom of the simplest $\sigma$ model

$$\mathcal{L} = \bar{\psi}[i\not{\partial} - g\not{A} - f(\sigma + i\eta''\gamma_5)]\psi - \tfrac{1}{2}F_{\mu\nu}D^{\mu\nu} - \tfrac{1}{2}\lambda^2 B_\mu B^\mu + \tfrac{1}{2}[(\partial_\mu\sigma)^2 + (\partial_\mu\eta'')^2] - \tfrac{1}{2}\mu^2[\sigma^2 + \eta''^2] - \tfrac{1}{4}h[\sigma^2 + \eta''^2]^2,$$
$$\mu^2 < 0. \quad (4.10)$$

The Feynman rules for this theory are familiar except for the photon modification mentioned above. All self-energy graphs of the $\eta''$ in which the gauge coupling $g$ is set equal to zero are known not to generate a mass. This follows since the axial-vector current is then conserved, and the Goldstone theorem requires the $\eta''$ to be massless. However, if the mechanism described in the previous section operates, the gauge-invariant massless $\eta''$ should disappear from the spectrum or become massive once $g$ is nonzero. The simplest suspicious graph in which the anomalous character of the axial-vector current appears occurs in fourth order in $g$ and is shown in Fig. 1. This process should yield a finite, nonzero mass shift. The expression for the graph reads

$$\Sigma(p) = \frac{2g^4}{(2\pi)^8}\int d^4q[q^2p^2 - (q\cdot p)^2]I_{00}{}^2(q + \tfrac{1}{2}p, -q + \tfrac{1}{2}p) \times \frac{\lambda^2}{(q + \frac{1}{2}p)^4}\frac{\lambda^2}{(q - \frac{1}{2}p)^4}, \quad (4.11)$$

where $I_{00}$ is the integral expression entering the evaluation of the triangle graph.[8] It tends to a constant as its arguments become small. Let $p$ be purely timelike and parametrize it $p = (p_0, 0)$. Then Eq. (4.11) becomes

$$\Sigma(p_0) \approx \frac{2g^4}{(2\pi)^8} p_0{}^2 I_{00}{}^2(0, 0) \times \int d^4q(q^2 - q_0{}^2)\frac{\lambda^2}{(q - \frac{1}{2}p)^4}\frac{\lambda^2}{(q - \frac{1}{2}p)^4}, \quad (4.12)$$

where $I_{00}(q + \frac{1}{2}p, -q + \frac{1}{2}p)$ has been replaced by the constant $I_{00}(0, 0)$. The justification for approximating $I_{00}$ comes from the fact that only the infrared region of the integral in Eq. (4.11) is significant when $p_0 \to 0$. For nonzero $p_0$ the integral converges to a finite constant times $p_0{}^{-2}$ as seen by dimensional analysis. Therefore,

$$\Sigma(p) \to \text{nonzero, finite constant} \quad (4.13)$$

as $p$ becomes soft. Thus we see that a finite proper self-energy is obtained in this theory of quark confinement.

This result contrasts sharply with the evaluation of this graph in ordinary quantum electrodynamics. In fact, if the gauge propagators are replaced by $g_{\mu\nu}/(q^2 + i\epsilon)$, then it is easy to see that the expression corresponding to Eq. (4.11) behaves as $p^2$ instead of the interesting constant.

It is not certain whether the Lagrangians of Eq. (4.3) and (4.6) define fully sensible theories. The real point that we should stress here is that anomalous infrared behavior of the gauge field can be the source of both quark confinement and $\eta''$ elimination.[13]

### B. Two-dimensional quantum electrodynamics

Our second example is the Schwinger model[6]—quantum electrodynamics in 1 space dimension and 1 time dimension. The quark-confining features of this theory have been discussed elsewhere.[1] The theory is most simply described in terms of a boson field constructed in terms of the currents of charged fermions. Since the electric current is conserved, it follows that it can be written as the curl of a scalar field,[1]

$$j_\mu = \epsilon_{\mu\nu}\partial^\nu\phi. \quad (4.14)$$

In one dimension the gauge-invariant axial-vector current is trivially related to the vector current

$$j_5^\mu = \epsilon^{\mu\nu}j_\nu = \partial^\mu\phi. \quad (4.15)$$

It can easily be shown that $\phi$ is a canonical field[1] satisfying

$$[\phi(z, t), \dot{\phi}(z', t)] = im^2\delta(z - z'), \quad m^2 = g^2/\pi. \quad (4.16)$$

Furthermore, the free fermion Lagrangian can be rewritten in terms of $\phi$,

$$\mathcal{L}_F = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi. \quad (4.17)$$

In the Coulomb gauge the gauge field $A_\mu$ is not an independent dynamical degree of freedom. In fact, the interaction between charges occurs through a one-dimensional Coulomb force. It reads

$$\tfrac{1}{2}g^2\int \rho(z)|z - z'|\rho(z')\,dz\,dz', \quad (4.18)$$

where $\rho(z)$ is the charge density operator $j^0(z)$. This interaction energy can be rewritten

$$\tfrac{1}{2}g^2\int \partial_1\phi(z)|z - z'|\partial_1\phi(z)\,dz\,dz', \quad (4.19)$$

where $\partial_1 = \partial/\partial z$. Thus the Hamiltonian becomes

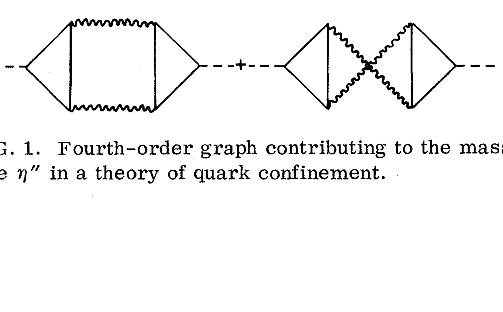

FIG. 1. Fourth-order graph contributing to the mass of the $\eta''$ in a theory of quark confinement.

$$H=\tfrac{1}{2}\int \dot{\phi}^2 dz+\tfrac{1}{2}\int (\partial_1\phi)^2 dz + \tfrac{1}{2}g^2\int \partial_1\phi(z)\,|z-z'|\,\partial_1\phi(z')\,dz\,dz' \ . \quad (4.20)$$

We shall be interested in the chiral symmetry of this theory. It is generated by

$$Q_5=\int \rho_5^0(z,t)dz=\int \dot{\phi}(z,t)dz \ . \quad (4.21)$$

From the commutation relation Eq. (4.16) it follows that a chiral transformation induces a translation of the field $\phi$,

$$\phi\to\phi+\text{const} \ . \quad (4.22)$$

From the structure of the Hamiltonian it is evident that this operation is a symmetry. Furthermore, Lowenstein and Swieca[14] have demonstrated that the vacuum of two-dimensional quantum electrodynamics is *not* invariant under chiral transformations. Therefore, if the Goldstone theorem applied, $\phi$ would have to be a massless particle. However, let us perform the integration in the Hamiltonian Eq. (4.20) by parts,

$$\tfrac{1}{2}g^2\int \partial_1\phi(z)\,|z-z'|\,\partial_1\phi(z')\,dz\,dz' = \tfrac{1}{2}g^2\int \phi^2(z)\,dz \ \text{(tentative)} \ . \quad (4.23)$$

We see that the Coulomb force is equivalent to adding a mass term to the free meson Hamiltonian. In this form the Hamiltonian describes massive excitations and has apparently lost is chiral symmetry. However, on closer inspection, we see that the integration by parts introduced the unwarranted assumption that $\phi(\infty)=0$. If $\phi(z=\infty,t)$ is not zero, the integration by parts gives

$$H=\tfrac{1}{2}\int [\dot{\phi}^2+(\partial_1\phi)^2]\,dz+\tfrac{1}{2}g^2\int [\phi(z,t)-\phi(\infty,t)]^2 dz \ . \quad (4.24)$$

In this form the Hamiltonian remains invariant under translation of $\phi$.

Let us define the field $\tilde{\phi}=\phi-\phi(\infty)$. The Hamiltonian then reads

$$H=\tfrac{1}{2}\int [\dot{\tilde{\phi}}^2+(\partial_1\tilde{\phi})^2]\,dz+\dot{\phi}(\infty)\int \dot{\tilde{\phi}}(z,t)dz + V\dot{\phi}^2(\infty)+\tfrac{1}{2}g^2\int \tilde{\phi}^2 dz \ , \quad (4.25)$$

where $V$ is the volume of space. The term $\dot{\phi}(\infty)\int\dot{\tilde{\phi}}(z,t)dz$ involves only the zero-momentum mode of $\phi$ and may be ignored. The chiral-symmetry operation is now given by

$$\tilde{\phi}(z)\to\tilde{\phi}(z),\ \ \phi(\infty)\to\phi(\infty)+\text{const} \ . \quad (4.26)$$

The field $\tilde{\phi}$ still describes a massive free excitation and is not involved in the realization of the symmetry. Denote $\pi(\infty)$ as the momentum conjugate to $\phi(\infty)$,

$$\pi(\infty)=2V\dot{\phi}(\infty) \ . \quad (4.27)$$

It is proportional to the chiral charge density. The term in the Hamiltonian involving $\dot{\phi}(\infty)$ becomes

$$\pi^2(\infty)/4V \ . \quad (4.28)$$

When $V\to\infty$, all states of the operator $\phi(\infty)$ become degenerate. In particular, the vacuum of the system is infinitely degenerate, but the transformed vacua are discrete states and are not connected to a continuum of energy levels as in the traditional Goldstone case.

This two-dimensional case explicitly shows how the Goldstone theorem for the $\eta''$ is circumvented in a theory whose infrared properties are sufficiently severe to confine quarks.

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