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Electron-electron scattering. II. Helicity cross sections for positron-electron scattering*

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The differential cross sections for polarized electron-positron scattering are calculated to order e^6 by using the five invariant amplitudes presented in a previous paper. The unpolarized result of Polovin is rederived. As an application of the helicity amplitudes the spin-momentum correlation for a polarized target positron is obtained in agreement with Fronsdal and Jaksic.

I. INTRODUCTION

The differential cross section for unpolarized electron-electron scattering, to order e^6, was calculated first by Redhead[1] and later by Polovin.[2] The spin-momentum correlation in electron-positron scattering in which the spin of only one of the particles is detected was calculated by Fronsdal and Jaksic.[3] However, the general polarization case has not been previously derived.

In an earlier paper[4] (called paper I), the five invariant amplitudes were obtained in spectral form. Here, we will apply these invariant amplitudes to calculate the helicity amplitudes for electron-positron scattering. (The corresponding results for electron-electron scattering will be presented in a subsequent communication.) Because of the infrared nature of charged-particle scattering, we will consider neither near-threshold nor forward scattering. However, these kinematical regions are correctly described in the results of paper I, in terms of a fictitious photon mass. This detailed structure cannot be measured directly and would be significant only in the application of the spectral forms to higher-order calculations.[5]

We present the helicity amplitudes in terms of the invariant amplitudes in Sec. II, and the explicit forms in Sec. III. In Sec. IV, we consider soft-photon contributions. The unpolarized differential cross section is calculated in Sec. V and the spin-momentum correlation is found in Sec. VI. Appendixes A and B contain the integrals necessary for the calculations of Sec. III while the invariant amplitudes are given in Appendix C.

II. HELICITY AMPLITUDES

This section is devoted to calculating the helicity amplitudes in terms of the invariant amplitudes. This is done by applying Eq. (I74) to an appropriate helicity state. (Here I refers to equations in paper I.) The two basic structures encountered are

$$F(1\,1';\,2\,2') = \sum_{i=1}^{5} M_2^i\, u_1^* \gamma^0 \Gamma_i\, u_2 u_{1'}^* \gamma^0 \Gamma_i\, u_{2'}$$

$$\equiv \sum_{i=1}^{5} M_2^i\, \Gamma_i(1\,2;\,1'2') \quad (1)$$

and

$$\tilde{F}(1\,1';\,2\,2') = \sum_{i=1}^{5} M_1^i\, u_1^* \gamma^0 \Gamma_i\, u_{1'}^* u_2 \gamma^0 \Gamma_i\, u_{2'}$$

$$\equiv \sum_{i=1}^{5} M_1^i\, \Gamma_i(1\,1';\,2\,2'). \quad (2)$$

For convenience, we will work in the center-of-mass system with $\vec{P}_2$ in the z direction and $\vec{P}_1$ in the x-z plane:

$$\vec{P}_1 = |\vec{P}_1|(\sin\theta, 0, \cos\theta).$$

An explicit representation for the Dirac spinor in terms of the helicity is[6]

$$u_{p\sigma} = \left[\left(\frac{p^0+m}{2m}\right)^{1/2} + \left(\frac{p^0-m}{2m}\right)^{1/2} i\gamma_5\sigma\right] v_\sigma ,$$

where

$$\gamma^0 v_\sigma = v_\sigma , \quad v_{2+} = \begin{pmatrix}1\\0\end{pmatrix}, \quad v_{2-} = \begin{pmatrix}0\\1\end{pmatrix},$$

$$v_{2'+} = \begin{pmatrix}0\\1\end{pmatrix}, \quad v_{2'-} = \begin{pmatrix}-1\\0\end{pmatrix},$$

$$v^*_{1+} = (\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta), \quad v^*_{1-} = (-\sin\tfrac{1}{2}\theta, \cos\tfrac{1}{2}\theta),$$

$$v^*_{1'+} = (-\sin\tfrac{1}{2}\theta, \cos\tfrac{1}{2}\theta), \quad v^*_{1'-} = (-\cos\tfrac{1}{2}\theta, -\sin\tfrac{1}{2}\theta).$$

The kinematical relationships are

$$m^2 s = (P_2 + P_{2'})^2 ,$$
$$m^2 t = (P_1 - P_2)^2 ,$$
$$m^2 u = (P_1 - P_{2'})^2 ,$$
$$p^0 = \tfrac{1}{2} m\sqrt{-s} ,$$
$$1 - \cos\theta = -\frac{2t}{s+4} ,$$
$$1 + \cos\theta = -\frac{2u}{s+4} ,$$
$$s + t + u + 4 = 0 ,$$

while some useful Dirac identities are[7]

$$\gamma^0\gamma_k = i\gamma_5\sigma_k , \quad \sigma_{ij} = \epsilon_{ijk}\sigma^k ,$$
$$v^*_\sigma i\gamma_5 v_{\sigma'} = 0 , \quad v^*_{-\sigma} = i\sigma\gamma_5 v_\sigma ,$$
$$v^*_{\sigma_1}\sigma_i v_{\sigma_2} v^*_{\sigma_1'}\sigma_i v_{\sigma_2'} = -v^*_{\sigma_1} v_{\sigma_2} v^*_{\sigma_1'} v_{\sigma_2'} + 2 v^*_{\sigma_1} v_{\sigma_2'} v^*_{\sigma_1'} v_{\sigma_2} .$$

With this information it is easy to find from Eq. (1) (dropping subscript 2)

$$\begin{aligned}
F(++;++) &= \tfrac{1}{2}sM^1 + \left(\frac{s+2}{2} - \frac{t}{s+4}\right)M^2 - \frac{u}{s+4}M^4 - \tfrac{1}{2}sM^5 ,\\
F(--;++) &= \tfrac{1}{2}(s-u)M^1 + \frac{t}{s+4}M^2 - \tfrac{1}{4}tM^3 + \frac{1}{2}\,\frac{s+2}{s+4}\,tM^4 + \tfrac{1}{2}(t+4)M^5 , \qquad (3)\\
F(-+;-+) &= -\tfrac{1}{2}uM^1 - \frac{1}{2}\,\frac{s+2}{s+4}uM^2 - \frac{u}{s+4}M^4 - \tfrac{1}{2}uM^5 ,\\
F(+-;-+) &= -\frac{t}{s+4}M^2 - \tfrac{1}{4}tM^3 + \frac{t}{s+4}M^4 ,\\
F(+-;++) &= -\frac{1}{2}\,\frac{(-stu)^{1/2}}{s+4}(M^2 - M^4) .
\end{aligned}$$

These independent helicity combinations occur 2, 2, 2, 2, and 8 times, respectively, in the total count of 16. The relative phases[8] are, for the first four F's,

$$F(\sigma_1\sigma_{1'};\sigma_2\sigma_{2'}) = +F(-\sigma_1 -\sigma_{1'}; -\sigma_2 -\sigma_{2'}) , \qquad (4)$$

and for the fifth,

$$\begin{aligned}
F(\sigma\,-\sigma;\sigma\sigma) &= +F(-\sigma\,\sigma;\sigma\sigma)\\
&= -F(-\sigma\,\sigma; -\sigma\,-\sigma)\\
&= -F(\sigma\sigma;\sigma\,-\sigma)\\
&= -F(\sigma\sigma; -\sigma\,\sigma) . \qquad (5)
\end{aligned}$$

We can calculate the results for $\tilde{F}$ [Eq. (2)] by means of a Fierz transformation.[9] For our basis set, we have

$$\Gamma_i(1\,1'; 2\,2') = \lambda_{ij}\,\Gamma_j(1\,2; 1'2') , \qquad (6)$$

where

$$\lambda_{ij} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 8 & 0 & -2\\ 0 & -2 & 8 & 6 & -2\\ 1 & 0 & 0 & 0 & -1\\ -2 & 2 & 0 & 2 & 0\\ -2 & 0 & -8 & 0 & -2\end{bmatrix} .$$

We then obtain (dropping subscript 1)

$$\begin{aligned}
\tilde{F}(++;++) &= \frac{t-u}{s+4}M^2 + \tfrac{1}{4}sM^3 + M^4 ,\\
\tilde{F}(--;++) &= \tfrac{1}{2}(u-t)M^1 + \frac{u-t}{s+4}M^2 + \tfrac{1}{4}sM^3 - \tfrac{1}{2}(s+2)M^4 - \tfrac{1}{2}(s+4)M^5 ,\\
\tilde{F}(-+;-+) &= \tfrac{1}{2}uM^1 + \frac{1}{2}\,\frac{su}{s+4}M^2 + \tfrac{1}{2}uM^5 , \qquad (7)\\
\tilde{F}(+-;-+) &= -\tfrac{1}{2}tM^1 - \frac{1}{2}\,\frac{st}{s+4}M^2 + \tfrac{1}{2}tM^5 ,\\
\tilde{F}(+-;++) &= \frac{(-stu)^{1/2}}{s+4}M^2 .
\end{aligned}$$

These relationships are in agreement with known results.[10]

III. HELICITY CROSS SECTIONS

We define the helicity amplitudes in terms of the probability amplitudes by

$$\langle 1_{p_1\sigma_1} 1_{p_{1'},\sigma_{1'}} | 1_{p_2\sigma_2} 1_{p_{2'},\sigma_{2'}}\rangle = 8\pi i\alpha(2\pi)^4\delta(p_1 + p_{1'} - p_2 - p_{2'})(d\omega_{p_1}\cdots d\omega_{p_{2'}})^{1/2} f(\sigma_1\sigma_{1'};\sigma_2\sigma_{2'}) , \qquad (8)$$

with

$$f(\cdots)=f^{(2)}(\cdots)+\frac{\alpha}{\pi}f^{(4)}(\cdots).$$

This yields the differential cross sections in the center-of-mass system

$$\left(\frac{d\sigma}{d\Omega}\right)_{\sigma_2\sigma_2'\to\sigma_1\sigma_1'}=-\frac{\alpha^2}{m^2 s}\,|f(\sigma_1\sigma_1';\sigma_2\sigma_2')|^2$$
$$=-\frac{\alpha^2}{m^2 s}\left\{[f^{(2)}(\sigma_1\sigma_1';\sigma_2\sigma_2')]^2+\frac{2\alpha}{\pi}f^{(2)}(\sigma_1\sigma_1';\sigma_2\sigma_2')\,\mathrm{Re}f^{(4)}(\sigma_1\sigma_1';\sigma_2\sigma_2')\right\}. \tag{9}$$

The lowest-order helicity amplitudes can be obtained from the lowest-order vacuum amplitude expression, Eq. (I63). They are

$$\begin{aligned}
f^{(2)}(++;++)&=\frac{s+2}{t}+4\frac{u+2}{s(s+4)},\\
f^{(2)}(--;++)&=-4\frac{u+2}{s(s+4)},\\
f^{(2)}(-+;-+)&=\frac{s+2}{t}+\frac{2s+t+6}{s+4},\\
f^{(2)}(+-;-+)&=\frac{t-2}{s+4},\\
f^{(2)}(+-;++)&=-\frac{(-stu)^{1/2}}{s+4}\left(\frac{1}{t}+\frac{2}{s}\right).
\end{aligned} \tag{10}$$

The fourth-order helicity amplitudes are obtained by using Eqs. (3) and (7) in conjunction with Eq. (I74). The necessary spectral integrals are easily done (see Appendixes A and B) and the integrated forms are given in Appendix C. The final results are

$$\begin{aligned}
f^{(4)}(++;++)={}&f^{(2)}(++;++)K\ln\lambda+\frac{2}{s}\left[\left(2-\frac{s+2u}{s+4}t\right)M(t)+\left(s+2+\frac{s-2u}{s+4}t\right)M(u)\right]\ln s\\
&+\left[(s+2)\left(-\frac{s+2}{t}+\frac{2}{s+4}\right)M(s)+\left(\frac{(s+2)^2}{s+4}\,\frac{u}{t}+\frac{2t}{s+4}\right)M(u)\right]\ln t-\frac{2}{s+4}\ln s\\
&+\left(-\frac{1}{2}\,\frac{s}{s+4}+\frac{s+2}{s+4}\,\frac{4}{t+4}\right)\ln t-\frac{2u}{s+4}M(s)-\left(4+\frac{t}{s+4}\right)M(t)-\frac{1}{2}\left(s+4+\frac{s}{s+4}t\right)M(u)\\
&+4\frac{s+2}{s+4}G(s)-\left[2\frac{s+2}{s+4}\,\frac{t}{t+4}+2+t+\frac{1}{2}\,\frac{s+2}{s(s+4)}t(t+2)\right]G(t)+\frac{s+2-2t}{s+4}D(s,t)\\
&+\frac{3s+10+2t}{s+4}D(s,u)-D(t,s)-\left[s+3+\tfrac{3}{2}t+\frac{1}{2}\,\frac{s+2}{s(s+4)}t(t+2)\right]D(t,u)+\frac{t-u}{s(s+4)}\mathfrak{L}(s)\\
&+\left(-\frac{1}{2}\,\frac{s+2}{t}+\frac{1}{s+4}\right)\mathfrak{L}(t),
\end{aligned} \tag{11}$$

$$\begin{aligned}
f^{(4)}(--;++)={}&f^{(2)}(--;++)K\ln\lambda-\frac{2}{s}\left[\left(2-\frac{s+2u}{s+4}t\right)M(t)+\left(s+2+\frac{s-2u}{s+4}t\right)M(u)\right]\ln s\\
&-2\left[\frac{s+2}{s+4}M(s)-\frac{u+2}{s+4}M(u)\right]\ln t+\frac{2}{s+4}\ln s-\frac{2}{s+4}\,\frac{t}{t+4}\ln t+\frac{2u}{s+4}M(s)-\frac{u}{s+4}M(t)\\
&+\frac{2u}{s+4}M(u)-4\frac{s+2}{s+4}G(s)-\frac{t}{s+4}\left(\frac{4}{t+4}+\frac{t+2}{s}\right)G(t)+2\frac{t+1}{s+4}D(s,t)+2\frac{u+1}{s+4}D(s,u)\\
&+D(t,s)+\left[1-\frac{t(t+2)}{s(s+4)}\right]D(t,u)+\frac{u-t}{s(s+4)}\mathfrak{L}(s)-\frac{1}{s+4}\mathfrak{L}(t),
\end{aligned} \tag{12}$$

$$
\begin{aligned}
f^{(4)}(-+;-+) &= f^{(2)}(-+;-+)K\ln\lambda - \left[\left(2+\frac{2-u}{s+4}t\right)M(t)+\left(s+2+\frac{2s+6+t}{s+4}t\right)M(u)\right]\ln s \\
&\quad +(s+2)\left[\frac{s+2}{s+4}\frac{u}{t}M(s)-\left(\frac{s+2}{t}+\frac{2s+6+t}{s+4}\right)M(u)\right]\ln t-\frac{1}{2}\frac{s}{s+4}\ln s \\
&\quad +\left(-\frac{1}{2}\frac{s}{s+4}+\frac{s+2}{s+4}\frac{4}{t+4}\right)\ln t+\frac{1}{2}\left(s+6+\frac{4t}{s+4}\right)M(s)+\frac{1}{2}\frac{s+2}{s+4}tM(t)-\frac{2u}{s+4}M(u) \\
&\quad -\left(s+\frac{1}{2}\frac{s(s+2)}{u}\right)G(s)-\left[2\frac{s+2}{s+4}\frac{t}{t+4}+\frac{t(t+2)}{2u}+2+t-\frac{t(t+2)}{s(s+4)}\right]G(t) \\
&\quad +\left[-\frac{1}{2}\frac{s^2-12}{s+4}+\frac{s+2}{s+4}t-\frac{s(s+2)}{2u}\right]D(s,t)-2\frac{u+1}{s+4}D(s,u)+\left[-\frac{t(t+2)}{2u}-\tfrac{1}{2}t+s+1\right]D(t,s) \\
&\quad +\left[-1+\frac{t(t+2)}{s(s+4)}\right]D(t,u)+\frac{1}{2}\frac{u}{s+4}\mathcal{L}(s)+\frac{1}{2}\frac{s+2}{s+4}\frac{u}{t}\mathcal{L}(t), \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
f^{(4)}(+-;-+) &= f^{(2)}(+-;-+)K\ln\lambda-\left[\frac{t(t+2)}{s+4}M(t)-\frac{u+2}{s+4}tM(u)\right]\ln s+\left[2\frac{s+2}{s+4}M(s)-2\frac{u+2}{s+4}M(u)\right]\ln t \\
&\quad -\frac{1}{2}\frac{s}{s+4}\ln s+\frac{2}{s+4}\frac{t}{t+4}\ln t+\frac{3t+u}{s+4}M(s)+\frac{u}{s+4}M(t)-\frac{1}{2}\left(s+4+\frac{s}{s+4}t\right)M(u) \\
&\quad -\left[s+\frac{1}{2}\frac{s(s+2)}{t}\right]G(s)+\frac{t}{s+4}\left(\frac{4}{t+4}+\frac{t+2}{s}\right)G(t)-2\frac{t+1}{s+4}D(s,t) \\
&\quad -\frac{1}{2}\left[\frac{s(s+2)}{t}+\frac{3s^2+12s+4}{s+4}+2\frac{s+2}{s+4}t\right]D(s,u)+\frac{t(t+2)}{s(s+4)}D(t,u)-\frac{1}{2}\frac{t}{s+4}\mathcal{L}(s)+\frac{1}{s+4}\mathcal{L}(t), \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
f^{(4)}(+-;++) &= f^{(2)}(+-;++)K\ln\lambda \\
&\quad +\frac{(-stu)^{1/2}}{s+4}\left\{\frac{2}{s}[(t+2)M(t)-(u+2)M(u)]\ln s+\frac{1}{t}[(s+2)M(s)-(u+2)M(u)]\ln t+\frac{1}{t+4}\ln t\right. \\
&\quad +\frac{1}{2}\frac{s-4}{s}M(s)+\frac{1}{2}\frac{u}{t}M(t)+M(u)+(s+3)\left(\frac{1}{t}-\frac{1}{u}\right)G(s)+\frac{1}{2}\left(\frac{4}{t+4}-\frac{t}{u}+\frac{t+6}{s}\right)G(t) \\
&\quad \left.+\frac{t+1}{u}D(s,t)-\frac{u+1}{t}D(s,u)+\frac{1}{2}\frac{s+4}{u}D(t,s)+\frac{1}{2}\frac{2-u}{s}D(t,u)+\frac{1}{s}\mathcal{L}(s)+\frac{1}{2}\frac{1}{t}\mathcal{L}(t)\right\}. \qquad (15)
\end{aligned}
$$

Here the coefficient of the photon-mass term in the individual amplitudes is the function K:

$$K=2[(s+2)M(s)+(t+2)M(t)-(u+2)M(u)-1]. \qquad (16)$$

The modified propagation function as well as the non-infrared-sensitive part of the electric form factor are contained in the function $\mathcal{L}$ (see Ref. 11):

$$
\begin{aligned}
\mathcal{L}(x) &= 4[1-2\Phi_x\coth 2\Phi_x+\tfrac{1}{2}\Phi_x\tanh\Phi_x-\tfrac{1}{2}(x+2)N(x)] \\
&\quad +2[(1-\tfrac{1}{3}\coth^2\Phi_x)(1-\Phi_x\coth\Phi_x)-\tfrac{1}{9}]. \qquad (17)
\end{aligned}
$$

The special functions that occur are

$$
\begin{aligned}
&\ln x=\ln|x|-\pi i\eta(-x), \\
&M(x)=\frac{\Phi_x}{\sinh 2\Phi_x}, \\
&\Phi_x=\tfrac{1}{2}\ln\left|\frac{(x+4)^{1/2}/x^{1/2}+1}{(x+4)^{1/2}/x^{1/2}-1}\right|-\tfrac{1}{2}\pi i\eta(-x), \\
&D(x,y)=M(y)\ln x+N(y), \qquad (18) \\
&N(x)=\frac{1}{\sinh 2\Phi_x}[-\Phi_x^2-2\Phi_x\ln(1+e^{-2\Phi_x}) \\
&\qquad +f(-e^{-2\Phi_x})+\tfrac{1}{12}\pi^2],
\end{aligned}
$$

and

$$G(x)=\frac{1}{\sinh 2\Phi_x}[f(e^{-2\Phi_x})+\Phi_x^2+\tfrac{1}{3}\pi^2].$$

In the above, $f(x)$ is the Spence function[12]:

$$f(x)=-\int_0^x\frac{dz}{z}\ln|1-z|.$$

We have essentially used the notations and special functions of Ref. 2 so as to facilitate comparison of the unpolarized cross section (see Secs. IV and V). The only differences in the definitions are found in Φ_x (Polovin uses just the real part and adds in $-\frac{1}{2}\pi i$ by hand when necessary), $G(x)$ (which agrees[13] for $x>0$ but differs for $x<0$), and $D(x,y)$ (which is not defined in Ref. 2).

IV. SOFT-PHOTON CONTRIBUTIONS

The vacuum amplitude that describes the skeletal interaction of four electrons and one photon is given by

$$\langle 0_+ | 0_- \rangle = \tfrac{1}{2} i \int (dx)(dy)(dz)\psi(x)\gamma^0 e\hat{q}\gamma^\mu \psi(x) D_+(x-y) \times \psi(y)\gamma^0 e\hat{q}\gamma_\mu G_+(y-z) e\hat{q}\gamma A(z)\psi(z). \tag{19}$$

When this is applied to the case of electron-positron scattering with the emission of one soft photon, we find

$$\frac{d\sigma}{d\Omega}(\text{inelastic})_{\sigma_2\sigma_2' \to \sigma_1\sigma_1'} = \left[\frac{d\sigma}{d\Omega}(\text{elastic})_{\sigma_2\sigma_2' \to \sigma_1\sigma_1'}\right]\frac{2\alpha}{\pi} J_0, \tag{20}$$

with

$$J_0 = 2\pi^2 \int d\omega_k \left(\frac{P_{2'}}{P_{2'} k} - \frac{P_2}{P_2 k} - \frac{P_{1'}}{P_{1'} k} + \frac{P_1}{P_1 k}\right)^2. \tag{21}$$

If ΔE is the minimum detectable energy (in the center-of-mass system) then J_0 is[14]

$$\begin{aligned} J_0 = {} & K \ln\frac{2\Delta E}{m\lambda} + 2\,\mathrm{Re}[\Phi_s \tanh\Phi_s + N(s)\cosh 2\Phi_s - \tfrac{1}{2}\pi^2 \coth 2\Phi_s] \\ & + 2\left[\frac{H(\theta_t)\cosh 2\Phi_t}{\sinh 2\Phi_s} - N(t)\cosh 2\Phi_t - \frac{2\Phi_t \,\mathrm{Re}\ln 2 \sinh\Phi_s}{\tanh 2\Phi_t} - (t \leftrightarrow u)\right], \end{aligned} \tag{22}$$

where

$$\begin{aligned} H(\theta) &= \frac{1}{\sin\frac{1}{2}\theta}\int_{\cos 1/2\theta}^{1} \frac{dx}{[x^2 - \cos^2(\frac{1}{2}\theta)]^{1/2}} \times \left\{\frac{\ln[\frac{1}{2}(1+\beta x)]}{1-\beta x} - \frac{\ln[\frac{1}{2}(1-\beta x)]}{1+\beta x}\right\}, \\ \beta &= \frac{(s+4)^{1/2}}{s^{1/2}}, \quad \theta_t = \theta, \quad \theta_u = \pi - \theta. \end{aligned} \tag{23}$$

The infrared-sensitive part of J_0 is such as to cancel the $\ln\lambda$ part of the individual helicity cross sections. The corresponding calculation for electron-electron scattering is very similar and the result agrees with that given by Polovin. However, his remarks on the soft-photon contribution for the electron-positron scattering case are misleading. For example, his prescription for crossing, $\Phi_a \leftrightarrow \Phi_b$ (essentially $s \leftrightarrow u$ in our variables), would imply

$$\frac{2\Phi_a}{\tanh\Phi_a} \to \frac{2\Phi_b}{\tanh\Phi_b},$$

or, in our notation,

$$\mathrm{Re}\frac{2\Phi_s}{\coth\Phi_s} \to \frac{2\Phi_u}{\tanh\Phi_u},$$

when, in fact, nothing happens to this term.

V. UNPOLARIZED DIFFERENTIAL CROSS SECTION

The unpolarized differential cross section can be written as

$$\frac{d\sigma}{d\Omega} = -\frac{\alpha^2}{m^2 s}\left(U + \frac{2\alpha}{\pi}\mathrm{Re}Y\right). \tag{24}$$

Here, the lowest-order result is

$$\begin{aligned} U &= \tfrac{1}{4}\sum [f^{(2)}(\sigma_1\sigma_1'; \sigma_2\sigma_2')]^2 \\ &= \frac{(s+2)^2}{t^2} + 2\frac{s^2-2}{s}\frac{1}{t} + \frac{3s^2+4}{s^2} + 2\frac{s+2}{s^2}t + \frac{t^2}{s^2}, \end{aligned} \tag{25}$$

which is the well-known result for Bhabha scattering.[15]

The sixth-order correction is given by

$$Y = \tfrac{1}{4}\sum_{\{\sigma\}} f^{(2)}(\sigma_1\sigma_1'; \sigma_2\sigma_2') f^{(4)}(\sigma_1\sigma_1'; \sigma_2\sigma_2'). \tag{26}$$

It is reasonably easy to obtain

$$\begin{aligned} Y = UK\ln\lambda + \frac{1}{4}\Bigg\{ & \left[4\frac{(s+2)^2}{t^2} + \frac{6s^2-8}{s}\frac{1}{t} + 6 + \frac{2t}{s}\right][-(s+2)M(s) + (u+2)M(u)]\ln t \\ & - \left[\frac{s+2}{t} - \frac{2(s-4)(u+2)}{s(s+4)}\right]\ln s + \left[\frac{(s+2)(s+10)}{t} + \frac{(s+8)(2s-2)}{s} + \frac{s-4}{s}t\right]M(s) \\ & - \left[\frac{s(s+2)}{t} + s - 6\right]M(u) - \left(8\frac{s+2}{t} + 8\frac{s-2}{t+4} + 4\frac{s^2+6s-2}{s} + 2\frac{2s+1}{s}t + \frac{t^2}{s}\right)G(t) \\ & + \left(4\frac{s+2}{t} + s + 4 + \frac{3s+4}{s}t + \frac{2t^2}{s}\right)D(s,t) - \left(2\frac{s^2+6s+8}{t} + \frac{3s^2+8s-16}{s} + \frac{s-4}{s}t\right)D(t,u) \\ & - \left[2\frac{(s+2)^2}{t^2} + \frac{3s^2-4}{s}\frac{1}{t} + 3 + \frac{t}{s}\right]\mathcal{L}(t) + (s \leftrightarrow t)\Bigg\}. \end{aligned} \tag{27}$$

This result is in complete agreement with Polovin except for the typographical errors mentioned in footnotes 11 and 13.

VI. SPIN-MOMENTUM CORRELATIONS

The simplest type of polarization experiment has the positron beam partly polarized and all other spins not measured. The differential cross section then is of the form

$$\frac{d\sigma}{d\Omega}=(1+\xi P)\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}}, \tag{28}$$

where ξ is the degree of polarization of the beam. The expression for P was calculated by Fronsdal and Jaksic[3] and we here rederive it to illustrate the application of the helicity amplitudes given in Sec. III.

Consider the positron beam fully polarized ($\xi=1$) in the direction $\vec{n}=(\cos\phi\sin\vartheta, \sin\phi\sin\vartheta, \cos\vartheta)$. The corresponding helicity amplitudes are

$$f(\sigma_1\sigma_{1'};(\vartheta,\phi,+),\sigma_{2'})=e^{-i\phi/2}\cos(\tfrac{1}{2}\vartheta)f(\sigma_1\sigma_{1'};+\sigma_{2'}) +e^{i\phi/2}\sin(\tfrac{1}{2}\vartheta)f(\sigma_1\sigma_{1'};-\sigma_{2'}), \tag{29}$$

which have been calculated by means of a rotation on the + helicity state ($|+\rangle$ refers to the z direction)[16]:

$$|\vartheta,\phi,+\rangle=e^{-i\phi\sigma_3/2}e^{-i\vartheta\sigma_2/2}|+\rangle.$$

Squaring and summing over undetected helicities and making use of the phase relations [Eqs. (4) and (5)] we find

$$\frac{1}{2}\sum_{\sigma_1\sigma_{1'}\sigma_{2'}}|f(\sigma_1\sigma_{1'};(\vartheta,\phi,+),\sigma_{2'})|^2 =\frac{1}{4}\sum_{\{\sigma\}}|f^{(2)}(\sigma_1\sigma_{1'};\sigma_2\sigma_{2'})|^2+\frac{\alpha}{\pi}\sin\phi\sin\vartheta\,\text{Im}\{f^{(2)}(5)[f^{(4)}(2)-f^{(4)}(4)-f^{(4)}(3)-f^{(4)}(1)] +f^{(4)}(5)[f^{(2)}(4)+f^{(2)}(3)+f^{(2)}(1)-f^{(2)}(2)]\}, \tag{30}$$

where the numbers in $f(\ )$ label the helicity amplitudes according to the order of appearance in Eqs. (3) and (7). The result can be written as

$$UP=\alpha\frac{\vec{n}\cdot\vec{P}_2\times\vec{P}_1}{|\vec{P}_2\times\vec{P}_1|}(-stu)^{1/2}\mathcal{A}, \tag{31}$$

with

$$\begin{aligned}\mathcal{A}=&\frac{1}{s+4}\frac{1}{t}+\frac{2}{s(s+4)}-\frac{1}{s^2}\frac{1}{s^{1/2}(s+4)^{1/2}}\left(4\frac{s^2+4s-2}{t}-2s-2-t\right)\\&-\frac{3}{2}\left[2\frac{s^2+7s+8}{s(s+4)}\frac{1}{t}+\frac{s+6}{s+4}\frac{1}{u}\right]\frac{1}{s^{1/2}(s+4)^{1/2}}\ln\left(-\frac{t}{s+4}\right)\\&-\frac{1}{2}\left[3\frac{s+2}{t^2}+\frac{3s^2+2s-24}{s(s+4)}\frac{1}{t}+3\frac{s-2}{s}\frac{1}{u}-\frac{8}{s(s+4)}\right]\text{Re}M(s)\\&-\frac{3}{2}\frac{1}{s}\left(\frac{4}{t}+\frac{s-2}{u}+2\right)M(t)-\frac{3}{2}\left(\frac{s+2}{t^2}+\frac{3s+2}{s}\frac{1}{t}+\frac{2}{s}\right)M(u).\end{aligned} \tag{32}$$

This is the same result as obtained in Ref. 3.

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APPENDIX A: y INTEGRALS

The following types of y integrals are encountered for the double-spectral form ($a=s,\ t,$ or u):

$$\int_{y_0}^{\infty}\frac{dy}{y}\frac{1}{y^{1/2}(y-y_0)^{1/2}}=\frac{1}{2\lambda^2}, \tag{A1}$$

$$\int_{y_0}^{\infty}\frac{dy}{y+a}\frac{1}{y^{1/2}(y-y_0)^{1/2}}=\frac{1}{a}\ln\frac{a}{\lambda^2}, \tag{A2}$$

$$\int_{y_0}^{\infty}\frac{dy}{y+x}\frac{1}{y^{1/2}(y-y_0)^{1/2}}=\frac{1}{x}\ln\frac{x}{\lambda^2}, \tag{A3}$$

$$\int_{y_0}^{\infty}\frac{dy}{(y+x)^2}\frac{1}{y^{1/2}(y-y_0)^{1/2}}=-\frac{1}{x^2}+\frac{1}{x^2}\ln\frac{x}{\lambda^2}, \tag{A4}$$

where [see Eq. (I55)]

$$y_0=4\lambda^2\frac{x+\lambda^2}{x}. \tag{A5}$$

The above integrals have been calculated for the case of nonforward scattering and away from threshold. Notice that because of Eq. (A1), the $-\lambda^2[y(x+y)]^{-1}(12y+8x)$ term in h_3 must be retained, while the other λ^2 terms in h_i [Eqs. (I25)–(I29)] can be neglected.

The remaining y integral is for the contact term in the photon channel. In terms of the special function $G(a)$ [see Eq. (18)] we obtain

$$\int_{4\lambda^2}^{\infty}\frac{dy}{y+a}\,\frac{(y-4\lambda^2)^{1/2}}{y^{1/2}}\,\frac{1}{(y-4)^2}\chi_4 = -\frac{1}{a+4}[\ln a+2G(a)]\,. \quad (A6)$$

The only nontrivial integration involved is[17] ($\gamma>0$)

$$2\gamma\int_0^{\infty}\frac{dt}{t^2+\gamma^2}\tan^{-1}t=\gamma\int_0^{\pi}\frac{\theta d\theta}{\gamma^2+1+(\gamma^2-1)\cos\theta} = f\left(\frac{\gamma-1}{\gamma+1}\right)-f\left(\frac{1-\gamma}{\gamma+1}\right)+\tfrac{1}{4}\pi^2\,. \quad (A7)$$

APPENDIX B: x INTEGRALS

The following integrals are involved in calculating the invariant amplitudes. Here, we let a and b stand for s, t, or u, and $a\neq b$. The special functions are all defined in Eq. (18). We have

$$\int_0^{\infty}\frac{dx}{x+4+b}\,\frac{1}{x^{1/2}(x+4)^{1/2}}=M(b)\,, \quad (B1)$$

$$\int_0^{\infty}\frac{dx}{x-a}\,\frac{1}{x^{1/2}(x+4)^{1/2}}=-\mathrm{Re}M(a)\,, \quad (B2)$$

$$\int_0^{\infty}\frac{dx}{(x-a)^2}\,\frac{1}{x^{1/2}(x+4)^{1/2}} = \frac{1}{a(a+4)}[-1+(a+2)\,\mathrm{Re}M(a)]\,, \quad (B3)$$

$$\int_0^{\infty}\frac{dx}{x+4}\,\frac{1}{x^{1/2}(x+4)^{1/2}}=\tfrac{1}{2}\,, \quad (B4)$$

$$\int_0^{\infty}\frac{dx}{x+4+b}\,\frac{1}{x^{1/2}(x+4)^{1/2}}\ln\frac{x}{\lambda^2}=-N(b)-M(b)\ln\lambda^2\,, \quad (B5)$$

$$\int_0^{\infty}\frac{dx}{x-a}\,\frac{1}{x^{1/2}(x+4)^{1/2}}\ln\frac{x}{\lambda^2}=-\ln\frac{a}{\lambda^2}\,\mathrm{Re}M(a)+G(a)\,, \quad (B6)$$

$$\int_0^{\infty}\frac{dx}{(x-a)^2}\,\frac{1}{x^{1/2}(x+4)^{1/2}}\ln\frac{x}{\lambda^2} = -\frac{a+2}{a(a+4)}\left[-\ln\frac{a}{\lambda^2}\,\mathrm{Re}M(a)+G(a)\right] + \frac{1}{a(a+4)}\ln\lambda^2-\frac{1}{a}\,\mathrm{Re}M(a)\,, \quad (B7)$$

$$\int_0^{\infty}\frac{dx}{x+4}\,\frac{1}{x^{1/2}(x+4)^{1/2}}\ln\frac{x}{\lambda^2}=-\tfrac{1}{2}\ln\lambda^2\,. \quad (B8)$$

The above integrals have all been reduced to the special functions by means of standard Spence-function identities[12] along with

$$f(x)-f(-x)=-f\left(\frac{2}{1+x}\right)+f\left(\frac{2x}{1+x}\right)+\tfrac{1}{4}\pi^2\,. \quad (B9)$$

It is to be noted that the $\mathrm{Re}M(a)$ terms always cancel in the calculation of any of the invariant amplitudes. Also, for $a>0$, some of these integrals have a singularity at $x=a$. We can handle this by excluding a small region of the x axis in the neighborhood of $x=a$: $a-\epsilon<x<a+\epsilon$. Then Eqs. (B3) and (B7) have terms of order $1/\epsilon$. Since the final form of the invariant amplitudes has no singularity at $x=a$, these terms cancel out and accordingly have been ignored.

The final x integration is for the contact terms in the electron channel. We have, for example,

$$\int_0^{\infty}\frac{dx}{x+4+b}\,\frac{1}{x^{1/2}(x+4)^{1/2}}\bar{\chi}_1=\tfrac{1}{2}M(b)\,. \quad (B10)$$

APPENDIX C: INVARIANT AMPLITUDES

The invariant amplitudes are [cf. Eqs. (I75) and (I76)]

$$M_2^i(s,t,u)=M_1^i(t,s,u)\,, \quad (C1)$$

$$M_1^1=\mathfrak{D}_1(s,t)-\mathfrak{D}_1(s,u)+I_1(s)\,, \quad (C2)$$

$$M_1^2=\mathfrak{D}_2(s,t)-\mathfrak{D}_2(s,u)+I_2(s)\,, \quad (C3)$$

$$M_1^3=\mathfrak{D}_3(s,t)+\mathfrak{D}_3(s,u)\,, \quad (C4)$$

$$M_1^4=\mathfrak{D}_4(s,t)+\mathfrak{D}_4(s,u)+I_4(s)\,, \quad (C5)$$

$$M_1^5=\mathfrak{D}_5(s,t)+\mathfrak{D}_5(s,u)\,. \quad (C6)$$

Here, I_i arise from the insertions

$$I_1(s)=-\tfrac{1}{2}M(s)\,, \quad (C7)$$

$$I_2(s)=\frac{1}{s}\,4(1-2\Phi_s\coth 2\Phi_s)\ln\lambda+\frac{1}{s}\,\mathfrak{L}(s)+\frac{s-4}{2s}M(s)\,, \quad (C8)$$

$$I_4(s)=\frac{t-u}{2s}M(s)\,. \quad (C9)$$

The contributions of the spectral forms are contained in $\mathfrak{D}_i(a,b)$:

$$\mathfrak{D}_i(a,b)=\int dx\,dy\,\frac{h_i}{\sqrt{\Delta}}\,\frac{1}{y+a}\,\frac{1}{x+4+b} + \int dy\,\frac{(y-4\lambda^2)^{1/2}}{y^{1/2}}\,\frac{1}{(y-4)^2}\,\frac{1}{y+a}\chi_i + \int dx\frac{1}{x^{1/2}(x+4)^{1/2}}\,\frac{1}{x+4+b}\bar{\chi}_i\,. \quad (C10)$$

In terms of the special functions of Eq. (18) we have

$$\mathfrak{D}_1(a,b)=\frac{1}{2}\frac{a+2}{a+b+4}M(b)+\frac{1}{2}\frac{1}{a+4}\left(\frac{a+2}{a+b+4}-\frac{2}{b}\right)\ln a+\frac{a}{a+b+4}\left(\frac{1}{a+4}-\frac{1}{2}\frac{a+2}{a+b+4}\right)G(a)+\frac{b+2}{a+b+4}\left(\frac{1}{2}\frac{a}{a+b+4}+\frac{1}{b}\right)D(a,b), \quad \text{(C11)}$$

$$\mathfrak{D}_2(a,b)=\frac{2b+4}{a}M(b)\ln\frac{a}{\lambda^2}-\frac{1}{2}M(b)+\frac{a+3}{a+b+4}G(a)-\frac{b+1}{a+b+4}D(a,b), \quad \text{(C12)}$$

$$\mathfrak{D}_3(a,b)=-\frac{4}{a}M(b)+\frac{4}{a+b+4}[G(a)+D(a,b)], \quad \text{(C13)}$$

$$\mathfrak{D}_4(a,b)=-\frac{1}{2}M(b)-\frac{1}{a+4}[\ln a+2G(a)]+\frac{3}{a+b+4}[G(a)+D(a,b)], \quad \text{(C14)}$$

$$\mathfrak{D}_5(a,b)=-\frac{1}{2}\frac{b+2}{a+b+4}M(b)+\frac{1}{2}\frac{1}{a+4}\left(\frac{a+2}{a+b+4}+\frac{2}{b}\right)\ln a+\frac{a}{a+b+4}\left(1-\frac{1}{2}\frac{a+2}{a+b+4}\right)G(a)+\frac{1}{a+b+4}\left(\frac{1}{2}\frac{a(b+2)}{a+b+4}-b-4-\frac{2}{b}\right)D(a,b). \quad \text{(C15)}$$

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