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Modified WKB approximation for phase shifts of an attractive singular potential

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The conventional WKB method for phase-shift calculations is known to fail for singular potentials at low incident energies. The modified WKB method can be applied successfully to singular potentials even at low energies since it is an extreme generalization of the conventional WKB method. As a special case, the modified WKB method can be reduced to the conventional WKB approximation and then agrees with the conventional WKB method in the high-energy region where the conventional WKB approximation is valid. Unlike the conventional WKB method, the modified WKB method can be applied to multi-turning-point problems without any difficulty. It is for these reasons that we present this method for determining the phase shifts of the attractive singular potential using a potential of the form $U(r)=-g^2r^{-4}$ as an example. We restrict ourselves to the low-energy, nonzero-orbital-angular-momentum case where there are three classical turning points in order to demonstrate the ability of this method to handle many turning points. The phase shifts obtained by this method agree with the numerical results.

I. INTRODUCTION

The conventional WKB approximations for phase-shift calculations fails at low energies, while yielding accurate results at high energies. This was pointed out in a recent review article by Frank, Land, and Spector[1] on singular potentials.

Since the conventional WKB approximation is a special case of the more general modified WKB approximation, we believe that the modified WKB method can be applied over the entire energy range without being hampered by the turning-point problems of the conventional WKB method. We choose to illustrate the problem by an attractive r^{-4} potential so that comparisons can be made with the exact results. This is done to verify the accuracy of the modified WKB method in regions where the conventional WKB approximation is known to fail. We are not necessarily confining ourselves to such a potential, nor do we intend to produce better results than the existing ones. We selected the $-r^{-4}$ potential because of the availability of the exact results. The modification of the WKB method, in order to deal with many turning points over all energies, is our main concern.

In order to avoid the singularity at the origin, a simple truncation is introduced such that,

$$u(r)=\frac{-g^2}{r^4}\,\theta(r-d)\ ,$$

where θ is the unit-step function and $g^2>0$.

The modified WKB method[2] approaches the problem by formulating a model potential qualitatively similar to the actual potential and whose Schrödinger equation can be solved exactly. Using the exact solutions as the bases of the approximation, one can obtain an approximation of the wave functions for the actual potential. The differences between model and actual potential are then treated as a WKB-type perturbation. The reader is referred to Ref. 2 for details. Lu and Measure[3] extended this method to the three-dimensional scattering case and removed an apparent divergence from the higher-order terms. However, in order to remove the divergence, another divergence is introduced at the maxima (and/or minima) of the potential which imposes conditions on the path of integration. Even so, this method was used to obtain excellent phase-shift results for the repulsive r^{-4} potential[4,5] (where there is only one classical turning point and hence no maxima or minima) and to the barrier-penetration problem[6] (where the method was limited to energies above the barrier). However, this method cannot be applied to the attractive r^{-4} case (since there are three classical turning points and a maxima as well) unless the divergence at the maxima is removed. We recently developed a method of removing this divergence which allows the inclusion of the higher-order terms and obtained the transmission coefficients for the one-dimensional single-barrier-penetration problem.[7] The reader is referred to Ref. 7 for details. By incorporating the singular integration method developed in Ref. 7 into the general modified WKB theory, the modified WKB approximation is valid even at low energies where there may be many turning points. We now extend this method to the solution of the three-dimensional problem here.

We choose our model potential as shown in Fig. 2; namely, a linearly increasing potential between a hard core and a centrifugal barrier. The slope of this potential and its turning points will be determined by the conditions of the actual potential. We restrict ourselves to the low-energy case (for $l \neq 0$), where there are three turning points in the problem. The results of the phase-shift calculation agree with the numerical results[9] as presented in Tables I and II.

In the beginning of Sec. II we present the model problem. It is exactly solvable, and it contains three turning points. In Sec. II B the zeroth-order approximation in $\hbar^2$ is discussed. The equations are basically simple. However, in Sec. II C, as we consider the first-order approximation in $\hbar^2$, we see that the formulas grow longer. It is further complicated as we use the method of singular integration by parts. However, we see that the principle involved is very straightforward indeed.

II. SOLUTION OF THE LOW-ENERGY SCATTERING PROBLEM

Since we restrict ourselves to the low-energy region and $l \neq 0$, we can simplify the problem by considering the case for small d, where the potential energy is far greater than the total energy and hence $\psi(r)$ is practically zero. Setting $x = kr$ and $\beta^2 = g^2 m/\hbar^2$, we obtain the dimensionless form

$$\psi(x)=0, \quad \text{for } x \leqslant kd$$
$$\left[\frac{d^2}{dx^2}+\frac{P_1^2(x)}{\hbar^2}\right]\psi(x)=0, \quad \text{for } x \geqslant kd \tag{1}$$

where

$$\psi(x)=(kr)R_l(kr)\,,$$
$$P_1^2(x)=T_1(x)=\hbar^2 t_1(x)=\hbar^2\left[1-\frac{2\alpha}{x^2}+\frac{(\beta k)^2}{x^4}\right], \tag{2}$$

and

$$\alpha=\tfrac{1}{2}[l(l+1)]\,. \tag{3}$$

The classical turning points correspond to the condition $P_1(x)=0$, and since $kr \geqslant 0$, we obtain (see Fig. 1)

$$x_1 = kd\,,$$
$$x_2=\{\alpha-[\alpha^2-(\beta k)^2]^{1/2}\}^{1/2}\,,$$
$$x_3=\{\alpha+[\alpha^2-(\beta k)^2]^{1/2}\}^{1/2}\,, \tag{4}$$
$$x_{\max}=\frac{\beta k}{\alpha^{1/2}}\,.$$

We now construct our model problem as follows:

$$\left[\frac{d^2}{ds^2}+\frac{P_2^2(s)}{\hbar^2}\right]\phi(s)=0\,, \tag{5}$$

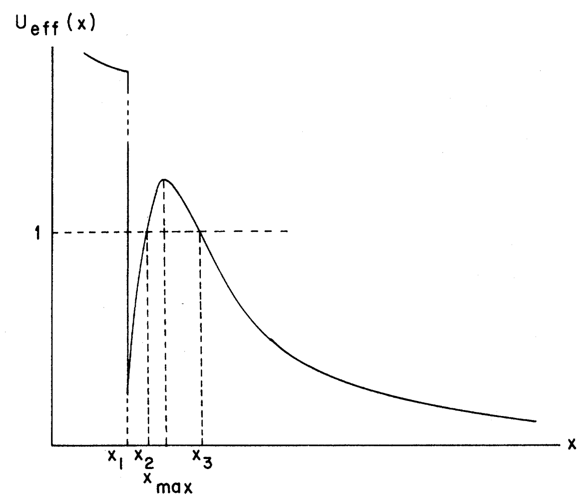

FIG. 1. $U_{\text{eff}} = -[(\beta k)^2/x^2]\,\theta(x-kd)+2\alpha/x^2$, where θ is the unit-step function.

where

$$\phi(s) = sR_l'(s),$$

$$P_2^2(s) = \hbar^2 t_2^2(s) \tag{6}$$

$$= \hbar^2[1 - V_{\text{eff}}(s)],$$

and

$$V_{\text{eff}}(s) = \begin{cases} \infty & \text{for } s \leqslant s_1, \\ as + b & \text{for } s_1 \leqslant s \leqslant s_2, \\ 2\alpha/s^2 & \text{for } s \geqslant s_2. \end{cases} \tag{7}$$

The turning points are designated s_1, s_4, and s_3, respectively (see Fig. 2). From the condition $P_2(s_4) = 0$ and the condition that the potential be the same at s_2 we obtain

$$a = (2\alpha - s_2^2)/[s_2^2(s_2 - s_4)], \tag{8a}$$

$$b = (s_2^3 - 2\alpha s_4)/[s_2^2(s_2 - s_4)] \tag{8b}$$

and from the condition $P_2(s_3) = 0$, we obtain

$$s_3 = (2\alpha)^{1/2}. \tag{8c}$$

The points s_1, s_2, and s_4 will be specified by the conditions of the problem later.

The solutions of Eq. (5) are as follows:

$$\phi(s) = 0, \quad \text{for } s \leqslant s_1 \tag{9a}$$

$$\phi(s) = c_1 \text{Ai}(y) + c_2 \text{Bi}(y), \quad \text{for } s_1 \leqslant s \leqslant s_2 \tag{9b}$$

$$\phi(s) = sA \cos\delta_l [J_l(s) - \eta_l(s) \tan\delta_l], \quad \text{for } s \geqslant s_2 \tag{9c}$$

where

$$y = a^{1/3}\left(s + \frac{b-1}{a}\right). \tag{10}$$

By equating the logarithmic derivatives at the boundaries s_1 and s_2, respectively, we eliminate the constants c_1, c_2, A and obtain

$$\tan\delta_s = \frac{J_l(s_2) + s_2 J_l'(s_2) - s_2 F J_l(s_2)}{\eta_l(s_2) + s_2 \eta_l'(s_2) - s_2 F \eta_l(s_2)}, \tag{11}$$

where

$$F = a^{1/3}\left[\frac{\text{Ai}'(y_2)\text{Bi}(y_1) - \text{Ai}(y_1)\text{Bi}'(y_2)}{\text{Ai}(y_2)\text{Bi}(y_1) - \text{Ai}(y_1)\text{Bi}(y_2)}\right] \tag{12}$$

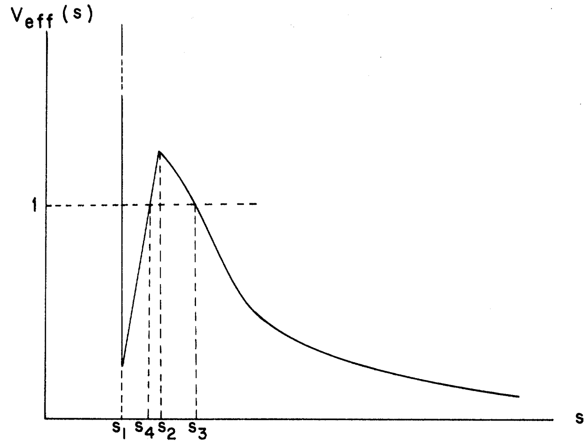

FIG. 2. The model potential as given by Eq. (7).

and y_1, y_2 are the values of y at s_1 and s_2, respectively.

A. The phase-shift connection formula

Now the solution of Eqs. (1) and (5) must be of the form

$$\psi_l(x) \underset{x \to \infty}{\sim} \sin(x - \tfrac{1}{2}\pi l + \delta_x), \tag{13}$$

$$\phi_l(s) \underset{s \to \infty}{\sim} \sin(s - \tfrac{1}{2}\pi l + \delta_s) \tag{14}$$

so that the phase shifts of the model problem are related to the phase shifts of the actual problem by

$$\delta_x = \delta_s + \lim_{\substack{x \to \infty \\ s \to \infty}} (s - x). \tag{15}$$

B. The zeroth-order approximation

To zeroth order in $\hbar^2$, the model problem is connected to the actual problem by (see Refs. 2 and 8)

$$\int_{x_1}^{x} P_1(x)dx = \int_{s_1}^{s} P_2(s)ds. \tag{16}$$

The path of integration may be subdivided and Eq. (16) becomes

$$\int_{x_1}^{x_2} P_1(x)dx + i\int_{x_2}^{x_{\max}} |P_1(x)|dx + i\int_{x_{\max}}^{x_3} |P_1(x)|dx + \int_{x_3}^{x} P_1(x)dx = \int_{s_1}^{s_4} P_2(s)ds + i\int_{s_4}^{s_2} |P_2(s)|ds$$

$$+ i\int_{s_2}^{s_3} |P_2(s)|ds + \int_{s_3}^{s} P_2(s)ds, \tag{17}$$

where $P_1^2(x)$ and $P_2^2(s)$ are negative in the range $x_2 \leq x \leq x_3$ and $s_4 \leq s \leq s_3$, respectively. By equating real and imaginary parts, we obtain

$$\int_{x_1}^{x_2} P_1(x)dx + \int_{x_3}^{x} P_1(x)dx = \int_{s_1}^{s_4} P_2(s)ds + \int_{s_3}^{s} P_2(s)ds \tag{18}$$

and

$$\int_{x_2}^{x_{\max}} |P_1(x)|dx + \int_{x_{\max}}^{x_3} |P_1(x)|dx = \int_{s_4}^{s_2} |P_2(s)|ds = \int_{s_2}^{s_3} |P_2(s)|ds\,. \tag{19}$$

Now Eqs. (18) and (19) are satisfied by the following conditions:

$$\int_{x_3}^{x} P_1(x)dx = \int_{s_3}^{s} P_2(s)ds\,, \tag{20a}$$

$$\int_{x_1}^{x_2} P_1(x)dx = \int_{s_1}^{s_4} P_2(s)ds\,, \tag{20b}$$

$$\int_{x_2}^{x_{\max}} |P_1(x)|dx = \int_{s_4}^{s_2} |P_2(s)|ds\,, \tag{20c}$$

$$\int_{x_{\max}}^{x_3} |P_1(x)|dx = \int_{s_2}^{s_3} |P_2(s)|ds\,. \tag{20d}$$

Thus Eqs. (20b), (20c), and (20d) specify points s_1, s_2, and s_4 to zeroth order in $\hbar^2$ while Eq. (20a) gives us the $\lim_{x\to\infty;\, s\to\infty}(s-x)$ term in Eq. (15).

C. The first-order approximation

To first order in $\hbar^2$, we obtain (see Ref. 3)

$$\int_{x_1}^{x} P_1\,dx + \frac{\hbar^2}{16}\oint\left(\frac{3P_1'^2}{P_1^3} - \frac{2P_1''}{P_1^2}\right)dx = \int_{s_1}^{s} P_2 ds + \frac{\hbar^2}{16}\oint\left(\frac{3P_2'^2}{P_2^3} - \frac{2P_2''}{P_2^2}\right)ds\,, \tag{21}$$

where the contour is taken from $\infty - i\epsilon$ around the lowest turning point back out to $\infty + i\epsilon$. Making repeated use of the integration by parts

$$\oint U dV = \lim_{\epsilon\to 0} UV\Big|_{\infty-i\epsilon}^{\infty+i\epsilon} - \int_{\infty-i\epsilon}^{\infty+i\epsilon} VdU = -\oint VdU\,, \tag{22}$$

Eq. (21) becomes

$$\int_{x_1}^{x} P_1 dx + \tfrac{1}{2}\oint G(t_1)t_1^{1/2}dx = \int_{s_1}^{s} P_2 ds + \tfrac{1}{2}\oint G(t_2)t_2^{1/2}ds\,, \tag{23}$$

where

$$G(t_i) = \tfrac{1}{12}\left(\frac{t_i''''}{t_i'^2} - \frac{4t_i'''t_i''}{t_i'^3} + 3\,\frac{t_i''^3}{t_i'^4}\right)\,. \tag{24}$$

While the divergences at the turning points have been removed, a divergence at $t'=0$ has been introduced if we convert the contour integrals into the ordinary definite ones. Using the method developed in Ref. 7 we convert the contour integrals in Eq. (23) to definite integrals, subdivide the intervals along the path of integration (which isolates the divergence), and after repeated integration by parts, Eq. (23) becomes

$$\int_{s_1}^{s_4} P_2\,ds + i\int_{s_4}^{s_2} U_2\,ds + i\left(1+\frac{1}{16\alpha}\right)\int_{s_2}^{s_3} U_2\,ds + \left(1+\frac{1}{16\alpha}\right)\int_{s_3}^{s} P_2\,ds$$
$$= \int_{x_1}^{x_2}[1+G(t_1)]\,t_1^{1/2}dx + i\int_{x_1}^{x_{\max}-\epsilon_1}[1+G(U_1)]\,U_1^{1/2}\,dx + i\int_{x_{\max}-\epsilon_1}^{x_{\max}+\epsilon_2} U_1^{1/2}\,dx + i\int_{x_{\max}+\epsilon_2}^{x_3}[1+G(U_1)]\,U_1^{1/2}\,dx$$
$$+ \frac{i}{12}\left[\left(\frac{U_1'''}{U_1'^2} - \frac{U_1''^2}{U_1'^3}\right)U_1^{1/2} + \frac{U_1''}{2U_1'U_1^{1/2}}\right]\Bigg|_{x_{\max}-\epsilon_1}^{x_{\max}+\epsilon_2} - \frac{i}{48}\int_{x_{\max}-\epsilon_1}^{x_{\max}+\epsilon_2}\frac{U_1''}{U_1^{3/2}}\,dx + \int_{x_3}^{x}[1+G(t_1)]\,t_1^{1/2}\,dx\,. \tag{25}$$

Notice that in Eq. (25) $P_2 = i\hbar U_2^{1/2}$, where U_2 is positive in the interval between s_4 and s_2 as well as between s_2 and s_3. On the right-hand side of Eq. (25) we see that $P_1 = i\hbar U_1^{1/2}$, where U_1 is positive between x_2 and x_3. We have applied integration by parts several times to Eq. (25) mainly to avoid the singularity where $U_1' = 0$ at $x = x_{\max}$. The choices of ϵ_1 and ϵ_2 are arbitrary insofar as they are different from zero and are positive. The final results shows that they are independent of these choices.

Repeating the procedure used in the zeroth-order approximation, we equate real and imaginary parts and obtain Eqs. (30) and (31), where

$$P_2 = \hbar t_2^{1/2} = i\hbar U_2^{1/2} = \begin{cases} \hbar(1-as-b)^{1/2}, & \text{for } s_1 \leq s \leq s_4 \\ i\hbar(as+b-1)^{1/2}, & \text{for } s_4 \leq s \leq s_2 \\ i\hbar\left(\frac{2\alpha}{s^2}-1\right)^{1/2}, & \text{for } s_2 \leq s \leq s_3 \\ \hbar\left(1-\frac{2\alpha}{s^2}\right)^{1/2}, & \text{for } s \geq s_3 \end{cases} \tag{26}$$

$$G(t_2) = \begin{cases} 0, & \text{for } s_1 \leq s \leq s_2 \\ \frac{1}{16\alpha}, & \text{for } s \geq s_2 \end{cases} \tag{27}$$

$$P_1 = \hbar t_1^{1/2} = i\hbar U_1^{1/2} = \begin{cases} \hbar\left[1-\frac{2\alpha}{x^2}-\frac{(\beta k)^2}{x^4}\right]^{1/2}, & \text{for } x_1 \leq x \leq x_2 \\ i\hbar\left[\frac{2\alpha}{x^2}+\frac{(\beta k)^2}{x^4}-1\right]^{1/2}, & \text{for } x_2 \leq x \leq x_3 \\ \hbar\left[1-\frac{2\alpha}{x^2}-\frac{(\beta k)^2}{x^4}\right]^{1/2}, & \text{for } x \geq x_3 \end{cases} \tag{28}$$

$$\epsilon_1 = \tfrac{1}{2}(x_{\max}-x_2), \quad \epsilon_2 = \tfrac{1}{2}(x_3-x_{\max}), \tag{29}$$

and

$$\alpha = \tfrac{1}{2}[l(l+1)],$$

$$\int_{s_1}^{s_4} P_2 ds + \left(1+\frac{1}{16\alpha}\right)\int_{s_3}^{s} P_2 ds = \int_{x_1}^{x_2}[1+G(t_1)]\,t_1^{1/2}dx + \int_{x_3}^{x}[1+G(t_1)]\,t_1^{1/2}dx \tag{30}$$

and

$$\int_{s_4}^{s_2} U_2 ds + \left(1+\frac{1}{16\alpha}\right)\int_{s_2}^{s_3} U_2 ds = \int_{x_2}^{x_{\max}-\epsilon_1}[1+G(U_1)]\,U_1^{1/2}dx + \int_{x_{\max}-\epsilon_1}^{x_{\max}+\epsilon_2} U_1^{1/2}dx + \int_{x_{\max}+\epsilon_2}^{x_3}[1+G(U_1)]\,U_1^{1/2}dx$$
$$+\tfrac{1}{12}\left[\left(\frac{U_1'''}{U_1'^2}-\frac{U_1''^2}{U_1'^3}\right)U_1^{1/2}+\frac{U_1''}{2U_1'U_1^{1/2}}\right]\Bigg|_{x_{\max}-\epsilon_1}^{x_{\max}+\epsilon_2} - \tfrac{1}{48}\int_{x_{\max}-\epsilon_1}^{x_{\max}+\epsilon_2}\frac{U_1''}{U_1^{3/2}}dx. \tag{31}$$

Now Eqs. (30) and (31) are satisfied by the following conditions:

$$\left[1+\frac{1}{16\alpha}\right]\int_{s_3}^{s} P_2 ds = \int_{x_3}^{x}[1+G(t_1)]\,t_1^{1/2}dx, \tag{32}$$

$$\int_{s_1}^{s_4} P_2 ds = \int_{x_1}^{x_2}[1+G(t_1)]\,t_1^{1/2}dx, \tag{33}$$

$$\int_{s_4}^{s_2} U_2^{1/2} ds = \int_{x_2}^{x_{\max}-\epsilon_1}[1+G(t_1)]U_1^{1/2}dx + \int_{x_{\max}-\epsilon_1}^{x_{\max}} U_1^{1/2}dx$$
$$-\tfrac{1}{12}\left[\left(\frac{U_1'''}{U_1'^2}-\frac{U_1''^2}{U_1'^3}\right)U_1^{1/2}+\frac{U_1''}{2U_1'U_1^{1/2}}\right]\Bigg|_{x=x_{\max}-\epsilon_1} - \tfrac{1}{48}\int_{x_{\max}-\epsilon}^{x_{\max}}\frac{U_1''}{U_1^{3/2}}dx, \tag{34}$$

and

TABLE I. Phase shifts δ_x for $l=2$ and various βk. (a) Modified WKB approximation to zeroth order in $\hbar^2$; (b) modified WKB approximation to first order in $\hbar^2$; (c) exact results.

	$\beta/d=2.8$			$\beta/d=3.2$			$\beta/d=3.6$			$\beta/d=4.0$		
βk	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
0.01	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
0.30	0.0024	0.0026	0.0027	0.0024	0.0026	0.0027	0.0024	0.0026	0.0027	0.0024	0.0026	0.0027
0.50	0.0066	0.0073	0.0076	0.0066	0.0073	0.0076	0.0066	0.0073	0.0076	0.0066	0.0073	0.0076
1.00	0.0269	0.0305	0.0317	0.0275	0.0308	0.0319	0.0280	0.0310	0.0322	0.0286	0.0313	0.0328
1.50	0.0634	0.0754	0.0763	0.0721	0.0777	0.0780	0.0718	0.0798	0.0804	0.0797	0.0823	0.0864

$$\left(1+\frac{1}{16\alpha}\right)\int_{s_2}^{s_3} U_2^{1/2}ds = \int_{x_{\max}}^{x_{\max}+\epsilon_2} U_1^{1/2}dx + \tfrac{1}{12}\left[\left(\frac{U_1'''}{U_1'^2} - \frac{U_1''^2}{U_1'^3}\right)U_1^{1/2} + \frac{U_1''}{2U_1'U_1^{1/2}}\right]\Bigg|_{x=x_{\max}+\epsilon_2}$$

$$-\tfrac{1}{48}\int_{x_{\max}}^{x_{\max}+\epsilon_2} \frac{U_1''}{U_1^{3/2}}dx + \int_{x_{\max}+\epsilon_2}^{x_3} [1+G(U_1)]U_1^{1/2}dx . \qquad (35)$$

Thus Eqs. (33), (34), and (35) specify the points s_1, s_2, and s_4 to first order in $\hbar^2$ and Eq. (32) (with some modification) gives us the $\lim_{x\to\infty;s\to\infty}(s-x)$ term in Eq. (15).

For $x \geqslant x_3$ and $s \geqslant s_3$, we substitute Eqs. (26) and (28) into Eq. (32) and obtain

$$\left(1+\frac{1}{16\alpha}\right)\int_{s_3}^{s}\left(1+\frac{2\alpha}{s^2}\right)^{1/2}ds = H_1 + \int_{s_3}^{x}\left\{[1+G(t_1)]\left[1-\frac{2\alpha}{x^2}+\frac{(\beta k)^2}{x^4}\right]^{1/2} - \left(1+\frac{1}{16\alpha}\right)\left(1-\frac{2\alpha}{x^2}\right)^{1/2}\right\}dx$$

$$+\left(1+\frac{1}{16\alpha}\right)\int_{s_3}^{x}\left(1-\frac{2\alpha}{x^2}\right)^{1/2}dx , \qquad (36)$$

where

$$H_1 = \int_{x_3}^{s_3}[1+G(t_1)]\left[1-\frac{2\alpha}{x^2}+\frac{(\beta k)}{x^4}\right]^{1/2}dx . \qquad (37)$$

Setting $y = 1/x$, Eq. (36) becomes

$$\left(1+\frac{1}{16\alpha}\right)\int_{s_3}^{s}\left(1+\frac{2\alpha}{s^2}\right)^{1/2}ds$$

$$= H_1 + H_2 + \left(1+\frac{1}{16\alpha}\right)\int_{s_3}^{x}\left(1+\frac{2\alpha}{x^2}\right)^{1/2}dx , \qquad (38)$$

where

$$H_2 = \int_{1/x}^{1/s_3}\left[[1+G(t_1)][1-2\alpha y^2+(\beta k)^2y^4]^{1/2} - \left(1+\frac{1}{16\alpha}\right)[1-2\alpha y^2]^{1/2}\right]\frac{dy}{y^2}.$$

Thus we obtain

$$\lim_{\substack{x\to\infty\\ s\to\infty}}(s-x) = \frac{16\alpha(H_1+H_2)}{1+16\alpha}$$

$$= \frac{8l(l+1)(H_1+H_2)}{1+8l(l+1)}, \qquad (39)$$

where

$$H_2 = \int_{0}^{1/s_3}\left[[1+G(t_1)][1-2\alpha y^2+(\beta k)^2y^4]^{1/2} - \left(1+\frac{1}{16\alpha}\right)[1-2\alpha y^2]^{1/2}\right]\frac{dy}{y^2}. \qquad (40)$$

The results obtained by this method are presented in Tables I and II.

III. DISCUSSION

In dealing with the three-dimensional attractive singular potential problem, we have restricted ourselves to the low-energy range where there are three turning points. Our main concern is to develop and extend the modified WKB method to the point where it can be applied successfully to multi-turning-point problems. For this reason, we limited ourselves to phase-shift calculations in order to compare results. Since the model-problem wave function is related to the actual-problem wave function by

$$\psi(x) = [s'(x)]^{-1/2}\phi(s(x)), \qquad (41)$$

we can obtain $\psi(x)$ without any difficulty.

TABLE II. Phase shifts δ_x for $l=3$ and various βk. (a) Modified WKB approximation to zeroth order in $\hbar^2$; (b) modified WKB approximation to first order in $\hbar^2$; (c) exact results.

	$\beta/d=3.6$			$\beta/d=4.0$		
βk	(a)	(b)	(c)	(a)	(b)	(c)
0.01	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
0.30	0.0008	0.0009	0.0009	0.0008	0.0009	0.0009
0.50	0.0023	0.0025	0.0025	0.0023	0.0025	0.0025
1.00	0.0094	0.0099	0.0101	0.0094	0.0099	0.0101
1.50	0.0214	0.0226	0.0229	0.0214	0.0226	0.0229
2.00	0.0384	0.0407	0.0415	0.0385	0.0408	0.0415
2.50	0.0610	0.0651	0.0663	0.0611	0.0653	0.0664
3.00	0.0895	0.0966	0.0985	0.0904	0.0973	0.0989
3.50	0.1248	0.1373	0.1393	0.1275	0.1392	0.1404
4.00	0.1681	0.1910	0.1904	0.1753	0.1960	0.1932

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Electron-electron scattering. II. Helicity cross sections for positron-electron scattering*

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The differential cross sections for polarized electron-positron scattering are calculated to order e^6 by using the five invariant amplitudes presented in a previous paper. The unpolarized result of Polovin is rederived. As an application of the helicity amplitudes the spin-momentum correlation for a polarized target positron is obtained in agreement with Fronsdal and Jaksic.

I. INTRODUCTION

The differential cross section for unpolarized electron-electron scattering, to order e^6, was calculated first by Redhead[1] and later by Polovin.[2] The spin-momentum correlation in electron-positron scattering in which the spin of only one of the particles is detected was calculated by Fronsdal and Jaksic.[3] However, the general polarization case has not been previously derived.

In an earlier paper[4] (called paper I), the five invariant amplitudes were obtained in spectral form. Here, we will apply these invariant amplitudes to calculate the helicity amplitudes for electron-positron scattering. (The corresponding results for electron-electron scattering will be presented in a subsequent communication.) Because of the infrared nature of charged-particle scattering, we will consider neither near-threshold nor forward scattering. However, these kinematical regions are correctly described in the results of paper I, in terms of a fictitious photon mass. This detailed structure cannot be measured directly and would be significant only in the application of the spectral forms to higher-order calculations.[5]

We present the helicity amplitudes in terms of the invariant amplitudes in Sec. II, and the explicit forms in Sec. III. In Sec. IV, we consider soft-photon contributions. The unpolarized differential cross section is calculated in Sec. V and the spin-momentum correlation is found in Sec. VI. Appendixes A and B contain the integrals necessary for the calculations of Sec. III while the invariant amplitudes are given in Appendix C.

II. HELICITY AMPLITUDES

This section is devoted to calculating the helicity amplitudes in terms of the invariant amplitudes. This is done by applying Eq. (I74) to an appropriate helicity state. (Here I refers to equations in paper I.) The two basic structures encountered are

$$F(1\,1';\,2\,2') = \sum_{i=1}^{5} M_2^i\, u_1^* \gamma^0 \Gamma_i\, u_2 u_{1'}^* \gamma^0 \Gamma_i\, u_{2'}$$

$$\equiv \sum_{i=1}^{5} M_2^i\, \Gamma_i(1\,2;\,1'2') \quad (1)$$

and

$$\tilde{F}(1\,1';\,2\,2') = \sum_{i=1}^{5} M_1^i\, u_1^* \gamma^0 \Gamma_i\, u_{1'}^* u_2 \gamma^0 \Gamma_i\, u_{2'}$$

$$\equiv \sum_{i=1}^{5} M_1^i\, \Gamma_i(1\,1';\,2\,2'). \quad (2)$$

For convenience, we will work in the center-of-mass system with $\vec{P}_2$ in the z direction and $\vec{P}_1$ in the x-z plane:

$$\vec{P}_1 = |\vec{P}_1|(\sin\theta, 0, \cos\theta).$$

An explicit representation for the Dirac spinor in terms of the helicity is[6]