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Gauge fields on a lattice. I. General outlook

R. Balian, J. M. Drouffe, and C. Itzykson
Service de Physique Théorique, Centre d'Études Nucléaires de Saclay, B.P. no. 2, 91190 Gif-sur-Yvette, France
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We present Wilson's model of gauge-field theory on a lattice, including a coupling to a matter field. The algebraic structure is surveyed for both commutative and noncommutative groups. Various regimes are suggested by mean-field theory according to the relative values of coupling constants. In particular the gauge field undergoes a first-order transition while the matter-field transition is of second order.

I. INTRODUCTION

The reasons for theorists' interest in gauge fields need not be explained here in detail. Apart from an aesthetical appeal, they seem to provide the most promising models for elementary-particle interactions. This seems to be the case not only in the realm of weak and electromagnetic forces, but also in the domain of strong interactions. Their dynamics in this regime is, however, poorly understood. Hence any indication of the strong-coupling region is *a priori* interesting.

Recently Wilson has introduced a model and various techniques that are novel in this game.[1] The purpose of this work is to comment on Wilson's model and present some numerical results.

It must be made clear that the model itself is to a large extent unrealistic. A noncovariant ultraviolet cutoff procedure is introduced which breaks Lorentz invariance (or rather Euclidean invariance after rotation to an imaginary time). Nevertheless, since it allows an investigation in an unfamiliar regime, it is interesting.

Let us first briefly present the main line of reasoning. Assume that a field theory is studied in the Euclidean region and replace continuous space-time points by a discrete lattice. According to standard practice the full contents of the theory can be thought in terms of a Feynman path integral of e^S, where S is the action, as a functional of classical field variables defined now on the discrete lattice. This is to be integrated over with an appropriate measure on the field variables. The analogy with statistical mechanics is evident. A set of values for the classical field corresponds to a configuration, and S appropriately scaled corresponds to "energy divided by temperature" for this configuration. The sum over configurations provides the analog of the partition function, the logarithm of which is essentially the free energy. Note that the E/T of the statistical interpretation is *not* the energy of the field theory. The latter can be recovered if one wishes through a study of the transfer matrix; we shall not enter into these matters here.

Discretization introduces a fundamental length in the problem, the inverse of which is a natural ultraviolet cutoff. The price paid is that Euclidean invariance is lost, and at best can be recovered when the spacing becomes immaterial. We shall see, however, in which precise sense this statement has to be made; a naive idea would be that no matter what the spacing a is, if we look at "soft phenomena," i.e., for distances $r \gg a$, the breaking of invariance ought not to be too disastrous. This, however, might be superficial, since, as we shall see by investigating more closely the types of distortions introduced by discretization, there exists some kind of built-in breaking of invariance which is hard to control even on a large scale.

If no external source is present the path integral describes the vacuum-to-vacuum amplitude. Hence the problem is: What is the stable vacuum; what are the elementary excitations? Of course, in order for us to be able to answer these questions, external sources have to be introduced at some later stage.

It is clear that with an ultraviolet cutoff present, the usual problems of renormalization of field theory disappear at first. This is, however, only an illusion if we look for small-distance phenomena where one would have to control the behavior as the spacing goes to zero. This is not, however, the purpose of the present approach at this stage; hence this aspect of the question will not be in the forefront. On the other hand, it is well known that gauge fields introduce long-range strong forces. In other words, the infrared problem is indeed catastrophic. This is what we really wish to investigate.

The first technical problem solved by Wilson was to present a gauge theory directly on the lattice that is not a straightforward and unimaginative replacement of a continuous space-time set of functions by their values at discrete points. The clue to this is in the original formulation of gauge invariance. By following essentially the steps that lead from a global invariance under an internal symmetry group to its local realization, one obtains Wilson's correct answer. What this amounts to is replacing a field with values in a *Lie algebra* (the continuous case) by a field variable taking its value in a *Lie group*. Even though this appears at first as a new technical difficulty, it has an unexpected bonus which is easily understood. Indeed the action is *gauge invariant*. This freedom of gauge is ordinarily a headache for quantization, or what is equivalent to formulating a Feynman path integral. For, notwithstanding the fact that the gauge functions vary from point to point, at each point they take values over an infinite interval, with a result that they yield infinite factors right at the beginning of the theory. The remedy to that is usually to *break gauge invariance* at the level of the Lagrangian and go through a complex procedure involving Ward identities to show that the final theory has the desirable covariance properties under gauge transformations. Here, however, the gauge degree of freedom takes its values in a Lie group which, for all practical purposes, is a *compact* group. If, furthermore, a spatial cutoff is introduced by considering a large but finite lattice, the gauge degrees of freedom will introduce no infinity. Consequently *no explicit breaking* of this invariance is needed.

This very interesting consequence of discretization suggests that new phenomena take place. To illustrate this possibility let us turn to a much simpler example that will, in fact, be discussed in detail below. Let us assume that instead of dealing with a local group, our dynamical system exhibits an internal global symmetry of the usual type at the level of the evolution equations. To be specific, let us think of a very simplified σ model. It has two degrees of freedom, the pion and the σ meson, with an O(2) invariance. We can think of the field as a two-component vector in some "isospin space." Let us further discretize the model and impose the restrictions that at each point this vector is of fixed length (to mimic the interaction) and that at neighboring sites the vectors interact by pairs through a term proportional to the scalar product (to mimic the kinetic term in the action). This model then has a global invariance. The scalar product is proportional to the square of the length of our vectors and can be thought of as the ratio between kinetic terms and interaction terms, hence inversely proportional to a fictitious coupling constant g. On the other hand, in the "statistical" interpretation this same strength can be described as inversely proportional to temperature T. The conclusion is that $T \propto g$. Now in the statistical view we can think of our vectors as giving the direction of two-dimensional magnets. Depending on a sign the whole system is of the type ferro- or antiferromagnetic. It turns out here that the correct sign is of the ferromagnetic type, whereupon at low temperature ($g \to 0$) "spins" tend to align. Statistical mechanics even suggests that a transition occurs. At low T (low coupling) an ordered phase is created. This means that the symmetry is spontaneously broken. A Goldstone long-range excitation is present (the massless pion) and a shorter-range branch is also present (the massive σ). As $T \to 0$ ($g \to 0$) the shorter-range excitation essentially disappears (the σ mass goes to infinity) and what are left are noninteracting transverse spin waves: the free massless pion field. Above the critical temperature T_c, however, disorder appears; the symmetry is really implemented in the ordinary way and we have a degenerate massive doublet of excitations. The region of the second-order phase transition is the most interesting from the statistical-mechanics point of view as well as for the particle interpretation. In this region we study the transition between the two possible types of symmetry. The specific dynamics as well as the precise value of the ultraviolet cutoff is indeed inessential since the "soft" scale is set by the correlation length, i.e., the inverse of the very small mass of the σ. In this region we might hope, as far as infrared behavior is concerned, that Euclidean invariance is restored.

The above description justifies the fact that we have devoted Sec. II of this work to this model. We call it the *scalar model*. The reason for this denomination is that in the limit of zero spacing it degenerates into the field theory for a massless scalar field corresponding to the spin waves of statistical mechanics. In this limit all quantities vary continuously from point to point and hence correspond to an almost perfect ordering. The model is then naturally associated with low temperatures or small coupling constants.

We shall then define a *gauge-field model* on the lattice to implement a *local* symmetry.

In the light of the previous discussion, it seems natural to expect that a transition takes place. At low temperature, gauge invariance would be spontaneously broken, corresponding to the usual "free field" case. For a large coupling constant, however, a disordered phase would appear, with local gauge invariance strictly enforced, and without any Goldstone "photon."

It is clear that any quantization of the photon field implies a choice of gauge even though physical quantities ought to be independent of this choice. However, the disordered phase where local invariance is strictly enforced is an unusual situation where quantities such as $\langle A_\mu(x)A_\nu(y)\rangle$ are obviously zero. Furthermore, there is some subtlety involved in choosing an order parameter. We find indication for a *first-order* transition in this system, in contradistinction with the previous case. Furthermore, the high-temperature phase exhibits interesting phenomena which will be described below. These support Wilson's idea that such a phase might provide a mechanism for charged particles binding, which would prevent them from escaping from each other.

We shall first present the various models, introducing notations and describing their main features. In the next parts the scalar, Abelian, and non-Abelian fields are studied in some detail. We shall also study the interesting *coupled system* to be introduced in the text. In each case we try to convince ourselves (and hopefully the reader) that a transition really takes place.

In particular we can be guided by mean field theory which is physically motivated as the dimension d of the lattice gets large. In all cases we find that the transition temperature, or coupling constant, grows linearly with d. Apart from trivial exact solutions in very low dimension (where no transition really takes place but ordering sets in as we approach $T=0$), one has no recourse but to turn to numerical calculations. These calculations involve an adaptation of the techniques familiar to devotees of the Ising model. Since this approach is by itself interesting, we shall postpone its discussion to a forthcoming paper.

It is a pleasant feature of the model to see how naturally non-Abelian gauge fields are incorporated in the formalism at a very minimal cost. Furthermore, insofar as the high-temperature expansion is concerned, they only add a touch of group theory to the bulk of the preceding numerical work.

Let us finally stress that it would certainly be very interesting to be able to formulate the analog of the present strong-coupling expansion in a more realistic field-theoretic case.

II. THE MODELS

A. Transition amplitudes as path integrals

The dynamics of a field theory is conveniently described by the vacuum-to-vacuum transition amplitude in the presence of suitably chosen external sources coupled to the system. These we denote collectively by J, and the above transition amplitude contains all information on the system. Let us call it $\langle 0|0\rangle_J$. Stability of the vacuum is included in the statement that for $J=0$, $\langle 0|0\rangle=1$, and that to first order $(\delta/\delta J)\langle 0|0\rangle_{J=0}=0$. If calculations indicate that this is not the case, this means most likely that the state called vacuum has not been correctly identified and the theory has to be modified accordingly. Even though infinities plague field theory, intuitive approaches can be used as formal tools at unsophisticated levels, to be made more precise at a later stage of calculation. Such a tool is provided by the Feynman path-integral formulation. Since by now the subject has become more familiar, we shall be using it without the usual apology.

In essence what it involves is the following. There exists an underlying classical field theory in terms of classical fields collectively denoted by $\phi(x)$ with an action integral $S(\phi)$. Classical equations of motion would emerge by requiring $S(\phi)$ to be stationary with respect to variations of ϕ: $\delta S(\phi)=0$. In quantum theory we evaluate transition amplitudes. These amplitudes involve summing over all possible "paths" the elementary contributions of the form

$$e^{iS(\phi)}\,.$$

A point of physics seldom stressed is that the only trace of the asymptotic states between which the amplitude is evaluated is essentially in the boundary conditions for large times on the fields ϕ. In particular for the vacuum-to-vacuum amplitude it is generally assumed that the field ϕ vanishes. This assumption might reveal itself faulty. Thus we write

$$\langle 0|0\rangle_J = A\int \mathfrak{D}\phi\, e^{iS(\phi,J)}\,. \tag{2.1}$$

The normalization factor A is chosen by requiring that for $J=0$, $\langle 0|0\rangle=1$. To be specific let us think in terms of a self-coupled scalar field ϕ where

$$S(\phi,J)=\int dx_0 d^3x\{\tfrac{1}{2}[(\partial_0\phi)^2-(\vec{\nabla}\phi)^2]-V(\phi,J)\}\ , \tag{2.2}$$

with V a polynomial in $\phi(x)$ and $J(x)$. We have distinguished spatial and time arguments in order to be able to jump immediately to a Euclidean space by assuming that an analytic continuation to pure imaginary times $(x_0\to -ix_0)$ can be performed. The underlying assumption is rather the reverse: that the Euclidean quantities stand a better chance to be well defined, in such a way that at a final stage an analytic continuation back to physical Minkowski space will be performed for meaningful quantities such as Green functions. Thus, keeping the same notations, we write

$$Z(J) = \int \mathfrak{D}\phi\, e^{S(\phi, J)} ,$$
$$S(\phi, J) = -\int d^4x[\tfrac{1}{2}(\partial\phi)^2 + V(\phi, J)] , \qquad (2.3)$$

with $(\partial\phi)^2$ standing now for the Euclidean square of the gradient of ϕ:

$$(\partial\phi)^2 = (\partial_0\phi)^2 + (\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 .$$

This expression has been written for simplicity as if we were dealing with a single scalar field $\phi(x)$. It is clear that (2.3) is not very well defined at this stage until we make precise (i) the test function space of classical fields and (ii) the integration procedure.

One way to proceed is to replace the continuum space-time by a discrete set of points of a lattice. We assume the latter to be hypercubical. That is, we restrict x to

$$x = \sum_0^3 x_i n_i a , \qquad (2.4)$$

with x_i an integer, $n_i \cdot n_j = \delta_{ij}$. The quantity a is the lattice spacing.

Any expression for S that tends in the limit $a \to 0$ to the one given in (2.3) is *a priori* a good candidate. As far as the potential term V is concerned, there is *a priori* no freedom. Furthermore, we shall for the moment suppress the external source J, and approximate V in such a way that it limits at every point x the range of values for $\phi(x)$ in a way to be made precise later on. We now concentrate on the kinetic term $[\partial\phi(x)]^2$ alone.

Our choice, which admittedly appears rather artificial at this stage, is the following. We set

$$S(\phi) = \sum_{(xx')} \frac{a^{d-4}}{g} \cos a[\phi(x) - \phi(x')] , \qquad (2.5)$$

with $\sum_{(xx')}$ meaning summation over all *pairs of neighboring lattice points*. We insist that the dimension of ϕ be the traditional one of inverse length, hence the occurrence of $a\phi(x)$ as a dimensionless quantity. We have furthermore introduced a new parameter d as the dimension of space-time. Up to now $d = 4$, but we shall allow d to be an arbitrary integer, and for some analytical calculations to be performed later d will take arbitrary real values. Finally, we have added a factor $1/g$ $(g > 0)$ as the ratio of the kinetic to the potential term in the action, the latter restricting the values of $\phi(x)$ to a range $-\pi/a$ to $+\pi/a$.

It is readily seen what the limit $a \to 0$ means. We can arrange the sum $\sum_{(x,x')}$ in the following way. First keeping x fixed we let x' run over $x' = x + an_i$, n_i being any one of the (positive) unit coordinate vectors. Then we sum over x. If a is small and if $\phi(x)$ varies smoothly from point to point we have approximately

$$\phi(x') \simeq \phi(x) + a\frac{\partial}{\partial x_i}\phi(x) ,$$

$$S(\phi) \simeq \frac{1}{g}\sum_x \{da^{d-4} - \tfrac{1}{2}a^d[\partial\phi(x)]^2\}$$
$$= \text{const} - \frac{1}{2g}\int d^dx\,[\partial\phi(x)]^2 .$$

Apart from an (infinite) irrelevant constant, we have thus as a limit the kinetic term of our previous action. However, for a small but finite a the restriction to $|a\phi| < \pi$ together with the higher-order terms in the expansion of the cosine lead to nontrivial interactions (essentially nonrenormalizable, since they involve higher and higher derivatives in the continuous limit where they are damped by powers of a^{2n}). Thus the length a plays a double role; it is an ultraviolet cutoff and scales the various interaction terms. Another way of identifying g as a coupling constant is to rescale ϕ into $\sqrt{g}\,\tilde{\phi}$ and replace the interaction $\theta(\pi^2 - a^2 g\tilde{\phi}^2)$ by a quantity proportional to $\exp[-\text{const} \times(-\tilde{\phi}^2 + ga^2\tilde{\phi}^4)]$.

B. The scalar model and global invariance

Our first discrete model is thus described by the action (2.5). Since we restrict the range of $|a\phi|$ to an interval of 2π and since the cosine is a periodic function, we can rescale ϕ into ϕ/a and write

$$Z = \int \prod_x \frac{d\phi(x)}{2\pi} \exp\Big(\beta \sum_{(x,x')} \cos[\phi(x) - \phi(x')]\Big), \qquad (2.6)$$

where β stands for $1/ga^{4-d}$. Each integral over ϕ runs in an interval of 2π and we have normalized Z in such a way that $Z = 1$ for $\beta \to 0$ (i.e., for $g \propto T \to \infty$). Let us recall that x stands for an arbitrary point on the lattice, i.e., an ordered set of d integers.

Finally, in order to give a meaning to the infinite integral (2.6), we introduce a spatial cutoff by restricting the lattice to a torus (imposing periodic boundary conditions) as follows: The variables x_i take integer values from 0 to $L - 1$ with $x_{i+L} \equiv x_i$. In this way we have $N \equiv L^d$ sites. We shall measure every quantity per site (i.e., per unit volume up to a proportionality constant a^d) and then let N grow infinitely large. In particular F will stand for

$$F = \lim_{N\to\infty} \frac{1}{N}\ln Z . \qquad (2.7)$$

It is related to the generating functional of connected vacuum-to-vacuum diagrams of field theory. We shall call it, by analogy with statistical mechanics, the free energy, even though it differs by a factor $-\beta$ from the traditional quantity.

A suggestive way to rewrite (2.6) is to introduce *a complex number of unit modulus* $k(x)\equiv e^{i\phi(x)}$ or *a unit two-dimensional vector* $\vec{k}(x)$ at each site. Thus we can write

$$\beta\sum_{(x,x')}\cos[\phi(x)-\phi(x')]\equiv\beta\,\mathrm{Re}\sum_{(x,x')}k^*(x)k(x')\equiv\beta\sum_{(x,x')}\vec{k}(x)\cdot\vec{k}(x')\ . \tag{2.8}$$

With this notation we observe that our model is identical with a classical planar Heisenberg model on a d-dimensional lattice. It has "spins" of unit length at each site (i.e., restricted to vary on a circle) and a "statistical mechanical energy" involving nearest neighbors

$$-\sum_{(x,x')}\vec{k}(x)\cdot\vec{k}(x'),$$

i.e., of ferromagnetic character, which mimics an exchange energy favoring alignment of the spins.

The N-integral (2.6) is obviously invariant under a global rotation of all spins

$$\vec{k}(x)\to R\vec{k}(x)\ \text{ or }\ k(x)\to e^{i\phi}k(x)\ , \tag{2.9}$$

where R is a rotation independent of x. In complex notation, it is written $e^{i\phi}$ with constant ϕ. It is known (folklore) that for $d>2$ such a system exhibits a transition.[2] For small β there exists a disordered phase exhibiting this O(2) symmetry. Within the context of field theory, this means that the vacuum is O(2) invariant. The wave excitations have a finite mass (a finite range) and possess the same symmetry. These will be identified with an ("isotopic") doublet of particles (the σ and π particles). Above some critical value β_c a different situation prevails. In this case the vacuum is no longer O(2)-invariant nor are the excitations. According to Goldstone's theorem[3] a long-range (zero-mass) excitation (the π) appears together with a finite-range one (the σ). We shall see that the range of the latter goes to zero (the mass goes to infinity) as $\beta\to\infty$ $(g\to 0)$ and we are then left with a scalar massless field which can be identified with the situation we started with in the preceding paragraph.

Mutatis mutandis we could generalize the model to an invariance group SO(n) by replacing the two-dimensional "spin" $\vec{k}$ by an n-dimensional vector of unit length. If we continue to call this vector $\vec{k}(x)$ the action will still be written

$$S=\beta\sum_{(x;x')}\vec{k}(x)\cdot\vec{k}(x')\ , \tag{2.10}$$

and the only change to make in the definition of Z is to replace the integration volume

$$\frac{d\phi(x)}{2\pi}$$

by the SO(n)-invariant normalized volume element $d^{n-1}\hat{k}(x)$ on the unit sphere in n-dimensional space, the normalization being such that

$$\int d^{n-1}\hat{k}=1\ .$$

The basic phase-transition phenomena we just described will be identical: below β_c, SO(n)-invariant vacuum and n-plet of degenerate massive excitations; above β_c, $(n-1)$-plet of massless "transverse" π's and a massive longitudinal σ.

Of course, the above statements will be made more precise as we proceed to explicit computations.

C. Local invariance, gauge field, and minimal coupling

It is clear that if we allow a rotation R [be it in SO(2) in the simplest case or in SO(n) in the generalized one] to depend on x, the action (2.10) will not be invariant. This is due to the fact that we couple nearest neighbors in (2.10). Thus in the transformation

$$\vec{k}(x)\to R(x)\vec{k}(x) \tag{2.11}$$

the coupling term $\vec{k}(x)\cdot\vec{k}(x')$ becomes

$$\vec{k}(x)\cdot\vec{k}(x')\to k(x)^T R^{-1}(x)R(x')k(x')\ ,$$

with $R^{-1}\equiv R^T$ for an orthogonal group.

In order to implement a local invariance one introduces, by analogy with the familiar continuous case, a gauge field. As the scalar field was a map, "point x on the lattice $\to$ point on the unit sphere in n-dimensional space," the gauge field is a map, "ordered link (x, x') on the lattice $\to A(x,x')$ element of the group SO(n)." To make precise what we mean by ordered link we introduce a *semi-order* on the lattice as follows. First we choose a positive sign on each axis of coordinate, then we say that

$$x'\geqslant x\ \text{ if for each } i\ (0\leqslant i\leqslant d-1),\ \ x_i'\geqslant x_i\ .$$

Each pair of neighbors on the lattice then defines an ordered link (x, x') by requiring that $x'>x$ (this is obviously possible since x and x' differ only in one coordinate).

To the link in the reverse order (x', x) the associated rotation is then taken to be the inverse one. In other words,

$$A(x',x)\equiv A^{-1}(x,x')\ . \tag{2.12}$$

Under a global rotation R we assume the transformation law to be

$$A(x, x') \to RA(x, x')R^{-1} ,$$

while under a *local* gauge transformation $R(x)$ we require that

$$A(x, x') \to R(x)A(x, x')R^{-1}(x') . \tag{2.13}$$

We notice immediately that (2.13) is compatible with (2.12) since

$$\begin{aligned} A(x', x) \equiv A^{-1}(x, x') &\to [R(x)A(x, x')R^{-1}(x')]^{-1} \\ &= R(x')A^{-1}(x, x')R^{-1}(x) \\ &= R(x')A(x', x)R^{-1}(x) . \end{aligned}$$

Having defined a gauge field $A(x, x')$, the coupling between nearest neighbors is changed in a *minimal* way through

$$\vec{k}(x) \cdot \vec{k}(x') \to k^T(x)A(x, x')k(x') . \tag{2.14}$$

The notation $k^T(x)$ is to indicate a row vector. If we combine (2.11) and (2.13) we see that this new coupling is obviously invariant under local transformations.

We pause to stress the analogy of the preceding procedure with conventional minimal coupling. The line of thought is entirely parallel and the results similar. In both cases the introduction of a gauge field is necessitated by a slight nonlocality of the "Lagrangian" generally in the kinetic term. In fact, we even recover the usual case by going to the continuous limit. Recall that the lattice spacing a had been eliminated by scaling. Then with $x' > x$ we have $x' = x + an_\mu$ for some direction n_μ kept fixed:

$$k(x') = k(x) + a\,\partial_\mu k(x) + \tfrac{1}{2}a^2\partial_\mu{}^2 k(x) + \cdots ,$$

while we may assume $A(x, x')$ sufficiently close to unity to write it as

$$A(x, x') = I + iea\mathcal{A}_\mu(x) - \tfrac{1}{2}e^2a^2\mathcal{A}_\mu(x)^2 + \cdots , \tag{2.15}$$

where $\mathcal{A}_\mu(x)$ is a convenient way to write $\mathcal{A}(x, x')$ and *belongs to the Lie algebra of* SO(n). To be more precise, it is a representative of this element in the representation which acts on the vectors $\vec{k}$. If we take a basis $\lambda^\alpha{}_{\beta\gamma}$ of this Lie algebra, we can write in detail

$$A(x, x')_{\beta\gamma} = \delta_{\beta\gamma} + iea\mathcal{A}_\mu(x)_\alpha\lambda^\alpha_{\beta\gamma} - \tfrac{1}{2}e^2a^2\mathcal{A}_\mu(x)_\alpha\mathcal{A}_\mu(x)_{\alpha'}\lambda^\alpha_{\beta\delta}\lambda^{\alpha'}_{\delta\gamma} + \cdots .$$

For each index α, $\mathcal{A}_\mu(x)_\alpha$ is thus a vector field. The "charge" e has been introduced according to usual practice. We rewrite formula (2.14) to second order in the lattice spacing. Remembering that $\vec{k}(x)^2 = 1$, it follows that $\vec{k}(x)\partial_\mu\vec{k}(x) = 0$ and $\vec{k}(x)\cdot\mathcal{A}\vec{k}(x) = 0$ due to the antisymmetry of $\mathcal{A}$ [since it belongs to the Lie algebra of SO(n) which preserves the norm $\vec{k}^2$]. Then

$$k^T(x)[I + iea\mathcal{A}_\mu(x) - \tfrac{1}{2}e^2a^2\mathcal{A}_\mu(x)^2 + \cdots][k(x) + a\partial_\mu k(x) + \tfrac{1}{2}a^2\partial_\mu{}^2k(x) + \cdots] = 1 + \tfrac{1}{2}a^2k(x)^T\{\partial_\mu{}^2 + 2ie\mathcal{A}_\mu(x)\partial_\mu + [ie\mathcal{A}_\mu(x)]^2\}k(x) .$$

Now

$$k^T(x)\partial_\mu{}^2k(x) = -\partial_\mu k(x)^T\partial_\mu k(x) ,$$

while due to the antisymmetry of $\mathcal{A}$

$$k^T(x)2ie\mathcal{A}_\mu(x)\partial_\mu k(x) = k(x)^Tie\mathcal{A}(x)\partial_\mu k(x) - ie[\partial_\mu k(x)]^T\mathcal{A}(x)k(x) .$$

If we introduce the covariant derivative D_μ as

$$(D_\mu k)(x) = (\partial_\mu - ie\mathcal{A}_\mu)k(x)$$

acting on a column vector we shall have for a row vector

$$(D_\mu k)(x)^T \equiv \partial_\mu k(x)^T + iek(x)^T\mathcal{A}(x) .$$

Finally then

$$k^T(x)A(x, x')k(x) = 1 - \tfrac{1}{2}a^2D_\mu k(x)^TD_\mu k(x) + \cdots . \tag{2.16}$$

Again apart from an inessential constant factor, if we sum over all pairs (x, x'), i.e., over all directions μ and over all x's, we recover the modified kinetic term in the presence of a gauge field $\mathcal{A}_\mu(x)_\alpha$ in the continuous model.

If $n = 2$ we have an *Abelian* gauge field and a situation very similar to electromagnetism. We leave it to the reader to rewrite (2.16) in this case with $k_1(x) = \cos\phi(x)$ and $k_2(x) = \sin\phi(x)$; the index α takes only one value as there is one generator only and $(\lambda_{\beta\gamma}) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ with β and γ taking two values only.

If $n > 2$ we have a typical *non-Abelian* theory. Later on, to present specific calculations, we shall choose, for instance, $n = 3$, with SO(3) $=$ SU(2)$/Z_2$, or $n \to \infty$.

The above construction for orthogonal groups can in fact be generalized to other representations or to other types of compact groups (in particular to unitary groups) without any difficulty.

We now return to our discrete lattice. Having exhibited the minimal coupling to the gauge field in order to build up in the theory a local invariance we stress now that formulas (2.11) and (2.13) define the gauge transformations. In particular (2.13) is the integrated form of the gauge transformation on the gauge field $A(x, x')$. The gauge group is thus the Nth tensor product of SO(n) groups, i.e., as long as the spatial cutoff N is kept finite, a compact group

$$\overset{N}{\otimes}\mathrm{SO}(n)\ .$$

Beyond minimal coupling it would seem natural to introduce in the action S an extra term designed to produce some dynamics for the gauge degrees of freedom. It is then also obvious that we want to preserve the local invariance. By inspection of (2.13) it is seen that a product

$$A(x_1x_2)A(x_2x_3)\cdots A(x_k x_1)$$

is gauge invariant. In this product the set $x_1x_2x_3\cdots x_k x_1$ constitutes a "closed curve" on the lattice, that is, a set of nearest neighbors (x_1x_2) $(x_2x_3)\cdots$ with the last point identical with the first. This is still not a number but an element of the group SO(n), or if we prefer, a matrix representative. In order to define an additional term in the action we proceed as follows. First we choose *a real irreducible character* of the group, χ. This means that we pick an irreducible representation of SO(n) in our case, such that the trace of the representatives is real. We denote this trace by χ. Then, instead of an arbitrary closed curve on the lattice, we choose the simplest one that we call (from the French) a *plaquette*. This is a set of four nearest neighbors (x_1x_2), (x_2x_3), (x_3x_4), (x_4x_1). Such a plaquette can be identified with a two-dimensional face of a hypercube on our lattice. It will be given, for instance, by

$$x_1=x\ ,$$
$$x_2=x+n_\mu\ ,$$
$$x_3=x+n_\mu+n_\nu\ ,$$
$$x_4=x+n_\nu\ .$$

The added contribution to the action due to the gauge field will be taken proportional to

$$\sum_{\text{plaquettes}}\chi(A(x_1x_2)A(x_2x_3)A(x_3x_4)A(x_4x_1))\ ,$$

where the sum runs over all distinct plaquettes of the lattice. By definition $\chi=\chi^*$. All irreducible representations of the compact group SO(n) can be taken to be equivalent to unitary ones, while the Hermitian conjugate of a unitary matrix is its inverse. The representative of an inverse is the inverse of a representative. Consequently

$$\begin{aligned}\chi(A(x_1, x_2)A(x_2, x_3)A(x_3, x_4)A(x_4, x_1))\\ =\chi(A^{-1}(x_4, x_1)A^{-1}(x_3, x_4)A^{-1}(x_2, x_3)A^{-1}(x_1, x_2))\\ =\chi(A(x_1, x_4)A(x_4, x_3)A(x_3, x_2)A(x_2, x_1))\ ,\end{aligned}$$

where we have used (2.12). This property means that the order in which we orient the closed curve $(x_1x_2x_3x_4x_1)$ is irrelevant.

Finally, to sum over all configurations we shall use the invariant measure on the group SO(n) which we denote by dA. We assume that it is normalized to unity:

$$\int dA=1\ .$$

To make notations shorter we use the symbols s for site, l for link, and p for plaquette. To each site is assigned an "isotopic" vector $\vec{k}$, to each ordered link a gauge field A and an interaction kAk, and to each plaquette a term $\chi(AAAA)$. Since $dA=dA^{-1}$ it is immaterial in fact how we orient the links, as long as we are consistent over all of the lattice.

The Abelian group SO(2) is a slight exception to the above formalism since all its irreducible representations are one-dimensional and complex. One has

$$A\equiv e^{i\alpha},$$

$$\chi(A)=e^{im\alpha},\quad m\text{ integer}\ .$$

In this case it is understood that we take for simplicity $m=1$ and replace χ by Reχ, i.e., $\chi(A)=\cos\alpha$.

We call β_l the coefficient in front of the interaction term kAk, and β_p the one of $\chi(AAAA)$ in the full action S, which we now write:

$$S=\beta_l\sum_l k^T(x)A(x, x')k(x')+\beta_p\sum_p\chi(A(x_1, x_2)A(x_2, x_3)A(x_3, x_4)A(x_4, x_1))\ , \tag{2.17}$$

$$Z=e^{NF}=\int\prod_s dk_x\prod_l dA(x, x')e^S\ . \tag{2.18}$$

As previously, we have assumed a spatial cutoff $N\to\infty$ and noticed that there are N sites and Nd links. The notation $Z=e^{NF}$ is a shorthand for

$$F=\lim_{N\to\infty}\frac{1}{N}\ln Z\ .$$

In the Abelian case of SO(2), writing $k=(\cos\phi, \sin\phi)$, $A=e^{i\alpha}$, the above formulas become

$$S_{\mathrm{Ab}}=\beta_l\sum_l \cos[\phi(x)-\phi(x')+\mathfrak{A}(x,x')]+\beta_p\sum_p\cos[\mathfrak{A}(x_1,x_2)+\mathfrak{A}(x_2,x_3)+\mathfrak{A}(x_3,x_4)+\mathfrak{A}(x_4,x_1)]\ , \qquad (2.17)_{\mathrm{Ab}}$$

$$Z_{\mathrm{Ab}}=e^{NF_{\mathrm{Ab}}}=\int^{N+dN}\prod_s \frac{d\phi(x)}{2\pi}\prod_l \frac{d\mathfrak{A}(x,x')}{2\pi}e^{S_{\mathrm{Ab}}}\ . \qquad (2.18)_{\mathrm{Ab}}$$

All integrals run over angles in an interval of 2π.

The case presented in Sec. II B corresponds to the freezing of all gauge degrees of freedom at the unit value $A=1$. Another interesting limit is obtained by taking $\beta_l=0$, in which case we have the pure Yang-Mills dynamics of the gauge field.

As for the interaction term, it is nice to observe that in the continuous limit one again recovers known formulas for the pure Yang-Mills interaction $\chi(AAAA)$. Again we restore the lattice spacing a. In the continuous limit, we write

$$A(x,x+an_\mu)=e^{iae\mathfrak{A}_\mu(x)}\ .$$

Then, the contribution of the plaquette $(\mu\nu)$ issued from the corner x is of the form

$$\chi(AAAA)=\chi(e^{ia^2e\mathfrak{F}_{\mu\nu}(x)})\,, \qquad (2.19)$$

where $\mathfrak{F}_{\mu\nu}$ belongs to the Lie algebra of the group.

In order to evaluate $\mathfrak{F}_{\mu\nu}$ to lowest order in a, we expand $\mathfrak{A}_\mu(x+an_\nu)$ and $\mathfrak{A}_\nu(x+an_\mu)$ in (19), and apply the Baker-Hausdorff formula:

$$e^Xe^Y=\exp(X+Y+\tfrac{1}{2}[X,Y]+\cdots)\ . \qquad (2.20)$$

This yields, dropping the variable x,

$$\begin{aligned}AAAA&=e^{iae\mathfrak{A}_\mu}e^{iae(\mathfrak{A}_\nu+a\partial_\mu\mathfrak{A}_\nu)}e^{-iae(\mathfrak{A}_\mu+a\partial_\nu\mathfrak{A}_\mu)}e^{-iae\mathfrak{A}_\nu}\\&\simeq\exp\{iae(\mathfrak{A}_\mu+\mathfrak{A}_\nu+a\partial_\mu\mathfrak{A}_\nu)-\tfrac{1}{2}a^2e^2[\mathfrak{A}_\mu,\mathfrak{A}_\nu]\}\\&\quad\times\exp\{-iae(\mathfrak{A}_\mu+\mathfrak{A}_\nu+a\partial_\nu\mathfrak{A}_\mu)-\tfrac{1}{2}a^2e^2[\mathfrak{A}_\mu,\mathfrak{A}_\nu]\}\\&\simeq\exp\{ia^2e(\partial_\mu\mathfrak{A}_\nu-\partial_\nu\mathfrak{A}_\mu)-a^2e^2[\mathfrak{A}_\mu,\mathfrak{A}_\nu]\}\ .\end{aligned}$$

Hence

$$\mathfrak{F}_{\mu\nu}=\partial_\mu\mathfrak{A}_\nu-\partial_\nu\mathfrak{A}_\mu+ie[\mathfrak{A}_\mu,\mathfrak{A}_\nu] \qquad (2.21)$$

is recovered as the generalization of the electromagnetic field.

Introducing the complete basis $\mathfrak{L}_\alpha$ of generators of the group, and their representatives L_α in the chosen irreducible representation, we write

$$\mathfrak{F}_{\mu\nu}=\mathfrak{F}^\alpha_{\mu\nu}\mathfrak{L}_\alpha\ ,$$

and (2.19) becomes

$$\begin{aligned}\chi(AAAA)&\simeq\mathrm{Tr}(1+ia^2e\mathfrak{F}^\alpha_{\mu\nu}L_\alpha-\tfrac{1}{2}a^4e^2\mathfrak{F}^\alpha_{\mu\nu}\mathfrak{F}^\beta_{\mu\nu}L_\alpha L_\beta)\\&=\text{dimension of the representation}\\&\quad-\tfrac{1}{2}a^4e^2\mathfrak{F}^\alpha_{\mu\nu}\mathfrak{F}^\beta_{\mu\nu}\,\mathrm{Tr}\,L_\alpha L_\beta\ .\end{aligned}$$

The term linear in $\mathfrak{F}$ has been dropped because L_α is traceless. The calculation is up to now entirely general. If the group is semisimple [as SO(n) or SU(n) is], we have by a proper choice of basis

$$\mathrm{Tr}\,L_\alpha L_\beta=\frac{\delta_{\alpha\beta}(\sum_\alpha\mathrm{Tr}\,L_\alpha^2)}{\sum_\alpha 1}\ ,$$

so that for any semisimple group and real irreducible representation the term $\chi(AAAA)$ in the action reduces in the continuous limit to

$$\chi(AAAA)=C_1-\tfrac{1}{2}a^4e^2C_2\sum_\alpha(\mathfrak{F}^\alpha_{\mu\nu})^2\ . \qquad (2.22)$$

For instance, for SU(2), we have $A=u_0+i\vec{u}\cdot\vec{\sigma}$ with $u_0^2+\vec{u}^2=1$ and $\vec{\sigma}$ the Pauli matrices. Hence u runs on the unit sphere in the 4-dimensional space, and $dA=(1/2\pi^2)d^4u\,2\delta(u^2-1)$. If we denote by ψ the angle of the associated rotation of SO(3) $=\mathrm{SU}(2)/Z_2$, we have $u_0=\cos\frac{1}{2}\psi$ and $|\vec{u}|=\sin\frac{1}{2}\psi$. Then we take for χ the trace in the representation of spin j, which yields

$$\chi(A)=\frac{\sin(2j+1)\frac{1}{2}\psi}{\sin(\frac{1}{2}\psi)}\ ,$$

and finally for a plaquette in the continuous limit

$$\chi(AAAA)=(2j+1)-\tfrac{1}{2}a^4e^2\tfrac{1}{3}j(j+1)\sum_{\alpha=1}^{3}(\mathfrak{F}^\alpha_{\mu\nu})^2\ . \qquad (2.23)$$

In any case, apart from an inessential constant, the sum over all the plaquettes reproduces up to a choice of scale (which as we see implies $\beta_p\propto 1/e^2$) the conventional action for the Yang-Mills field

$$-\tfrac{1}{4}\int dx\,\mathfrak{F}^2\ .$$

This applies obviously also to the commutative case.

Thus the action (2.17) is a good candidate on our discrete lattice for a matter field (k) coupled to a gauge field (A). The usual Yang-Mills construction might seem even simpler in that case.

While the discretization in the matter-field case yields a model well known in statistical mechanics, it does not seem to be so in the gauge-field case to our knowledge. Interactions involve four links at once, and on the other hand there is a very large built-in local invariance. As we proceed further we shall see that the kind of "stuff" that this might represent is in a sense more like a *liquid* in its disordered phase. Again a phase transition occurs for some β_p^c. Beyond this value an ordered phase corresponding to the "breaking" of gauge invariance comes into play. A long-range excitation is present corresponding to the usual massless Yang-Mills boson. It might perhaps, at first, seem surprising that, at least for $\beta_p \to \infty$ corresponding to the continuous limit discussed above, gauge invariance is spontaneously broken. On second thought, however, one realizes that any known quantization procedure in the usual theory breaks gauge invariance, which is restored by asking "gauge-invariant questions." Thus, while the discrete case does not involve any breaking at the level of the action, this occurs in a natural way in its ordered (large-β_p) phase.

Correspondingly, for $\beta_p < \beta_p^c$ only gauge-invariant quantities have nonvanishing expectation values. Typical of these are expressions of the form (we assume for the moment that $\beta_l = 0$)

$$\langle \chi(A(x_1, x_2)A(x_2, x_3)\cdots A(x_k, x_1))\rangle \ ,$$

where the points $(x_1, x_2)\cdots(x_k, x_1)$ form a closed curve on the lattice. In the commutative case and in the continuous limit this expression is essentially

$$\left\langle \exp\left[ie \oint_C dx_\mu \mathcal{A}_\mu(x)\right]\right\rangle \ .$$

(In the noncommutative case a T-ordering symbol along the curve should appear in front of the exponential.) This average might be taken as a representative of the effect of a closed loop for a charged particle interacting with the gauge field. Now for $\beta_p \gg \beta_p^c$ (or $e \to 0$) we expect lowest order in perturbation theory to be a reliable guide, in which case one would have

$$\left\langle \exp\left[ie \oint_C dx_\mu \mathcal{A}_\mu(x)\right]\right\rangle \sim \exp\left[-e^2 \oint_C \oint_C dx_{1\mu}\Delta_{\mu\nu}(x_1 - x_2)dx_{2\nu}\right] \ ,$$

with $\Delta_{\mu\nu}$ the free massless propagator (in Euclidean space) which behaves like $\delta_{\mu\nu}/|x_1 - x_2|^{d-2}$. Thus the above exponential decreases most likely as exp(−const × length of C), up to logarithms. A proper evaluation requires the ultraviolet cutoff.

We shall see, however, that for e large enough or $\beta_p < \beta_p^c$ the corresponding expectation value behaves rather like

$$\exp(-\text{const} \times \text{minimal area enclosed by } C) \ .$$

This is an indication that in the strong-coupling limit a pair of oppositely charged particles have a very hard time to separate themselves in the presence of the gauge field. The long-range part of the forces seem to have built up a strong attractive barrier. This will be elaborated later.

Let us add a remark on the structure of "interactions" provided by the Yang-Mills Lagrangian in its discrete version. For that purpose consider the commutative case of SO(2) where in an appropriate scale the relevant term is for each plaquette

$$\mathcal{L}_p = \cos(\mathcal{A}_{12} + \mathcal{A}_{23} + \mathcal{A}_{34} + \mathcal{A}_{41}) \ .$$

Let us identify $e^{i\mathcal{A}}$ with a unit two-dimensional vector that we denote by $\vec{A}$. We set

$$e^{i\mathcal{A}_{12}} \to \vec{A}^1 \ ,$$
$$e^{i\mathcal{A}_{23}} \to \vec{A}^3 \ ,$$
$$e^{-i\mathcal{A}_{34}} = e^{i\mathcal{A}_{43}} \to \vec{A}^2 \ ,$$
$$e^{-i\mathcal{A}_{41}} = e^{i\mathcal{A}_{14}} \to \vec{A}^4 \ ,$$

and find

$$\begin{aligned}\mathcal{L}_p &= \mathrm{Re}[(A_1^1 + iA_2^1)(A_1^3 + iA_2^3)(A_1^2 - iA_2^2)(A_1^4 - iA_2^4)]\\ &= \mathrm{Re}\{[A_1^1A_1^2 + A_2^1A_2^2 + i(A_2^1A_1^2 - A_1^1A_2^2)]\\ &\qquad \times [A_1^3A_1^4 + A_2^3A_2^4 + i(A_2^3A_1^4 - A_1^3A_2^4)]\}\\ &= (\vec{A}^1\cdot\vec{A}^2)(\vec{A}^3\cdot\vec{A}^4) - (\vec{A}^2\times\vec{A}^1)\cdot(\vec{A}^4\times\vec{A}^3)\\ &= (\vec{A}^1\cdot\vec{A}^2)(\vec{A}^3\cdot\vec{A}^4) - (\vec{A}^1\cdot\vec{A}^3)(\vec{A}^2\cdot\vec{A}^4)\\ &\quad + (\vec{A}^1\cdot\vec{A}^4)(\vec{A}^2\cdot\vec{A}^3) \ . \end{aligned} \tag{2.24}$$

In other words, $\mathcal{L}_p$ is a Pfaffian,

$$\mathcal{L}_p = \mathrm{Pf}\begin{Bmatrix} (\vec{A}^1\cdot\vec{A}^2) & (\vec{A}^1\cdot\vec{A}^3) & (\vec{A}^1\cdot\vec{A}^4)\\ & (\vec{A}^2\cdot\vec{A}^3) & (\vec{A}^2\cdot\vec{A}^4)\\ & & (\vec{A}^3\cdot\vec{A}^4)\end{Bmatrix} \ , \tag{2.24'}$$

or the root of the antisymmetric determinant obtained by completing the Pfaffian. This alternative expression (useful in the sequel) shows even in the Abelian case the complexity of the "interaction." If to each ordered link we associate a "spin" $\vec{A}$ we see how these spins combine four by four. The disadvantage of this notation is that one loses track of the local SO(2) invariance.

Finally, we have only considered up to now continuous gauge groups. However, since we are directly working on the group and not on its Lie algebra we can extend these models to include discrete groups. In particular we can look at the case when this group is

$$\overset{N}{\otimes} Z_2$$

(Z_2 being the group with two elements ±1). It is the natural extension to $n = 1$ of the previous models described by Eqs. (2.17) and (2.18). More precisely, $k(x)$ takes the values ±1 as well as $A(x, y)$, while $\chi(AAAA) \equiv AAAA$. As a result, when one freezes $A(x, y)$ to the value unity, one recovers the ordinary Ising model. Thus one obtains a gauge-invariant generalization of the Ising model with a discrete gauge group with no continuous version. It turns out that precise and nontrivial results can be obtained in this case, which we shall present in another paper of this series.

III. EXACT SOLUTIONS IN LOW DIMENSION

It is well known from the Ising case that some statistical models are soluble in small dimension. The analogy here is for the scalar or σ model for $d=1$. No explicit solution corresponding to the Onsager one is available for $d=2$. This is unfortunate, since for the gauge field the model only makes sense for $d\geq 2$. For the latter, the solution can be found for $d=2$. We shall thus present these various solutions but will be unable here to treat an example of the coupled case (2.17), (2.18).

Apart from being useful in that one gets some familiarity with the manipulation of the expressions, these soluble cases are expected to behave near $\beta\to\infty$ as will the more "realistic" ones in higher dimension near their critical point. Hence for these models $\beta=\infty$ or $T=0$ (or zero coupling) can be identified in a certain sense with a critical point.

At the other extreme when d gets very large we shall see that we can essentially treat the coupled model exactly. This will be postponed until the next section. Numerical calculations at intermediate d will be presented in another article.

A. Scalar model for $d = 1$

For $d=1$, the "free energy" is defined by

$$Z=e^{NF}=\int\prod_{0}^{N-1}\frac{d\phi_i}{2\pi}\exp\left[\beta\sum_{0}^{N-1}\cos(\phi_{i+1}-\phi_i)\right]. \tag{3.1}$$

More generally, for SO(n), we have to integrate the vector k_i associated to each site over a unit sphere:

$$Z=e^{NF}=\int\prod_{0}^{N-1}d^{\,n-1}k_i\exp\left(\beta\sum_{0}^{N-1}k_i\cdot k_{i+1}\right). \tag{3.2}$$

This model is well known in the context of statistical mechanics.[2] The "free energy" is given by

$$F=\ln[\Gamma(\tfrac{1}{2}n)(\beta)^{1-n/2}I_{n/2-1}(\beta)]\ , \tag{3.3}$$

where $I_\nu(x)$ is the modified Bessel function. In particular, for $n=2$, we have

$$F=u(\beta)=\ln I_0(\beta)\ , \tag{3.4}$$

a function which will play a crucial role in the remaining part of this work. The main properties of $u(x)$ and its first two derivatives are displayed in Table I. No transition, of course, occurs in this system, the nearest singularity being the complex zero of I_0, $\beta=\pm i\,2.405$.

More interesting is the behavior of the correlation function[2]

TABLE I. The function $u(x)=\ln I_0(x)$ and its first two derivatives.

Function		x small	x large
$u(x)$	even	$(\frac{1}{2}x)^2-\frac{1}{4}(\frac{1}{2}x)^4+\cdots$	$x-\frac{1}{2}\ln(2\pi x)+\frac{1}{8x}-\frac{4}{(8x)^2}+\cdots$
$u'(x)$	odd	$(\frac{1}{2}x)-\frac{1}{2}(\frac{1}{2}x)^3+\cdots$	$1-\frac{1}{2x}-\frac{1}{8x^2}-\frac{1}{8x^3}+\cdots$
$u''(x)$	even	$\frac{1}{2}-\frac{3}{4}(\frac{1}{2}x)^2+\cdots$	$\frac{1}{2x^2}+\frac{1}{4x^3}+\frac{3}{8x^4}+\cdots$

$$G(r,\beta)=\langle k_i\cdot k_{i+r}\rangle=\left(\frac{dF}{d\beta}\right)^r\ , \tag{3.5}$$

which behaves for β large as

$$G(r,\beta)\sim\exp\left(-r\,\frac{n-1}{2\beta}\right)\ . \tag{3.6}$$

At low temperature, the correlation range $2\beta/(n-1)$ becomes very large. In other words, the mass $\mu=(n-1)/2\beta$ goes to zero as if the system were more and more ordered near $\beta=\infty$. An ordered regime with zero mass will set in below some critical temperature for $d>2$, as discussed later.

The generalization to $n>2$ acquires an additional interest because an exact solution exists in another limit. The latter is the Stanley[4] limit obtained by keeping d fixed and letting $n\to\infty$. One obtains then a nontrivial behavior with a transition.

B. Abelian gauge field for $d = 2$

We turn now to a pure Abelian gauge field. That is, we set $\beta_l=0$ and drop the index p on β_p. Such a model only makes sense for $d\geq 2$, hence we expect that it is simple for $d=2$. This is indeed the case. Recall the formulas $(2.17)_{\rm Ab}$ and $(2.18)_{\rm Ab}$:

$$Z=e^{NF}=\int\prod_{l}\frac{d\mathcal{A}_{ij}}{2\pi}\,e^S\ ,$$
$$S=\beta\sum_{p}\cos(\mathcal{A}_{12}+\mathcal{A}_{23}+\mathcal{A}_{34}+\mathcal{A}_{41})\ . \tag{3.7}$$

To be specific assume the lattice to be a square of $N=L^2$ lattice sites. Note that while there are $2N$ links there are N plaquettes. If we *do not* assume periodic boundary conditions it is readily seen that the plaquette variables $\mathcal{A}_{12}+\mathcal{A}_{23}+\mathcal{A}_{34}+\mathcal{A}_{41}$ are independent. Consequently, taking them among the integration variables, we obtain

$$e^{NF}=I_0(\beta)^N\ .$$

Hence

$$F=u(\beta) \tag{3.8}$$

exactly as before for the scalar model and $n=2$, $d=1$.

As we discussed in Sec. II the interesting correlations are expressed not in terms of Green's functions but in terms of averages of gauge-invariant quantities of the type

$$\mathcal{C} = \left\langle \exp\left(i \sum_C \mathcal{A}_{ij}\right)\right\rangle = Z^{-1}\int^{2N}\prod_l \frac{d\mathcal{A}_{ij}}{2\pi} \exp\left(i \sum_C \mathcal{A}_{ij} + S\right) , \quad (3.9)$$

where $\sum_C \mathcal{A}_{ij}$ means a sum along a simple closed curve on the lattice. We assume the curve not to be self-intersecting, that is, to separate an internal region from an external one. Now due to the fact that $\exp(i\mathcal{A}_{ij}) = \exp(-i\mathcal{A}_{ji})$ one sees (as in the traditional proof of Cauchy's theorem) that

$$\exp\left(i \sum_C \mathcal{A}_{ij}\right) = \exp\left(i \sum_p \mathcal{A}_p\right) ,$$

where $\mathcal{A}_p$ is a shorthand notation for the sum of four $\mathcal{A}$'s pertaining to a plaquette and the sum runs over all internal plaquettes enclosed by C. This manipulation has been possible (i) because of the topology of the plane (a simple nonintersecting curve defines an interior and an exterior), and (ii) because of the Abelian character of the field. The number of plaquettes enclosed by C is nothing but the *area* $s(C)$ bounded by this curve. Consequently

$$\mathcal{C} = \left[\frac{\int_0^{2\pi} \frac{d\mathcal{A}}{2\pi} \exp(\beta\cos\mathcal{A} + i\mathcal{A})}{\int_0^{2\pi} \frac{d\mathcal{A}}{2\pi} \exp(\beta\cos\mathcal{A})}\right]^{s(C)} = \exp\left(-s(C)\ln \frac{1}{u'(\beta)}\right) . \quad (3.10)$$

We observe a similitude with the behavior of the correlation function in the scalar case for $d = 1$. The average value $\mathcal{C}$ decreases exponentially with the area enclosed by the curve C. This is what we expected in general in the disordered phase.

Note that the coefficient of this decrease behaves exactly as we discussed in the preceding paragraph. Namely, as $\beta \to \infty$ it goes to zero. This type of "binding" becomes less and less effective as we approach the "pseudocritical point" $\beta = \infty$.

C. Non-Abelian gauge field for $d = 2$

Let us see how far one can go in the non-Abelian gauge field case for the simple topology of the plane. The function to be computed first is

$$Z = e^{NF} = \int \prod_l dA \exp\left[\beta \sum_p \chi(A_{12}A_{23}A_{34}A_{41})\right] , \quad (3.11)$$

where we recall that dA stands for the normalized invariant measure on the group with

$$dA = dAB = dBA = dA^{-1}$$

for a fixed element B of the compact group. Again this allows us to consider on a square of $N = L^2$ lattice points the plaquette variables $A_p = A_{12}A_{23}A_{34}A_{41}$ as independent ones. To see this one might start from the edges of the square and integrate successively on the free variables pertaining to the boundary links. This leads to

$$e^F = \int dA\, e^{\beta\chi(A)} . \quad (3.12)$$

Qualitatively, F will have the same properties as in the Abelian case. To illustrate this point consider the case of the group SO(3) and the jth character (cf. Sec. IIC). Let θ be the polar angle on the unit sphere in four-dimensional space; then

$$e^F = \frac{2}{\pi} \int_0^\pi d\theta \sin^2\theta \exp\left(\beta \frac{\sin(2j+1)\theta}{\sin\theta}\right). \quad (3.13)$$

For j arbitrary this is still quite complicated, while for $j = \frac{1}{2}$, corresponding to the simplest character, we have

$$e^F = \frac{2}{\pi} \int_0^\pi d\theta \sin^2\theta \exp(2\beta\cos\theta) ,$$

$$= \frac{1}{\pi} \int_0^\pi d\theta(1 - \cos 2\theta)\exp(2\beta\cos\theta)$$

$$= I_0(2\beta) - I_2(2\beta) = \frac{I_1(2\beta)}{\beta} .$$

Finally then

$$F = \ln\frac{I_1(2\beta)}{\beta} \quad [\mathrm{SO}(3),\ j = \tfrac{1}{2}] . \quad (3.14)$$

A general expression can in fact be obtained for any SO(n) and any character χ. The handling of the average

$$\mathcal{C} = \left\langle \chi\left(\prod_C A_{ij}\right)\right\rangle , \quad (3.15)$$

where $\Pi_C A_{ij}$ denotes an ordered product along a simple closed curve C, is not as simple due to the noncommutativity of the group. The "Cauchy trick" does not seem to apply simply for this reason. Nevertheless, it is also possible to prove that $\ln\mathcal{C}$ is proportional to the area enclosed by C.

IV. MEAN-FIELD APPROXIMATION

At the extreme opposite of the low-dimensionality case discussed in the previous section, one expects another type of simplification when $d \to \infty$. For $d \to \infty$ the number of neighbors "interacting" with a spin of our scalar model, say, grows like

d. Hence it might be expected that their over-all effect is equivalent to a mean field to be determined consistently. This is the mean-field approximation. We shall show, using Peierls's inequality,[5] that one obtains in this way a lower bound on F. This bound is assumed exact as $d\to\infty$. We do not know a rigorous proof of this fact. However, a power-series expansion in $1/d$ is readily obtained, as we shall see in a forthcoming paper.

While the mean-field approximation is a standard device in statistical mechanics, we do not expect every reader to be familiar with it. Furthermore, some caution has to be exercised owing to gauge invariance. Consequently we proceed by steps. First we present the mean field in detail in the scalar model. We then extend it, without special care, first to the pure Abelian gauge field, then to the coupled system. (We ignore in this section non-Abelian gauge fields.) Finally we refine the analysis to justify this rather blunt procedure in order to meet possible objections on the role of gauge invariance.

Our most interesting result is the phase diagram (Fig. 1) obtained for the coupled system. Perturbation theory is possible around the mean-field approximation. As a consequence, the over-all picture obtained here is likely to be close to the exact solution except for fine details, especially near critical curves.

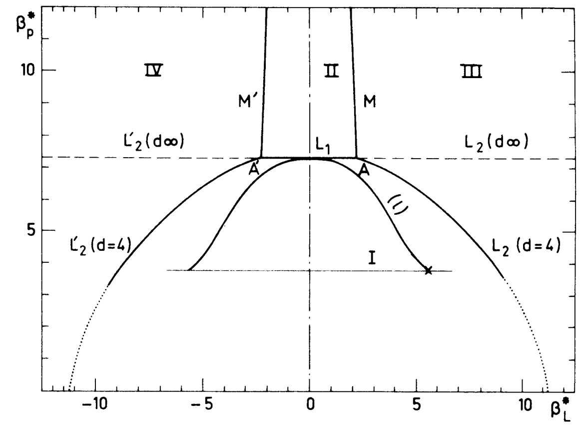

FIG. 1. The phase diagram for the coupled system. The phases are (I) disordered $H=K=0$; (II) Yang-Mills order $H=0$, $K\neq 0$; (III) "ferromagnetic" order $H\neq 0$, $K\neq 0$; (IV) "antiferromagnetic" arder $H\neq 0$, $K\neq 0$. The curves (L_2) and (L_2') (drawn for $d=4$) are speculative as one moves away from the triple points A, A'. The curve (l) arises from the poor mean-field approximation to the "B model" of Sec. IV D.

A. Scalar model

We want to compute

$$Z=e^{NF}=\int\prod_{0}^{N-1}d\hat{k}\,\exp\Big(\beta\sum_{(ij)}\vec{k}_i\cdot\vec{k}_j\Big) \qquad (4.1)$$

for d very large. A given $\vec{k}_i$ interacts with $\sum_{j(i)}\vec{k}_j$, where $j(i)$ denotes the $2d$ neighbors of the site i.

The average

$$\frac{1}{2d}\sum_{j(i)}\vec{k}_j$$

can be expected to behave like some mean field when $d\to\infty$. To present the matter on a firmer basis, one proceeds as follows. If we give ourselves the N values of the two-dimensional vectors $\vec{k}_i$ we call it a configuration $\{k_i\}$. The configuration space is thus

$$\overset{N}{\otimes} S_1\,,$$

the Cartesian product of N unit circles. This is a compact space (identical with an N-torus) on which we can define normalized measures

$$d\mu\{k_i\}\,,\qquad \int_{\overset{N}{\otimes}S_1} d\mu\{k_i\}=1\,.$$

Rather than the actual measure

$$\frac{1}{Z}\prod_i d\hat{k}_i e^S\,,$$

the above discussion suggests to introduce the easily tractable one

$$d\mu\{k_i\}=\frac{\prod_0^{N-1}d\hat{k}_i\exp(\vec{H}\cdot\vec{k}_i)}{\int\prod_0^{N-1}d\hat{k}_i\exp(\vec{H}\cdot\vec{k}_i)}\,, \qquad (4.2)$$

where $\vec{H}$ represents a mean field (up to a scaling factor) which remains to be determined. We shall assume this field to be uniform for the moment.

Now we have on the measure μ Peierls's inequality (due to the convexity of the exponential function),

$$\langle e^A\rangle\geq e^{\langle A\rangle}\,, \qquad (4.3)$$

so that we may write

$$\int d\mu\,\exp\Big(\beta\sum_{(ij)}\vec{k}_i\cdot\vec{k}_j-\sum_i\vec{H}\cdot\vec{k}_i\Big)\geq\exp\Big[\int d\mu\,\Big(\beta\sum_{(ij)}\vec{k}_i\cdot\vec{k}_j-\sum_i\vec{H}\cdot\vec{k}_i\Big)\Big]\,. \qquad (4.4)$$

Inserting (4.1) and (4.2) in (4.4), we finally obtain

$$F \geq \mathcal{F} + \frac{1}{N}\left(\beta \sum_{(ij)} \langle \vec{k}_i \cdot \vec{k}_j \rangle - \vec{H} \cdot \sum_i \langle \vec{k}_i \rangle\right) , \quad (4.5)$$

where

$$e^{\mathcal{F}} = \int d\hat{k}\, e^{\vec{H}\cdot\vec{k}} \quad (4.6)$$

arises from the denominator of (4.2), and where the averages $\langle\ \rangle$ are taken on the measure (4.2).

The right-hand side of (4.5) is a function of $\vec{H}$. We can maximize it over $\vec{H}$, in which case we recognize a usual form of the minimization of $-F/\beta$, the ordinary free energy in statistical mechanics. Thus we write in final form

$$F \geq \sup_{\vec{H}} \left[\beta \frac{1}{N} \sum_{(ij)} \langle \vec{k}_i \cdot \vec{k}_j \rangle + \left(\mathcal{F}(\vec{H}) - \frac{1}{N}\sum_i \langle \vec{H}\cdot\vec{k}_i \rangle\right)\right] . \quad (4.7)$$

Up to a constant the first term is proportional to energy, the second to entropy. Both terms are readily evaluated.

We have

$$\mathcal{F}(\vec{H}) = \ln\left(\int \frac{d\phi}{2\pi}\, e^{|\vec{H}|\cos\phi}\right) = u(H), \quad H \equiv |\vec{H}|$$

$$\langle \vec{k}_i \rangle = \frac{\vec{H}}{H} u'(H) \equiv \hat{H} u'(H) , \quad (4.8)$$

$$\langle \vec{k}_i \cdot \vec{k}_j \rangle = \langle \vec{k}_i \rangle \cdot \langle \vec{k}_j \rangle = u'(H)^2 .$$

Finally, using the fact that we have N sites on the lattice and Nd links or pairs of interacting neighbors,

$$F \geq \sup_{\vec{H}} \{\beta d u'(H)^2 + [u(H) - H u'(H)]\} . \quad (4.9)$$

The idea is now that for $d \to \infty$, the right-hand side is in fact the value of F; thus in this limit we replace the inequality by an equality sign. The natural scale of inverse "temperature" or inverse coupling constant thus appears to be βd.

For small H the right-hand side of (4.9) exhibits the competition of two terms in H^2, the first with a positive coefficient and the second with a negative one, while for H large the second dominates and gets large and negative like $-\frac{1}{2}\ln H$ (see Table I). Consequently, for small β the maximum is reached for $H=0$ (in which case it is zero), while for β large it is obtained for some finite value H_{eff} at which point F is positive. The dividing line is obtained by requiring the coefficient of H^2 for small H to vanish. That is,

$$(\tfrac{1}{2}H)^2(\beta_c d - 1) = 0 .$$

Thus the critical value of β is

$$\beta_c d = 1 . \quad (4.10)$$

For $\beta d < 1$,

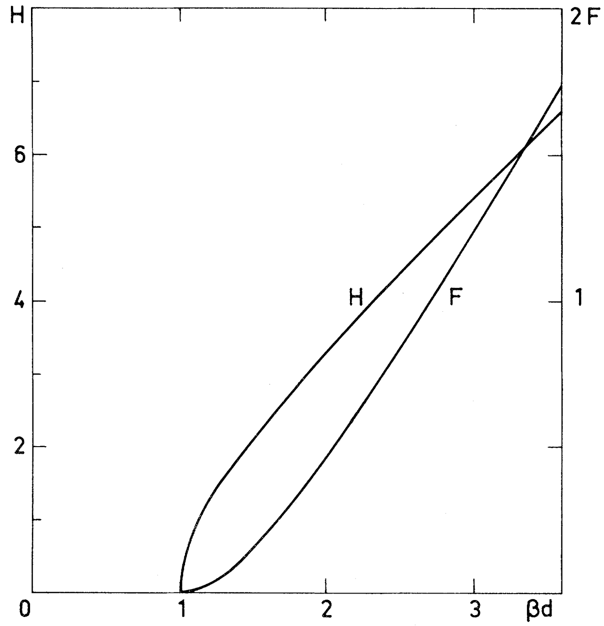

FIG. 2. The functions F and H_{eff} of the scalar model in the mean-field theory ($n=2$).

$$H_{\text{eff}} = 0 \text{ and } F = 0 ;$$

for $\beta_c d = 1 + \epsilon$,

$$H_{\text{eff}} = 2(2\epsilon)^{1/2} + \cdots \text{ and } F = \epsilon^2 + \cdots ;$$

finally, for $\beta_c d > 1$, H_{eff} is given by

$$2\beta d u'(H_{\text{eff}}) = H_{\text{eff}} , \quad (4.11)$$

and F is positive:

$$F = u(H_{\text{eff}}) - \tfrac{1}{2} H_{\text{eff}}\, u'(H_{\text{eff}}) . \quad (4.12)$$

This behavior is summarized in Fig. 2. Since beyond the critical points H_{eff} starts from a zero value and since F and $\partial F/\partial\beta$ are continuous, we find a typical *second-order* transition.

Within the mean-field approximation it is also possible to compute the inverse correlation length μ. Consider the response of the system to the coupling to an external field Q_i. We write

$$Z(\beta, Q_i) = \int \prod_0^{N-1} d\hat{k}_i \exp\left(\beta \sum_{(ij)} \vec{k}_i \cdot \vec{k}_j + \sum_i \vec{Q}_i \cdot \vec{k}_i\right) . \quad (4.13)$$

The connected two-point function is

$$G_{\mu\nu}(i,j;\beta) = \frac{\partial^2}{\partial Q_i^\mu \partial Q_j^\nu} \ln Z(\beta, Q_i)\bigg|_{Q_i=0} . \quad (4.14)$$

Within the mean-field approximation we take a varying mean field $\vec{H}_i$ and find ($\hat{H}_i = \vec{H}_i/H_i$)

$$\ln Z(\beta, Q_i) = \operatorname*{Sup}_{\vec{H}_i}\left\{\sum_{ij} \beta \hat{H}_i \cdot \hat{H}_j u'(H_i) u'(H_j) + \sum_i [u(H_i) + (\vec{Q}_i - \vec{H}_i)\cdot \hat{H}_i u'(H_i)]\right\}. \tag{4.15}$$

Q_i is infinitesimal since we are only interested in the second derivative in Q at $Q=0$. The condition for a maximum in H is obtained by setting the gradient in H of the right-hand side equal to zero. This is

$$\beta\left[\sum_{j(i)} (I - \hat{H}_i \otimes \hat{H}_i) \frac{\hat{H}_j}{H_i} u'(H_i)u'(H_j) + (\hat{H}_j \cdot \hat{H}_i)\hat{H}_i u''(H_i)u'(H_j)\right] + u'(H_i)(I - \hat{H}_i \otimes \hat{H}_i)\frac{\vec{Q}_i}{H_i} + (\vec{Q}_i - \vec{H}_i)\cdot \hat{H}_i \hat{H}_i u''(H_i) = 0 . \tag{4.16}$$

Not only is $\vec{Q}_i$ very small, but since we are interested in long-wavelength excitations, we can assume its direction to vary very slowly around a mean direction. For $\beta d > 1$ its main effect will be to drag the mean field along this direction. We can thus speak of longitudinal and transverse excitations (with respect to this direction in "isospin space"). As we can verify using (4.16) and in accordance with Goldstone's theorem the transverse excitation (the π) is of infinite range, $\mu_\pi = 0$ $(\beta d > 1)$. The longitudinal excitations are obtained as follows. Take the scalar product of (4.16) with $\hat{H}_i$:

$$\sum_{j(i)} \beta(\hat{H}_j \cdot \hat{H}_i)^2 u''(H_i)u'(H_j) + u''(H_i)\hat{H}_i \cdot (\vec{Q}_i - \vec{H}_i) = 0 . \tag{4.17}$$

Since $\vec{Q}_i$ is small we can write for the longitudinal part

$$\vec{H}_i = \vec{H} + \sum_s \chi_{is}\vec{Q}_s ,$$

with χ_{is} essentially identical with the longitudinal part of the Green's function (4.14). Expanding now (4.17) to first order in Q we find (with $|\vec{H}| = H_{\rm eff}$ and suppressing the index "eff")

$$\beta \sum_{j(i)}\left[u'''(H)u'(H)\sum_s \chi_{is}\hat{H}\cdot\vec{Q}_s + u''(H)^2 \sum_s \chi_{js}\hat{H}\cdot\vec{Q}_s\right] + u''(H)\hat{H}\cdot\vec{Q}_i + [Hu'''(H) - u''(H)]\sum_s \chi_{is}\hat{H}\cdot\vec{Q}_s = 0 . \tag{4.18}$$

The elementary solution of this equation fulfills then

$$2\beta d\left\{[u'''(H)u'(H) + u''(H)^2]\chi_{is} + u''(H)^2 \frac{1}{2d}\sum_{j(i)} (\chi_{js} - \chi_{is})\right\} - [u''(H) - Hu'''(H)]\chi_{is} + u''(H)\delta_{is} = 0 .$$

This is a second-order difference equation on the lattice with $\sum_{j(i)}(\chi_{js} - \chi_{is})$ playing the role of the Laplacian. For i very far from s it can be approximated by an ordinary partial differential equation.

Recalling (4.11) and expressing everything as functions of H instead of β we find with Δ_d the Laplacian in d dimensions

$$\left\{\Delta_d + \frac{2d}{u''(H)^2}\left[u'''(H)u'(H) + u''(H)^2 - \frac{u'(H)}{H}[u''(H) - Hu'''(H)]\right]\right\}\chi(x, x') = -\frac{2d}{u''(H)}\frac{u'(H)}{H}\delta(x - x') . \tag{4.19}$$

This is a typical free-particle wave equation. If we call

$$\mu_\sigma{}^2 = -\frac{2d}{u''(H)^2}\left[u'''(H)u'(H) + u''(H)^2 - \frac{u'(H)}{H}[u''(H) - Hu'''(H)]\right], \quad \beta d > 1 \tag{4.20}$$

$\mu_\sigma{}^2$ is an effective-mass square for longitudinal excitations. Asymptotically for large $|x - x'|$ the behavior of χ is

$$\chi(x, x') \sim \text{const} \times \frac{\exp(-\mu_\sigma |x - x'|)}{|x - x'|^{d-2}} . \tag{4.21}$$

As $H \to 0$ or $\beta d \to 1$ we find from (4.20)

$$\mu_\sigma{}^2 \sim 28d(\beta d - 1), \quad \beta d > 1 . \tag{4.22}$$

For $\beta d < 1$ the Goldstone solution disappears, H_i is of order Q_i. We again expand (4.16) to first order in Q with

$$\vec{H}_i = \sum_s \chi_{is}\vec{Q}_s ,$$

and we find

$$\tfrac{1}{2}\beta \sum_{j(i)} \chi_{js}\vec{Q}_s - \sum_s \chi_{is}\vec{Q}_s + \vec{Q}_i = 0 . \tag{4.23}$$

Thus χ satisfies

$$\beta d\left[\frac{1}{2d}\sum_{j(i)} (\chi_{js} - \chi_{is})\right] - (1 - \beta d)\chi_{is} = -\delta_{is} . \tag{4.24}$$

The same comments as above apply to

$$\mu_{\sigma,\pi}{}^2 = \frac{1-\beta d}{\beta d}\,, \quad \beta d < 1\,. \tag{4.25}$$

The two masses are now degenerate. We observe in the vicinity of $\beta d = 1$ a change of slope by comparing (4.22) and (4.25). These results, however, are rough, and in the vicinity of $d=4$ one knows from Wilson's theory of critical exponents[6] that modifications appear.

Finally, by studying the expression (4.20), one can show that for $\beta d \to \infty$ (limit of zero coupling) $\mu_\sigma{}^2 \to \infty$ (in fact, $\mu_\sigma{}^2$ grows like β), as we stated earlier.

In Fig. 3 we have given the global picture for the masses as functions of β.

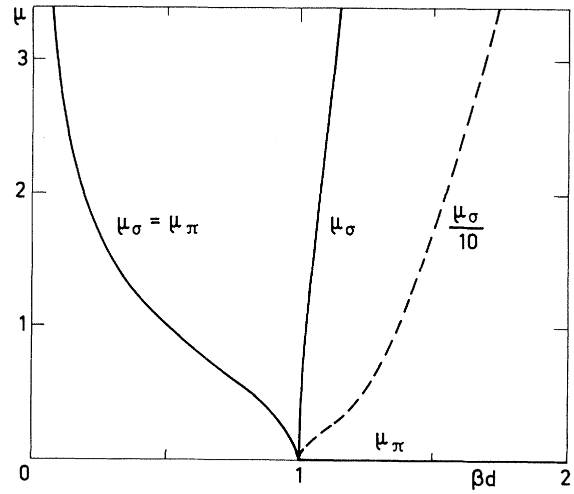

FIG. 3. Behavior of μ_σ and μ_π as functions of βd. Owing to the sharp rise of μ_σ beyond the critical point, we have also plotted on the same scale $\frac{1}{10}\mu_\sigma$.

B. Pure gauge field

We now undertake a parallel discussion for the Abelian gauge-field models $(2.17)_{\text{Ab}}$ and $(2.18)_{\text{Ab}}$. It helps to use the "spins" introduced at the end of Sec. II C with the expression (2.24) for a plaquette contribution.

A nonvanishing mean field will correspond to a nonvanishing average value $\langle A_{ij}\rangle$. This in turn implies breaking of gauge invariance. If we have a uniform mean field, $\langle A_{ij}\rangle$ will be independent of the link (ij). Of course, this is not the most general way in which gauge invariance can be broken. However, it is good enough for our purpose, and we shall not yet elaborate this subtle point further.

We denote by $\vec{K}$ the mean field associated with the spins $\vec{A}_{ij}$. This is again a two-dimensional vector. If we recall that there are $Nd(d-1)/2$ plaquettes on the lattice while there are Nd links, we readily find

$$F \geq \operatorname*{Sup}_{\vec{K}} \{\tfrac{1}{2}\beta d(d-1)[u'(K)]^4 + d[u(K) - Ku'(K)]\}\ . \tag{4.26}$$

As in the previous paragraph we take the right-hand side of (4.26) as representative of F for $d\to\infty$. We note, however, that the situation is different from the scalar case due to the occurrence of u'^4 instead of u'^2 in the "energy" term. For small as well as large K values, the entropy term $u(K) - Ku'(K)$ always dominates irrespective of the value of β. The expression $F(K,\beta)$ to be maximized always has a local maximum at $K=0$ and possibly a second maximum at some other finite value of K. For various values of β the expression $F(K,\beta)$ is sketched in Fig. 4 as function of K. For some critical value β_c the relevant maximum at $K=0$, where $F=0$, jumps to $F=0$ at some critical finite field K_c. As β gets larger than β_c the maximum is obtained for some $K > K_c$ with $F>0$. This sudden jump in K will correspond to a discontinuity in $dF/d\beta$ at β_c and hence is characteristic of a *first-order transition.*

First-order transitions are in fact common in thermodynamical systems even though mostly second-order ones have been scrutinized recently[6] due to interesting fluctuation phenomena. However, it is to some point slightly surprising to find this behavior for the gauge field. We believe this is due to the very different invariance that is broken when "order" sets in. As we said earlier, and as will appear more clearly in the next paragraph, the kind of transition that we observe can be roughly compared to a liquid-solid one.

Next we evaluate K_c and β_c. They are obtained from the pair of equations expressing that $F=0$ and $\partial F/\partial K = 0$ excluding the solution $K=0$. Thus

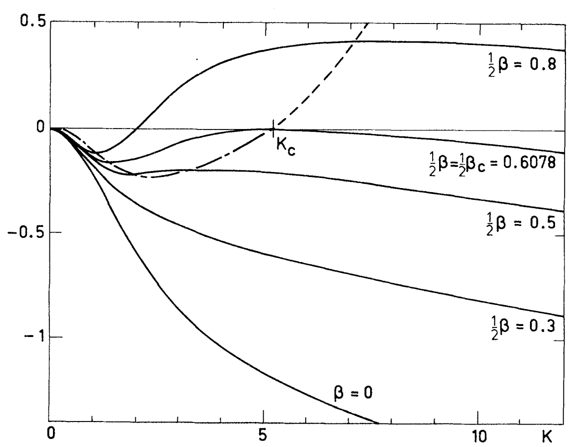

FIG. 4. The function $F(K,\beta)$ for the Abelian Yang-Mills field as a function of the mean field K.

from (4.26)

$$\tfrac{1}{2}\beta_c(d-1)[u'(K_c)]^4 - [u(K_c) - K_c u'(K_c)] = 0 \ ,$$
$$2\beta_c(d-1)[u'(K_c)]^3 - K_c = 0 \ . \tag{4.27}$$

These equations can be rewritten

$$\beta_c(d-1) = \frac{K_c}{2[u'(K_c)]^3} \ ,$$
$$4u(K_c) - 3K_c u'(K_c) = 0 \ . \tag{4.28}$$

The second equation yields K_c; the first one then gives β_c. They can be solved numerically. The curve $y(x) = u(x) - \frac{3}{4}xu'(x)$ is drawn in Fig. 5. We find

$$K_c = 5.32 \ ,$$
$$\tfrac{1}{2}\beta_c(d-1) = 1.82 \ . \tag{4.29}$$

C. Coupled system

It is interesting to treat now the complete system of 2-dimensional unit spins $\vec{k}_i$ coupled to an Abelian gauge field. The complete action is a sum

$$S_1 + S_2 \ ,$$
$$S_1 = \beta_p \sum_p \cos(\alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41}) \ , \tag{4.30}$$
$$S_2 = \beta_l \sum_l k_i^T A_{ij} k_j \ .$$

The partial results of the previous two paragraphs indicate a competition between a second-order transition in the (σ, π) system and a first-order one of the gauge field.

It is indeed clear from the outset that one has the following limiting behaviors. If $\beta_l = 0$ we recover clearly the pure gauge field with its first-order transition. Next if $\beta_p \to \infty$ then the gauge

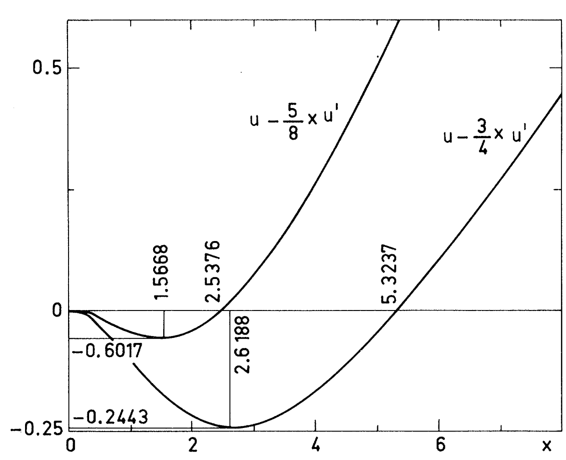

FIG. 5. The curves $u(x) - \frac{3}{4}xu'(x)$ and $u(x) - \frac{5}{8}xu'(x)$.

field A_{ij} is quenched to a pure gauge $A_{ij} = O_i O_j^{-1}$. This results in the Abelian as well as non-Abelian case (where it is in fact slightly less trivial) from the conditions $\chi(AAAA)$ = maximum value on any plaquette. Thus redefining $k_i \to O_i k_i$ one now recovers the scalar model with its second-order transition. Finally, if $\beta_p = 0$ one can in fact compute F exactly, which turns out to be analytic in β_l on this line. Indeed each k_i can be written $O_i q$ with q a fixed unit vector and O_i belongs to SO(n). We change variables from A_{ij} to $A_{ij} = O_i B_{ij} O_j^{-1}$; thus the integral over k_i is trivial, and now so is the one on the independent B_{ij}'s, with the result

$$F = d \ln \left(\int dB \exp(\beta_l q^T B q) \right) \ .$$

We assume for the moment that β_p and $\beta_l \geq 0$, and call $\vec{H}$ and $\vec{K}$ respectively the mean fields associated to the $\vec{k}$ and A degrees of freedom. By using the method of the previous two paragraphs we obtain inequalities analogous to (4.7) and (4.26) and take the right-hand side as the value of F for $d \to \infty$; thus

$$F = \sup_{H,K} \{ \beta_p \tfrac{1}{2} d(d-1)[u'(K)]^4 + d[u(K) - Ku'(K)] + \beta_l d[u'(H)]^2 u'(K) + u(H) - Hu'(H) \} \ . \tag{4.31}$$

Owing to gauge invariance no reference to the relative direction of $\vec{H}$ and $\vec{K}$ remains; only their lengths appear. The expression to be maximized, which we shall call $F(H, K)$, is the sum of two terms pertaining to the gauge field and the $\vec{k}$ field, respectively. These resemble very much the terms we already studied. The only difference and the only place where the coupling occurs is in the replacement of $\beta_l d[u'(H)]^2$ by $\beta_l d u'(K)[u'(H)]^2$. Thus $\beta_l u'(K)$ [recall from Table I that $0 \leq u'(K) \leq 1$] plays the role of an effective coupling.

We have to maximize $F(H, K)$ in the quadrant $H \geq 0$, $K \geq 0$. Now the innocent looking replacement $\beta_l \to \beta_l u'(K)$ has the consequence that for H and K small enough the negative quadratic form

$$d[u(K) - Ku'(K)] + u(H) - Hu'(H) \simeq [d(\tfrac{1}{2}K)^2 + (\tfrac{1}{2}H)^2]$$

always dominates no matter what β_l and β_p are. Hence the origin (where $H = K = F = 0$) is always a local maximum and in fact *the* maximum for β_p, β_l small enough.

Consider the surface $(F(H, K), H, K)$ in a three-dimensional space. It is symmetric with respect to the plane $H = 0$. Its interaction with this plane looks like the curves sketched in Fig. 4. Since $F(H, K)$ is an even function of H, the extremum which appears as β_p grows is either a local maximum of the surface or a saddle point. Clearly, when β_l is small enough this is necessarily a local maximum. Then it will come in competition with the maximum at $H=K=0$ when F will cross the value 0, i.e., for $\frac{1}{2}(d-1)\beta_p=1.82$ as obtained in equations (4.28) and (4.29). Thus we find in the β_l, β_p plane a first-order transition along a segment of line (a segment only since β_p varies from zero to some fixed value as we shall shortly see). We shall find it more convenient to use the variables $2\beta_p(d-1)$ and $2\beta_l d$, and will compute below the limiting value of β_l along this segment. Thus we have a transition line (L_1) along

$$\beta_p^*=2\beta_p(d-1)=7.29\ ,$$
$$0\leqslant\beta_l^*=2\beta_l d\leqslant 2.22\ ,$$
$$(L_1)\to\text{first-order transition.}\qquad(4.32)$$

The limiting value for β_l is obtained through the following consideration. The extremum found above in the plane $H=0$ at $K=K_c=5.32$ competing with the one at the origin is a true maximum as long as β_l is small enough. We can study its nature by examining the curvature in the H direction at this point. We observe that it is negative for small β_l and increases as β_l increases. It vanishes for the value $2\beta_l d=2.22$ as was stated in (4.32).

Beyond this value the above extremum becomes a saddle point. Thus two new maxima at two opposite and nonvanishing values of H appear. Again by adjusting β_l and β_p we can bring these maxima at $F=0$. This defines a new first-order transition line (L_2) obtained by requiring that

$$F(H,K)=0\ ,\qquad \frac{\partial F}{\partial H}(H,K)=0,\qquad \frac{\partial F}{\partial K}(H,K)=0\ .\qquad H,K\neq 0\quad(4.33)$$

These equations can be written as

$$\tfrac{1}{2}\beta_p d(d-1)[u'(K)]^4+\beta_l du'(K)[u'(H)]^2+d[u(K)-Ku'(K)]+[u(H)-Hu'(H)]=0\ ,\qquad(4.34\text{a})$$

$$2\beta_p d(d-1)[u'(K)]^3+\beta_l d[u'(H)]^2-dK=0,\qquad(4.34\text{b})$$

$$2\beta_l du'(H)u'(K)-H=0\ .\qquad(4.34\text{c})$$

We can simplify them slightly by replacing (4.34a) by a combination (4.34a) $-\frac{1}{4}u'(K)$ (4.34b) $-\frac{3}{8}u'(H)$ $\times$(4.34c); thus we find

$$d[u(K)-\tfrac{3}{4}Ku'(K)]+[u(H)-\tfrac{5}{8}Hu'(H)]=0\ ,\qquad(4.35\text{a})$$

$$2\beta_p d(d-1)=\frac{dKu'(K)-\frac{1}{2}Hu'(H)}{[u'(K)]^4}\ ,\qquad(4.35\text{b})$$

$$2\beta_l d=\frac{H}{u'(H)u'(K)}\qquad(4.35\text{c})$$

$$(L_2)\to\text{first-order transition .}$$

Equation (4.35a) defines a curve in the (H,K) plane. Given this curve, (4.35b) and (4.35c) represent parametrically (L_2) in the β_l, β_p plane.

We have already plotted $y(x)=u(x)-\frac{3}{4}xu'(x)$ in Fig. 5. The quantity $y(x)=u(x)-\frac{5}{8}xu'(x)$ has a very similar shape and is reproduced on the same figure. From these two figures, we can sketch the plot of the critical curve in (H,K) space corresponding to (L_2). This is drawn in Fig. 6. The point A, $H=0$, $K=y_1=5.32$ is d-independent and corresponds to the extremity of (L_1) and the starting point of (L_2). Let us study (L_2) in the vicinity of this point. The two small quantities are H and $(K-y_1)$. We expand equations (4.35) around A in these two infinitesimals:

$$d[u(K)-\tfrac{3}{4}Ku'(K)]=\tfrac{1}{4}d[u'(y_1)-3y_1u''(y_1)](K-y_1)-\tfrac{1}{8}d[2u''(y_1)+3y_1u'''(y_1)](K-y_1)^2+\cdots,$$

$$u(H)-\tfrac{5}{8}Hu'(H)=-\tfrac{1}{4}(\tfrac{1}{2}H)^2+\tfrac{3}{8}(\tfrac{1}{2}H)^4+\cdots.$$

Thus (4.35a) yields

$$(K-y_1)=\frac{1}{d[u'(y_1)-3y_1u''(y_1)]}\left(\frac{H}{2}\right)^2+\left\{\frac{2u''(y_1)+3y_1u'''(y_1)}{2d^2[u'(y_1)-3y_1u''(y_1)]^3}-\frac{3}{2d[u'(y_1)-3y_1u''(y_1)]}\right\}\left(\frac{H}{2}\right)^4+\cdots.$$

From this and (4.35c):

$$2\beta_l d=\frac{2}{u'(y_1)}+\left(\frac{H}{2}\right)^2\frac{1}{u'(y_1)}\left\{1-\frac{2}{du'(y_1)[u'(y_1)-3y_1u''(y_1)]}\right\}+\cdots.\qquad(4.36)$$

The quantity $2/u'(y_1)=2.22$, is just the value of $2\beta_l d$ at the extremity of (L_1). The coefficient of $(\frac{1}{2}H)^2$ is positive for d large enough. Similarly from (4.35b),

$$2\beta_p(d-1)=\frac{y_1}{[u'(y_1)]^3}-\left(\frac{H}{2}\right)^4\frac{1}{d[u'(y_1)]^4}\left[1+O\left(\frac{1}{d}\right)\right]+\cdots\ . \tag{4.37}$$

We see similarly that $y_1/[u'(y_1)]^3=7.29$ corresponds to the value of $2\beta_p(d-1)$ at the extremity of (L_1); then the coefficient of $(\frac{1}{2}H)^4$ is negative if $d\to\infty$. Comparing (4.36) and (4.37) we see that (L_2) has a parabolic shape towards negative β_p and positive β_l in the vicinity of A. Thus (L_1) and (L_2) join together smoothly [the tangents are identical at (A)] and constitute a unique first-order transition curve (L). We shall return later to this very smooth behavior. Note also that the curve (L_2) flattens as d grows (the curvature goes to zero).

In the vicinity of B in Fig. 6, $H\to y_2=2.54$, $K\to 0$,

$$\begin{aligned}\beta_l^*&=2\beta_l d\sim\frac{2}{K}\,\frac{y_2}{u'(y_2)}\to\infty\ ,\\ \beta_p^*&=2\beta_p(d-1)\sim\frac{2}{(\frac{1}{2}K)^2}-\frac{1}{2d}\,\frac{y_2u'(y_2)}{(\frac{1}{2}K)^4}\to-\infty\ .\end{aligned} \tag{4.38}$$

This has a quartic shape in the β_l, β_p plane extending in the region $\beta_p\to-\infty$. We remark a very nonuniform behavior as d grows. The leading negative term in β_p is proportional to $1/d$.

The shape of this curve (L_2) is quite nonphysical in the region $\beta_l^*\to\infty$, since in the mean time, it crosses the axis $\beta_p^*=0$ where as we know no transition occurs. Furthermore, if we strictly adhere to the limit $d\to\infty$, then (L_2) degenerates into $\beta_p^*=7.29$ given in (4.32). A correct treatment will be to include in a systematic fashion *all* $1/d$ corrections, which presumably alter the shape of

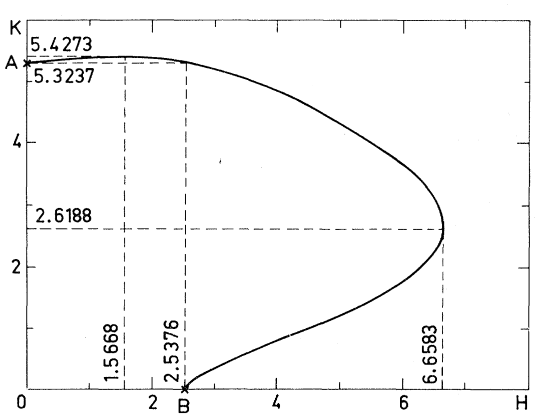

FIG. 6. The critical curve in (H,K) space corresponding to the first-order transition along (L_2), drawn for $d=4$. It is given by the implicit equation $d[u(K)-\frac{3}{4}Ku'(K)]+[u(H)-\frac{5}{8}Hu'(H)]=0$.

(L_2), bending it downwards, but not enough to cross the axis $\beta_p^*=0$. This is why in Fig. 1, where the phase diagram has been drawn in the mean-field approximation for $d=4$, the curve (L_2) appears as dotted in the large β_l^* direction.

This is, however, not the end of the story. Returning to point A we have followed two new bumps emerging in the surface $F(H,K)$ and reaching the plane $F=0$. But at A the local maximum was becoming a saddle point. Thus instead of comparing the new bumps $H\neq 0$, $K\neq 0$, with the maximum at $H=K=0$, we could have followed in the plane $H=0$ the transformation of the maximum into a saddle point at which point the maximum with $F>0$ divides itself into two new other ones. This clearly defines a new curve (M) corresponding to a *second order* transition. The equations governing this mechanism are

$$\begin{aligned}&\frac{\partial F}{\partial K}(H=0,K)=0\ ,\\ &\frac{\partial^2 F}{\partial H^2}(H=0,K)=0\ .\end{aligned} \tag{4.39}$$

Explicitly we find from (4.31) a parametric expression for (M) in terms of K:

$$\begin{aligned}\beta_p^*&=2\beta_p(d-1)=\frac{K}{u'(K)^3}\ ,\\ \beta_l^*&=2\beta_l d=\frac{2}{u'(K)}\ ,\end{aligned}$$

$$(M)\to\text{second-order transition}\ . \tag{4.40}$$

The reader might notice that Eqs. (4.40) are obtainable from (4.35b) and (4.35c) by taking the limit $H=0$. The curve (M) starts from A for the value K_c and goes to infinity in the positive β_p direction with an asymptote at $\beta_l d=1$. This value corresponds to the transition in the $\vec{k}$ system alone, because the gauge field is frozen at its unit value (in the group) when $\beta_p\to\infty$ (or $T_p=0$). For amusement we can easily complete the picture in the $\beta_l<0$ half plane by symmetry. The difference is that the region bounded by (L_2') and (M') in which $H,K\neq 0$ corresponds rather to an antiferromagnetic ordering where at successive sites the effective field $\vec{H}_i$ jumps from a value $\vec{H}$ to $-\vec{H}$.

A careful study, the details of which need not be presented here, shows that we have exhausted all possible transitions.

On the whole, the phase diagram of the coupled Abelian system in the mean-field approximation is represented on Fig. 1. The phase I is fully disordered $(H=K=0)$; both the gauge field and the particle field have vanishing values. The first-order transition line (L) separates this phase from the other phases II, III, IV, in which the gauge invariance is spontaneously broken ($K \neq 0$ and $\langle A \rangle \neq 0$). In such phases, we have assumed $\vec{K}$ to be a constant on all links of the lattice, and found an overall rotational degeneracy in the solution; however, the actual degeneracy is much larger owing to gauge invariance, since independent rotations may be performed on each site of the lattice. In the phase II, rotational invariance of the field k remains unbroken ($H=0$, $\langle k \rangle = 0$). The second-order transition line (M) separates this phase from the fully ordered phase III in which all invariances are spontaneously broken, giving rise to a massless π and a massive σ. This phase III may be considered as "ferromagnetic," since if the self-consistent field $\vec{K}$ for the gauge field is taken as uniform, the order parameter $\langle \vec{k}_i \rangle$ is constant over the lattice. (Similarly, the symmetrical phase IV is "antiferromagnetic" with the same uniform choice for $\vec{K}$.) Clearly, the coupling $\beta_l \sum_l k_i A_{ij} k_j$ between neighboring fields k_i, k_j may become effective only when $\langle A_{ij} \rangle$ is nonzero. Thus the occurrence of a transition for the gauge field, leading (for β_p large enough) to a nonzero value for $\langle A_{ij} \rangle$, is a prerequisite to the ordering of the particle field k. This ordering takes place [along (M) or along (L_2)] when $\langle A_{ij} \rangle$ already has a finite value.

This phase diagram presents a great analogy with the (p, T) phase diagram of a material which may become magnetic, with β_p playing the role of pressure and β_l the role of inverse temperature. The disordered phase I is the equivalent of a liquid. When pressure (β_p) is increased, it crystal lizes (phases II, III, IV) through a first-order transition, $\langle A_{ij} \rangle$ playing the role of the lattice order parameter in the solid. If the k_i are interpreted as the atomic spins, the term $\sum_l k_i A_{ij} k_j$ has the same features as an exchange interaction, which becomes effective only in the crystalline phase. We thus have a second-order magnetic transition (M) or (M') between the nonmagnetic crystal II and the ferro- (or antiferro-) magnetic crystal III (or IV) at low temperature ($1/|\beta_l|$ small). The curve (M) is very steep, because once it is settled, the crystalline order is not very sensitive to pressure, and the exchange interaction between spins does not vary much with pressure. In the field-theoretical interpretation, the situation is the same, since along (M) the value of $\langle A \rangle = u'(K)$ varies only from 0.90 to its maximum value 1.

The triple point A is the most interesting feature of the phase diagram. The fact that the slope of (L) remains continuous across A may easily be understood by standard thermodynamic arguments, since (M) is a second-order transition line. It reflects the fact that second-order effects are weak compared to first-order ones. Although the phase diagram has been established in the mean-field approximation, expected to be exact for infinite dimensionality, experience in statistical mechanics suggests that the qualitative features will remain unchanged by a more refined treatment.

D. Validity of mean-field theory

In the gauge-invariant theories one may at first question the validity of the qualitative results obtained through mean-field theory. The latter needs in fact to be made more precise. Indeed the mean field is a conjugate variable to some order parameter. *Order* can be defined as the prevalent situation as "temperature" goes to zero, i.e., here when β_p and $\beta_l \to \infty$. From the structure of the action S it is seen that for arbitrary O_i this amounts to $k_i = O_i q$ (fixed q), $A_{ij} = O_i O_j^{-1}$. Our previous treatment is seen to imply a choice among the highly degenerate "vacuum states." One could argue that it would be more reasonable to break gauge invariance first by integration over some subset of variables.

For instance, returning to the general expression for the action we might perform a change of variables as follows. For fixed $k_i = O_i q$ we set $A_{ij} = O_i B_{ij} O_j^{-1}$. It is then possible to take as new variables B's and O's and integrate over O's. The result is a B model

$$Z = \int \prod_{(ij)} dB_{ij}\, e^S ,$$
$$S = \beta_l \sum_l q^T B_{ij} q + \beta_p \sum_p \chi(B_{12}B_{23}B_{34}B_{41}) , \tag{4.41}$$

for which apparently no reference remains to the particle field.

Application of mean-field theory to (4.41) would yield the first-order transition line (l) of Fig. 1. Nevertheless, large fluctuations would be unavoidable for β_l small as we did not yet really cope with the genuine gauge problem. Moreover, this curve has an end point. Thus there is a seemingly continuous path in the diagram between phases. The clue to this apparent paradox is presumably that is not a good order parameter since it is always expected to be nonvanishing.

An example of this type of situation would be to take in the customary Ising model a variational parameter proportional to $S_i S_j = b_{ij}$ [with (ij) a link on the lattice].

Thus mean-field theory should only be applied in such models where

(i) a realistic order parameter expected to have a discontinuous behavior owing to some symmetry of the problem is indeed identified, and

(ii) the vacuum is not too degenerate in order to avoid wild fluctuations which would spoil any attempt to improve the approximation by a perturbation expansion.

In order to meet these qualitative prerequisites we can further proceed starting from Eq. (4.41) by defining a "Coulomb gauge" in the following way. We choose some direction, denoted the time axis, to play a particular role. To each lattice point we let correspond the timelike link that starts from the point. For the sake of clarity let T_i be the corresponding B_{ij} variables while $\tilde{S}_{ij}$ denote the other spacelike B_{ij} variables. Let further D stand for the unit time displacement operator on the lattice. It is easily seen that T_i can be written as $O_{Di}O_i^{-1}$. One can then change variables from T_i and $\tilde{S}_{ij}$, to $k_i = O_i q$ and $S_{ij} = O_i \tilde{S}_{ij} O_j^{-1}$. By integrating over the little groups of q one recovers "matter variables" k_i and a subset S_{ij} of the previous gauge field variables A_{ij} as follows:

$$Z = \int \prod_i d\hat{k}_i \int \prod_{(ij)} dS_{ij}\, e^S ,$$

$$S = \beta_l\Big(\sum_i k_i^T k_{Di} + \sum_{(ij)} k_i^T S_{ij} k_j\Big) + \beta_p\Big(\sum_{(ij)} \chi(S_{ij}S_{Dj,Di}) + \sum_p \chi(S_{12}S_{23}S_{34}S_{41})\Big) . \tag{4.42}$$

All summations are carried over spacelike links and spacelike plaquettes. Clearly one could have performed the steps leading to (4.42) directly. The remaining gauge arbitrariness is now only a "surface" effect with a group

$$\overset{N(d-1)/d}{\otimes} \quad \mathrm{SO}(n)$$

and of course the infinite-volume limit ($N\to\infty$) is to be taken first. As a result this arbitrariness is presumably irrelevant.

We see that (4.42) amounts to restricting all timelike links to the unit value. Application of mean-field theory now yields for the Abelian case $n=2$

$$F = \operatorname*{Sup}_{(H,K)} F(H,K),$$

$$\begin{aligned} F(H,K) = {} & \tfrac{1}{2}\beta_p(d-1)(d-2)[u'(K)]^4 + \beta_p(d-1)[u'(K)]^2 \\ & + (d-1)[u(K) - Ku'(K)] \\ & + \beta_l(d-1)[u'(H)]^2 u'(K) \\ & + \beta_l[u'(H)]^2 + u(H) - Hu'(H) , \end{aligned} \tag{4.43}$$

to be compared with (4.31). There are clearly differences, which, however, are washed out by taking the variables β_p^* and β_l^* as previously and letting $d\to\infty$. One then recovers exactly the results of the previous section in this limit and qualitatively the same phase diagram for finite and large enough d. One may remark the presence of a new quadratic term in $[u'(K)]^2$. Thus there is a possibility for a second-order transition in the pure gauge field for low dimension. However, this term is doubtful even as a zeroth-order approximation. Hence one feels more confident about the mean-field approximation and it is suggested to collect systematically all $1/d$ corrections.

V. CONCLUSION

In this first paper we have only used rather unsophisticated mathematical means, and did not touch upon several problems pertaining to the noncommutative case. However, we have disclosed a very rich variety of interesting phenomena in this Wilson model. Apart from its speculative application to new binding modes in the domain of particle theory, it would certainly be amusing to find some physical system to which the thermodynamical version would apply. In the next papers we shall present perturbative and diagrammatic expansions and further develop the study of this model.

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[1]This model was presented by K. Wilson in a seminar at Orsay during the summer of 1973. E. Brezin communicated to us later some fragments of a manuscript by Wilson, which has now been published [Phys. Rev. D 10, 2445 (1974)].

[2]A useful review of phase-transition phenomena including a large wealth of references is in H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Clarendon, Oxford, 1971).

[3]J. Goldstone, Nuovo Cimento 19, 154 (1961).

[4]H. E. Stanley, Phys. Rev. 176, 718 (1968).

[5]R. E. Peierls, Phys. Rev. 54, 918 (1938).

[6]K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974).