Quantum electrodynamics and radiation reaction: Nonrelativistic atomic frequency shifts and lifetimes*

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We present a quantum electrodynamic treatment of radiative corrections in atoms which is patterned after Lorentz's classical work on radiation damping. Expressions for both radiative lifetimes and frequency shifts are calculated through second order in the electric charge for a fictitious two-level model atom and for a spinless one-electron atom with an infinite number of arbitrarily spaced energy levels. In order to apply the classical ideas of Lorentz to quantum-electrodynamic problems of this kind we work directly with the relevant dynamical variables of the atom and field. The calculations are carried out entirely in the Heisenberg picture by recognizing the importance of radiation reaction. The quantized-field operator equations are integrated with the aid of a Markov approximation. The part of the integrated field that arises from the atomic electron current operator, the radiation-reaction field, is shown to be solely responsible for the atom's linewidths and frequency shifts. It is clear that it is unnecessary to invoke vacuum fluctuations at any stage. The usual quantum electrodynamic exponential decay law is found to govern the expectation values of the energy and dipole moment of the atom as well as the radiated-field amplitude. The theory nevertheless remains unitary. The Heisenberg operator commutation relations are shown to be valid at all times, and the Markov approximation is justified for times longer than a reciprocal transition frequency.

I. INTRODUCTION

A. Historical background

The classical interaction of any charge distribution with electromagnetic radiation is completely described by the Maxwell-Lorentz equations.[1] In particular, the motion of a particle with charge e is governed by the field acting on it, according to the Lorentz force law:

$$\frac{d\vec{p}_e}{dt} = e\left[\vec{E}(\vec{r}_e, t) + \frac{\vec{v}_e}{c} \times \vec{B}(\vec{r}_e, t)\right] . \tag{1.1}$$

At the same time, a field is determined by the solution of the appropriate Maxwell wave equation in which the particle current acts as source:

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int \frac{[\vec{J}_T] d^3r'}{|\vec{r}' - \vec{r}|} + \vec{A}^H(\vec{r}, t), \tag{1.2}$$

where $[\vec{J}_T]$ is the retarded transverse part of the total current $\vec{J}$,

$$\vec{J}(\vec{r}', t) = e\vec{v}_e(t)\delta^3(\vec{r}' - \vec{r}_e), \tag{1.3}$$

and $\vec{A}^H$ is the homogeneous solution, independent of the current $\vec{J}$.

If there are no other charges in the universe, then $\vec{A}^H = 0$. If Eq. (1.2) is then used to find the fields in Eq. (1.1), the resulting problem of the mutual coupling of the particle and its own field is one of the oldest problems of 20th-century physics for which no very general solutions are known.

Modern particle theory can be said to begin in Lorentz's attempts to formulate a consistent theory of the electron, starting from the equations above. Our interest in Lorentz's work stems from his treatment of bound electrons, in which the potential importance of radiative corrections is recognized. That is, Lorentz saw that since an electron could not escape itself, it could never be in a region free of field. Thus corrections imposed by the presence of its own field on any hypothetical field-free motion were necessary to allow for in any computation of orbits. It will be helpful in approaching our quantum-electrodynamic calculations to recall Lorentz's classical results.

The familiar expansion[2] of $[\vec{J}_T]$ in powers of the retardation time $|\vec{r}' - \vec{r}|/c$ leads to a power series for $\vec{A}(\vec{r}, t)$, from which one easily finds

$$e\vec{E}(\vec{r}_e, t) = -\tfrac{4}{3} m_{e-e} \frac{d^2\vec{r}_e}{dt^2} + \frac{2}{3}\frac{e^2}{c^3}\frac{d^3\vec{r}_e}{dt^3} + \cdots . \tag{1.4}$$

Here the "electromagnetic mass" is defined by $m_{e-e}c^2 = W_{e-e}$ where W_{e-e} is the electron's Coulomb self-energy, which is infinite if the electron is given no internal structure.[1] As a consequence the nonrelativistic force law obeyed by a classical "atomic" electron is

$$m\frac{d^2\vec{r}_e}{dt^2} = -m\omega_0{}^2\vec{r}_e - \tfrac{4}{3}m_{e-e}\frac{d^2\vec{r}_e}{dt^2} + \frac{2}{3}\frac{e^2}{c^3}\frac{d^3\vec{r}_e}{dt^3} + \cdots . \tag{1.5}$$

Here the restoring force $-m\omega_0{}^2\vec{r}_e$ has been added, following Lorentz, to give binding and oscillation at the natural frequency ω_0.

It is natural to assume that the electromagnetic corrections to the natural oscillation of the electron are small. Atoms are fairly stable objects. Furthermore it is customary[1] to ignore terms beyond $d^3\vec{r}_e/dt^3$ on the grounds that their contribution is vanishingly small for a vanishingly small electron. Consequently, one feels safe in using the free-oscillation solution

$$\vec{r}_e(t') \sim \vec{r}_e(t)e^{\pm i\omega_0(t'-t)} \tag{1.6}$$

to reduce the equation of motion (1.5) to

$$\frac{d^2\vec{r}_e}{dt^2} + \frac{2}{3}\frac{e^2\omega_0{}^2}{mc^3}\frac{d\vec{r}_e}{dt} + \left(1 - \frac{4}{3}\frac{m_{e-e}}{m}\right)\omega_0{}^2\vec{r}_e = 0. \tag{1.7}$$

The obvious result is that the electronic oscillation occurs at the shifted frequency

$$\omega_0' = \left(1 - \frac{4}{3}\frac{m_{e-e}}{m}\right)^{1/2}\omega_0 , \tag{1.8a}$$

and the electronic energy decays exponentially with lifetime τ_0, where

$$\frac{1}{\tau_0} = \frac{2}{3}\frac{e^2\omega_0{}^2}{mc^3} . \tag{1.8b}$$

Thus radiative self-interactions lead to observable effects. Radiative self-damping establishes a natural linewidth $1/\tau_0$ to the atomic emission line.

The mechanism causing the radiative corrections displayed in Eqs. (1.8) can sensibly be called radiation reaction, even though the frequency shift is due to the electron's interaction with its own nonradiative Coulomb field. Our convention, in both classical and quantum contexts, will be to make the term "radiation reaction" serve for all types of electromagnetic self-interaction. Similarly, the term "source field" will be used to denote any field acting on the electron which has the same electron in question as its source. In the classical problem Eq. (1.4) gives the source field.

Radiative damping was recognized by Slater[3] in 1924 to be closely related to the spontaneous emission introduced into electrodynamics by Einstein[4] in 1917. Using the new quantum theory, excited-state probabilities were shown[5] to decay exponentially by Landau in 1927 and by Bloch in 1928. Landau found that quantum-mechanical decay resulted from a reaction term in the quantum equations of motion analogous to the third term on the right-hand side of Eq. (1.5), showing that Slater's idea was well founded. In Landau's equations, however, there was no electromagnetic mass term analogous to the second term on the right-hand side of Eq. (1.5).

Weisskopf and Wigner[6] in 1930, in model calculations of linewidths in atomic spectra, connected a Lorentzian natural lineshape with the exponential decay of probabilities in spontaneous emission. Their Schrödinger-picture calculation, as well as that of Bloch, is not related in any obvious way to Lorentz's classical work. In general, working in the Schrödinger or interaction pictures with state amplitudes and transition probabilities makes it difficult to draw detailed classical analogies.

Two decades after Landau's calculation the famous Lamb-Retherford experiment was performed.[7] Bethe showed,[8] by implementing Kramer's mass-renormalization ideas, that the measured departure from the Dirac prediction for the hydrogen spectrum could be attributed to the atomic electron's interaction with the vacuum radiation field. Subsequent workers concentrated, for the most part, on increasing the numerical precision of more and more sophisticated calculations of such radiative corrections. Apart from the quasiclassical heuristic calculation of Welton[9] in 1948, in which a connection between vacuum field fluctuations and atomic level shifts was established, practically no attention was devoted to problems of overall meaning and interpretation.[10] This is all the more surprising in light of the well-known[11] failure of Welton's quasiclassical method when applied to the correction to the electron's anomalous magnetic moment,[12] a radiative correction equally as fundamental as the Lamb shift.

One senses that something like physical intuition is absent in modern quantum electrodynamics (QED).[10,11] It has even been suggested[13] by Dirac that conventional methods of calculation may be so anti-intuitive as to conceal solutions to the logical and mathematical problems of which QED has more than its share. While staying within the usual QED framework, Dirac[13] has been able to avoid certain divergences of Schrödinger-picture QED which cannot be absorbed into a redefinition of the electron's charge or mass, by simply working in the Heisenberg picture. Using a Heisenberg-operator constants-of-the-motion method, Dirac[13] has explicitly calculated the correction to the electron's anomalous magnetic moment and the Lamb shift.

Recently, however, there have been a number of renewed attempts to avoid the divergences as well as the conceptual indirectness of quantum electrodynamics. Jaynes and co-workers[14] have gone so far as to construct a new nonquantized-field "neoclassical" theory of electrodynamics

which is based on the old Schrödinger interpretation of quantum mechanics. The primary feature of their approach is that a nonoperator coupling of radiation and matter is employed. Nevertheless, natural decay rates and other radiative corrections may be calculated, with some results similar to those of QED, directly from their coupled radiation-matter equations. In the neoclassical theory radiative corrections arise in a very natural way from radiation reaction. Vacuum fluctuations are nonexistent in the nonquantized neoclassical theory. In 1968 Series[15a] showed that a quantum-mechanical *operator* reaction field, together with an *ad hoc* rule of operator ordering, can be used to calculate radiative corrections in atoms. More recently, Lama and Mandel[15b] have shown that an antisymmetrization postulate allows the calculation of the correct lifetime but not of a level shift in a model two-level atom, again using an operator source field.

Other recent work on source-field or reaction-field quantum electrodynamics is due to Bullough[16] and to Nesbet.[17] Moniz and Sharp[18] have also used the Heisenberg picture to reconsider the problems of preacceleration and runaway solutions which have plagued classical analyses of the free electron's self-interaction.

If we reexamine recent attempts and compare them with earlier work, we find it reasonable to suspect that a fully quantum-electrodynamic source-field approach to atom-field problems can lead to calculations of atom level shifts.

That a source-field approach to QED is not completely straightforward to construct has already been pointed out.[14-17] Its advantages may be substantial, however, in providing a clearer and conceptually simpler framework within which radiative corrections of all kinds can be interpreted. In the following sections we sketch the development of a quantum-electrodynamic source-field approach to atomic radiation theory which relies completely on the time evolution of the theory's various dynamical variables. By working in the Heisenberg picture we also gain the advantage of strong classical analogies at every stage.

B. Organization of paper

In Sec. II we illustrate the basic ideas and utility of our approach to quantum electrodynamics by concentrating on the problem of spontaneous emission from a fictitious two-level atom. By considering in Sec. III spontaneous emission from a more realistic atom, one with an infinite number of arbitrarily spaced levels, we are able to show which features of the two-level-atom treatment are also valid for a real atom. In every case radiative corrections are seen to arise from radiation reaction in the emission process. Vacuum fluctuations need play no role. Of course, the decay rates and frequency shifts found are identical with those obtained from stationary-state Schrödinger-picture perturbation theory.

In Secs. II and III the Heisenberg equations cannot be integrated exactly, of course. The approximation used is similar to the classical approximation used in going from Eq. (1.5) to Eq. (1.7). It has three distinct elements in practice. First the Maxwell equation for the field amplitude is integrated formally but exactly to all orders in the coupling constant with the aid of the exact Liouville operator. Next a Markovian assumption allows certain time-dependent coefficients to be discarded after a time longer than the period of the unperturbed atom's principal mode of oscillation. Finally the Liouville operator is expanded in powers of the coupling constant, and the low-order terms of interest are used in integrating the atomic operator equations. In Sec. IV we establish the validity of the Markov approximation, and verify the consistency of our coupling-constant expansion by showing that the Markov-approximated theory remains unitary through second order in the coupling constant.

Section V is devoted to a discussion of our main results, and remarks on the source-field approach to radiative corrections. Finally, several appendixes contain calculations too cumbersome for the main text.

II. TWO-LEVEL-ATOM SPONTANEOUS EMISSION

A. The two-level Hamiltonian

The largest radiative corrections in atoms are associated with the lowest electric dipole transitions. In this section we will assume that an atom consists of nothing but this lowest transition. In other words, our atom has only two energy eigenstates, and the electric dipole matrix element between these two states is not zero. By integrating approximately the Heisenberg equations of motion for this truncated atom we will be able to identify a frequency shift and a decay rate for this single spontaneous transition. These are the analogs of the Lamb shift and the Einstein spontaneous decay coefficient which we will study in a more realistic many-level atom in Sec. III. As we will see, the basic physical approximations which suggest themselves in considering two-level-atom spontaneous emission will remain valuable when we study a many-level atom.

In a real one-electron atom the Hamiltonian appropriate to the electron-nucleus relative motion is, in the dipole approximation,

$$H=\frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2+V(r)+H_f\ , \tag{2.1}$$

where H_f is the electromagnetic-field Hamiltonian, m is the electron's mass, and $\vec{A}$ is the vector potential evaluated at the center of the atom, the origin for our coordinate system. We have assumed that the electron is spinless and that the potential V which binds the electron to the nucleus is a central potential, for convenience. The vector potential (in the Coulomb gauge) and H_f can both be written in terms of photon creation and destruction operators in the usual way:

$$\vec{A}(\vec{r})=\sum_\lambda\left(\frac{2\pi\hbar c^2}{\omega_\lambda \mathcal{V}}\right)^{1/2}\hat{\epsilon}_\lambda\left(a_\lambda e^{i\vec{k}_\lambda\cdot\vec{r}}+a_\lambda^\dagger e^{-i\vec{k}_\lambda\cdot\vec{r}}\right) \tag{2.2}$$

and

$$H_f=\sum_\lambda \hbar\omega_\lambda a_\lambda^\dagger a_\lambda\ , \tag{2.3}$$

where the mode index λ specifies both wave vector and polarization, and where the usual commutation relation is satisfied:

$$[a_\lambda, a_{\lambda'}^\dagger]=\delta_{\lambda\lambda'}\ . \tag{2.4}$$

The general Hamiltonian H given in Eq. (2.1) can easily be converted into its two-level-atom equivalent simply by constructing the operator which has the same matrix elements as H between the two lowest eigenstates of the atom, and zero matrix elements between all other states. If we denote the two lowest states by $|1\rangle$ and $|2\rangle$, and the corresponding unperturbed energies by E_1 and E_2, then the operator $H_a^{(2\times2)}$ which is identical with the unperturbed atomic Hamiltonian H_a,

$$H_a=\frac{1}{2m}p^2+V\ , \tag{2.5}$$

in the 2×2 space of the two lowest states and is zero everywhere else has the obvious matrix representation

$$H_a^{(2\times2)}=\begin{bmatrix} \vdots & \vdots & \vdots \\ \cdots 0 & 0 & 0 \\ \cdots 0 & E_2 & 0 \\ \cdots 0 & 0 & E_1 \end{bmatrix}. \tag{2.6}$$

The matrix representation for the 2×2 interaction Hamiltonian $H_i^{(2\times2)}$, which is identical with the actual interaction Hamiltonian H_i in the space of the states $|1\rangle$ and $|2\rangle$, and is zero everywhere else, may be constructed equally easily, as follows.

First, we write the actual H_i in the form

$$H_i=-\left(\frac{e}{mc}\right)\vec{p}\cdot\vec{A}\ , \tag{2.7}$$

ignoring the A^2 term because it makes no contribution to frequency shifts.[19] Next, the matrix elements of the momentum operator $\vec{p}$, computed using eigenstates of H_a, are trivially related to the corresponding matrix elements of the relative coordinate operator $\vec{r}$. For example,

$$\langle2|\vec{p}|1\rangle=im\omega_0\langle2|\vec{r}|1\rangle\ , \tag{2.8}$$

where we have abbreviated

$$E_2-E_1=\hbar\omega_0\ . \tag{2.9}$$

If we denote by the real number d the magnitude of the electric dipole 2-1 matrix element, and by $\hat{d}$ its direction:

$$\hat{d}d=e\langle2|\vec{r}|1\rangle\ , \tag{2.10}$$

and let the scalar quantity A stand for the projection of $\vec{A}$ along $\hat{d}$ evaluated at the center of the atom:

$$A\equiv\sum_\lambda\left(\frac{2\pi\hbar c^2}{\omega_\lambda\mathcal{V}}\right)^{1/2}(\hat{\epsilon}_\lambda\cdot\hat{d})(a_\lambda+a_\lambda^\dagger), \tag{2.11}$$

then the 2-1 and 1-2 matrix elements of H_i are

$$\langle2|H_i|1\rangle=-i\left(\frac{\omega_0 d}{c}\right)A,\quad \langle1|H_i|2\rangle=i\left(\frac{\omega_0 d}{c}\right)A\ . \tag{2.12}$$

The matrix representation for $H_i^{(2\times2)}$ is obviously

$$H_i^{(2\times2)}=\begin{bmatrix} \vdots & \vdots & \vdots \\ \cdots 0 & 0 & 0 \\ \cdots 0 & 0 & -i(\omega_0 d/c)A \\ \cdots 0 & +i(\omega_0 d/c)A & 0 \end{bmatrix}. \tag{2.13}$$

Of course, in computing commutators in order to work out the atom's operator equations of motion the infinite number of rows and columns that are zero in $H_a^{(2\times2)}$ and $H_i^{(2\times2)}$ have no influence. Thus we may as well ignore them from the outset, and treat both $H_a^{(2\times2)}$ and $H_i^{(2\times2)}$ as operators on a two-dimensional vector space, the space spanned $|1\rangle$ and $|2\rangle$. In that case combinations of the Pauli σ matrices can be used to represent all of the atom's operators. For example, we find

$$H_a^{(2\times2)}=\tfrac{1}{2}\hbar\omega_0\sigma_3 \tag{2.14}$$

and

$$H_i^{(2\times2)}=+\left(\frac{\omega_0 d}{c}\right)A\sigma_2\ , \tag{2.15}$$

where a term proportional to the 2×2 unit matrix, has been eliminated from the expression for $H_a^{(2\times2)}$ by choosing $\frac{1}{2}(E_2+E_1)=0$, so that the zero of the unperturbed atomic energy lies halfway between

E_1 and E_2. Since the only atomic operator in H_i is the momentum operator $\vec{p}$, expression (2.15) is equivalent to a Pauli matrix representation for $\vec{p}^{(2\times2)}$, the 2×2 version of $\vec{p}$. In addition, the corresponding 2×2 representation for the dipole moment operator

$$e\vec{r}^{(2\times2)} = \hat{d}d\sigma_1 , \tag{2.16}$$

will frequently be of use.[20] Note one consequence of the two-level truncation: $\vec{r}^{(2\times2)}$ and $\vec{p}^{(2\times2)}$ do not have a c-number commutator. As a result, the two-level atom does not exhibit the dipole sum rules expected of real atoms.

B. Equations of motion and solution for the field

Having reduced the atomic operators to operators in the 2×2 space of the lowest electric dipole transition, it is simple to find their equations of motion. We will drop the superscripts "(2×2)" for simplicity hereafter. The Hamiltonian which governs the field and the two-level atom is

$$H = \tfrac{1}{2}\hbar\omega_0\sigma_3 + \left(\frac{\omega_0 d}{c}\right)A\sigma_2 + H_f \tag{2.17}$$

and the Heisenberg equations of motion for the operators in the problem are

$$\dot{\sigma}_1 = -\omega_0\sigma_2 + 2\left(\frac{\omega_0 d}{\hbar c}\right)A\sigma_3 , \tag{2.18a}$$

$$\dot{\sigma}_2 = \omega_0\sigma_1 , \tag{2.18b}$$

$$\dot{\sigma}_3 = -2\left(\frac{\omega_0 d}{\hbar c}\right)A\sigma_1 , \tag{2.18c}$$

and

$$\dot{a}_\lambda = -i\omega_\lambda a_\lambda - i\left(\frac{\omega_0 d}{\hbar c}\right)g_\lambda\sigma_2 , \tag{2.19}$$

where we have abbreviated

$$g_\lambda \equiv \left(\frac{2\pi\hbar c^2}{\omega_\lambda \mathcal{V}}\right)^{1/2}\hat{\epsilon}_\lambda \cdot \hat{d} \tag{2.20}$$

for convenience. Equations (2.18a) and (2.18b) are the quantum analogs to the classical Lorentz force equation since σ_1 and σ_2 are the two-level analogs of $\vec{r}$ and $\vec{p}$. Equations (2.18c) and (2.19) are the atomic energy operator and the field-mode operator equations of motion, respectively. Equation (2.19) embodies the Maxwell equations, which are the same in both classical and quantum electrodynamics.

In Eqs. (2.18) it is the electromagnetic field in the form of A that drives the atomic variables, modifying their unperturbed time evolution. We therefore begin an approximate solution of these operator equations by formally integrating (2.19). With the choice of ordinary retarded boundary conditions, appropriate to our quantum-mechanical problem, we easily find

$$a_\lambda(t) = a_\lambda(0)e^{-i\omega_\lambda t} - i\left(\frac{\omega_0 d}{\hbar c}\right)g_\lambda\int_0^t dt'\sigma_2(t')e^{-i(\omega_\lambda - i\epsilon)(t-t')} , \tag{2.21}$$

where we chose to begin the problem at $t=0$.

The solution for $a_\lambda(t)$ is written as a sum of two contributions, one from the free evolution of the field operator and the other from the radiating atomic source. Because we have in mind a spontaneous transition, which starts with the field in the vacuum state, we designate the free-field part of $a_\lambda(t)$ as the "vacuum" part of the mode operator. In the same way, it is sensible to label the contribution from the radiating atom as the "source" part of the mode operator. Thus we write

$$a_\lambda(t) = a_\lambda^v(t) + a_\lambda^s(t) , \tag{2.22}$$

where

$$a_\lambda^v(t) = a_\lambda(0)e^{-i\omega_\lambda t} \tag{2.23a}$$

and

$$a_\lambda^s(t) = -i\left(\frac{\omega_0 d}{\hbar c}\right)g_\lambda\int_0^t dt'\sigma_2(t')e^{-i(\omega_\lambda - i\epsilon)(t-t')} . \tag{2.23b}$$

The atomic excited-state lifetime is much longer than the time scale of the dipole oscillation, which Eqs. (2.18a) and (2.18b) show to be on the order of ω_0^{-1}. If we write τ_0 for the excited-state lifetime, then typically $\omega_0\tau_0 \sim 10^6$. Thus it seems sensible, at least in first approximation, to assume that $\sigma_2(t')$ is oscillating freely without interaction in order to carry out the integral in Eq. (2.23b). This is easiest to do after rewriting σ_2 in terms of the non-Hermitian operators $\sigma_\pm$,

$$\sigma_\pm \equiv \frac{\sigma_1 \pm i\sigma_2}{2} , \tag{2.24}$$

which serve as the energy-raising and -lowering operators for the atom. By combining Eqs. (2.18a) and (2.18b) the corresponding equations for σ_+ and σ_- are easily seen to be

$$\dot{\sigma}_+ = i\omega_0\sigma_+ + \left(\frac{\omega_0 d}{\hbar c}\right)A\sigma_3 \tag{2.25a}$$

and

$$\dot{\sigma}_- = -i\omega_0\sigma_- + \left(\frac{\omega_0 d}{\hbar c}\right)A\sigma_3 . \tag{2.25b}$$

The free evolution of the atom's raising and lowering operators is given by

$$\sigma_\pm(t') = \sigma_\pm(t)e^{\pm i\omega_0(t'-t)} . \tag{2.26}$$

The use of the appropriate combination of these

in Eq. (2.23b) for $\sigma_2(t')$ allows $a_\lambda^s(t)$ to be found explicitly:

$$a_\lambda^s(t) = -i\left(\frac{\omega_0 d}{\hbar c}\right) g_\lambda [\sigma_-(t) I^*(\omega_\lambda - \omega_0;\, t) - \sigma_+(t) I^*(\omega_\lambda + \omega_0;\, t)], \quad (2.27)$$

where we have adopted the abbreviation

$$I(x;\, t) = \lim_{\epsilon \to 0} -i \int_0^t d\tau\, e^{i(x + i\epsilon)\tau} \quad (2.28)$$

for the integrals which remain when the approximation embodied in Eq. (2.26) is used to carry out the integration on the right-hand side of Eq. (2.23b).[21]

The field-mode operator solution, to within the approximation implied by the use of Eq. (2.26) in Eq. (2.23b), is now contained in Eqs. (2.23a) and (2.27). We comment briefly in Sec. IV on the nature of the approximation involved, and ways in which it can be improved. For the present it is most convenient to move directly toward the lowest-order solution to the coupled Eqs. (2.18) and (2.19) which describe the atom. In Sec. II C we obtain these solutions by substituting the field-mode operator solution into the equations for the atomic operators.

C. Spontaneous emission solutions, frequency shifts, and decay rates

Equation (2.25a) for the non-Hermitian operator $\sigma_+(t)$ takes the form

$$\dot{\sigma}_+ = i\omega_0 \sigma_+ + \left(\frac{\omega_0 d}{\hbar c}\right) \sum_\lambda g_\lambda (a_\lambda^\dagger \sigma_3 + \sigma_3 a_\lambda), \quad (2.29)$$

when the vector potential is written in terms of the individual mode operators $a_\lambda(t)$, and $a_\lambda^\dagger(t)$. Furthermore, in order to simplify later elements of the calculations, we have adopted a normal rule of ordering for the field operators.

The question of operator ordering deserves a brief comment. Since the field and the atom constitute separate physical systems at time $t = 0$, the Heisenberg operators for the field commute with the Heisenberg operators for the atom at that time. What is more, because the Heisenberg equation of motion implies a unitary time evolution of the operators, the field and atom operators must commute with each other for all equal times. Thus the operator product $[a_\lambda(t) + a_\lambda^\dagger(t)]\sigma_3(t)$, which appears on the right-hand side of Eqs. (2.25a) and (2.25b), can be written in many different ways:

$$\begin{aligned}(a_\lambda + a_\lambda^\dagger)\sigma_3 &= a_\lambda \sigma_3 + \sigma_3 a_\lambda^\dagger \\ &= \sigma_3 a_\lambda + a_\lambda^\dagger \sigma_3 \\ &= 118 a_\lambda \sigma_3 - 117 \sigma_3 a_\lambda + 2\sigma_3 a_\lambda^\dagger - a_\lambda^\dagger \sigma_3 \\ &= \cdots, \end{aligned} \quad (2.30)$$

if the time argument of the Heisenberg operators is t in every case. Out of this infinity of ways of writing all products of equal-time commuting operators, the so-called normal ordering of field operators is most convenient and will enable us to draw classical analogies most easily in interpreting our results. In this ordering the field operator a_λ is placed to the right of all atomic operators, and the field operator $a_\lambda^\dagger$ is placed to the left of all atomic operators, if the operators involved have equal-time arguments and thus commute. Normal ordering is used explicitly in writing Eq. (2.29), and is the first of the examples on the right-hand side of Eq. (2.30).

Of course, after the field-mode operator is broken up into vacuum and source parts, as in Eq. (2.22), the freedom to reorder operators is lost. Only the full mode operator $a_\lambda(t)$ can be unambiguously identified as a field operator at all times, representing a degree of freedom distinct from the atomic degrees of freedom. The individual parts, $a_\lambda^s(t)$ and $a_\lambda^v(t)$, of the field-mode operator do not necessarily commute separately with the atomic operators at time t or with each other. The question of the meaning to be associated with the various possible types of ordering which might be adopted at the beginning of the calculation in place of normal ordering has been discussed recently.[22]

Finally, the use of Eqs. (2.22), (2.23a), and (2.27) and their Hermitian conjugates in Eq. (2.29), lead to the relatively simple equation

$$\dot{\sigma}_+ = i\omega_0 \sigma_+ + i\left(\frac{\omega_0 d}{\hbar c}\right)^2 \sum_\lambda g_\lambda^2 \{\sigma_+ [-I(\omega_\lambda - \omega_0;\, t) + I^*(\omega_\lambda + \omega_0;\, t)] + \sigma_- [I^*(\omega_\lambda - \omega_0;\, t) - I(\omega_\lambda + \omega_0;\, t)]\} + \left(\frac{\omega_0 d}{\hbar c}\right) : A^v \sigma_3 : \quad (2.31)$$

and an equally simple equation for σ_-. In order to put Eq. (2.31) into final form we have used the operator identities

$$\sigma_3 \sigma_\pm = \pm \sigma_\pm \quad (2.32)$$

and have adopted the convention that the colon brackets $: (\cdots) :$ indicate that the quantity $(\cdots)$ inside the brackets is to be taken in normally ordered form. The "vacuum" field A^v defined by

$$A^v(t) = \sum_\lambda g_\lambda [a_\lambda^v(t) + a_\lambda^{v\dagger}(t)] \quad (2.33)$$

is obviously the free-field solution for the vector potential.

It is at this stage of the calculation that radiation reaction and vacuum fluctuations first appear separately. The final term in Eq. (2.31) comes from the free radiation field, the part independent of the source. In QED it is an operator, representing in this problem the vacuum, and it cannot simply be set equal to zero as in the classical case. The other interaction terms in Eq. (2.31) arise from radiation reaction. As they are proportional to the variables of the atom, they clearly show how the source reacts back on itself, modifying its own evolution.

The sums over field modes remaining in Eqs. (2.31) can be performed in the usual way. In the limit of a very large quantization volume we may make the replacement

$$\frac{1}{\mathcal{V}}\sum_{\lambda} \to \left(\frac{1}{2\pi}\right)^3 \int d^3k_\lambda \sum_{\text{pol}} \tag{2.34}$$

and use the identity

$$\sum_{\text{pol}} (\hat{\epsilon}_\lambda \cdot \hat{a})(\hat{\epsilon}_\lambda \cdot \hat{b}) = \hat{a}\cdot\hat{b} - (\hat{k}\cdot\hat{a})(\hat{k}\cdot\hat{b}) \tag{2.35}$$

to perform the polarization sum and the angular part of the integration immediately.

The final frequency integral is time-dependent because the I functions are time-dependent. However, we show in Appendix A that, after a time t long enough to satisfy $\omega_0 t \gg 1$, the integrals can be evaluated using the familiar relation

$$I(\omega_\lambda - \omega_0;\, t) \to \mathrm{P}\left(\frac{1}{\omega_\lambda - \omega_0}\right) - i\pi\delta(\omega_\lambda - \omega_0), \tag{2.36}$$

where P and δ stand for a Cauchy principal part and Dirac delta function. The simplification of Eq. (2.31) which results from this "long time" limit corresponds physically to the fact that the atom itself is a simple object only when viewed for times longer than ω_0^{-1}. Over shorter intervals it is not even possible to say what excited level the atom is in, much less what its transition processes are like.

The final atomic operator equations take the form

$$\dot{\sigma}_+ = i\omega_0\sigma_+ + \left(i\Delta - \frac{1}{2\tau_0}\right)\sigma_+ - \left(i\Delta + \frac{1}{2\tau_0}\right)\sigma_- + \left(\frac{\omega_0 d}{\hbar c}\right) :A^v\sigma_3: , \tag{2.37a}$$

$$\dot{\sigma}_- = -i\omega_0\sigma_- - \left(i\Delta + \frac{1}{2\tau_0}\right)\sigma_- + \left(i\Delta - \frac{1}{2\tau_0}\right)\sigma_+ + \left(\frac{\omega_0 d}{\hbar c}\right) :A^v\sigma_3: , \tag{2.37b}$$

and

$$\dot{\sigma}_3 = -\frac{1}{\tau_0} - \frac{1}{\tau_0}\sigma_3 - 2\left(\frac{\omega_0 d}{\hbar c}\right) :A^v\sigma_1: . \tag{2.37c}$$

The last of these, the atom's energy operator equation, was obtained from Eq. (2.18c) in the same way that the first two equations above were obtained from Eqs. (2.18a) and (2.18b).

The two constants, Δ and $1/\tau_0$, which appear in Eqs. (2.37) are abbreviations for the final integrals involving principal part and delta functions:

$$\Delta = -\frac{8\pi}{3\hbar c}\left(\frac{\omega_0 d}{2\pi c}\right)^2 \int_0^\infty \left(\frac{\mathrm{P}}{\omega_\lambda - \omega_0} - \frac{\mathrm{P}}{\omega_\lambda + \omega_0}\right)\omega_\lambda\, d\omega_\lambda \tag{2.38a}$$

and

$$\frac{1}{\tau_0} = \frac{4}{3}\left(\frac{\omega_0^2 d^2}{\hbar c^3}\right)\int_0^\infty [\delta(\omega_\lambda - \omega_0) + \delta(\omega_\lambda + \omega_0)]\,\omega_\lambda\, d\omega_\lambda . \tag{2.38b}$$

The integral defining Δ is logarithmically divergent at high frequencies. The upper limit of the integration may be understood to be roughly $\Lambda \approx mc^2/\hbar$, the frequency at which our nonrelativistic calculation becomes unreliable in any event.

The three Eqs. (2.37) are clearly the quantum-electrodynamic Bloch equations for a two-level system. The terms proportional to A^v on the right-hand sides of the equations contribute the quantum fluctuations associated with the quantized field. These fluctuations and the atom-field correlations embodied in operator products like $:A^v\sigma_3:$ can in some cases be ignored. In any event, one could take the expectation of each equation in a coherent state of the field. The normally ordered product $:A^v\sigma_3:$ would factorize in the process, and the result would be the standard semiclassical Bloch equations for a two-level atom in the presence of the external field given by the coherent state expectation of A^v.

Our approach must avoid factorizations and decorrelations. The only "external" field present is the vacuum field, whose relative fluctuations are large. The first useful information contained in Eqs. (2.37) regarding spontaneous emission becomes apparent when their expectation value is taken. Such an expectation is computed using the density matrix

$$\rho = \rho_a |\text{vac}\rangle\langle\text{vac}| , \tag{2.39}$$

where ρ_a is an arbitrary atomic density matrix, and the vacuum projector is the field density matrix appropriate to the no-photon condition. In defining the states of interest we keep in mind that the time $t = 0$ has special significance. It is at $t = 0$ that the atomic and field systems are conceptually distinct in our decay process; so necessarily the states are referred to the operators at $t = 0$.

This means, for example, that $|\text{vac}\rangle$ is defined such that $a_\lambda(0)|\text{vac}\rangle$ vanishes, whereas $a_\lambda(t)|\text{vac}\rangle$ is generally not zero for times $t \neq 0$.

Vacuum expectations of normally ordered field operators at $t=0$ vanish identically, so all of the terms in Eqs. (2.37) involving A^ν drop out when the expectation is computed. The equations for the expectations of σ_+, σ_-, and σ_3 are linear first-order differential equations:

$$\langle \dot{\sigma}_+ \rangle = i(\omega_0 + \Delta)\langle \sigma_+ \rangle - \frac{1}{2\tau_0}\langle \sigma_+ \rangle - \left(i\Delta + \frac{1}{2\tau_0}\right)\langle \sigma_- \rangle , \tag{2.40a}$$

$$\langle \dot{\sigma}_- \rangle = -i(\omega_0 + \Delta)\langle \sigma_- \rangle - \frac{1}{2\tau_0}\langle \sigma_- \rangle - \left(-i\Delta + \frac{1}{2\tau_0}\right)\langle \sigma_+ \rangle , \tag{2.40b}$$

and

$$\langle \dot{\sigma}_3 \rangle = -\frac{1}{\tau_0} - \frac{1}{\tau_0}\langle \sigma_3 \rangle . \tag{2.40c}$$

The counter-rotating terms, those proportional to $\langle \sigma_- \rangle$ in the $\langle \dot{\sigma}_+ \rangle$ equation, and proportional to $\langle \sigma_+ \rangle$ in the $\langle \dot{\sigma}_- \rangle$ equation, introduce only small corrections to the exponentially decaying oscillations of $\langle \sigma_+ \rangle$ and $\langle \sigma_- \rangle$. The solution for $\langle \sigma_3 \rangle$ is probably the oldest result in QED:

$$1 + \langle \sigma_3(t) \rangle = [\, 1 + \langle \sigma_3(0) \rangle \,] e^{-t/\tau_0} . \tag{2.40d}$$

The monotonic exponential decay of excited-state energy was implicit in Einstein's earliest studies.[4]

Our principal results[23] are contained in Eqs. (2.40). Neglecting the small influences of the counter-rotating terms, it is obvious by inspection that $\omega_0 + \Delta$, not ω_0, is the actual dipole oscillation frequency. Similarly, τ_0 is clearly the lifetime associated with the spontaneous transition being treated. We may tentatively call Δ the transition's Lamb shift. Equation (2.38b) shows τ_0^{-1} to be the correct Einstein A coefficient. It is also obvious that these radiative corrections arise entirely from the reaction of the atomic source field back on the atom. The vacuum field A^ν makes no contribution to the final equations. The normal ordering of field operators allows radiative corrections to be interpreted in a very classical way.

D. Line shape

The exponential nature of the decay of the two-level atom's energy toward the ground state, i.e., the decay of $\langle \sigma_3(t) \rangle$ to -1, given in (2.40d), suggests that the shape of the emission line associated with the decay is Lorentzian. That this is correct may be shown by computing the spectral distribution of the radiation[24] by using the atom's dipole-dipole correlation function.

The operator equations (2.37a) and (2.37b) may be simplified slightly in order to construct the desired correlation function as directly as possible. In working out the expectation value equations (2.40), we kept only those contributions of second or lower order in the coupling constant, and then obtained our principal results by neglecting counter-rotating terms. In this case the same end is achieved by simply dropping from the $\dot{\sigma}_-$ equation those terms proportional to σ_+ and $a_\lambda^\dagger(0)$ and those terms proportional to σ_- and $a_\lambda(0)$ in the $\dot{\sigma}_+$ equation. This may be recognized to be a rotating-wave approximation. For definiteness we will occasionally refer to it as the rotating-wave approximation (RWA) at the second level. (The usual rotating-wave approximation, what we might call the RWA at the first level, is made directly in the Hamiltonian. It has *not* been used in this paper. We have commented elsewhere[25] on the difficulties associated with the first level RWA, and the errors it creates in calculations of frequency shifts.)

After we have made the RWA at the second level, Eqs. (2.37a) and (2.37b) become

$$\dot{\sigma}_+ = i\omega_0\sigma_+ + \left(i\Delta - \frac{1}{2\tau_0}\right)\sigma_+ + \frac{\omega_0 d}{\hbar c}\alpha^\dagger(t) \tag{2.41a}$$

and

$$\dot{\sigma}_- = -i\omega_0\sigma_- - \left(i\Delta + \frac{1}{2\tau_0}\right)\sigma_- + \frac{\omega_0 d}{\hbar c}\alpha(t), \tag{2.41b}$$

where $\alpha(t)$ denotes the normally ordered inhomogeneous term in Eq. (2.37b) after the second level RWA has been invoked:

$$\alpha(t) \equiv -\left(\frac{\omega_0 d}{\hbar c}\right)\sum_\lambda g_\lambda \sigma_3(t) a_\lambda(0) e^{-i\omega_\lambda t} . \tag{2.42}$$

By simply letting $t \to t+\tau$ in (2.41a), and then by writing the time derivative with respect to $t+\tau$ simply as a time derivative with respect to τ we find

$$\frac{d}{d\tau}\sigma_+(t+\tau) = i\omega_0\sigma_+(t+\tau) + \left(i\Delta - \frac{1}{2\tau_0}\right)\sigma_+(t+\tau) + \frac{\omega_0 d}{\hbar c}\alpha^\dagger(t+\tau) . \tag{2.43}$$

After multiplying (2.43) from the right with $\sigma_-(t)$ and taking expectation values in a spontaneous-emission state, we find the equation of motion for the dipole-dipole correlation function,

$$\frac{d}{d\tau}\langle \sigma_+(t+\tau)\sigma_-(t) \rangle = i\omega_0\langle \sigma_+(t+\tau)\sigma_-(t) \rangle + i\left(\Delta - \frac{1}{2\tau_0}\right)\langle \sigma_+(t+\tau)\sigma_-(t) \rangle , \tag{2.44}$$

which has the obvious solution

$$\langle\sigma_+(t+\tau)\sigma_-(t)\rangle = \langle\sigma_+(t)\sigma_-(t)\rangle \times \exp\left[i\left(\omega_0+\Delta+\frac{i}{2\tau_0}\right)\tau\right]. \quad (2.45)$$

It is now clear that the Lorentzian lineshape associated with spontaneous emission follows from our radiation reaction approach to the problem. The Fourier transform of relation (2.45) shows that the peak in the emitted spectrum is located at $\omega_0+\Delta$, shifted slightly from the noninteracting atomic resonance frequency ω_0. The half-width at half maximum of the Lorentzian is $1/2\tau_0$ as it should be. The importance of the correlation function in Eq. (2.45) is not that it confirms what the solutions to Eqs. (2.40) suggested, but that the conclusions about lineshape which can be drawn from it are independent of the average values $\langle\sigma_\pm(t)\rangle$, etc. For example, if the atomic part of the spontaneous-emission state shown in (2.39) is chosen to be the pure excited state, the correlation function does not vanish. This is not true of the mean-value solutions to Eqs. (2.40).

E. Remarks

The two-level atom model used in our calculations is of course artificial. Even so, there is strong confirmation of the suggestions made in the Introduction concerning the conceptual simplicity of the Heisenberg picture in identifying radiative corrections. That Δ is to be interpreted as a frequency shift is already obvious in Eqs. (2.40). The Heisenberg equation method simply does not allow questions to be asked about unobservable effects such as energy-level shifts. It is only differences in energy-level shifts, that is, frequency shifts, that can be observed.

One consequence, in the two-level-atom model, is that there are no renormalizations required. That is, the infinite mass shifts whose effect must be subtracted out of every level-shift calculation automatically cancel out of the frequency shift in a two-level atom. We will show in Sec. III, where we treat a realistic many-level atom, that this cancellation is fortuitous, as it should be expected to be. In other words, we have made no attempts at the beginning of the problem to isolate the consequences of the atom's interaction with the radiation field on the atom's *inertial* properties. Thus the reduced electron mass appearing in various formulas must still be the bare mass, and require renormalization. There is no reason in principle why the inertial consequences of radiation reaction could not also be treated by the Heisenberg equation method.

The frequency shift Δ given in Eq. (2.38a) is infinite because the integral diverges logarithmically at the upper limit. This logarithmic divergence is well known in nonrelativistic calculations of radiative frequency shifts.[3] It appears in any nonrelativistic calculation of the Lamb shift. For the purpose of comparing our expression for the two-level atom frequency shift with the same two-level atom shift calculation in some other way, say by taking the difference of first and second level energy shifts computed by time-independent perturbation theory, it is sufficient to compare the integrands of the divergent integrals which occur. It is easy to show that the expression (2.38a) for Δ agrees with the value computed in other ways if the same interaction Hamiltonian (2.15) is used in all computations.[26]

III. MULTILEVEL-ATOM SPONTANEOUS EMISSION

A. Multilevel Hamiltonian

Let us consider a spinless nonrelativistic one-electron atom which undergoes a single spontaneous transition[20] from the first excited atomic state to the ground state with the emission of a photon. The operators whose dynamics best exhibit the characteristics of this spontaneous-emission process are those that initially have matrix elements only in the 2×2 space of the first excited and ground states of the atom. These "principal transition operators" can be written as extensions of the Pauli 2×2 operators, and are defined at $t=0$ by

$$S_3 \equiv \begin{bmatrix} \vdots & \vdots & \vdots \\ \cdots & 0 & 0 \\ \cdots & 0 & \\ \cdots & 0 & \end{bmatrix} \sigma_3, \qquad S_\pm \equiv \begin{bmatrix} \vdots & \vdots & \vdots \\ \cdots & 0 & 0 \\ \cdots & 0 & \\ \cdots & 0 & \end{bmatrix} \sigma_\pm . \quad (3.1a)$$

It is convenient to work in the basis in which the unperturbed atomic Hamiltonian is diagonal, in which case it has the matrix representation

$$H_a = \begin{bmatrix} \ddots & & & \\ & E_\alpha & & \\ & & \ddots & \\ & & & E_2 \\ & & & & E_1 \end{bmatrix} . \quad (3.1b)$$

Here E_1 and E_2 are the energy eigenvalues for the ground and first excited states, respectively, and ω_0, the frequency of the principal transition, is given by $E_2-E_1\equiv\hbar\omega_0$.

The full Hamiltonian, including the radiation-field Hamiltonian H_f, is

$$H=\frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2+V(r)+H_f$$
$$=H_a-\frac{e}{mc}\vec{A}\cdot\vec{p}+\frac{e^2}{2mc^2}\vec{A}\cdot\vec{A}+H_f\,.$$

In the dipole approximation, and with the neglect of the A^2 term (which makes no contribution to atomic frequency shifts[19]), H can be written

$$H=H_a+H_f-\frac{1}{c}\int d^3r'(\vec{A}_+\cdot\vec{J}_T+\vec{J}_T\cdot\vec{A}_-)\,. \tag{3.2}$$

The creation (+) and destruction (−) parts of the vector potential operator are

$$\vec{A}_\mp(\vec{r}',t)=\sum_\lambda\left(\frac{2\pi\hbar c^2}{\omega_\lambda\mathcal{V}}\right)^{1/2}\hat{\epsilon}_\lambda a_\lambda^\mp(t)\,e^{\pm i\vec{k}_\lambda\cdot\vec{r}'}\,, \tag{3.3}$$

and $\vec{J}_T$ is the transverse part of the total current density $\vec{J}=(e/m)\vec{p}\delta^3(\vec{r}')$. In this notation the usual Hermitian conjugate creation and destruction field-mode operators, $a_\lambda^\dagger$ and a_λ, are denoted by a_λ^+ and a_λ^-, respectively.

The basic commutation relations which will determine the dynamics in the Heisenberg picture are

$$[r_j(t),p_l(t)]=i\hbar\delta_{jl} \tag{3.4a}$$

and

$$[a_\lambda^-(t),a_{\lambda'}^+(t)]=\delta_{\lambda\lambda'}\,. \tag{3.4b}$$

For all equal times the field-mode operators must commute with all atomic operators. Unequal-time commutators of all kinds are generally unknown and depend upon the system's interactions.

It is convenient to decompose $\vec{J}$ into plane waves:

$$\vec{J}=\frac{1}{2}\frac{e}{m}\vec{p}\,\frac{1}{\mathcal{V}}\sum_\lambda e^{-i\vec{k}_\lambda\cdot\vec{r}'}\,. \tag{3.5a}$$

If we recall that $\sum_\lambda$ implies a wave vector *and* polarization sum, and use (2.35) to rewrite (3.5a), we find

$$\vec{J}=\frac{1}{2}\frac{e}{m}\frac{1}{\mathcal{V}}\sum_\lambda\hat{k}_\lambda(\hat{k}_\lambda\cdot\vec{p})e^{-i\vec{k}_\lambda\cdot\vec{r}'}$$
$$+\frac{e}{m}\frac{1}{\mathcal{V}}\sum_\lambda\hat{\epsilon}_\lambda(\hat{\epsilon}_\lambda\cdot\vec{p})e^{-i\vec{k}_\lambda\cdot\vec{r}'}\,. \tag{3.5b}$$

The atomic matrix elements of the transverse part of $\vec{J}$ are simply

$$[\vec{J}_T]_{\alpha\beta}=\frac{e}{m}\frac{1}{\mathcal{V}}\sum_\lambda\hat{\epsilon}_\lambda(\hat{\epsilon}_\lambda\cdot\vec{p})_{\alpha\beta}e^{-i\vec{k}_\lambda\cdot\vec{r}'}\,. \tag{3.6}$$

Since only the transverse part of the current appears in subsequent calculations, we will henceforth drop the subscript T on $\vec{J}_T$ for convenience.

If $\vec{J}$ is allowed to have matrix elements only in the 2×2 part of the space, then we recover the results found earlier in Sec. II. Of course $\vec{J}$ actually has matrix elements which couple the 2×2 part of the space with the remainder of the space. Modifications of our two-level-atom results, for example those due to virtual transitions not present in a two-level atom, must arise from these matrix elements. Since a virtual transition consists of the atom undergoing a non-energy-conserving transition to some higher-lying excited level and then returning to the 2×2 space, the transverse current must couple to itself.

To see this coupling it is only necessary to consider the Heisenberg equations of motion [implied by the Hamiltonian given in Eq. (3.2)] for any one of the three principal transition operators S defined in Eq. (3.1a):

$$i\hbar\dot{S}=[S,H_a]-\frac{1}{c}\int d^3r\,\vec{A}_+\cdot[S,\vec{J}]-\frac{1}{c}\int d^3r[S,\vec{J}]\cdot\vec{A}_-\,, \tag{3.7a}$$

and for the creation and destruction parts of the vector potential:

$$\Box\vec{A}_\mp(\vec{r},t)=-\frac{2\pi}{c}\vec{J}(\vec{r},t)$$
$$\mp\frac{i}{\pi c^2}\int d^3r'\,\frac{d\vec{J}(\vec{r}',t)/dt}{|\vec{r}-\vec{r}'|^2}\,. \tag{3.7b}$$

The wave equation (3.7b) is derived explicitly in Appendix B. The two terms on the right-hand side of Eq. (3.7b) arise because the total vector potential has been decomposed into creation and destruction parts: $\vec{A}=\vec{A}_++\vec{A}_-$. Obviously the inhomogeneous parts of the solutions to the wave equations for $\vec{A}_+$ and $\vec{A}_-$ will be proportional to the transverse current. Substitution of these solutions into Eqs. (3.7a) then leads to explicit current-current coupling, and, as an immediate consequence, to the dependence of the atomic dynamics on all of the atom's transitions, not just the principal transition. Because we assume the atomic state vector to have components only in the 2×2 part of the space initially, these other transitions outside this part of the space are considered virtual transitions. In a Schrödinger picture calculation they make the primary contribution to the energy-level shift and they will have a similar effect here.

B. Solution for the field

In order to integrate the wave equation for the (+) and (−) parts of the vector potential we use Fourier transform methods. The result is

$$\vec{A}_\mp(\vec{r},t)=\vec{A}_\mp^{\,v}(\vec{r},t)+\vec{A}_\mp^{\,s}(\vec{r},t),$$

where the retarded source-field, or inhomogeneous, part of the solution is given by

$$\vec{A}^{s}_{\mp}(\vec{r},t)=\frac{2\pi}{\mathcal{V}}\sum_{\lambda}\frac{e^{-i\vec{k}_\lambda\cdot\vec{r}}}{|\vec{k}_\lambda|}\int_0^t dt'\sin[\omega_\lambda(t-t')]\left[\vec{J}_\lambda(t')\pm\frac{i}{\omega_\lambda}\frac{d\vec{J}_\lambda(t')}{dt}\right], \tag{3.8a}$$

where $\omega_\lambda=|\vec{k}_\lambda|c$. To write it in this form we have used Eq. (3.6) and have introduced the abbreviation

$$\vec{J}_\lambda(t)=\frac{e}{m}\hat{\epsilon}_\lambda\hat{\epsilon}_\lambda\cdot\vec{p}(t). \tag{3.8b}$$

$\vec{A}^{v}_{\mp}(\vec{r},t)$ is the free-field, or homogeneous, or "vacuum" solution, and can be written explicitly in terms of the field creation and destruction operators at the initial time:

$$\vec{A}^{v}_{\mp}(\vec{r},t)=\sum_\lambda\left(\frac{2\pi\hbar c^2}{\omega_\lambda\mathcal{V}}\right)^{1/2}\hat{\epsilon}_\lambda a^{\mp}_\lambda(0)e^{\pm i(\vec{k}_\lambda\cdot\vec{r}-\omega_\lambda t)}. \tag{3.9}$$

In order to simplify further the writing of the solution (3.8a), we introduce the Liouville operator L. Its definition,

$$L\hat{O}\equiv[H,\hat{O}], \tag{3.10a}$$

taken together with the Heisenberg equation $i\hbar d\hat{O}/dt=[\hat{O},H]$, implies that the time dependence of any operator $\hat{O}$ can be found formally, but explicitly and exactly, to be

$$\hat{O}(t)=e^{iLt/\hbar}\hat{O}(0). \tag{3.10b}$$

The use of Eq. (3.10b) and its first derivative allows the source-field part of the vector potential to be written compactly as

$$\vec{A}^{s}_{\mp}(\vec{r},t)=\frac{2\pi}{\mathcal{V}}\sum_\lambda\frac{e^{-i\vec{k}_\lambda\cdot\vec{r}}}{(\omega_\lambda/c)}\left[\int_0^t ds\,\sin\omega_\lambda s\,e^{-iLs/\hbar}\left(1\mp\frac{L}{\hbar\omega_\lambda}\right)\right]\vec{J}_\lambda(t), \tag{3.11}$$

following the change of integration variables $s=t-t'$. Note that $\vec{J}_\lambda$ depends only on the final time t. The fact that the current has been integrated is concealed in L.

Our present interest is only in the initial reaction back of the field on the atom. Higher-order corrections due to the atom radiating and having its evolution modified by the radiation field, subsequently reradiating and having its motion remodified, etc., will be neglected. This means that in Eq. (3.11) we can simply replace L by $L_0=[H_a,\ ]$. When L_0 operates on an atomic operator, its action is very straightforward:

$$[H_a,\vec{J}]_{\alpha\beta}=E_{\alpha\beta}\vec{J}_{\alpha\beta}, \tag{3.12}$$

where $E_{\alpha\beta}=E_\alpha-E_\beta$.

We found for the two-level atom (see Appendix A) that the frequency dependence of g_λ^2 in (2.31) was so weak that for t sufficiently greater than ω_0^{-1} we could replace $I(\omega-\omega_0;t)$ by $[P/(\omega-\omega_0)]-i\pi\delta(\omega-\omega_0)$. The same argument also applies to the multilevel atom, showing the existence of a time t after which we can formally replace the square bracket in (3.11) by the time-independent expression

$$\lim_{\epsilon\to0}\frac{1}{\omega_\lambda\pm L_0/\hbar\mp i\epsilon}=\frac{P}{\omega_\lambda\pm L_0/\hbar}\pm i\pi\delta(\omega_\lambda\pm L_0/\hbar). \tag{3.13}$$

Equation (3.12) implies that this time t should be greater than $\hbar/E_{\alpha\beta}$. However, $E_{\alpha\beta}$ varies from matrix element to matrix element. Since we remember that the important matrix elements of $\vec{J}$ are those which couple the 2×2 space with the remainder of the space, we find that the smallest value of $E_{\alpha\beta}/\hbar$ is approximately ω_0, as either α or β always refers to either state 1 or state 2. This means that the source-field solution is valid for $t>1/\omega_0$.

Since the frequency shift is a result of the principal-part term in (3.13), we see that a well-defined shift in ω_0 does not appear naturally until $t>1/\omega_0$. This result is consistent with the energy-time uncertainty relation since it is apparent that we must wait a time long enough for the atomic transition frequency ω_0 to be established in order to be able to discuss the shift in that frequency.

C. Spontaneous emission solutions, frequency shifts, and decay rates

Our approach in solving the atomic operator equations of motion for the multilevel atom is the same as that used in Sec. II C. We substitute, in normal order, our approximate solution for the field found in Sec. III B into Eqs. (3.7a) and take expectation values in a state appropriate to spontaneous emission. The resulting equations of motion are easily solved.

To illustrate our method let us begin by substituting the solution for $\vec{A}_\pm$ obtained in Sec. III B into Eq. (3.7a):

$$i\hbar\dot{S}=[S,H_a]-\frac{1}{c}\frac{2\pi}{v}\sum_{\lambda}\left(\frac{c}{\omega_\lambda}\right)\left\{[S,\vec{J}_{-\lambda}]\cdot\left(\frac{1}{\omega_\lambda+L_0/\hbar-i\epsilon}\vec{J}_\lambda\right)+\left(\frac{1}{\omega_\lambda-L_0/\hbar+i\epsilon}\vec{J}_\lambda\right)\cdot[S\cdot\vec{J}_{-\lambda}]\right\}-\frac{1}{c}:\left(\int d^3r[S,\vec{J}]\cdot\vec{A}^v\right):, \tag{3.14}$$

where $\vec{A}^v$ is defined in (3.9) and the limit $\epsilon\to 0$ is implicitly assumed. It should be clear that the operator in the curly brackets in (3.14) spans the entire atomic space, coupling the S operators to all of the other levels of the atom at all times $t>0$. If in (3.14) we separate the bracketed operator into two parts, one part in the 2×2 space and one part outside the 2×2 space, then it can be clearly seen that part of the coupling is due to 2×2 space S operators. To understand the coupling due to the part of the bracketed operator which lies outside the 2×2 space we may visualize this operator's own equation of motion, which is analogous in form to (3.14). The expectation values of all operators lying outside of the 2×2 space vanish at $t=0$ by assumption. The evolution of these operator expectation values therefore must be due to their coupling with the 2×2 part of the source field, and consequently proportional to e^2. Substitution of such a result into the $\langle S\rangle$ equations of motion leads to a contribution of order e^4 which must be neglected in order to be consistent with our previous lowest order (order e^2) source-field approximations of Sec. III B.

The physical separation of the operator in the curly brackets in (3.14) into a part proportional to an S operator and a part remaining outside the 2×2 space can be formally accomplished by first removing the time dependence of the bracket, using the Liouville operator:

$$\{\text{bracket in (3.14)}\}=e^{iLt/\hbar}\left\{[S,\vec{J}_{-\lambda}]\cdot\left(\frac{1}{\omega_\lambda+L_0/\hbar-i\epsilon}\vec{J}_\lambda\right)+\left(\frac{1}{\omega_\lambda-L_0/\hbar+i\epsilon}\vec{J}_\lambda\right)\cdot[S,\vec{J}_{-\lambda}]\right\}_0 . \tag{3.15a}$$

Here the zero on the bracket on the right-hand side of (3.15a) indicates evaluation at $t=0$. The α,β matrix element of this bracket is simply

$$\{\ \}_{0\,\alpha\beta}=\sum_{\gamma}^{\infty}\left\{[S,\vec{J}_{-\lambda}]_{\alpha\gamma}\cdot\left(\frac{1}{\omega_\lambda+E_{\gamma\beta}/\hbar-i\epsilon}(\vec{J}_\lambda)_{\gamma\beta}\right)+\left(\frac{1}{\omega_\lambda-E_{\alpha\gamma}/\hbar+i\epsilon}(\vec{J}_\lambda)_{\alpha\gamma}\right)\cdot[S,\vec{J}_{-\lambda}]_{\gamma\beta}\right\}_0 , \tag{3.15b}$$

where the γ summation is over all atomic states. By restricting both α and β to be in the 2×2 space we find the important 2×2 part of this operator. After substituting the 2×2 operator part, in the form of S operators, into (3.14) and taking vacuum expectation values, we find the self-consistent set of equations

$$\langle\dot{S}_3\rangle=-\frac{1}{\tau_0}(\langle S_3\rangle+1) \tag{3.16a}$$

and

$$\langle\dot{S}_\pm\rangle=\left[\pm i(\omega_0+\Delta_2-\Delta_1)-\frac{1}{2\tau_0}\right]\langle S_\pm\rangle+\left(\mp i\Delta-\frac{1}{2\tau_0}\right)\langle S_\mp\rangle, \tag{3.16b}$$

where $1/\tau_0$ is the Einstein spontaneous decay coefficient defined in (2.38b). Furthermore,

$$\hbar\Delta_n=-\frac{\alpha\hbar^2}{(2\pi m)^2}\sum_{\text{pol}}\int\frac{d^3k}{k}\sum_{\beta}^{\infty}\frac{|\hat{\epsilon}_\lambda\cdot\vec{p}_{n\beta}|^2}{\hbar\omega-E_{n\beta}} \tag{3.16c}$$

is exactly the same as the usual time-independent perturbation theory shift in the nth energy level,[8] and Δ is the two-level-atom frequency shift defined in (2.38a). It is interesting to notice that the two-level-atom shift appears in this calculation only in the counter-rotating term in (3.16b), in contrast to its appearance in Eqs. (2.40a) and (2.40b). The integration symbol $\int$ implies a Cauchy principal value.

The shift Δ needs no renormalization since the linearly divergent parts of the integral identically cancel. The shift $\Delta_2-\Delta_1$, however, must be renormalized. This can be accomplished easily by mass counterterm methods[19] where the unrenormalized energy eigenvalues E_n contain a term,

$$\left\langle n\left|\frac{p^2}{2m}\frac{\delta m}{m}\right|n\right\rangle ,$$

which identically cancel the linearly divergent expression in Δ_n when $\delta m=\frac{4}{3}(\alpha\hbar/\pi c)\int dk$. The renormalization scheme for the frequency shift in the Heisenberg picture is therefore precisely the same as that used in ordinary perturbation theory.

If we consider the time evolution of the system, we find that the expectation of the atomic energy operator $\langle S_3\rangle$ decays exponentially at a rate equal to the Einstein decay coefficient. The expectations of the dipole moment operators $\langle S_+\rangle$ and $\langle S_-\rangle$ de-

cay at a rate equal to one-half the Einstein decay coefficient and oscillate at a frequency equal to the shifted transition frequency. These results are known for the two-level-atom spontaneous-emission problem.[23b] The significant effect of including the higher-lying energy levels has been to obtain the complete nonrelativistic multilevel-atom frequency shift $\Delta_2 - \Delta_1$.

We can recover the two-level-atom results of Sec. II by simply restricting our Hilbert space to the essential states 1 and 2. This means that $S_j \to \sigma_j$ and $\sum_\beta$ in (3.16c) becomes a sum only over states 1 and 2. Equations (3.16a) and (3.16b) and $\Delta_2 - \Delta_1$ reduce to the two-level-atom equations of motion (2.40) and the two-level frequency shift Δ, respectively.

D. Generalization to include spontaneous emission from higher lying levels

Our approach to the multilevel atom can be generalized to treat spontaneous emission from energy levels higher than the second. If we restrict our attention to the m lowest-lying energy levels, then we can generalize the S operators to span that restricted $m \times m$ space. The field operator which we calculated in Sec. III B is still valid and can be substituted into these new S equations of motion. Taking expectation values and neglecting the operator parts which represent the levels above the restricted space, we obtain the linear first-order coupled differential equations

$$i\hbar\langle\dot{S}_{ij}\rangle = -E_{ij}\langle S_{ij}\rangle - \frac{2\pi}{\mathcal{V}}\sum_\lambda \frac{1}{\omega_\lambda}\left\{\sum_n^m\sum_l^\infty\left[\langle S_{in}\rangle\frac{(\vec{J}_\lambda)_{jl}\cdot(\vec{J}_\lambda)_{ln}}{\omega_\lambda - E_{nl}/\hbar - i\epsilon} - \langle S_{nj}\rangle\frac{(\vec{J}_{-\lambda})_{nl}\cdot(\vec{J}_\lambda)_{li}}{\omega_\lambda - E_{nl}/\hbar + i\epsilon}\right]\right.$$
$$\left. + \sum_n^m\sum_l^m\left[\langle S_{nl}\rangle(\vec{J}_{-\lambda})_{ni}\cdot(\vec{J}_\lambda)_{jl}\left(\frac{1}{\omega_\lambda - E_{ni}/\hbar + i\epsilon} - \frac{1}{\omega_\lambda - E_{lj}/\hbar - i\epsilon}\right)\right]\right\}, \tag{3.17}$$

where the two indices on S refer to the location of the "matrix element" operator at $t=0$. For example, in this notation the 2×2 operators in Eq. (3.1a) are written

$$S_3 \to S_{22} - S_{11}, \tag{3.18a}$$

$$S_+ \to S_{21}, \tag{3.18b}$$

and

$$S_- \to S_{12}. \tag{3.18c}$$

In order to visualize easily the physics involved we can restrict our attention to the diagonal operators S_{ii}. If we simply neglect coupling to off-diagonal operators as being small, then we find from Eq. (3.17) that

$$i\hbar\langle\dot{S}_{ii}\rangle = -\frac{2\pi}{\mathcal{V}}\sum_\lambda\frac{1}{\omega_\lambda}(2\pi i)\left[\sum_l^\infty(\vec{J}_{-\lambda})_{il}\cdot(\vec{J}_\lambda)_{li}\delta(\omega_\lambda - E_{il}/\hbar)\langle S_{ii}\rangle - \sum_l^m(\vec{J}_{-\lambda})_{li}\cdot(\vec{J}_\lambda)_{il}\delta(\omega_\lambda - E_{li}/\hbar)\langle S_{ll}\rangle\right]. \tag{3.19}$$

By defining the Einstein decay coefficient A_{ij} for a transition from state i to state j as

$$A_{ij} \equiv \frac{(2\pi)^2}{\hbar\mathcal{V}}\sum_\lambda\frac{1}{\omega_\lambda}(\vec{J}_{-\lambda})_{ij}\cdot(\vec{J}_\lambda)_{ji}\delta(\omega_\lambda - E_{ij}/\hbar), \tag{3.20a}$$

then we find for Eq. (3.19)

$$\langle\dot{S}_{ii}\rangle = -\sum_l^m A_{il}\langle S_{ii}\rangle + \sum_l^m A_{li}\langle S_{ll}\rangle. \tag{3.20b}$$

This rate equation expresses the fact that the occupation of the state i is depleted by the spontaneous decay into all the lower lying levels and is increased by the spontaneous decay into it from all the higher lying levels. This equation was first derived by Einstein[4] using energy balance arguments and later by Landau and Bloch[5] using Schrödinger-picture quantum electrodynamics.

In addition, Eq. (3.20b) implies the conservation of probability since a solution to the equation

$$\sum_i^m\langle\dot{S}_{ii}\rangle = 0 \tag{3.21a}$$

is

$$\sum_i^m\langle S_{ii}\rangle = 1. \tag{3.21b}$$

E. Recovery of the Lorentz force equation

The equation of motion (3.17) embodies (to order e^2) all our knowledge of single-atom spontaneous emission. It relates each single atomic spontaneous transition to all the other spontaneous transitions. It is interesting to show the relation between this quantum transition description of spontaneous emission and that of Lorentz which

we discussed in the Introduction.

To do this we simply generate the equations of motion for $\langle \vec{r} \rangle$ and $\langle \vec{p} \rangle$ from (3.17) by using the obvious identities

$$\langle \vec{r} \rangle \equiv \sum_{i,j}^{\infty} \langle S_{ij} \rangle \vec{r}_{ij} \tag{3.22a}$$

and

$$\langle \vec{p} \rangle \equiv \sum_{i,j}^{\infty} \langle S_{ij} \rangle \vec{p}_{ij} , \tag{3.22b}$$

where $\vec{r}_{ij}$ and $\vec{p}_{ij}$ are the i,j atomic matrix elements of $\vec{r}$ and $\vec{p}$, respectively. The sums bounded by m in Eq. (3.17) are here extended to infinity since we are now considering the entire atom.

The equation of motion for $\langle \vec{p} \rangle$ upon substitution of (3.17) into (3.22b) becomes

$$i\hbar \left\langle \frac{d\vec{p}}{dt} \right\rangle = -\sum_{i,j}^{\infty} E_{ij} \vec{p}_{ij} \langle S_{ij} \rangle - \frac{e^2}{m^2} \frac{2\pi}{\mathcal{V}} \sum_{\lambda} \frac{1}{\omega_\lambda} \sum_{n,l,i,j}^{\infty} \left[\langle S_{in} \rangle \frac{\vec{p}_{ij} (\hat{\epsilon} \cdot \vec{p}_{jl})(\hat{\epsilon} \cdot \vec{p}_{ln})}{\omega_\lambda - E_{nl}/\hbar - i\epsilon} - \langle S_{nj} \rangle \frac{\vec{p}_{ij} (\hat{\epsilon} \cdot \vec{p}_{ni})(\hat{\epsilon} \cdot \vec{p}_{li})}{\omega_\lambda - E_{nl}/\hbar + i\epsilon} \right.$$
$$\left. + \langle S_{nl} \rangle \frac{\vec{p}_{ij} (\hat{\epsilon} \cdot \vec{p}_{ni})(\hat{\epsilon} \cdot \vec{p}_{jl})}{\omega_\lambda - E_{ni}/\hbar + i\epsilon} - \langle S_{nl} \rangle \frac{\vec{p}_{ij} (\hat{\epsilon} \cdot \vec{p}_{ni})(\hat{\epsilon} \cdot \vec{p}_{jl})}{\omega_\lambda - E_{ij}/\hbar - i\epsilon} \right] , \tag{3.23}$$

where we have explicitly substituted for $\vec{J}_\lambda$ from (3.8b). In the first term in the square bracket in (3.23) the index j appears only on $\vec{p}_{ij}$ and $\hat{\epsilon} \cdot \vec{p}_{jl}$. Since the sum over j is complete we can simply write

$$\sum_{j}^{\infty} \vec{p}_{ij} (\hat{\epsilon} \cdot \vec{p}_{jl}) = (\vec{p} \hat{\epsilon} \cdot \vec{p})_{il} .$$

Each separate term in the square bracket in (3.23) has an index of this type which can be summed over and removed. By interchanging dummy variables in the second and fourth terms we can rewrite (3.23) as

$$i\hbar \left\langle \frac{d\vec{p}}{dt} \right\rangle = -\sum_{i,j}^{\infty} E_{ij} \vec{p}_{ij} \langle S_{ij} \rangle - \frac{e^2}{m^2} \frac{2\pi}{\mathcal{V}} \sum_{\lambda} \frac{1}{\omega_\lambda} \sum_{n,l,i}^{\infty} \left[\langle S_{in} \rangle \frac{(\vec{p} \hat{\epsilon} \cdot \vec{p})_{il} (\hat{\epsilon} \cdot \vec{p}_{ln})}{\omega_\lambda - E_{nl}/\hbar - i\epsilon} - \langle S_{nl} \rangle \frac{(\hat{\epsilon} \cdot \vec{p}_{ni})(\hat{\epsilon} \cdot \vec{p}\, \vec{p})_{il}}{\omega_\lambda - E_{ni}/\hbar + i\epsilon} \right.$$
$$\left. + \langle S_{nl} \rangle \frac{(\hat{\epsilon} \cdot \vec{p}_{ni})(\vec{p} \hat{\epsilon} \cdot \vec{p})_{il}}{\omega_\lambda - E_{ni}/\hbar + i\epsilon} - \langle S_{in} \rangle \frac{(\hat{\epsilon} \cdot \vec{p}\, \vec{p})_{il} (\hat{\epsilon} \cdot \vec{p}_{ln})}{\omega_\lambda - E_{nl}/\hbar - i\epsilon} \right] , \tag{3.24}$$

where the entire term in square brackets trivially vanishes. Since

$$-\sum_{i,j}^{\infty} E_{ij} \vec{p}_{ij} \langle S_{ij} \rangle = \sum_{ij}^{\infty} [\vec{p}, H_a]_{ij} \langle S_{ij} \rangle$$
$$= -i\hbar \langle \vec{\nabla} V \rangle , \tag{3.25}$$

where $H_a = p^2/2m + V$, we find for Eq. (3.24) the usual Ehrenfest relation[27]:

$$\left\langle \frac{d\vec{p}}{dt} \right\rangle = -\langle \vec{\nabla} V \rangle . \tag{3.26}$$

The calculation of the analogous equation for $\langle d\vec{r}/dt \rangle$ will be derived in Appendix C. The result is simply

$$m \left\langle \frac{d\vec{r}}{dt} \right\rangle = \langle \vec{p} \rangle - \frac{e^2}{m} \left(\frac{2}{3\pi c^3} \right) \left[2\Lambda \langle \vec{p} \rangle - \pi \left\langle \frac{d\vec{p}}{dt} \right\rangle \right] , \tag{3.27a}$$

where $\Lambda \approx mc^2/\hbar$ is the nonrelativistic cutoff. If we define m_{e-e} as $m_{e-e} \equiv e^2 \Lambda / \pi c^3$, then (3.27a) becomes

$$m \left\langle \frac{d\vec{r}}{dt} \right\rangle = \langle \vec{p} \rangle \left(1 - \frac{4}{3} \frac{m_{e-e}}{m} \right) + \frac{2e^2}{3mc^3} \left\langle \frac{d\vec{p}}{dt} \right\rangle . \tag{3.27b}$$

We can compare our results with those of Lorentz by simply considering spontaneous emission from an elastically bound electron, $V \equiv \frac{1}{2} m \omega_0^2 r^2$. Upon substitution for V into Eq. (3.26) we find

$$\left\langle \frac{d\vec{p}}{dt} \right\rangle = -m\omega_0^2 \langle \vec{r} \rangle . \tag{3.28}$$

If we differentiate Eq. (3.27b) and eliminate derivatives of $\langle \vec{p} \rangle$ by using (3.28), then we obtain an equation for $\langle d^2\vec{r}/dt^2 \rangle$:

$$\left\langle \frac{d^2\vec{r}}{dt^2} \right\rangle + \frac{1}{\tau_0} \left\langle \frac{d\vec{r}}{dt} \right\rangle + (\omega_0')^2 \langle \vec{r} \rangle = 0 , \tag{3.29}$$

where ω_0' and τ_0 are defined by Eqs. (1.8a) and (1.8b), respectively. Equation (3.29) is formally identical with the classical Eq. (1.7). We should point out that in this same context, it appears to have been Landau[5] who first showed the connection between equations like (3.17) and the equations of classical physics.

IV. COMMUTATION RELATIONS

A. Approximate field operator solution

The source-field approach to atomic quantum electrodynamics can be consistent only if the

canonical commutation relations are preserved in time. It is not obvious that this is the case after the field-mode operator equations have been integrated approximately (see Bullough, Ref. 16, for comments on this point). In this section we demonstrate that the atomic and field operator commutation relations remain consistent with their canonical values assumed at $t=0$. Our arguments will be based on a consistency condition since a direct calculation of the commutation relations would involve our previous knowledge of non-equal-time commutators, which are unknown in general. We therefore demand that equal-time atom-field commutators vanish, and then from this condition we show that the separate atom-atom and field-field equal-time commutation relations are preserved.

For simplicity we will consider the two-level-atom model of Sec. II. For easy reference we will repeat the Heisenberg equations of motion:

$$\dot{\sigma}_1 = -\omega_0\sigma_2 + 2\left(\frac{\omega_0 d}{\hbar c}\right)\sum_\lambda g_\lambda (a_\lambda + a_\lambda^\dagger)\sigma_3 , \tag{4.1a}$$

$$\dot{\sigma}_2 = \omega_0\sigma_1 , \tag{4.1b}$$

$$\dot{\sigma}_3 = -2\left(\frac{\omega_0 d}{\hbar c}\right)\sum_\lambda g_\lambda (a_\lambda + a_\lambda^\dagger)\sigma_1 , \tag{4.1c}$$

and

$$\dot{a}_\lambda = -i\omega_\lambda a_\lambda - i\left(\frac{\omega_0 d}{\hbar c}\right) g_\lambda \sigma_2 . \tag{4.1d}$$

At this point one can easily verify the validity of the equal-time canonical commutation relations for all time. However, it is the commutation relations which follow from our approximated equations of motion which are in doubt, not those which follow from the exact equations.

Our use in Secs. II and III of the source-field method was restricted to making Born and Markov approximations to Eq. (4.1d). We are now interested in commutators valid to second order in the coupling strength, so we must recalculate the source-field solution to that order. We begin by writing the exact solution to (4.1d) as

$$a_\lambda(t) = a_\lambda(0)e^{-i\omega_\lambda t} - i\left(\frac{\omega_0 d}{\hbar c}\right) g_\lambda \int_0^t dt' \sigma_2(t') e^{-i(\omega_\lambda - i\epsilon)(t-t')} . \tag{4.2}$$

With the aid of the Liouville operator $\sigma_2(t')$ can be written as

$$\sigma_2(t') = e^{-i(L/\hbar)(t-t')}\sigma_2(t) . \tag{4.3}$$

Upon substituting (4.3) into (4.3) and considering t sufficiently large (see Appendix A), we find

$$a_\lambda(t) = a_\lambda(0)e^{-i\omega_\lambda t} - \left(\frac{\omega_0 d}{\hbar c}\right) g_\lambda \frac{1}{\omega_\lambda + L/\hbar - i\epsilon}\sigma_2(t) , \tag{4.4}$$

where $1/(\omega_\lambda + L/\hbar - i\epsilon)$ means $P/(\omega_\lambda + L/\hbar) + i\pi\delta(\omega_\lambda + L/\hbar)$ in the infinite-volume limit.

Since (4.4) only has real meaning when integrated over all modes, in the construction of the field operator A, for example, we must specify the class of field operators under consideration. In particular we would like to show that the relevant field operators which appear in the atomic equations of motion obey the canonical commutation relations. We will therefore restrict our attention to the class of near-zone field operators in which $\vec{A}_\pm(0,t)$ is included. All of these operators may be derived from

$$\vec{F}(\vec{r},t) \equiv \sum_\lambda f(\omega_\lambda)\hat{\epsilon}_\lambda g_\lambda a_\lambda(t) e^{-i\vec{k}_\lambda\cdot\vec{r}} , \tag{4.5}$$

where $f(\omega_\lambda)$ is any reasonably well-behaved slowly varying function of ω_λ which approaches zero sufficiently rapidly for $\omega_\lambda > \Lambda$. Λ is the frequency above which the nonrelativistic theory becomes invalid. Our interest in the atom's own source field, or self-reaction field, restricts $\vec{r}$ to be in the near zone of the radiating atomic distribution. Thus $\exp(-i\vec{k}_\lambda\cdot\vec{r})$ can be written $1 - i\vec{k}_\lambda\cdot\vec{r} - \frac{1}{2}(\vec{k}_\lambda\cdot\vec{r})^2$ with sufficient accuracy in the limit $\vec{r}\to 0$.

We now expand $a_\lambda(t)$ in Eq. (4.4) to second order in the coupling constant. Our method consists of expanding the Liouville expression $1/(\omega_\lambda + L/\hbar - i\epsilon)$ in (4.4) in powers of L_i. The operator expansion is

$$\frac{1}{\omega_\lambda + L/\hbar - i\epsilon} = \frac{1}{\omega_\lambda + L_0/\hbar - i\epsilon} - \frac{1}{\omega_\lambda + L_0/\hbar - i\epsilon}\,\frac{L_i}{\hbar}\,\frac{1}{\omega_\lambda + L_0/\hbar - i\epsilon} + \cdots . \tag{4.6}$$

After substituting (4.6) into (4.4) and performing the indicated operations on $\sigma_2(t)$, the approximate field-mode operator solution, to second order in the coupling constant, becomes

$$a_\lambda(t) = a_\lambda(0)e^{-i\omega_\lambda t} - \frac{h_\lambda}{\omega_\lambda + \omega_0}\left[\frac{\omega_\lambda\sigma_2(t) + i\omega_0\sigma_1(t)}{\omega_\lambda - \omega_0 - i\epsilon}\right] + \frac{2h_\lambda\omega_0}{(\omega_\lambda + \omega_0)(\omega_\lambda - \omega_0 - i\epsilon)}\sum_{\lambda'} h_{\lambda'}\left[\frac{a_{\lambda'}^\dagger(0)e^{i\omega_{\lambda'}t}\sigma_3(t)}{\omega_\lambda + \omega_{\lambda'} - i\epsilon} + \frac{\sigma_3(t)a_{\lambda'}(0)e^{-i\omega_{\lambda'}t}}{\omega_\lambda - \omega_{\lambda'} - i\epsilon}\right] , \tag{4.7}$$

where $h_\lambda \equiv (\omega_0 d/\hbar c) g_\lambda$. In order to be consistent, we have replaced $a_\lambda(t)$ and $a_\lambda^\dagger(t)$ in the second-order term of (4.7) by their free-field solutions $a_\lambda(0)e^{-i\omega_\lambda t}$ and $a_\lambda^\dagger(0)^{i\omega_\lambda t}$, respectively.

If we had computed the solution for $a_\lambda(t)$ to third order in the coupling constant and substituted this solution into (4.1a) and (4.1c), then the radiative corrections could have been calculated to fourth order. This process of calculating higher order radiative corrections can in principle be done to an arbitrarily high order.

B. Condition on field-mode solution

Any field operator should commute with any atomic operator at any equal time t since the field and the atom bring different degrees of freedom to the interaction. This is obviously true at $t=0$. Such commutation will generally not be true if the field and atom operators are evaluated at *different* times t and t'. We shall demand that the equal-time atom-field commutation relations for $\omega_0 t \gg 1$ be satisfied by our approximate field-mode operator (4.7) and the atomic operators. This imposes a condition on $a_\lambda^v(t) = a_\lambda(0)e^{-i\omega_\lambda t}$, the free-field or vacuum part of $a_\lambda(t)$, which we now derive.

Let us consider the equal-time commutation relation

$$[\sigma_1(t), a_\lambda(t)] = 0 . \tag{4.8}$$

Upon substitution of (4.7) into (4.8) we obtain the condition (for $\omega_0 t \gg 1$)

$$[\sigma_1(t), a_\lambda(0)]e^{-i\omega_\lambda t} = +\frac{i2h_\lambda\omega_\lambda\sigma_3(t)}{(\omega_\lambda+\omega_0)(\omega_\lambda-\omega_0-i\epsilon)} + \frac{i4h_\lambda\omega_0}{(\omega_\lambda+\omega_0)(\omega_\lambda-\omega_0-i\epsilon)}\sum_{\lambda'} h_{\lambda'}\left[\frac{a_{\lambda'}^\dagger(0)e^{i\omega_{\lambda'}t}\sigma_2(t)}{\omega_\lambda+\omega_{\lambda'}-i\epsilon} + \frac{\sigma_2(t)a_{\lambda'}(0)e^{-i\omega_{\lambda'}t}}{\omega_\lambda-\omega_{\lambda'}-i\epsilon}\right] , \tag{4.9}$$

where we have used the equal-time properties of the Pauli matrices. Since it will be shown that the vanishing of the equal-time atom-field commutators, such as (4.8), leads to the preservation of the equal-time properties of the Pauli matrices, it is consistent to have used these properties in obtaining (4.9). Upon considering the equal-time commutation relations

$$[\sigma_2(t), a_\lambda(t)] = 0 \tag{4.10a}$$

and

$$[\sigma_3(t), a_\lambda(t)] = 0 , \tag{4.10b}$$

we obtain (for $\omega_0 t \gg 1$) conditions similar to (4.9):

$$[\sigma_2(t), a_\lambda(0)]e^{-i\omega_\lambda t} = \frac{2h_\lambda\omega_0\sigma_3(t)}{(\omega_\lambda+\omega_0)(\omega_\lambda-\omega_0-i\epsilon)} - \frac{i4h_\lambda\omega_0}{(\omega_\lambda+\omega_0)(\omega_\lambda-\omega_0-i\epsilon)}\sum_{\lambda'} h_{\lambda'}\left[\frac{a_{\lambda'}^\dagger(0)e^{i\omega_{\lambda'}t}\sigma_1(t)}{\omega_\lambda+\omega_{\lambda'}-i\epsilon} + \frac{\sigma_1(t)a_{\lambda'}(0)e^{-i\omega_{\lambda'}t}}{\omega_\lambda-\omega_{\lambda'}-i\epsilon}\right] \tag{4.11a}$$

and

$$[\sigma_3(t), a_\lambda(0)]e^{-i\omega_\lambda t} = -\frac{2h_\lambda[i\omega_\lambda\sigma_1(t)+\omega_0\sigma_2(t)]}{(\omega_\lambda+\omega_0)(\omega_\lambda-\omega_0-i\epsilon)} . \tag{4.11b}$$

In order to demonstrate the preservation of the equal-time properties of the Pauli matrices we will only need to use (4.8) and (4.10) coupled with the atomic equations of motion [(4.11a)–(4.11c)]. However, to show the preservation of the equal-time field operators commutation relations we will make explicit use of the conditions (4.9) and (4.11) coupled with (4.7). For the atomic operators we still retain the atomic equations of motion (4.1a)–(4.1c) and can therefore use them in order to demonstrate the atomic operator's commutator properties. However, for $\omega_0 t \gg 1$, the field operator has been replaced by the approximate operator solution (4.7). This means we must use that explicit operator solution in order to verify the field operators' commutation relations.

C. Atomic operator's commutator properties

The simplest way to show that the atomic operator properties are preserved in the face of our approximate integration (4.7) of the field equation is to use the atomic operator equations of motion. We can construct from these equations the equations of motion for the canonical commutators. The first derivative of the commutator $[\sigma_1, \sigma_2]$ is $[\dot\sigma_1, \sigma_2] + [\sigma_1, \dot\sigma_2]$. Upon substitution from (4.1a) and (4.1b) we obtain

$$\frac{d}{dt}[\sigma_1, \sigma_2] = B[\sigma_2, \sigma_3] , \tag{4.12}$$

where

$$B \equiv -2\left(\frac{\omega_0 d}{\hbar c}\right)\sum_\lambda g_\lambda (a_\lambda^\dagger + a_\lambda)\,. \tag{4.13}$$

Since B is proportional to $A(0, t)$ which commutes with all the atomic operators at time t, B acts like a c number in (4.12). By repeating this procedure for the commutators $[\sigma_2, \sigma_3]$ and $[\sigma_3, \sigma_1]$ we obtain the additional equations

$$\frac{d}{dt}[\sigma_2, \sigma_3] = -\omega_0[\sigma_3, \sigma_1] - B[\sigma_1, \sigma_2] \tag{4.14a}$$

and

$$\frac{d}{dt}[\sigma_3, \sigma_1] = \omega_0[\sigma_2, \sigma_3]. \tag{4.14b}$$

In (4.12) and (4.14), if we make the identification

$$[\sigma_j, \sigma_k]/2i \to \sigma_l \epsilon_{jkl}\,, \tag{4.15}$$

where ϵ_{jkl} is the Levi-Civita tensor, then we obtain the atomic equations of motion (4.1a)–(4.1c). Since these sets of equations, (4.1a)–(4.1c) along with (4.12) and (4.14), can be made equivalent, and initial-time operator quantities set equal, we recognize that (4.15) is identically satisfied for all subsequent times.

Let us now consider the anticommutation relations of the Pauli matrices. The derivatives of these anticommutators vanish since they are all constant. A Taylor series expansion of the anticommutators about $t = 0$ would therefore be expected to have all the initial-time derivatives vanishing. This is not difficult to verify by explicit calculation. Thus the atomic commutators and anticommutators are preserved in time as a simple consequence of the vanishing of the atom-field equal-time commutators.

D. Field operator's commutation relations

In order to complete our consistency argument we must show the preservation of our field-mode operators' commutation relations:

$$[a_\lambda(t), a_{\lambda'}^\dagger(t)] = \delta_{\lambda\lambda'} \tag{4.16}$$

and

$$[a_\lambda(t), a_{\lambda'}(t)] = 0. \tag{4.17}$$

Our approach will be to substitute the expression (4.7) for $a_\lambda(t)$ and its Hermitian conjugate into (4.16) and (4.17) and to calculate, to second order in the coupling constant, the commutators. The mode-operator commutation relations will not be satisfied trivially. In the interesting infinite-volume limit they become distributions and must be treated as such. Thus we will deal with appropriately smoothed functions, like $\vec{\mathbf{F}}(\vec{\mathbf{r}}, t)$ given in Eq. (4.5), where the smoothing function, $f(\omega_\lambda)$ in the case of Eq. (4.5), obeys relatively weak restrictions. Our method is therefore restricted to the class of near-zone fields discussed earlier, and thus our results will not be the most general. For reaction-field theory, however, they are entirely adequate.

Let us begin by considering the commutation relation (4.16). Upon substitution of (4.7) and its Hermitian conjugate into (4.16a) we find, to second order in the coupling constant,

$$[a_\lambda(t), a_{\lambda'}^\dagger(t)] = \delta_{\lambda\lambda'} + 2h_\lambda h_{\lambda'}\omega_0\sigma_3(t)\left[\frac{1}{(\omega_\lambda' + \omega_0)(\omega_\lambda' - \omega_0 + i\epsilon)(\omega_\lambda' - \omega_\lambda + i\epsilon)} + \frac{1}{(\omega_\lambda + \omega_0)(\omega_\lambda - \omega_0 - i\epsilon)(\omega_\lambda - \omega_\lambda' - i\epsilon)}\right.$$
$$\left. + \frac{\omega_\lambda' + \omega_\lambda}{(\omega_\lambda' + \omega_0)(\omega_\lambda + \omega_0)(\omega_\lambda' - \omega_0 + i\epsilon)(\omega_\lambda - \omega_0 - i\epsilon)}\right]\,. \tag{4.18}$$

In deriving (4.18) we have made explicit use of (4.9), (4.11), and the usual equal-time properties of the Pauli matrices. The second-order correction to $\delta_{\lambda\lambda'}$ in (4.18) does not obviously vanish.

In order to demonstrate the preservation of the commutation relations we will consider the commutator of two different arbitrary near-zone field operators:

$$[\vec{\mathbf{F}}_1(\vec{\mathbf{r}}, t), \vec{\mathbf{F}}_2(\vec{\mathbf{r}}', t)] = \sum_\lambda \sum_{\lambda'} \hat{\epsilon}_\lambda \hat{\epsilon}_{\lambda'} g_\lambda g_{\lambda'} f_1(\omega_\lambda) f_2(\omega_{\lambda'}) e^{-i\vec{k}_\lambda \cdot \vec{r}} e^{i\vec{k}_{\lambda'} \cdot \vec{r}'} [a_\lambda(t), a_{\lambda'}^\dagger(t)]\,, \tag{4.19}$$

where $f_1(\omega_\lambda)$ and $f_2(\omega_\lambda)$ are different smoothing functions, but satisfy the criteria discussed in IV A. After going to the infinite-volume limit and defining the quantities

$$\vec{\mathrm{h}}(\omega, \vec{\mathbf{r}}, \vec{\mathbf{d}}) \equiv \int d\Omega \sum_{\text{pol}} \hat{\epsilon}_\lambda (\hat{\epsilon}_\lambda \cdot \hat{d}) e^{i\vec{k} \cdot \vec{r}}\,, \tag{4.20a}$$

$$\vec{\mathbf{F}}(\omega) \equiv \omega f_1(\omega) \vec{\mathrm{h}}(\omega, \vec{\mathbf{r}}, \vec{\mathbf{d}})\,, \tag{4.20b}$$

and

$$\vec{G}(\omega) \equiv \omega f_2(\omega)\vec{h}(\omega, -\vec{r}', \vec{d}), \tag{4.20c}$$

we can write the second-order term in (4.18) as

$$\int_0^\Lambda d\omega\, \vec{F}(\omega) \int_0^\Lambda d\omega' \vec{G}(\omega')[\quad], \tag{4.21}$$

where the square brackets in (4.21) are the same as the square brackets in (4.18). We have neglected the constants and the $\sigma_3(t)$ which appear in front of the second-order term (4.21) since this term will be shown to vanish.

If we replace $1/(x \pm i\epsilon)$ by $P/x \mp i\pi\delta(x)$, (4.21) can be rewritten as

$$\int_0^\Lambda d\omega\, \vec{F}(\omega) \int_0^\Lambda d\omega' \vec{G}(\omega')\left[\frac{(\omega+\omega')}{(\omega'+\omega_0)(\omega+\omega_0)}\frac{1}{(\omega'-\omega_0)}\frac{1}{(\omega-\omega_0)} - \frac{1}{(\omega'+\omega_0)}\frac{1}{(\omega'-\omega_0)}\frac{1}{(\omega-\omega')} - \frac{1}{(\omega+\omega_0)}\frac{1}{(\omega-\omega_0)}\frac{1}{(\omega'-\omega)}\right]$$
$$-\frac{\pi^2}{2\omega_0}\vec{F}(\omega_0)\vec{G}(\omega_0), \tag{4.22}$$

where the integration symbol $\int\!\!\!\!\!\!-$ implies a Cauchy principal value. The imaginary part of (4.21) vanishes identically. By placing the function in the integral in (4.22) over a common denominator and looking at the region of integration which is common to all three terms, we find that the numerator vanishes. This means that we must integrate only over the frequency regions which are not common to all three terms. These regions are small strips of width 2ϵ which exist due to the principal-part restrictions in each separate term. Expression (4.22) can now be more explicitly written as

$$\int_0^\Lambda d\omega \int_{\omega-\epsilon}^{\omega+\epsilon} d\omega' \vec{F}(\omega)\vec{G}(\omega')\frac{(\omega+\omega')}{(\omega'+\omega_0)(\omega+\omega_0)}\frac{1}{(\omega'-\omega_0)}\frac{1}{(\omega-\omega_0)}$$
$$-\int_0^\Lambda d\omega' \int_{\omega_0-\epsilon}^{\omega_0+\epsilon} d\omega\, \vec{G}(\omega')\vec{F}(\omega)\frac{1}{(\omega'+\omega_0)}\frac{1}{(\omega'-\omega_0)}\frac{1}{(\omega-\omega')}$$
$$-\int_0^\Lambda d\omega \int_{\omega_0-\epsilon}^{\omega_0+\epsilon} d\omega' \vec{F}(\omega)\vec{G}(\omega')\frac{1}{(\omega+\omega_0)}\frac{1}{(\omega-\omega_0)}\frac{1}{(\omega'-\omega)} - \frac{\pi^2}{2\omega_0}\vec{F}(\omega_0)\vec{G}(\omega_0), \tag{4.23}$$

where the principal part ϵ's are all taken to be equal. We will explicitly calculate the first integral in Eq. (4.23) in order to demonstrate the method of calculation for the remaining two integrals.

A diagram exhibiting the region of integration in Fig. 1 shows that this first integral can be broken up into six separate integrals. This first integral in (4.23) now looks like

$$\left(\int_0^\epsilon d\omega \int_0^{\omega+\epsilon} d\omega' + \int_\epsilon^{\omega_0-2\epsilon} d\omega \int_{\omega-\epsilon}^{\omega+\epsilon} d\omega' + \int_{\omega_0-2\epsilon}^{\omega_0-\epsilon} d\omega \int_{\omega-\epsilon}^{\omega_0-\epsilon} d\omega'\right.$$
$$\left. + \int_{\omega_0+\epsilon}^{\omega_0+2\epsilon} d\omega \int_{\omega_0+\epsilon}^{\omega+\epsilon} d\omega' + \int_{\omega_0+2\epsilon}^{\Lambda-\epsilon} d\omega \int_{\omega-\epsilon}^{\omega+\epsilon} d\omega' + \int_{\Lambda-\epsilon}^{\Lambda} d\omega \int_{\omega-\epsilon}^{\Lambda} d\omega'\right)\left(\vec{F}(\omega)\vec{G}(\omega')\frac{(\omega+\omega')}{(\omega'^2-\omega_0^2)(\omega^2-\omega_0^2)}\right). \tag{4.24}$$

These six integrals can be readily integrated after recognizing that $\vec{F}(\omega)$ and $\vec{G}(\omega')$ are smooth functions of ω and ω'. The integrals in (4.24) are explicitly calculated in Appendix D. The result is that (4.24) reduces to

$$\frac{\vec{G}(\omega_0)\vec{F}(\omega_0)}{\omega_0}\left[\lim_{\alpha\to 0}\lim_{\epsilon\to 0}\left(\int_{2\epsilon}^{\alpha}\frac{dx}{x}\ln\frac{x+\epsilon}{x-\epsilon} + \int_\epsilon^{2\epsilon}\frac{dx}{x}\ln\frac{x+\epsilon}{\epsilon}\right)\right]. \tag{4.25}$$

By following the analogous steps which led to (4.25), the remaining two integrals in (4.23) can be shown to be identical with (4.25). Therefore (4.23) becomes simply

$$\frac{3\vec{G}(\omega_0)\vec{F}(\omega_0)}{\omega_0}\left[\lim_{\alpha\to 0}\lim_{\epsilon\to 0}\left(\int_{2\epsilon}^{\alpha}\frac{dx}{x}\ln\frac{x+\epsilon}{x-\epsilon} + \int_\epsilon^{2\epsilon}\frac{dx}{x}\ln\frac{x+\epsilon}{\epsilon}\right)\right] - \frac{\pi^2}{2\omega_0}\vec{G}(\omega_0)\vec{F}(\omega_0). \tag{4.26}$$

In Appendix E we evaluate the two integrals in (4.26) and show that they have the sum $\pi^2/6$. Upon substitution of this result in (4.26) we find that the second-order term is identically zero. *The first canonical commutation relation (4.16) is therefore preserved through second order in the coupling constant.*

Let us now consider (4.17) . If we follow the same procedures as we did to obtain (4.21), then we ob-

tain the second-order term

$$\int_0^\Lambda d\omega \int_0^\Lambda d\omega' \vec{F}(\omega)\vec{G}(\omega')\left[\frac{(\omega+\omega')}{(\omega'+\omega_0)(\omega+\omega_0)(\omega'-\omega_0-i\epsilon)(\omega-\omega_0-i\epsilon)}\right.$$

$$\left.+\frac{1}{(\omega'+\omega_0)(\omega'-\omega_0-i\epsilon)(\omega'+\omega-i\epsilon)}-\frac{1}{(\omega+\omega_0)(\omega-\omega_0-i\epsilon)(\omega'+\omega-i\epsilon)}\right]. \tag{4.27}$$

The zeroth-order term for the commutator (4.17) vanishes as it should. Making the replacement $1/(x+i\epsilon) \to P/x - i\pi\delta(x)$ we can reduce (4.27) to

$$\int_\epsilon^\Lambda d\omega \int_\epsilon^\Lambda d\omega' \vec{F}(\omega)\vec{G}(\omega')\left[\frac{(\omega'-\omega)}{(\omega'+\omega_0)(\omega+\omega_0)}\frac{1}{(\omega'-\omega_0)}\frac{1}{(\omega-\omega_0)}\right.$$

$$\left.+\frac{1}{(\omega'+\omega_0)(\omega'+\omega)}\frac{1}{(\omega'-\omega_0)}-\frac{1}{(\omega+\omega_0)(\omega'+\omega)}\frac{1}{(\omega-\omega_0)}\right]. \tag{4.28}$$

All delta function terms cancel identically. The lower limit ϵ's were introduced to take into account the point $\omega=\omega'=0$. The integration of $1/(\omega+\omega')$, however, does not cause any difficulty. This can be seen from the integral

$$\lim_{\epsilon\to 0}\int_\epsilon^\Lambda d\omega \int_\epsilon^\Lambda d\omega' \frac{1}{\omega+\omega'}\ , \tag{4.29}$$

which is finite.

By placing the function in the integral in (4.28) over a common denominator and looking at the region of integration which is common to all three terms, we find that the numerator vanishes. This means that we must integrate only over the frequency regions which are not common to all three terms. These regions are small strips of width 2ϵ due to the missing principal parts in the second and third terms of (4.28). Equation (4.28) can be more explicitly written as

$$\int_{\omega_0-\epsilon}^{\omega_0+\epsilon} d\omega \int_\epsilon^\Lambda d\omega' \vec{F}(\omega)\vec{G}(\omega')\frac{1}{(\omega'+\omega_0)(\omega'+\omega)}\frac{1}{(\omega'-\omega_0)} - \int_\epsilon^\Lambda d\omega \int_{\omega_0-\epsilon}^{\omega_0+\epsilon} d\omega' \vec{F}(\omega)\vec{G}(\omega')\frac{1}{(\omega+\omega_0)(\omega'+\omega)}\frac{1}{(\omega-\omega_0)}, \tag{4.30}$$

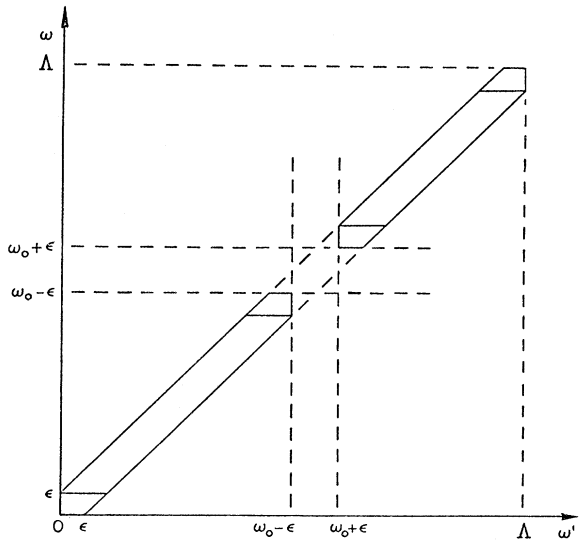

FIG. 1. The six regions along the diagonal enclosed by solid borders comprise the integration region for the first integral in Eq. (4.23).

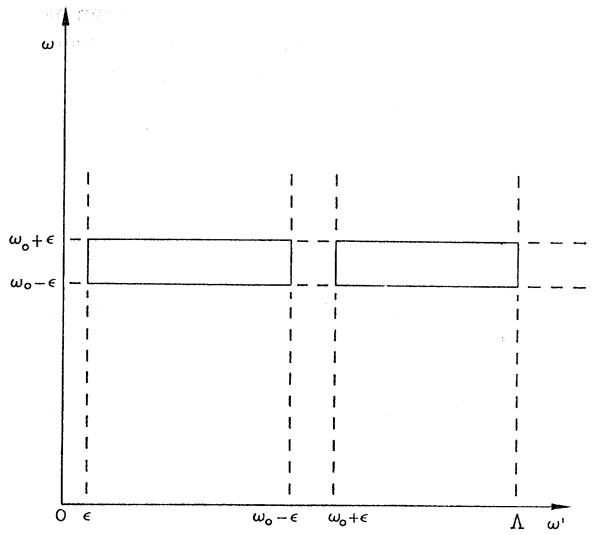

FIG. 2. The two regions with solid borders comprise the integration region for the first integral in Eq. (4.30).

where the principal-part ϵ's are all taken to be equal. Since the integrals are formally identical if we interchange ω and ω', we need only integrate the first one. A diagram indicating the region of integration in Fig. 2 shows that the integral should be broken up into two separate integrals. The first integral in (4.30) looks like

$$\left(\int_{\omega_0-\epsilon}^{\omega_0+\epsilon} d\omega \int_{\epsilon}^{\omega_0-\epsilon} d\omega' + \int_{\omega_0-\epsilon}^{\omega_0+\epsilon} d\omega \int_{\omega_0+\epsilon}^{\Lambda} d\omega'\right) \times \left(\frac{\vec{F}(\omega)\vec{G}(\omega')}{(\omega'+\omega_0)(\omega'+\omega)(\omega'-\omega_0)}\right) . \quad (4.31)$$

These integrals are carried out in Appendix F. The result is that (4.31) vanishes, which immediately shows that the second-order term (4.27) is equal to zero. *The canonical commutation relation (4.17) is therefore preserved to second order in the coupling constant.*

E. Relation to the work of Nesbet and Bullough

In this section we comment on the recent work of Bullough[16] and Nesbet,[17] who have raised questions about the viability of the canonical commutation relations. Their results might be taken to contradict ours and must be examined carefully.

Nesbet has suggested that a QED source-field description of two-level-atom spontaneous emission must be inconsistent because the operators representing the source field belong to a Fermi, not Bose, algebra. To remedy what he considered to be a fundamental problem of QED, he proposed a new "semiquantized" version of electrodynamics in which only fermion fields would be quantized.

However, the problem noticed by Nesbet can be resolved more simply. In calculating the source-field operator solution for the vector potential,

$$\vec{A}^s(\vec{r}, t) = \frac{1}{c}\int \frac{[\vec{J}_T(\vec{r}', t)]}{|\vec{r}-\vec{r}'|} d^3r' , \quad (4.32a)$$

from Maxwell's wave equation,

$$\Box \vec{A} = -\frac{4\pi}{c}\vec{J}_T , \quad (4.32b)$$

it is improper to object to the operator $\vec{A}^s$ on the grounds that it belongs to the algebra of the source current $\vec{J}_T$, and may not be a Bose field. Only the total field

$$\vec{A}(\vec{r}, t) = \vec{A}^v(\vec{r}, t) + \vec{A}^s(\vec{r}, t) \quad (4.33)$$

needs to be Bose field, and we have just shown in Sec. IV D that the total field operators remain Bose operators for all time.

It may still appear mildly paradoxical that both $\vec{A}$ and the homogeneous or "vacuum" part $\vec{A}^v$ of the total can both be Bose operators, while $\vec{A}^s$ is not. The explanation lies in the simple fact that, while $\vec{A}^v(\vec{r}, t)$ commutes with atomic operators at $t=0$, it fails to do so for $t>0$. This very natural result, which follows from the coupled atomic and field operator equations, is what allows the commutation relations to be preserved for all time.

Bullough's objections, however, are more serious. Bullough found (Ref. 16, see especially pp. 127–129 and Appendix 2), in the context of developing a QED nonlinear radiation reaction theory, that canonical commutation relations were violated by the approximately integrated total-field operator solutions. Since we have shown that the commutation relations are preserved by our approximately integrated field operator solutions, it is interesting to examine Bullough's argument.

Bullough evaluated the equal-time commutation relations between the approximately integrated electric field operator and all the atomic operators. He found that upon letting $t \to 0$ the commutation relations did not reduce to the usual canonical result. In our calculation, using the creation and destruction parts of the field operators, we could not allow $t \to 0$ in the commutation relations (4.9) and (4.11) since our approximate source-field solution in (4.9) and (4.11) was valid only for $\omega_0 t \gg 1$. If we had allowed $t \to 0$ formally in (4.9) and (4.11), then we also would have found an obvious inconsistency which is directly related to the inconsistency pointed out by Bullough.

We can illustrate Bullough's point by considering the component of the total vector potential operator in the direction of the dipole moment evaluated at the center of the atom,

$$A(0, t) = \sum_{\lambda} g_\lambda (a_\lambda + a_\lambda^\dagger), \quad (4.34)$$

where a_λ is found from (2.23a) and (2.27) substituted into (2.22). In the infinite-volume limit, after integrating over angles, we find

$$A(0,t) = A^v(0,t) + \frac{2\omega_0 d}{3\pi c^2}\left\{\sigma_-(t)\int_0^\infty \omega d\omega \left[\int_0^\infty dt' (e^{-i(\omega-\omega_0-i\epsilon)t'} - e^{i(\omega+\omega_0+i\epsilon)t'}) - \int_t^\infty dt' (e^{-i(\omega-\omega_0-i\epsilon)t'} - e^{i(\omega+\omega_0+i\epsilon)t'})\right] + \text{H.c.}\right\} , \quad (4.35)$$

where we have separated the time integrals into a constant long-time part and a time-dependent part. Obviously the curly-bracketed source-field contribution vanishes as $t \to 0$.

Caution must be exercised, however, if the integrals are cut off. By using the regulator function $\exp(-\omega/\Lambda)$ as in Appendix A, we obtain

$$A(0,t) = A^{v}(0,t) + \eta\left\{\sigma_{-}(t)\left[\pi\omega_0 - 2i\Lambda + \int_t^{\infty} dt'\, e^{(-\epsilon+i\omega_0)t'}\left(\frac{1}{(t'-i/\Lambda)^2} - \frac{1}{(t'+i/\Lambda)^2}\right)\right] + \text{H.c.}\right\}, \tag{4.36}$$

where $\eta \equiv 2\omega_0 d/3\pi c^2$. The entire curly brackets in (4.36) still vanish identically at $t=0$, as they must. Furthermore, if $\Lambda t \gg 1$ in (4.36), then the integral in the brackets becomes very small and can be neglected, and all of our results for frequency shifts and linewidths follow automatically.

However, having introduced the regulator function, it is possible to obtain inconsistent results by incautious interchanges of limiting processes, or by failing to treat the entire source field as a unit. For example, if $\Lambda \to \infty$ inside the integrand in (4.36), then the integrand itself vanishes identically. This is intimately connected with the singularity of the integrated term $2i\Lambda$ in the same limit. Obviously, if $\Lambda = \infty$ there is no way to interpret $2i\Lambda$, and there is no way for the source field then to vanish at $t=0$. In effect, the product Λt has the value appropriate to the Λ limit, or the t limit, whichever is taken first.

In the end the question is not really a mathematical one. It is for good physical reasons that an upper frequency cutoff is supplied in nonrelativistic calculations of radiative corrections. In a proper relativistic calculation such a cutoff is supplied naturally, and is roughly $mc^2/\hbar$ in size.

Bullough's atom-field commutation relations which involve total-field operator solutions such as

$$A(0,t) = A^{v}(0,t) + \eta\{\sigma_{-}(t)(\pi\omega_0 - 2i\Lambda) + \text{H.c.}\} \tag{4.37}$$

are valid, we now recognize, only for $t \gg \Lambda^{-1}$. This is, in fact, only a necessary condition on their validity. Closer inspection shows that, for finite $\Lambda \approx mc^2/\hbar$, we must also take into account the size of ω_0 with respect to Λ, finding (4.37) to be valid only for $t > \omega_0^{-1} \gg \Lambda^{-1}$. It is not surprising that the use of the result (4.37) leads to inconsistencies for times outside that range. For times such that $t \ll \omega_0^{-1}$ the exact result (4.35) must be used. Moreover, as is pointed out in Appendix A, only for $t \gg \omega_0^{-1} \gg \Lambda^{-1}$ is it sensible to speak of the atom as if it had measurable transition frequencies, let alone frequency shifts and widths.

V. REMARKS AND CONCLUSIONS

We have attempted to show that a more directly physical derivation than is customary may be given for some of the nontrivial results of quantum electrodynamics. It may be worthwhile to explain what the term "more directly physical" is intended to mean here. To take the example of the Lamb shift, it seems sensible that a theory should be prepared to describe the emitted radiation that is detected in a Lamb shift experiment. A "direct" route to a description of the radiation would have four obvious steps: (1) Calculate the power radiated by the atom into a given mode of the field by first calculating the radiated electric and magnetic fields. (2) Calculate the fields by solving Maxwell's equations, the source current in Maxwell's equations being simply the radiating atom's dipole moment (in the dipole approximation). (3) The atomic dipole moment itself is found by solving its Heisenberg equation, which depends on the details of the atom (details built into the atom's Hamiltonian) and the nature of any fields present, including its own radiation field. (4) In the end a self-consistency condition must be met: The atomic dipole moment that generates the emitted radiation must be the same as the moment that reacts to the same radiation.

Such a sequence of calculations is conceptually so straightforward as to be dull. This is, of course, one of its advantages. It is difficult to get the physics wrong. However, as we have remarked, there have been strong doubts expressed about the practical reliability of this "source-field" approach. We have shown here that, at least for problems of atomic spontaneous emission, those doubts appear to be groundless.

For both a fictitious two-level atom and for a spinless nonrelativistic one-electron atom, we have used the Heisenberg operator source-field method to find correct order-e^2 quantum-electrodynamic radiative corrections. These include radiative lifetimes, Lorentzian lineshapes, exponential decay of excited-state energies, and Lamb shifts. Recently Lai, Knight, and Eberly have also used the source-field approach in computing radiative corrections in a free-electron problem.[12] In other recent applications of the operator source-field method, Milonni[28] has studied strong-field resonance fluorescence, and Lai[29] has derived the Low-Goldberger–Gell-Mann low-energy theorem and has studied Compton scattering. The

work of Bullough[16] and of Moniz and Sharp[18] has already been mentioned.

Finally, we should emphasize in conclusion that throughout this paper we have attempted to maintain contact, at least in spirit, with Lorentz's classical work on radiation damping. In fact it might be fair to say that our results merely show that Lorentz's *ideas* were entirely correct. His results for radiative corrections in atoms were not right simply because he did not have quantum theory to work with. In this connection we can point again to the remarkable feature of normally-ordered quantum calculations: All radiative corrections come from the reaction of the atomic source field back on the atom, and none from vacuum fluctuations.[22]

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APPENDIX A

In Eqs. (2.37a) and (2.37b) the coefficients of the operators σ_- and σ_+ are independent of time. These coefficients replace the time-dependent coefficients in Eq. (2.31). In this appendix we will justify the approximation involved in the replacement.

Let us denote by $C(t)$ one typical coefficient in Eq. (2.31):

$$C(t)\equiv\sum_\lambda g_\lambda{}^2[I(\omega_\lambda-\omega_0;\,t) -I^*(\omega_\lambda+\omega_0;\,t)]. \tag{A1}$$

In the infinite-volume limit, after performing the integration over angles and after using the definitions of the I functions in (2.28), we obtain

$$C(t)=\beta\int_0^\infty \omega d\omega\left[\int_0^t d\tau\,(e^{i(\omega-\omega_0+i\epsilon)\tau} +e^{-i(\omega+\omega_0-i\epsilon)\tau})\right], \tag{A2}$$

where $\beta\equiv-(i2\hbar/3\pi c)$. The corresponding time-independent coefficient used in Eqs. (2.37a) and (2.37b) is $C(\infty)$.

The difference $\delta C\equiv C(\infty)-C(t)$ can easily be shown to be small as follows. If we first introduce a cutoff function $\exp(-\omega/\Lambda)$ in the frequency integration (taking $\Lambda\to\infty$ at the end of the calculation), we may write

$$\delta C=\lim_{\substack{\epsilon\to0\\ \Lambda\to\infty}}\beta\int_t^\infty d\tau\,e^{-(\epsilon+i\omega_0)\tau}\int_0^\infty \omega d\omega\,e^{-\omega/\Lambda}(e^{i\omega\tau}+e^{-i\omega\tau}), \tag{A3}$$

which is the same as

$$\delta C=-\lim_{\substack{\epsilon\to0\\ \Lambda\to\infty}}\beta\int_t^\infty d\tau\,e^{-(\epsilon+i\omega_0)\tau}\left[\frac{1}{(\tau+i/\Lambda)^2}+\text{c.c.}\right]. \tag{A4}$$

Note that for times t such that $\omega_0 t\gg1$ we automatically have $\omega_0\tau\gg1$, so that the exponential in (A4) is highly oscillatory. The integral itself will be very small as a consequence. A sequence of partial integrations leads to an asymptotic series which exhibits the smallness of δC. After taking the ϵ and Λ limits and inserting the value of β, we find

$$\delta C=-\left(\frac{i2\hbar}{3\pi c}\right)(i2\omega_0)\left[\left(\frac{1}{\omega_0 t}\right)^2+i2\left(\frac{1}{\omega_0 t}\right)^3 +O((\omega_0 t)^{-4})\right]e^{-i\omega_0 t}. \tag{A5}$$

Before drawing conclusions it is necessary to compare δC with $C(\infty)$:

$$C(\infty)=-\left(\frac{i2\hbar}{3\pi c}\right)(i2\omega_0)\left[-i\frac{\pi}{2}+\lim_{\Lambda\to\infty}\left(\ln\frac{\Lambda}{\omega_0}+O(\omega_0/\Lambda)\right)\right]. \tag{A6}$$

In a real atom, with a finite upper cutoff $\Lambda\gg\omega_0$ supplied by relativistic effects, it is known that the terms in the bracket in the expression for $C(\infty)$ are roughly equal and of order unity, and thus much larger than the corresponding terms in δC for any time t such that $\omega_0 t\gg1$. It is, of course, physically sensible to consider atomic transitions only on time scales much greater than $\omega_0{}^{-1}$. For shorter times the atom's level structure is not well defined. Thus in the present case the replacement of the time-dependent coefficients in Eqs. (2.31) by the time-independent coefficients in Eqs. (2.37a) and (2.37b) is well justified.

APPENDIX B

In this appendix we will derive the wave equation for the creation and destruction parts of the vector potential (3.7b) from the Hamiltonian,

$$H=H_a+\sum_\lambda\hbar\omega_\lambda a_\lambda^+ a_\lambda^- -\frac{1}{c}\int\vec{\mathrm{J}}\circ\vec{\mathrm{A}}\,d^3r'. \tag{B1}$$

The creation and destruction parts of the vector potential, electric field, and magnetic field are defined to be

$$\vec{A}_{\pm}(\vec{r},t)\equiv\sum_{\lambda}\left(\frac{2\pi\hbar c^2}{\omega_\lambda \mathcal{V}}\right)^{1/2}\hat{\epsilon}_\lambda a_\lambda^{\pm}(t)e^{\mp i\vec{k}_\lambda\cdot\vec{r}}, \tag{B2a}$$

$$\vec{E}_{\pm}(\vec{r},t)\equiv\mp i\sum_{\lambda}\left(\frac{2\pi\hbar\omega_\lambda}{\mathcal{V}}\right)^{1/2}\hat{\epsilon}_\lambda a_\lambda^{\pm}(t)e^{\mp i\vec{k}_\lambda\cdot\vec{r}}, \tag{B2b}$$

and

$$\vec{B}_{\pm}(\vec{r},t)\equiv\vec{\nabla}\times\vec{A}_{\pm}(\vec{r},t), \tag{B2c}$$

where

$$[a_\lambda^-(t),a_{\lambda'}^+(t)]=\delta_{\lambda\lambda'}. \tag{B3}$$

The Heisenberg equations of motion are found in the usual way from the Hamiltonian (B1), and the equation of motion for $\vec{A}_-$ is

$$i\hbar\frac{d\vec{A}_-(\vec{r},t)}{dt}=-i\hbar c\vec{E}_-(\vec{r},t)-\frac{1}{c}\int[\vec{A}_-(\vec{r},t),\vec{A}_+(\vec{r}',t)]\cdot\vec{J}(\vec{r}',t)d^3r'. \tag{B4}$$

From (B2a) and (B3) it follows that

$$[\vec{A}_-(\vec{r},t),\vec{A}_+(\vec{r}',t)]=\sum_{\lambda}\left(\frac{2\pi\hbar c^2}{\omega_\lambda \mathcal{V}}\right)\hat{\epsilon}_\lambda\hat{\epsilon}_\lambda e^{i\vec{k}_\lambda\cdot(\vec{r}-\vec{r}')}. \tag{B5}$$

Upon substitution of (B5) into (B4), and with the use of the relations

$$\int d^3r'\vec{A}_T(\vec{r}')\cdot\vec{B}_T(\vec{r}-\vec{r}')\equiv\int d^3r'\vec{A}(\vec{r}')\cdot\vec{B}_T(\vec{r}-\vec{r}') \tag{B6}$$

and

$$\frac{1}{\mathcal{V}}\sum_{\lambda}\frac{e^{i\vec{k}_\lambda\cdot(\vec{r}-\vec{r}')}}{|\vec{k}_\lambda|}\xrightarrow[\mathcal{V}\to\infty]{}\frac{1}{\pi^2|\vec{r}-\vec{r}'|^2}, \tag{B7}$$

we obtain for (B4)

$$\frac{d\vec{A}_-(\vec{r},t)}{dt}=-c\vec{E}_-(\vec{r},t)+\frac{i}{\pi}\int d^3r'\frac{\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|^2}. \tag{B8a}$$

The Hermitian conjugate equation for $\vec{A}_+$ is simply

$$\frac{d\vec{A}_+(\vec{r},t)}{dt}=-c\vec{E}_+(\vec{r},t)-\frac{i}{\pi}\int d^3r'\frac{\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|^2}. \tag{B8b}$$

Let us consider the equation of motion for $\vec{E}_-$:

$$i\hbar\frac{d\vec{E}_-(\vec{r},t)}{dt}=i\hbar c\vec{\nabla}\times[\vec{\nabla}\times\vec{A}_-(\vec{r},t)]-\frac{1}{c}\int[\vec{E}_-(\vec{r}',t),\vec{A}_+(\vec{r}',t)]\cdot\vec{J}(\vec{r}',t)d^3r'. \tag{B9}$$

From (B2b) and (B3) we find

$$[\vec{E}_-(\vec{r},t),\vec{A}_+(\vec{r}',t)]=\frac{i2\pi\hbar c}{\mathcal{V}}\sum_{\lambda}\hat{\epsilon}_\lambda\hat{\epsilon}_\lambda e^{i\vec{k}_\lambda\cdot(\vec{r}-\vec{r}')}. \tag{B10}$$

Upon substitution of (B10) into (B9), and with the use of (B6) and

$$\frac{1}{2\mathcal{V}}\sum_{\lambda}e^{i\vec{k}_\lambda\cdot(\vec{r}-\vec{r}')}\xrightarrow[\mathcal{V}\to\infty]{}\delta^3(\vec{r}-\vec{r}'), \tag{B11}$$

we obtain for (B9)

$$\frac{d\vec{E}_-(\vec{r},t)}{dt}=c\vec{\nabla}\times[\vec{\nabla}\times\vec{A}_-(\vec{r},t)]-2\pi\vec{J}(\vec{r},t). \tag{B12}$$

In the Coulomb gauge we can rewrite (B12) as

$$\frac{d\vec{E}_-}{dt}=-c\nabla^2\vec{A}_--2\pi\vec{J}. \tag{B13a}$$

The equation for $\vec{E}_+$ is simply

$$\frac{d\vec{E}_+}{dt}=-c\nabla^2\vec{A}_+-2\pi\vec{J}. \tag{B13b}$$

After taking the time derivative of Eqs. (B8a) and (B8b) we can substitute for $d\vec{E}_\pm/dt$ from (B13a) and (B13b) to find the wave equation for the creation and destruction parts of the vector potential:

$$\Box\vec{A}_\pm=-\frac{2\pi}{c}\vec{J}\pm\frac{i}{\pi c^2}\int d^3r'\frac{d\vec{J}(\vec{r}',t)/dt}{|\vec{r}-\vec{r}'|^2}, \tag{B14}$$

which is the form used in the text, Eq. (3.7b).

APPENDIX C

We will derive from Eqs. (3.17) and (3.22a) the equation of motion for $\langle d\vec{r}/dt\rangle$, Eq. (3.27a). By substituting (3.17) into (3.22a) we find

$$i\hbar\left\langle\frac{d\vec{r}}{dt}\right\rangle=-\sum_{i,j}E_{ij}\vec{r}_{ij}\langle S_{ij}\rangle-\frac{e^2}{m^2}\frac{2\pi}{\mathcal{V}}\sum_{\lambda}\frac{1}{\omega_\lambda}\sum_{\substack{n,l\\i,j}}^{\infty}\Bigg[\langle S_{in}\rangle\frac{\vec{r}_{ij}(\hat{\epsilon}\cdot\vec{p}_{jl})(\hat{\epsilon}\cdot\vec{p}_{ln})}{\omega_\lambda-E_{nl}/\hbar-i\epsilon}-\langle S_{nj}\rangle\frac{\vec{r}_{ij}(\hat{\epsilon}\cdot\vec{p}_{nl})(\hat{\epsilon}\cdot\vec{p}_{li})}{\omega_\lambda-E_{nl}/\hbar+i\epsilon}$$
$$+\langle S_{nl}\rangle\frac{\vec{r}_{ij}(\hat{\epsilon}\cdot\vec{p}_{ni})(\hat{\epsilon}\cdot\vec{p}_{jl})}{\omega_\lambda-E_{ni}/\hbar+i\epsilon}-\langle S_{nl}\rangle\frac{\vec{r}_{ij}(\hat{\epsilon}\cdot\vec{p}_{ni})(\hat{\epsilon}\cdot\vec{p}_{jl})}{\omega_\lambda-E_{lj}/\hbar-i\epsilon}\Bigg], \tag{C1}$$

where we have explicitly substituted for $\vec{J}_\lambda$ from (3.8b). In the first term in the square brackets in (C1) the index j appears only on $\vec{r}_{ij}$ and $\hat{\epsilon}\cdot\vec{p}_{jl}$. Since the sum over j is complete we can simply write

$$\sum_j \vec{r}_{ij}\hat{\epsilon}\cdot\vec{p}_{jl} = (\vec{r}\hat{\epsilon}\cdot\vec{p})_{il} .$$

Each separate term in the square brackets in (C1) has an index of this type which can be summed over and removed. By interchanging dummy variables in the second and fourth terms we can rewrite (C1) as

$$i\hbar\left\langle\frac{d\vec{r}}{dt}\right\rangle = -\sum_{i,j}^{\infty} E_{ij}\vec{r}_{ij}\langle S_{ij}\rangle - \frac{e^2}{m^2}\frac{2\pi}{v}\sum_\lambda\frac{1}{\omega_\lambda}\sum_{n,l,i}^{\infty}\left[\langle S_{in}\rangle\frac{(\vec{r}\hat{\epsilon}\cdot\vec{p})_{il}(\hat{\epsilon}\cdot\vec{p}_{ln})}{\omega_\lambda - E_{nl}/\hbar - i\epsilon} - \langle S_{nl}\rangle\frac{(\hat{\epsilon}\cdot\vec{p}_{ni})(\hat{\epsilon}\cdot\vec{p}\,\vec{r})_{il}}{\omega_\lambda - E_{ni}/\hbar + i\epsilon}\right.$$
$$\left. + \langle S_{nl}\rangle\frac{(\hat{\epsilon}\cdot\vec{p}_{ni})(\vec{r}\hat{\epsilon}\cdot\vec{p})_{il}}{\omega_\lambda - E_{ni}/\hbar + i\epsilon} - \langle S_{in}\rangle\frac{(\hat{\epsilon}\cdot\vec{p}\,\vec{r})_{il}(\hat{\epsilon}\cdot\vec{p}_{ln})}{\omega_\lambda - E_{nl}/\hbar - i\epsilon}\right] . \quad \text{(C2)}$$

In the brackets in (C2) the four terms can be combined to form two terms proportional to the matrix elements of the commutator $[\vec{r}, \vec{p}]$,

$$[\vec{r}, \hat{\epsilon}\cdot\vec{p}]_{jl} = \hat{\epsilon} i\hbar\delta_{jl} . \quad \text{(C3)}$$

After substituting (C3) into (C2) and using the relation

$$-\sum_{i,j}^{\infty} E_{ij}\vec{r}_{ij}\langle S_{ij}\rangle = \sum_{i,j}^{\infty}[\vec{r}, H_a]_{ij}\langle S_{ij}\rangle = i\hbar\frac{\langle\vec{p}\rangle}{m} , \quad \text{(C4)}$$

we obtain for $\langle d\vec{r}/dt\rangle$

$$\left\langle\frac{d\vec{r}}{dt}\right\rangle = \frac{\langle\vec{p}\rangle}{m} - \frac{e^2}{m^2}\frac{2\pi}{v}\sum_\lambda\frac{1}{\omega_\lambda}\sum_{i,n}^{\infty}\left[\frac{\hat{\epsilon}\hat{\epsilon}\cdot\vec{p}_{in}\langle S_{in}\rangle}{\omega_\lambda - E_{ni}/\hbar - i\epsilon} + \frac{\hat{\epsilon}\hat{\epsilon}\cdot\vec{p}_{ni}\langle S_{ni}\rangle}{\omega_\lambda - E_{ni}/\hbar + i\epsilon}\right] . \quad \text{(C5)}$$

We can now combine the two terms in the square brackets in (C5) after interchanging the dummy indices i and n in the second term. In the continuum limit, after performing the integration over angles, we simply find

$$\left\langle\frac{d\vec{r}}{dt}\right\rangle = \frac{\langle\vec{p}\rangle}{m} - \frac{e^2}{m^2}\left(\frac{2}{3\pi c^2}\right)\sum_{i,n}^{\infty}\vec{p}_{in}\langle S_{in}\rangle\int_0^\Lambda \omega d\omega\left[\frac{1}{(\omega - E_{ni}/\hbar - i\epsilon)} + \frac{1}{(\omega - E_{in}/\hbar + i\epsilon)}\right] , \quad \text{(C6)}$$

where Λ is the nonrelativistic cutoff proportional to $mc^2/\hbar$. The integral in (C6) can be evaluated to be

$$\int_0^\Lambda \omega d\omega[\text{bracket in (C6)}] = 2\Lambda + O\left(\frac{E_{in}}{\Lambda}\right) - i\pi\frac{E_{in}}{\hbar} , \quad \text{(C7)}$$

where the term proportional to $O(E_{in}/\hbar)$ is negligible. Upon substituting (C7) into (C6) and using the relation

$$-i\sum_{j,n}^{\infty}\vec{p}_{jn}\langle S_{jn}\rangle\frac{E_{in}}{\hbar} = -\left\langle\frac{d\vec{p}}{dt}\right\rangle , \quad \text{(C8)}$$

we easily arrive at the equation for $\langle d\vec{r}/dt\rangle$,

$$\left\langle\frac{d\vec{r}}{dt}\right\rangle = \frac{\langle\vec{p}\rangle}{m} - \frac{e^2}{m^2}\left(\frac{2}{3\pi c^3}\right)\left[2\Lambda\langle\vec{p}\rangle - \pi\left\langle\frac{d\vec{p}}{dt}\right\rangle\right] , \quad \text{(C9)}$$

which is simply Eq. (3.27a).

APPENDIX D

We will reduce the integrals in (4.24) to the form of (4.25) by making use of the well-behaved nature of $\vec{F}(\omega)$ and $\vec{G}(\omega)$. The first integral

$$\int_0^\epsilon d\omega\int_0^{\omega+\epsilon} d\omega'\vec{F}(\omega)\vec{G}(\omega')\frac{(\omega+\omega')}{(\omega'^2 - \omega_0^2)(\omega^2 - \omega_0^2)} \quad \text{(D1a)}$$

for small ϵ can be put into the form

$$\frac{\vec{F}(0)\vec{G}(0)}{\omega_0^4}\int_0^\epsilon d\omega\int_0^{\omega+\epsilon} d\omega'(\omega+\omega') , \quad \text{(D1b)}$$

which vanishes in the limit $\epsilon \to 0$.

The second integral

$$\int_\epsilon^{\omega_0 - 2\epsilon} d\omega\int_{\omega-\epsilon}^{\omega+\epsilon} d\omega'\vec{F}(\omega)\vec{G}(\omega')\frac{(\omega+\omega')}{(\omega'^2 - \omega_0^2)(\omega^2 - \omega_0^2)} \quad \text{(D2a)}$$

for small ϵ is equivalent to

$$\int_\epsilon^{\omega_0 - 2\epsilon} d\omega\int_{\omega-\epsilon}^{\omega+\epsilon} d\omega'\frac{\vec{F}(\omega)\vec{G}(\omega)\,2\omega}{(\omega+\omega_0)^2(\omega - \omega_0)}\frac{1}{(\omega' - \omega_0)} \quad \text{(D2b)}$$

since ω' is essentially equal to ω. After performing the ω' integration, we obtain

$$\int_0^{\omega_0-2\epsilon} d\omega \frac{\vec{F}(\omega)\vec{G}(\omega)\,2\omega}{(\omega+\omega_0)^2(\omega-\omega_0)} \ln\left|\frac{\omega-\omega_0+\epsilon}{\omega-\omega_0-\epsilon}\right| . \qquad \text{(D2c)}$$

As long as ω is not equal to ω_0 the limit $\epsilon \to 0$ forces the integrand to be zero. However, as ω nears ω_0 the integrand does not vanish. This means that the integral will only contribute in the region of integration near the upper limit. If we define some parameter $\alpha > 2\epsilon$ which we can let go to zero after $\epsilon \to 0$, we can write (D2c) as

$$\lim_{\alpha\to 0}\lim_{\epsilon\to 0}\int_{\omega_0-\alpha}^{\omega_0-2\epsilon} d\omega \frac{\vec{F}(\omega)\vec{G}(\omega)\,2\omega}{(\omega+\omega_0)^2(\omega-\omega_0)} \ln\frac{\omega_0-\omega-\epsilon}{\omega_0-\omega+\epsilon} . \qquad \text{(D2d)}$$

By considering the well-behaved nature of the functions $\vec{F}(\omega)$ and $\vec{G}(\omega)$ we can write (D2d) as

$$\frac{\vec{F}(\omega_0)\vec{G}(\omega_0)}{2\omega_0}\lim_{\alpha\to 0}\lim_{\epsilon\to 0}\int_{\omega_0-\alpha}^{\omega_0-2\epsilon}\frac{d\omega}{\omega-\omega_0}\ln\frac{\omega_0-\omega-\epsilon}{\omega_0-\omega+\epsilon} , \qquad \text{(D2e)}$$

which upon changing variables becomes

$$\frac{\vec{F}(\omega_0)\vec{G}(\omega_0)}{2\omega_0}\lim_{\alpha\to 0}\lim_{\epsilon\to 0}\int_{2\epsilon}^{\alpha}\frac{dx}{x}\ln\frac{x+\epsilon}{x-\epsilon} . \qquad \text{(D2f)}$$

The third integral

$$\int_{\omega_0-2\epsilon}^{\omega_0-\epsilon} d\omega \int_{\omega-\epsilon}^{\omega_0-\epsilon} d\omega' \vec{F}(\omega)\vec{G}(\omega')\frac{(\omega+\omega')}{(\omega'^2-\omega_0^2)(\omega^2-\omega_0^2)} \qquad \text{(D3a)}$$

for small ϵ is equivalent to

$$\frac{\vec{F}(\omega_0)\vec{G}(\omega_0)}{2\omega_0}\int_{\omega_0-2\epsilon}^{\omega_0-\epsilon} d\omega \int_{\omega-\epsilon}^{\omega_0-\epsilon} d\omega' \frac{1}{(\omega-\omega_0)}\frac{1}{(\omega'-\omega_0)} . \qquad \text{(D3b)}$$

Upon integration (D3b) reduces to

$$\frac{\vec{F}(\omega_0)\vec{G}(\omega_0)}{2\omega_0}\int_{\omega_0-2\epsilon}^{\omega_0-\epsilon} d\omega \frac{1}{(\omega-\omega_0)}\ln\frac{\epsilon}{\omega_0-\omega+\epsilon} , \qquad \text{(D3c)}$$

which after changing variables becomes

$$\frac{\vec{F}(\omega_0)\vec{G}(\omega_0)}{2\omega_0}\int_{\epsilon}^{2\epsilon}\frac{dx}{x}\ln\frac{x+\epsilon}{\epsilon} . \qquad \text{(D3d)}$$

The fourth integral is done in the same way as the third, and the result equals (D3d). The fifth is done in the same way as the second and the result equals (D2f). The sixth vanishes as did the first. We therefore recover (4.25) for the sum of all six integrals.

APPENDIX E

We will show that the two integrals in (4.26) are equal to $\pi^2/6$. Upon changing variables in the two integrals in (4.26) with the substitution $y = \epsilon/x$ we obtain

$$\int_{\epsilon/\alpha}^{1/2}\frac{dy}{y}\ln\frac{1+y}{1-y}+\int_{1/2}^{1}\frac{dy}{y}\ln\frac{1+y}{y} . \qquad \text{(E1a)}$$

If we simply take the limit $\epsilon \to 0$ in (E1a), we find

$$\int_0^{1/2}\frac{dy}{y}\ln\frac{1+y}{1-y}+\int_{1/2}^{1}\frac{dy}{y}\ln\frac{1+y}{y} , \qquad \text{(E1b)}$$

which now is independent of α. By adding and subtracting the same term we can complete the integration of the first integral in (E1b) up to the limit one:

$$\int_0^1\frac{dy}{y}\ln\frac{1+y}{1-y}+\int_{1/2}^{1}\frac{dy}{y}\ln\frac{1-y}{y} . \qquad \text{(E2)}$$

By changing variables in the second integral in (E2) with the substitution $z = -1 + 1/y$ we find for (E2)

$$\int_0^1\frac{dy}{y}\ln\frac{1+y}{1-y}+\int_0^1\frac{dz}{1+z}\ln z , \qquad \text{(E3)}$$

both of which can be found in the tables to give $\pi^2/6$.

APPENDIX F

We will show that (4.31) reduces to zero. Let us us begin by considering the well-behaved nature of $\vec{F}(\omega)$. We can reduce (4.31) to

$$\left(\int_{\omega_0-\epsilon}^{\omega_0+\epsilon} d\omega\int_{\epsilon}^{\omega_0-\epsilon} d\omega'+\int_{\omega_0-\epsilon}^{\omega_0+\epsilon} d\omega\int_{\omega_0+\epsilon}^{\Lambda} d\omega'\right)\times\frac{\vec{F}(\omega_0)\vec{G}(\omega')}{(\omega'^2-\omega_0^2)(\omega+\omega')} . \qquad \text{(F1)}$$

If we perform the ω integration in (F1), we obtain

$$\left(\int_{\epsilon}^{\omega_0-\epsilon} d\omega'+\int_{\omega_0+\epsilon}^{\Lambda} d\omega'\right)\frac{\vec{F}(\omega_0)\vec{G}(\omega')}{(\omega'^2-\omega_0^2)}\ln\frac{\omega'+\omega_0+\epsilon}{\omega'+\omega_0-\epsilon} . \qquad \text{(F2)}$$

As with (D2c) we find that the integral vanishes for $\omega' \neq \omega_0$. We can write (F2) as

$$\frac{\vec{F}(\omega_0)\vec{G}(\omega_0)}{2\omega_0}\left(\int_{\omega_0-\alpha}^{\omega_0-\epsilon} d\omega'+\int_{\omega_0+\epsilon}^{\omega_0+\alpha} d\omega'\right)\times\frac{1}{(\omega'-\omega_0)}\ln\frac{\omega'+\omega_0+\epsilon}{\omega'+\omega_0-\epsilon} , \qquad \text{(F3)}$$

where $\alpha > \epsilon$. The first integral in (F3) can be performed, with the result

$$\frac{\vec{F}(\omega_0)\vec{G}(\omega_0)}{2\omega_0}\ln\left(\frac{2\omega_0+\epsilon}{2\omega_0-\epsilon}\right)\ln\left(\frac{\epsilon}{\alpha}\right) , \qquad \text{(F4)}$$

which goes to zero as ϵ goes to zero. The second integral in (F3) can also be evaluated. It is found to be

$$\frac{\vec{F}(\omega_0)\vec{G}(\omega_0)}{2\omega_0}\ln\left(\frac{2\omega_0+\epsilon}{2\omega_0-\epsilon}\right)\ln\left(\frac{\alpha}{\epsilon}\right) , \qquad \text{(F5)}$$

which also goes to zero as ϵ goes to zero. The limit $\alpha \to 0$ is consequently unimportant. The integral (4.31) therefore vanishes.

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[1]F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965).

[2]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), pp. 578–608.

[3]*Sources of Quantum Mechanics*, edited by B. L. van der Waerden (Dover, New York, 1968), p. 12.

[4]A. Einstein, Phys. Z. 18, 121 (1917).

[5]*Collected Papers of L. D. Landau*, edited by D. ter Haar (Gordon and Breach, New York, 1967), pp. 8–18; F. Bloch, Phys. Z. 29, 58 (1928).

[6]V. Weisskopf and E. Wigner, Z. Phys. 63, 54 (1930).

[7]W. E. Lamb and R. C. Retherford, Phys. Rev. 72, 241 (1957).

[8]H. A. Bethe, Phys. Rev. 72, 339 (1947).

[9]T. A. Welton, Phys. Rev. 74, 1157 (1948).

[10]R. P. Feynman, in *The Quantum Theory of Fields*, Proceedings of the Twelfth Conference on Physics at the Univ. of Brussels, 1961 (Interscience, New York, 1963).

[11]V. F. Weisskopf, *Physics in the 20th Century* (MIT Press, Cambridge, Mass., 1972), especially pp. 97–111 and 120–127; Rev. Mod. Phys. 21, 305 (1949).

[12]S. B. Lai, P. L. Knight, and J. H. Eberly [Phys. Rev. Lett. 32, 494 (1974)] show that the source-field approach may be used successfully in a low-energy treatment of the electron's magnetic moment anomaly; see also P. L. Knight, S. B. Lai, and J. H. Eberly, Bull. Am. Phys. Soc. 18, 27 (1973).

[13]P. A. M. Dirac, *Lectures on Quantum Field Theory* (Belfer Graduate School of Science, Yeshiva University, New York, 1966).

[14]M. D. Crisp and E. T. Jaynes, Phys. Rev. 179, 1253 (1969); C. R. Stroud, Jr. and E. T. Jaynes, Phys. Rev. A 1, 106 (1970).

[15](a) G. W. Series, in *Optical Pumping and Atomic Line Shape*, edited by T. Skalinski (Panstwowe Wdawnictwo Naukowe, Warszawa, 1969), p. 25; Bull. Am. Phys. Soc. 18, 1523 (1973). (b) W. L. Lama and L. Mandel, in *Coherence and Quantum Optics*, edited by L. Mandel and E. Wolf (Plenum, New York, 1973), p. 273.

[16]R. K. Bullough, in *Coherence and Quantum Optics*, edited by L. Mandel and E. Wolf (Plenum, New York, 1973), p. 121.

[17]R. K. Nesbet, Phys. Rev. Lett. 27, 553 (1971).

[18]E. J. Moniz and D. H. Sharp, Phys. Rev. D 10, 1133 (1974).

[19]E. A. Power, *Introductory Quantum Electrodynamics* (Longmans, Green, London, 1964), pp. 117–124.

[20]For definiteness, we specifically consider a linear electric-dipole transition.

[21]The approximate integration of the source part of the Heisenberg field operator has recently played a role in work on many-atom cooperative emission effects: M. Dillard and H. Robl, Phys. Rev. 184, 312 (1969); R. H. Lehmberg, Phys. Rev. A 2, 883 (1970); J. H. Eberly and N. E. Rehler, *ibid.* 2, 1607 (1970); N. E. Rehler and J. H. Eberly, *ibid.* 3, 1735 (1971).

[22]I. R. Senitzky, Phys. Rev. Lett. 31, 955 (1973); P. W. Milonni, J. R. Ackerhalt, and W. A. Smith, *ibid.* 31, 958 (1973); P. W. Milonni and W. A. Smith (unpublished).

[23]These results have been reported recently. See (a) I. R. Senitzky, Phys. Rev. A 6, 1175 (1972), specifically p. 1182; (b) J. R. Ackerhalt, P. L. Knight, and J. H. Eberly, Phys. Rev. Lett. 30, 456 (1973).

[24]R. H. Lehmberg, Phys. Rev. A 2, 889 (1970).

[25]J. R. Ackerhalt, J. H. Eberly, and P. L. Knight, in *Coherence and Quantum Optics*, edited by L. Mandel and E. Wolf (Plenum, New York, 1973), p. 635.

[26]See L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, to be published), Chap. 7.

[27]L. I. Schiff, *Quantum Mechanics*, third edition (McGraw-Hill, New York, 1968), pp. 28 and 179.

[28]P. W. Milonni, Ph.D. thesis, Univ. of Rochester, 1974 (unpublished).

[29]S. B. Lai (unpublished).