Effective potential for the O(N) model to order $1/N$ †

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The effective potential of the O(N) model is calculated to the next-to-leading order in the $1/N$ expansion in one, two, three, and four space-time dimensions. In one and two dimensions the vacuum is symmetric, as expected. In three dimensions the radiative corrections of order $1/N$ do not affect the position of the minimum of the effective potential. In four dimensions, the contribution to the effective potential of order $1/N$ is complex everywhere, suggesting that the $1/N$ expansion fails.

I. INTRODUCTION

Recently several authors[1-3] have approximated the effective potential by the leading term in the $1/N$ expansion, where N is the number of fields. They found that this infinite set of loop diagrams revealed interesting aspects of the effective potential which were not present in the one- and two-loop approximations. In this paper we evaluate in one, two, three, and four space-time dimensions the next-to-leading terms for a $\lambda\Phi^4$ theory possessing an O(N) symmetry. We follow closely the approach of Coleman, Jackiw, and Politzer.[1]

The model we wish to study is described by the Lagrangian density

$$\mathcal{L}(\Phi)=\tfrac{1}{2}\partial_\mu\Phi_i\partial^\mu\Phi_i-\tfrac{1}{2}\mu^2\Phi_i\Phi_i-\frac{\lambda}{4!N}(\Phi_i\Phi_i)^2\,, \tag{1.1}$$

where the repeated Latin subscripts are to be summed from one to N. The mass-squared term μ^2 may be either positive or negative, but the coupling constant λ is required to be positive for a stable vacuum to exist.

It has been shown by Jackiw[4] that the effective potential $V(\phi)$ for this theory is given by the tree approximation,

$$V_{\text{tree}}(\phi)=\tfrac{1}{2}\mu^2\phi^2+\frac{\lambda}{4!N}\phi^4\,,$$

plus the sum of all connected one-particle irreducible vacuum graphs for the theory described by the shifted Lagrangian density

$$\hat{\mathcal{L}}(\Phi_i(x);\phi_i)=\mathcal{L}(\Phi_i(x)+\phi_i)+V_{\text{tree}}(\phi)+\mu^2\Phi_i(x)\phi_i+\frac{\lambda}{3!N}\Phi_j(x)\phi_j\phi^2\,, \tag{1.2}$$

where ϕ is the vacuum expectation value of $\Phi(x)$. Figure 1 shows typical diagrams which contribute to the effective potential.

The effective potential to order $(1/N)^m$ is found by considering only those diagrams which have a combinatorical factor, due to summing over internal lines, which is greater than or equal to $(N)^{l-m}$, where l is the number of loops in the diagram. (The tree approximation has $l=0$ and a combinatorical factor of one, hence it contributes to the order-one term.) However, if ϕ is treated as order $\sqrt{N}$, the term of order $(1/N)^m$ can be identified directly with the contributions proportional to $(1/N)^{m-1}$. The next-to-leading-order contributions are most easily found by the second procedure; therefore we introduce a rescaled vacuum expectation value in order to manifest the $1/N$ dependence. In all subsequent calculations we refer only to the rescaled field, thus no confusion should result if henceforth we use the notation ϕ for the rescaled vacuum expectation value; that is,

$$\langle\Phi(x)\rangle=\sqrt{N}\phi\,.$$

The order-one term is then proportional to N, and the order-$1/N$ term is proportional to one.

II. DERIVATION OF THE FORMAL EXPRESSION

The contribution to the effective potential to order $1/N$ can be calculated by a straightforward though tedious direct summation of diagrams similar to those shown in Figs. 1(a) and 1(b).[5] However, the evaluation is simplified considerably by using a combinatorical trick discussed by Coleman, Jackiw, and Politzer.[6]

The dynamics remains unchanged if an extra field $\chi(x)$ is introduced into the Lagrangian density (1.1) in the following manner:

$$\mathcal{L}(\Phi,\chi)=\mathcal{L}(\Phi)+\frac{3N}{2\lambda}\left(\chi-\frac{\lambda}{6N}\Phi_i\Phi_i-\mu^2\right)^2. \tag{2.1}$$

The Euler-Lagrange equation for $\chi(x)$ is merely an equation of constraint. The effective potential $V(\phi,\hat{\chi})$ of this modified theory reduces to the effective potential of the original theory if $\hat{\chi}$ satisfies the requirement

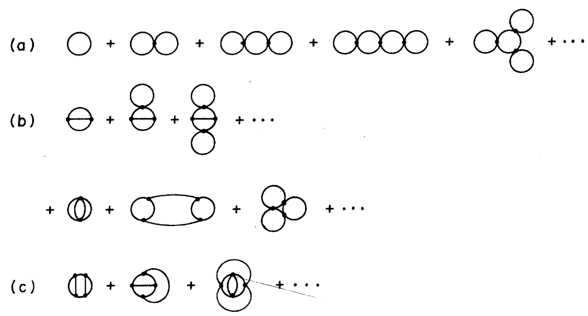

FIG. 1. Typical contributions to $V(\phi)$: (a) Contributions of order one. (b) Contributions of order $1/N$. (c) Contributions of order $1/N^2$ or higher.

$$\frac{\partial V(\phi,\hat{\chi})}{\partial\hat{\chi}}=0\,. \tag{2.2}$$

We use (2.2) to define $\hat{\chi}$ as a function of ϕ.

Although the introduction of $\chi(x)$ does not alter the dynamics of the full theory, it does lead to a new perturbation series in which the $1/N$ expansion has a simple diagrammatic interpretation. The modified Lagrangian density can be written as

$$\mathcal{L}(\Phi,\chi)=\tfrac{1}{2}\partial_\mu\Phi_i\partial^\mu\Phi_i+\frac{3N\chi^2}{2\lambda}-\frac{\chi\Phi_i\Phi_i}{2}-\frac{3N\mu^2\chi}{\lambda}+\frac{3N\mu^4}{2\lambda}. \tag{2.3}$$

The effective potential $V(\phi,\hat{\chi})$ can then be calculated by the method of Jackiw by shifting the fields as follows:

$$\begin{aligned}\Phi_i(x)&=\Phi_i(x)\,,\quad i=2\text{ to }N\\ \Phi_1(x)&=\sigma(x)+\sqrt{N}\phi\,,\\ \chi(x)&=\tilde{\chi}(x)+\hat{\chi}\,.\end{aligned} \tag{2.4}$$

Then the tree approximation is

$$V_{\text{tree}}(\phi,\hat{\chi})=N\left(-\frac{3\hat{\chi}^2}{2\lambda}+\frac{\hat{\chi}\phi^2}{2}+\frac{3\mu^2\hat{\chi}}{\lambda}\right), \tag{2.5}$$

where a term independent of ϕ and $\hat{\chi}$ has been dropped. The shifted Lagrangian density with constant and linear terms deleted is found to be

$$\begin{aligned}\mathcal{L}(\Phi,\sigma,\tilde{\chi};\phi,\hat{\chi})=&\tfrac{1}{2}\partial_\mu\Phi_j\partial^\mu\Phi_j-\tfrac{1}{2}\hat{\chi}\Phi_j\Phi_j\\ &+\tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma-\tfrac{1}{2}\hat{\chi}\sigma^2\\ &+\frac{3N\tilde{\chi}^2}{2\lambda}-\tfrac{1}{2}\tilde{\chi}(\sigma^2+\Phi_j\Phi_j)-\tilde{\chi}\sigma\phi\sqrt{N}\,,\end{aligned} \tag{2.6}$$

where the sum over repeated Latin indices goes from two to N. The nonzero components of the inverse propagator matrix $D^{-1}{}_{\alpha\beta}$ then have the following momentum-space representation:

$$\begin{aligned}iD^{-1}{}_{jl}(\phi,\hat{\chi};k^2)&=\delta_{jl}(k^2-\hat{\chi})\,,\\ iD^{-1}{}_{\sigma\sigma}(\phi,\hat{\chi};k^2)&=k^2-\hat{\chi}\,,\\ iD^{-1}{}_{\chi\chi}(\phi,\hat{\chi},k)&=\frac{3N}{\lambda}\,,\\ iD^{-1}{}_{\chi\sigma}(\phi,\hat{\chi};k)&=iD^{-1}{}_{\sigma\chi}(\phi,\hat{\chi};k)=-\phi\sqrt{N}\,.\end{aligned} \tag{2.7}$$

The relevant propagators are

$$\begin{aligned}D_{ij}&=i\delta_{ij}(k^2-\hat{\chi}+i\epsilon)^{-1}\,,\\ D_{\sigma\sigma}&=i(k^2-\hat{\chi}-\tfrac{1}{3}\lambda\phi^2+i\epsilon)^{-1}\,,\\ D_{\chi\chi}&=\frac{i\lambda}{3N}\,\frac{(k^2-\hat{\chi})}{(k^2-\hat{\chi}-\frac{1}{3}\lambda\phi^2+i\epsilon)},\\ D_{\chi\sigma}&=\frac{i\lambda\phi}{3\sqrt{N}}(k^2-\hat{\chi}-\tfrac{1}{3}\lambda\phi^2+i\epsilon)^{-1}\,.\end{aligned} \tag{2.8}$$

The only interaction term in the Lagrangian density is

$$-\tfrac{1}{2}\tilde{\chi}(\sigma^2+\Phi_j\Phi_j)\,. \tag{2.9}$$

The Feynman rules are summarized in Fig. 2; typical contributions to the effective potential in this perturbation series are shown in Fig. 3.

The diagrams which contribute to the effective potential to order $1/N$ can be determined easily if we keep in mind the behavior in $1/N$ of the propagators. We associate with the propagators the following factors: (1) a factor of $1/N$ with $D_{\chi\chi}$ from its definition; (2) a factor or order N with a closed loop of Φ fields, due to the summation over possible internal fields; (3) a factor $1/\sqrt{N}$ with $D_{\chi\sigma}$ from its definition. Note that all multiloop diagrams which involve the $\chi\sigma$ propagator are of order $1/N^2$ or higher [see Fig. 3(e)].

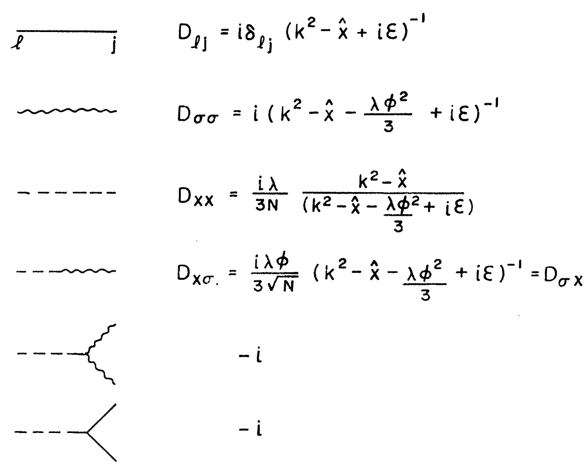

FIG. 2. Feynman rules for the theory involving the χ field.

The only diagram of order one is the closed loop shown in Fig. 3(a). The one-loop contribution involving $D_{\chi\chi}$, $D_{\chi\sigma}$, and $D_{\sigma\sigma}$, which we have diagrammatically represented in Fig. 3(b) as a pair of one-loop diagrams, is of order $1/N$. The only other contributions of order $1/N$ must have a closed Φ loop (a factor N) associated with each $\chi\chi$ propagator (a factor $1/N$). Only closed loops formed by alternating $\chi\chi$ propagators and closed Φ loops as shown in Fig. 3(c) have this behavior. There are no further contributions to order $1/N$.

The contributions to Figs. 3(a), 3(b), and 3(c) in l space-time dimensions are readily evaluated: For Fig. 3(a) we have

$$-\tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}(N-1)\ln(k^2-\hat{\chi}+i\epsilon)\,; \tag{2.10}$$

for Fig. 3(b),

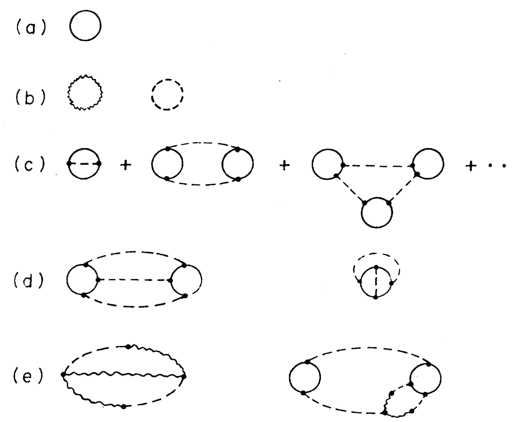

FIG. 3. Typical contributions to $V(\varphi,\,\hat{\chi})$. (a) The order-one contribution. (b) The single-loop order-$1/N$ contribution. (c) Typical multiloop order-$1/N$ contributions. (d) Typical higher-order terms. (e) Higher-order terms involving $D_{\chi\sigma}$.

$$-\tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln\det\begin{pmatrix}k^2-\hat{\chi} & -\phi\sqrt{N}\\ -\phi\sqrt{N} & 3N/\lambda\end{pmatrix}=-\tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln(k^2-\hat{\chi}-\tfrac{1}{3}\lambda\phi^2+i\epsilon)+\text{constant}\,; \tag{2.11}$$

and for Fig. 3(c),

$$i\int\frac{d^lk}{(2\pi)^l}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{(k^2-\hat{\chi})B(\hat{\chi},k^2)}{k^2-\hat{\chi}-\frac{1}{3}\lambda\phi^2+i\epsilon}\right)^n=-\tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln\left(\frac{(k^2-\hat{\chi})[1-B(\hat{\chi},k^2)]-\frac{1}{3}\lambda\phi^2+i\epsilon}{k^2-\hat{\chi}-\frac{1}{3}\lambda\phi^2+i\epsilon}\right), \tag{2.12}$$

where, for convenience, we have defined

$$B(\hat{\chi},k^2)\equiv\tfrac{1}{6}\lambda\int\frac{d^lp}{(2\pi)^l}\frac{i}{(p^2-\hat{\chi}+i\epsilon)[(k+p)^2-\hat{\chi}+i\epsilon]}. \tag{2.13}$$

Adding together the contributions from (2.5) and (2.10)–(2.13) we find

$$\begin{aligned}V(\phi,\hat{\chi})=N&\left(-\frac{3\hat{\chi}^2}{2\lambda}+\frac{\hat{\chi}\phi^2}{2}+\frac{3\mu^2\hat{\chi}}{\lambda}-\tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln(k^2-\hat{\chi}+i\epsilon)\right)\\ &-\tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln\left(\frac{(k^2-\hat{\chi})[1-B(\hat{\chi},k^2)]-\frac{1}{3}\lambda\phi^2+i\epsilon}{k^2-\hat{\chi}+i\epsilon}\right).\end{aligned} \tag{2.14}$$

The effective potential of the original theory is then found by evaluating (2.14) at $\hat{\chi}=\overline{X}(\phi)$ which is defined as the solution to

$$\left.\frac{\partial V(\phi,\hat{\chi})}{\partial\hat{\chi}}\right|_{\hat{\chi}=\overline{X}(\phi)}=0\,. \tag{2.15}$$

We find

$$\begin{aligned}\overline{X}(\phi)=\mu^2+\tfrac{1}{6}\lambda\phi^2+\tfrac{1}{6}\lambda\int\frac{d^lk}{(2\pi)^l}\frac{i}{k^2-\overline{X}+i\epsilon}+\frac{\lambda}{6N}\int\frac{d^lk}{(2\pi)^l}\frac{i(k^2-\overline{X})\partial B(\overline{X},k^2)/\partial\overline{X}}{(k^2-\overline{X})[1-B(\overline{X},k^2)]-\frac{1}{3}\lambda\phi^2+i\epsilon}\\ +\frac{\lambda\phi^2}{3}\frac{\lambda}{6N}\int\frac{d^lk}{(2\pi)^l}\frac{i}{(k^2-\overline{X}+i\epsilon)\{(k^2-\overline{X})[1-B(\overline{X},k^2)]-\frac{1}{3}\lambda\phi^2+i\epsilon\}}.\end{aligned} \tag{2.16}$$

For comparison with previous evaluations, the effective potential to order one is found by solving (2.15) to order one. If we write $\overline{X}(\phi)=X(\phi)+O(1/N)$ we find

$$X(\phi)=\mu^2+\tfrac{1}{6}\lambda\phi^2+\tfrac{1}{6}\lambda\int\frac{d^lk}{(2\pi)^l}\frac{i}{k^2-X(\phi)+i\epsilon}, \tag{2.17}$$

which agrees with the results of previous works.

Furthermore, as a consequence of the defining equation for $\overline{X}(\phi)$, Eq. (2.15), we know that $\partial V/\partial \hat{\chi}|_X$ is proportional to one and $(\overline{X}-X)$ is proportional to $1/N$. Hence we have

$$V(\phi,\overline{X}(\phi)) = V(\phi, X(\phi)) + (\overline{X}-X)\frac{\partial V(\phi,X)}{\partial X}\bigg|_{X(\phi)} + O\left(\frac{1}{N}\right)$$
$$= V(\phi, X(\phi)) + O\left(\frac{1}{N}\right). \qquad (2.18)$$

Thus the effective potential to order $1/N$ is given by $V(\phi,\hat{\chi})$ defined by (2.14) but with $\hat{\chi}$ defined only by the leading-order equation (2.16), that is, $\hat{\chi}=X(\phi)$. Of course this simplification is limited to the order-$1/N$ calculation. In order to find $V(\phi)$ to higher orders, higher-order approximations to $\overline{X}(\phi)$ must be used.

To order $1/N$, the following expressions for the effective potential and its derivative are sufficient:

$$V(\phi) = N\left(\frac{-3X^2}{2\lambda} + \frac{X\phi^2}{2} + \frac{3\mu^2 X}{\lambda} - \tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln(k^2 - X + i\epsilon)\right)$$
$$-\tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln[1-B(X,k^2)] - \tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln\left(1-\frac{\lambda\phi^2}{3}\frac{1}{(k^2-X)[1-B(X,k^2)]}\right), \qquad (2.19)$$

$$2\frac{dV(\phi)}{d\phi^2} = NX(\phi) + \frac{d}{d\phi^2}\left[i\int\frac{d^lk}{(2\pi)^l}\ln\left(\frac{k^2-X+i\epsilon}{(k^2-X)[1-B(X,k^2)]-\frac{1}{3}\lambda\phi^2+i\epsilon}\right)\right]$$
$$= NX(\phi) + \tfrac{1}{3}\lambda\int\frac{d^lk}{(2\pi)^l}\frac{i}{(k^2-X)[1-B(X,k^2)]-\frac{1}{3}\lambda\phi^2+i\epsilon}$$
$$+\frac{\partial X}{\partial\phi^2}\left(\int\frac{d^lk}{(2\pi)^l}\frac{i(k^2-X)\partial B(X,k^2)/\partial X}{(k^2-X)[1-B(X,k^2)]-\frac{1}{3}\lambda\phi^2+i\epsilon}\right.$$
$$\left.+\tfrac{1}{3}\lambda\phi^2\int\frac{d^lk}{(2\pi)^l}\frac{i}{(k^2-X+i\epsilon)\{(k^2-X)[1-B(X,k^2)]-\frac{1}{3}\lambda\phi^2+i\epsilon\}}\right), \qquad (2.20)$$

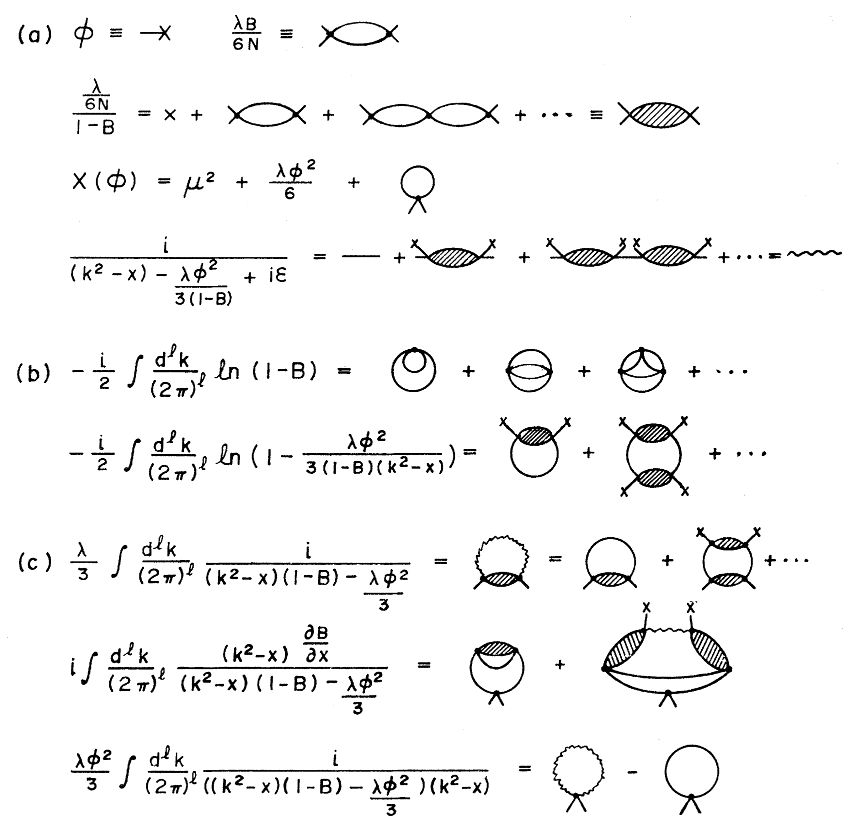

FIG. 4. Diagrammatic interpretation of the order-$1/N$ contributions to $V(\phi, X(\phi))$: (a) Definition of some recurring expressions. (b) Order-$1/N$ contributions to $V(\phi, X(\phi))$. (c) Order-$1/N$ contributions to $2dV/d\phi^2$.

where

$$X(\phi)=\mu^2+\tfrac{1}{6}\lambda\phi^2+\tfrac{1}{6}\lambda\int\frac{d^lk}{(2\pi)^l}\frac{i}{k^2-X+i\epsilon}. \quad (2.21)$$

It is helpful to interpret the expressions appearing in the above evaluations of $V(\phi)$ and $2dV/d\phi^2$ in terms of diagrams involving only the original Φ fields. The χ field appears only as $X(\phi)$, and it acts as a ϕ dependent mass term in the $\Phi\Phi$ propagator,

$$i[k^2-X(\phi)+i\epsilon]^{-1}.$$

The diagrammatic interpretation of several of the functions occurring in $V(\phi)$ and $2dV/d\phi^2$ is given in Fig. 4. In Fig. 4(a) we give the interpretation of many of the recurring expressions. Note that $\lambda/[6N(1-B)]$ plays the role of the complete, to order $1/N$, four-point function used in the calculation of $V(\phi)$, and

$$i\left(k^2-X-\frac{\lambda\phi^2}{3}\frac{1}{1-B(X,k^2)}\right)^{-1}$$

is a modified propagator. The representation of two of the terms in $V(\phi)$ is shown in Fig. 4(b), while three terms appearing in $2dV/d\phi^2$ are shown in Fig. 4(c).

III. RENORMALIZED RESULTS

A. The renormalization procedure

The expressions (2.21)–(2.23) derived in the previous section are formal ones; in general they must be renormalized. We can determine how renormalization modified the results by returning to the original Lagrangian density and reexpressing it as

$$\mathcal{L}_R(\Phi)=\tfrac{1}{2}Z\partial^\mu\Phi_i\partial_\mu\Phi_i-\tfrac{1}{2}\bar{\mu}^2\Phi_i\Phi_i-\frac{\bar{\lambda}}{4!N}(\Phi_i\Phi_i)^2, \quad (3.1)$$

where Φ_i is now the renormalized field and we define

$$Z=1-z,$$
$$\bar{\mu}^2=\mu^2+A, \quad (3.2)$$
$$\bar{\lambda}=\lambda+C.$$

The renormalized mass and coupling constant are μ^2 and λ, respectively, while A, z, and C are counterterms designed to cancel infinities. Both A and C may have order-one terms, but z is of order $1/N$.

The combinatoric trick now requires using the new Lagrangian density

$$\mathcal{L}_R(\Phi,\chi)=\mathcal{L}_R(\Phi)+\frac{3N}{2\bar{\lambda}}\left(\chi-\frac{\bar{\lambda}\Phi_i\Phi_i}{6N}-\bar{\mu}^2\right)^2. \quad (3.3)$$

By the simple substitutions

$$\Phi_i\to\sqrt{Z}\,\Phi_i,\quad \chi\to\frac{\chi}{Z}$$
$$\lambda\to\bar{\lambda}\frac{1}{Z^2},\quad \mu^2\to\bar{\mu}^2\frac{1}{Z} \quad (3.4)$$

the original Lagrangian density (2.1) involving $\chi(x)$ is converted to the Lagrangian density (3.3) involving renormalized quantities and counterterms. Similarly the effective potential in terms of the renormalized quantities is easily obtained by making the aforementioned substitutions (3.4) into the expression for $V(\phi,\chi)$ derived previously (2.14):

$$V(\phi,\hat{\chi})=N\left[-\frac{3\hat{\chi}^2}{2\bar{\lambda}}+\frac{\hat{\chi}\phi^2}{2}+\frac{3\bar{\mu}^2\hat{\chi}}{\bar{\lambda}}-\tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln\left(k^2-\frac{\hat{\chi}}{Z}+i\epsilon\right)\right]$$
$$-\tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln\left[1-\bar{B}\left(\frac{\hat{\chi}}{Z},k^2\right)\right]-\tfrac{1}{2}i\int\frac{d^lk}{(2\pi)^l}\ln\left(1-\frac{\frac{1}{3}\bar{\lambda}\phi^2}{(k^2-\hat{\chi}/Z+i\epsilon)[1-\bar{B}(\hat{\chi}/Z,k^2)]}\right), \quad (3.5)$$

where

$$\bar{B}(\hat{\chi},k^2)=\tfrac{1}{6}\bar{\lambda}\int\frac{d^lp}{(2\pi)^l}\frac{i}{(p^2-\hat{\chi}+i\epsilon)[(k+p)^2-\hat{\chi}+i\epsilon]}. \quad (3.6)$$

As before the effective potential $V(\phi)$ is found by evaluating $V(\phi,\hat{\chi})$ at $\hat{\chi}=\bar{X}(\phi)$ determined by $\partial V/\partial\hat{\chi}|_{\bar{X}}=0$.

Since we are interested in $V(\phi)$ only to order $1/N$, it is sufficient to expand $\bar{\lambda}$ and $\bar{\mu}^2$ in powers of $1/N$ as follows:

$$\bar{\mu}^2=\mu_N{}^2+\mu_1{}^2+O\left(\frac{1}{N^2}\right),$$
$$\frac{1}{\bar{\lambda}}=\frac{1}{\lambda_N}\left[1+\delta+O\left(\frac{1}{N^2}\right)\right], \quad (3.7)$$

where $\mu_1{}^2$ and δ are of order $1/N$. Then, continuing as in the previous section, we define $X(\phi)$ by the order-one equation for $\bar{X}(\phi)$; that is,

$$X(\phi)=\mu_N{}^2+\tfrac{1}{6}\lambda_N\phi^2+\tfrac{1}{6}\lambda_N\int\frac{d^lk}{(2\pi)^l}\frac{i}{k^2-X(\phi)+i\epsilon}. \quad (3.8)$$

Finally, to order $1/N$ the derivative of the effective potential is given by

$$2\frac{dV}{d\phi^2}=NX(\phi)+\tfrac{1}{3}\lambda_N\int\frac{d^lk}{(2\pi)^l}\frac{i}{(k^2-X)[1-\overline{B}(X,k^2)]-\frac{1}{3}\lambda_N\phi^2+i\epsilon}$$
$$+\frac{\partial X}{\partial\phi^2}\left(-\frac{6NX\delta}{\lambda_N}+\frac{6N}{\lambda_N}(\mu_N{}^2\delta+\mu_1{}^2)+iNz\int\frac{d^lk}{(2\pi)^l}\frac{k^2}{(k^2-X+i\epsilon)^2}+i\int\frac{d^lk}{(2\pi)^l}\frac{(k^2-X)\partial\overline{B}(X,k^2)/\partial X}{(k^2-X)[1-\overline{B}(X,k^2)]-\frac{1}{3}\lambda_N\phi^2}\right.$$
$$\left.+\tfrac{1}{3}\lambda_N\phi^2\int\frac{d^lk}{(2\pi)^l}\frac{i}{\{(k^2-X)[1-\overline{B}(X,k^2)]-\frac{1}{3}\lambda_N\phi^2\}(k^2-X)}\right). \qquad (3.9)$$

The renormalization program will be successful if all the infinities arising from integrals in (3.8) are canceled by $\mu_N{}^2$ and λ_N, so that $X(\phi)$ is finite, and the infinities occurring in the four integrals in (3.9) are canceled by the terms involving δ, $\mu_1{}^2$, and z.

B. One dimension

No infinities need be absorbed into renormalization constants in one dimension (time), hence the formal expressions (2.22) and (2.23) are correct as they stand. We find

$$B(X,k^2)=\frac{\lambda}{6\sqrt{X}}\frac{1}{k^2-4X+i\epsilon}, \qquad (3.10)$$

which, after a Wick rotation to Euclidean momentum space, leads to

$$2\frac{dV}{d\phi^2}=NX(\phi)+\frac{d}{d\phi^2}\left[\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\ln\left(\frac{(k^2+4X)(k^2+X)+\frac{1}{3}\lambda\phi^2(k^2+4X)+(\lambda/6\sqrt{X})(k^2+X)}{(k^2+X)(k^2+4X)}\right)\right], \qquad (3.11)$$

where $X(\phi)$ is the real solution of

$$X(\phi)=\mu^2+\tfrac{1}{6}\lambda\phi^2+\tfrac{1}{12}\lambda/[X(\phi)]^{1/2}. \qquad (3.12)$$

Performing the integration, we can further express (3.11) as

$$2\frac{dV}{d\phi^2}=NX(\phi)+\frac{d}{d\phi^2}[(m_+{}^2)^{1/2}+(m_-{}^2)^{1/2}-3\sqrt{X}], \qquad (3.13)$$

with

$$m_\pm{}^2=\tfrac{7}{2}X-\mu^2\pm[(\tfrac{7}{2}X-\mu^2)^2-4X^2-\tfrac{4}{3}X\lambda\phi^2-\tfrac{1}{6}\lambda\sqrt{X}]^{1/2}. \qquad (3.14)$$

Recall that there is a positive real solution for $X(\phi)$ for all ϕ irrespective of the sign of μ^2. Therefore the effective potential has an absolute minimum at the origin whenever N is sufficiently large. For given values of λ and μ^2, (3.13) can be used to determine the values of N for which the order-one terms can be neglected. We do not pursue this application any further here; we merely note that the leading-order result remains valid for large N.

When the tree approximation has a displaced minimum, the radiative corrections of order one fill in the dip, and restore the symmetry. Further corrections of order $1/N$ do not alter the position of the minimum. This result is reasonable since the propagator no longer has a vanishing mass term and no new infrared divergences occur.

C. Two dimensions

The only infinities which occur in two space-time dimensions can be canceled by the mass counterterms. The infinite part of the order-one counterterm is completely determined by the definition of $X(\phi)$. We can write

$$X(\phi)=\mu_N{}^2+\tfrac{1}{6}\lambda\phi^2+\tfrac{1}{6}i\lambda\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2-X+i\epsilon}$$
$$=\mu^2+\tfrac{1}{6}\lambda\phi^2-\frac{\lambda}{24\pi}\ln\frac{X}{|\mu^2|}, \qquad (3.15)$$

if we define

$$\mu_N{}^2=\mu^2-\mathrm{Re}\left(\tfrac{1}{6}\lambda\int\frac{d^2k}{(2\pi)^2}\frac{i}{k^2-\mu^2+i\epsilon}\right). \qquad (3.16)$$

It is necessary to take the real part of the integral to ensure that the counterterm remains real even when $\mu^2<0$. This form for the counterterm was chosen since it is a natural generalization to $\mu^2<0$ of the counterterm which preserves for $\mu^2>0$ the tree-approximation result $X(\phi=0)=\mu^2$. (Of course we could have used an arbitrary positive mass squared M^2 and defined

$$\mu_N{}^2=\mu^2-\tfrac{1}{6}\lambda\int\frac{d^2k}{(2\pi)}\frac{i}{k^2-M^2+i\epsilon}, \qquad (3.17)$$

which does not involve taking a real part, but then μ^2 loses its interpretation for $\mu^2>0$ as the value of X at $\phi = 0$.)

As in the one-dimensional theory, $X(\phi)$ has a real positive solution for all μ^2, hence, insofar as the order-$1/N$ terms are small, the effective potential has an absolute minimum at the origin. That is, no infrared divergences occur in order $1/N$ which could affect the position of the minimum for sufficiently large N. The order-$1/N$ result which we now derive can be used to find the values of N for given λ and μ^2 for which the leading-order term indeed dominates.

To complete the evaluation of $2dV/d\phi^2$ to order one, we use the following expressions for $B(X,k^2)$ and $\partial B/\partial X$ for Euclidean momenta:

$$B(X,k^2) = \frac{-\lambda}{24\pi\sqrt{k^2}} \frac{1}{(X+\frac{1}{4}k^2)^{1/2}} \ln\left(\frac{[(X+\frac{1}{4}k^2)^{1/2}+\frac{1}{2}\sqrt{k^2}\,]^2}{X}\right), \tag{3.18}$$

$$\frac{\partial B(X,k^2)}{\partial X} = \frac{\lambda}{24\pi}\left[\frac{2}{X(k^2+4X)} + \frac{1}{2\sqrt{k^2}\,(X+\frac{1}{4}k^2)^{3/2}} \ln\left(\frac{\{(X+\frac{1}{4}k^2)^{1/2}+(\frac{1}{4}k^2)^{1/2}\}^2}{X}\right)\right]. \tag{3.19}$$

The derivative of $X(\phi)$ is found to be

$$\frac{\partial X}{\partial \phi^2} = \frac{\frac{1}{6}\lambda}{1+\lambda/24\pi X}. \tag{3.20}$$

Substituting these expressions into the renormalized equation for $2dV/d\phi^2$ we find, in terms of integrals over Euclidean momenta,

$$2\frac{dV}{d\phi^2} = NX(\phi) + \frac{\partial X}{\partial\phi^2}\left[\frac{6N\mu_1^2}{\lambda} + 2\left(1+\frac{\lambda}{24\pi X}\right)\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2} - \int\frac{d^2k}{(2\pi)^2}\frac{(k^2+X)\partial B(X,k^2)/\partial X}{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2} - \tfrac{1}{3}\lambda\phi^2\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+X)\{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2\}}\right]. \tag{3.21}$$

All the infinite terms are contained in

$$\frac{6N\mu_1^2}{\lambda} + 2\left(1+\frac{\lambda}{24\pi X}\right)\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+X)[1-B(X,k^2)]} - \int\frac{d^2k}{(2\pi)^2}\frac{\partial B(X,k^2)/\partial X}{1-B(X,k^2)}. \tag{3.22}$$

If we choose

$$\mu_1^2 = -\frac{\lambda}{3N}\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2+|\mu^2|}\frac{1}{1-B(|\mu^2|,k^2)} + \delta\tilde{\mu}_1^2, \tag{3.23}$$

where $\delta\tilde{\mu}_1^2$ is a finite constant which can be chosen to meet a selected renormalized condition, then (3.22) becomes

$$\frac{6N\delta\tilde{\mu}_1^2}{\lambda} + \frac{\frac{1}{3}\lambda}{\partial X/\partial\phi^2}\int\frac{d^2k}{(2\pi)^2}\left(\frac{1}{(k^2+X)[1-B(X,k^2)]} - \frac{1}{k^2+|\mu^2|}\frac{1}{1-B(|\mu^2|,k^2)}\right) - \int\frac{d^2k}{(2\pi)^2}\left(\frac{\partial B(X,k^2)/\partial X}{1-B(X,k^2)} - \frac{\lambda}{12\pi X}\frac{1}{1-B(|\mu^2|,k^2)}\frac{1}{k^2+|\mu^2|}\right). \tag{3.24}$$

However, in the limit of large k^2 we find

$$\frac{\partial B}{\partial X} - \frac{\lambda}{12\pi X}\frac{1}{k^2+|\mu^2|} \sim O\left(\frac{1}{k^4}\right); \tag{3.25}$$

thus each of the three terms in (3.24) is finite. The final form for the derivative of the effective potential in two dimensions is

$$2\frac{dV(\phi)}{d\phi^2} = NX(\phi) + \tfrac{1}{3}\lambda\int\frac{d^2k}{(2\pi)^2}\left(\frac{1}{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2} - \frac{1}{(k^2+|\mu^2|)[1-B(|\mu^2|,k^2)]}\right) + \frac{\partial X}{\partial\phi^2}\left[\frac{6N\delta\tilde{\mu}_1^2}{\lambda} - \int\frac{d^2k}{(2\pi)^2}\left(\frac{(k^2+X)\partial B(X,k^2)/\partial X}{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2} - \frac{\lambda}{12\pi X}\frac{1}{[1-B(|\mu^2|,k^2)](k^2+|\mu^2|)}\right) - \tfrac{1}{3}\lambda\phi^2\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+X)\{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2\}}\right]. \tag{3.26}$$

For the leading-order approximation, the counterterm was determined entirely for $\mu^2>0$ by the requirement $X(\phi=0)=0$. Here it is no longer simple to find a μ^2-dependent integral expression for $\delta\tilde{\mu}_1^2$ which preserves $2dV/d\phi^2=\mu^2$; hence $\delta\tilde{\mu}_1^2$ is left as an arbitrary finite constant.

D. Three dimensions

In three space-time dimensions all the infinities can still be absorbed into the mass renormalization. Choosing the counterterm to be

$$\mu_N^2-\mu^2=-\mathrm{Re}\left(\tfrac{1}{6}\lambda\int\frac{d^3k}{(2\pi)^3}\frac{i}{k^2-\mu^2+i\epsilon}\right), \quad (3.27)$$

we find

$$X(\phi)=\mu_N^2+\tfrac{1}{6}\lambda\phi^2+\tfrac{1}{6}\lambda\int\frac{d^3k}{(2\pi)^3}\frac{i}{k^2-X+i\epsilon}$$
$$=\mu^2+\tfrac{1}{6}\lambda\phi^2-\frac{\lambda\sqrt{X}}{24\pi}+\frac{\lambda}{24\pi}\sqrt{\mu^2}\,\Theta(\mu^2), \quad (3.28)$$

where $\Theta(\mu^2)=1$ if $\mu^2>0$, and zero otherwise.

In Euclidean momentum space the functions needed in order to evaluate the order-$1/N$ contribution to the effective potential are

$$B(X,k^2)=-\frac{\lambda}{24\pi\sqrt{k^2}}\sin^{-1}\left(\frac{1}{(1+4X/k^2)^{1/2}}\right), \quad (3.29)$$

$$\frac{\partial B(X,k^2)}{\partial X}=\frac{\lambda}{24\pi\sqrt{X}}\frac{1}{k^2+4X}, \quad (3.30)$$

$$\frac{\partial X}{\partial\phi^2}=\frac{\lambda}{6}\left(1+\frac{\lambda}{48\pi\sqrt{X}}\right)^{-1}. \quad (3.31)$$

The renormalization of the order-$1/N$ terms in the three-dimensional theory follows the two-dimensional procedure closely. By choosing

$$\mu_1^2=-\frac{\lambda}{3N}\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+|\mu^2|)[1-B(|\mu^2|,k^2)]}+\delta\tilde{\mu}_1^2,$$

we find that

$$2\frac{dV}{d\phi^2}=NX(\phi)+\tfrac{1}{3}\lambda\int\frac{d^3k}{(2\pi)^3}\left(\frac{1}{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2}-\frac{1}{(k^2+|\mu^2|)[1-B(|\mu^2|,k^2)]}\right)$$
$$+\frac{\partial X}{\partial\phi^2}\left[\frac{6N\delta\tilde{\mu}_1^2}{\lambda}-\int\frac{d^3k}{(2\pi)^3}\left(\frac{(k^2+X)\partial B(X,k^2)/\partial X}{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2}-\frac{\lambda}{24\pi\sqrt{X}}\frac{1}{(k^2+|\mu^2|)[1-B(|\mu^2|,k^2)]}\right)\right.$$
$$\left.-\tfrac{1}{3}\lambda\phi^2\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+X)\{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2\}}\right]. \quad (3.32)$$

If $\mu^2>0$, the equation which defines $X(\phi)$ has a real positive solution, hence the minimum of $V(\phi)$ must be at the origin. However, if μ^2 is negative, $X(\phi)$ is real and positive only if $\frac{1}{6}\lambda\phi^2$ is larger than $-\mu^2$. At $\lambda\phi^2=-6\mu^2$, we find $X(\phi)=0$, while for $\frac{1}{6}\lambda\phi^2$ less than $-\mu^2$ no solution exists.[7] We interpret this behavior as a symmetry-breaking minimum of $V(\phi)$ at $\lambda\phi^2=-6\mu^2$. Since the mass appearing in the propagator vanishes at $\lambda\phi^2=-6\mu^2$, higher-order radiative corrections may have a significant effect on the position of the minimum. In the limit of X going to zero, $\partial X/\partial\phi^2$ behaves as $\sqrt{X}$. Hence for vanishing X we find

$$2\frac{dV}{d\phi^2}=\tfrac{1}{3}\lambda\int\frac{d^3k}{(2\pi)^3}\left(\frac{1}{k^2[1-B(0,k^2)]+\frac{1}{3}\lambda\phi^2}-\frac{1}{(k^2+|\mu^2|)[1-B(|\mu^2|,k^2)]}\right)$$
$$-\int\frac{d^3k}{(2\pi)^3}\left(\frac{k^2(\partial X/\partial\phi^2)\partial B(X,k^2)/\partial X|_{X=0}}{k^2[1-B(0,k^2)]+\frac{1}{3}\lambda\phi^2}-\frac{\lambda}{3}\frac{1}{(k^2+|\mu^2|)[1-B(|\mu^2|,k^2)]}\right)$$
$$=0,$$

since

$$\left.\frac{\partial X}{\partial\phi^2}\frac{\partial B}{\partial X}\right|_{X=0}=\tfrac{1}{3}\lambda k^2.$$

Therefore the minimum remains at $\phi^2=-6\mu^2/\lambda$ irrespective of the choice of $\delta\tilde{\mu}_1^2$. (ϕ^2 is determined only to order $1/\sqrt{N}$ by the order-$1/N$ approximation to V.)

Although, strictly speaking, $X(\phi)$ exists only for $\lambda\phi^2>-6\mu^2$, it is possible to define an analytic continuation of $2dV/d\phi^2=X$ by writing the solution to (3.28) for $\mu^2<0$ as

$$[X(\phi)]^{1/2}=-\frac{\lambda}{48\pi}+\left[\left(\frac{\lambda}{48\pi}\right)^2+\mu^2+\tfrac{1}{6}\lambda\phi^2\right]^{1/2}.$$

This continuation leads to a real value for $2dV/d\phi^2$ if $\mu^2+\frac{1}{6}\lambda\phi^2$ is greater than $-(\lambda/48\pi)^2$. It has been speculated[1] that in higher orders the analytic continuation of $V(\phi)$ should become complex for $\lambda\phi^2<-6\mu^2$. However, to order $1/N$, $V(\phi)$ remains real wherever $X(\phi)$ is real and positive since the integrals in (3.32) are real whenever X is positive.

E. Four dimensions

In four dimensions wave function renormalization and coupling constant renormalization are needed in addition to mass renormalization. The order-one expression for $X(\phi)$ involves only coupling constant and mass renormalization; it is given by

$$X(\phi)=\mu_N{}^2+\tfrac{1}{6}\lambda_N\phi^2+\tfrac{1}{6}\lambda_N\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-X+i\epsilon}. \tag{3.33}$$

If we define

$$\mu_N{}^2=\mu^2-\mathrm{Re}\left(\tfrac{1}{6}\lambda_N\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\epsilon}\right), \tag{3.34}$$

and

$$\frac{1}{\lambda_N}=\frac{1}{\lambda}+\mathrm{Re}\left(\tfrac{1}{6}i\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\mu^2+i\epsilon)^2}\right), \tag{3.35}$$

the equation satisfied by $X(\phi)$ becomes

$$X(\phi)=\mu^2+\tfrac{1}{6}\alpha\phi^2+\frac{\alpha}{96\pi^2}X\ln\frac{X}{|\mu^2|}, \tag{3.36}$$

with $\alpha=\lambda/(1+\lambda/96\pi^2)$. This choice of renormalization constants preserves, for $\mu^2>0$, both $X(\phi=0)=\mu^2$ and $dX/d\phi^2|_{\phi=0}=\frac{1}{6}\lambda$.

It is possible for both renormalized quantities, μ^2 and α, to assume either positive or negative values. For our purposes it is sufficient to examine only the simple choice $\mu^2>0$, $\lambda>0$.

The renormalization procedure is complicated not only by the additional renormalization constants but also by the quadratic degree of divergence of some of the integrals. In the appendix we give the details of the cancellation of infinities. The final renormalized expression is

$$\begin{aligned}2\frac{dV}{d\phi^2}=&NX(\phi)+\int_f\frac{d^4k}{(2\pi)^4}\frac{\lambda_N}{3}\frac{1}{(k^2+X)[1-\overline{B}(X,k^2)]}-\tfrac{1}{3}\lambda\tfrac{1}{3}\lambda\phi^2\int_f\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+X)^2[1-B(X,k^2)]^2}\\&+(\tfrac{1}{3}\lambda\phi^2)^2\tfrac{1}{3}\lambda\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+X)^2[1-B(X,k^2)]^2\{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2\}}\\&+\frac{\partial X}{\partial\phi^2}\bigg[-\int_f\frac{d^4k}{(2\pi)^4}\frac{\partial\overline{B}(X,k^2)}{\partial X}\frac{1}{1-\overline{B}(X,k^2)}+\tfrac{1}{3}\lambda\phi^2\int_f\frac{d^4k}{(2\pi)^4}\frac{\partial B(X,k^2)}{\partial X}\frac{1}{(k^2+X)[1-B(X,k^2)]^2}\\&-(\tfrac{1}{3}\lambda\phi^2)^2\int\frac{d^4k}{(2\pi)^4}\frac{\partial B(X,k^2)}{\partial X}\frac{1}{(k^2+X)[1-B(X,k^2)]^2\{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2\}}+aX+b\\&-\tfrac{1}{3}\lambda\phi^2\int_f\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+X)^2[1-B(X,k^2)]}\\&+(\tfrac{1}{3}\lambda\phi^2)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+X)^2[1-B(X,k^2)]\{(k^2+X)[1-B(X,k^2)]+\frac{1}{3}\lambda\phi^2\}}\bigg],\end{aligned} \tag{3.37}$$

where the subscript f means the finite part of the integral as defined in the Appendix. The constants a and b are finite; they are determined by specifying the renormalized values to order $1/N$ of $2dV/d\phi^2$ and $2d^2V/(d\phi^2)^2$ or $X(\phi)$ and $dX/d\phi^2$ at some appropriate value of ϕ.

In the integrals in (3.37) which involve only logarithmic divergences, the divergent term $\overline{B}(X,k^2)$ has been eliminated in favor of the finite function

$$B(X,k^2)=\frac{-\lambda}{96\pi^2}\left\{2+\ln\frac{|\mu^2|}{X}-\left(\frac{4X}{k^2}+1\right)^{1/2}\ln\left(\frac{k^2}{X}\left[\left(\frac{X}{k^2}+\frac{1}{4}\right)^{1/2}+\frac{1}{2}\right]^2\right)\right\} \tag{3.38}$$

by using the relationship

$$\lambda_N[1-\overline{B}(X,k^2)]^{-1}=\lambda[1-B(X,k^2)]^{-1}. \tag{3.39}$$

It has been noted elsewhere[1] that the O(N) model in four dimensions is inconsistent in the leading-order approximation when a symmetry-breaking solution exists. The σ propagator has a tachyon pole in it. Furthermore the effective potential becomes complex if ϕ becomes too large. Similar difficulties arise when we examine the next-to-leading term in the effective potential, even for $\mu^2>0$, $\lambda>0$.

The order-one definition of $X(\phi)$

$$(X-\mu^2)=\tfrac{1}{6}\alpha\phi^2+\frac{\alpha}{96\pi^2}X\ln\frac{X}{|\mu^2|} \tag{3.40}$$

has a real solution for $\phi^2 > 0$ but less than

$$\phi_{\max}{}^2 = \mu^2 \left(\frac{\alpha}{96\pi^2} e^{96\pi^2/\lambda} - 1 \right) . \tag{3.41}$$

As ϕ^2 increases from zero, the solution to (3.40) increases monotonically from $X = \mu^2$ at $\phi = 0$ to $X = \mu^2 e^{96\pi^2/\lambda}$ at $\phi_{\max}{}^2$. As ϕ^2 increases beyond $\phi_{\max}{}^2$, no real solution for $X(\phi)$ exists. The next-to-leading-order calculations can now be used to suggest whether this behavior is due to keeping an inadequate number of terms in the $1/N$ expansion, or if the expansion itself is at fault.

The $1/N$ terms involve the σ propagator in an external field ϕ, which has the form

$$\left(k^2 + X + \frac{\lambda_N \phi^2}{3} \frac{1}{1 - \overline{B}(X, k^2)} \right)^{-1} \tag{3.42}$$

in Euclidean momentum space. Although $1 - \overline{B}(X, k^2)$ is positive definite, it is also infinite; thus we must use the renormalized expression

$$\left(k^2 + X + \frac{\lambda \phi^2}{3[1 - B(X, k^2)]} \right)^{-1} . \tag{3.43}$$

However, $1 - B(X, k^2)$ is not always positive. Indeed, on the one hand, we have

$$1 - B(X, 0) = 1 + \frac{\lambda}{96\pi^2} \ln \frac{X}{\mu^2} ,$$

which is positive for $\lambda > 0$ and $X > \mu^2$, while, on the other hand, as $k^2 \to \infty$,

$$[1 - B(X, k^2)] \sim 1 - \frac{\lambda}{96\pi^2} \left(\ln \frac{k^2}{\mu^2} - 2 \right) ,$$

which is negative for large values of k^2.

Thus $[1 - B(X, k^2)]^{-1}$ and the σ propagator (3.43) have poles in them. The pole in the propagator is the well-known Landau ghost. Its presence in the defining equation for $V(\phi)$ to order $1/N$ destroys the reality of the effective potential. Keeping the order-$1/N$ terms in the $1/N$ expansion in four dimensions only aggravates the problem of complexity rather than alleviating it.

It is not difficult to locate a possible reason for the apparent breakdown of the $1/N$ expansion in four dimensions. Unlike the lower dimensional theories, in the four-dimensional theory $B(X, k^2)$ increases as $\ln(k^2/\mu^2)$ for large k^2. However, there are many contributions to the four-point function of higher order in $1/N$ but which increase more rapidly than $\ln(k^2/\mu^2)$ as $k^2 \to \infty$. For sufficiently large k^2, then, the higher-order contributions dominate. In the $1/N$ expression for $V(\phi)$ we must integrate over all k^2; consequently the higher-order terms are not negligible. In the order-one approximation only $B(X, 0)$ is required; thus the crucial momentum dependence does not enter. In one, two, and three dimensions $B(X, k^2)$ vanishes as $k^2 \to \infty$, and the higher-order corrections vanish even faster. Hence the momentum integration does not invalidate the $1/N$ expansion.

In order to surmount this difficulty, if indeed it is the fault of the perturbation expansion and not of the theory itself, it is necessary to find a better approximation to the four-point function which incorporates more of the nonlinear structure of the full theory.

IV. CONCLUSIONS

We have been able to find an expression for the next-to-leading terms in the $1/N$ expansion of the effective potential, and to associate with them a diagrammatic interpretation. These improved approximations yield information on three points: The minimum of the effective potential in three dimensions is not affected by radiative corrections of order $1/N$: rather it remains at the position determined by the order-one equations; the analytic continuation of $V(\phi)$ is real to order $1/N$ whenever the order-one result is real; and the order-$1/N$ contribution to the effective potential in four dimensions is complex for all ϕ, suggesting that the $1/N$ expansion breaks down.

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APPENDIX

Herein we discuss the renormalization of the derivative of the effective potential in four dimensions. We must be careful in our renormalization procedure to handle correctly the quadratic self-energy divergences. First, we isolate the possibly troublesome divergences by expanding the denominators as follows:

$$\frac{1}{p^2 + X} = \frac{1}{p^2 + \mu^2} \sum_{n=0}^{\infty} \left(\frac{\mu^2 - X}{p^2 + \mu^2} \right)^n ,$$
$$\frac{1}{1 - \overline{B}(X, k^2)} = \frac{1}{1 - \overline{B}(\mu^2, k^2)} \times \sum_{n=0}^{\infty} \left(\frac{\overline{B}(X, k^2) - \overline{B}(\mu^2, k^2)}{1 - \overline{B}(\mu^2, k^2)} \right)^n . \tag{A1}$$

Then the quadratic divergences involve only the first terms in the expansion in Eq. (A1), and they are identical to those encountered in the unshifted theory. Since these terms no longer involve X or ϕ dependence we can appeal to the well-known results of the unshifted theory in order to renormalize them. For this reason we discuss only $\mu^2 > 0$, $\lambda > 0$.

The remaining contributions involve nothing worse than logarithmic divergences. We assume that it is permissible first to do the order-one internal renormalization,

$$\frac{\lambda_N}{1-\overline{B}(X,k^2)} = \frac{\lambda}{1-B(X,k^2)},$$

then to make the subtractions defining the completely renormalized quantity.

We are interested only in the finite part of $2dV/d\phi^2$, not in the exact form of the renormalization counterterms. We therefore define the finite part of an integral by subtracting a term with the same high-energy behavior, but with a denominator which is independent of X and ϕ. The renormalization program is successful if the sum of all these subtraction terms can be absorbed into the available counterterms.

For our discussion it is convenient to define the following functions. We know that

$$B(X,k^2) = -\frac{\lambda}{96\pi^2}\left\{2+\ln\frac{\mu^2}{X} - \left(1+\frac{4X}{k^2}\right)^{1/2}\ln\left[\frac{k^2}{X}\left(\left(\frac{X}{k^2}+\frac{1}{4}\right)^{1/2}+\frac{1}{2}\right)^2\right]\right\}; \tag{A2}$$

hence, if we define

$$\delta B(X,k^2) = \frac{\lambda}{48\pi^2}\frac{1}{k^2+\mu^2}\left[X\ln\frac{k^2}{X} - \mu^2\ln\frac{k^2}{\mu^2} + (X-\mu^2)\right], \tag{A3}$$

we have, for large k^2

$$[B(X,k^2) - B(\mu^2,k^2)] \sim \delta B(X,k^2) + O\left(\frac{1}{k^4}\right). \tag{A4}$$

Similarly for the expression

$$\frac{\partial B(X,k^2)}{\partial X} = \frac{\lambda}{48\pi^2 k^2}\frac{1}{(1+4X/k^2)^{1/2}}\ln\left[\frac{k^2}{X}\left(\left(\frac{X}{k^2}+\frac{1}{4}\right)^{1/2}+\frac{1}{2}\right)^2\right] \tag{A5}$$

we are led to define

$$\delta\frac{\partial B}{\partial X} = \frac{\lambda}{48\pi^2}\frac{1}{k^2+\mu^2}\ln\frac{k^2}{X}, \tag{A6}$$

so that for large k^2

$$\frac{\partial B}{\partial X}(X,k^2) \sim \delta\frac{\partial B}{\partial X} + O\left(\frac{1}{k^4}\right). \tag{A7}$$

We also need the function

$$\frac{\partial G(X,k^2)}{\partial X} \equiv \tfrac{1}{3}\lambda\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+X)^2[(p+k)^2+\mu^2]}$$
$$= \frac{\lambda}{48\pi^2}\left[\frac{1}{2k^2}\ln\frac{\mu^2}{X} - \frac{(k^2+X-\mu^2)}{k^2[(k^2+X-\mu^2)^2+4k^2\mu^2]^{1/2}}\ln\left(\frac{4\mu^2 X}{\{[(k^2+X-\mu^2)^2+4k^2\mu^2]^{1/2}+k^2+X+\mu^2\}^2}\right)\right]. \tag{A8}$$

Then, with the definition

$$\delta\frac{\partial(B-G)}{\partial X} = \frac{\lambda}{48\pi^2}\frac{1}{(k^2+\mu^2)^2}\left(X-\mu^2+(2X-\mu^2)\ln\frac{X}{\mu^2} - 2(X-\mu^2)\ln\frac{k^2}{\mu^2}\right) \tag{A9}$$

we have for large k^2

$$\left(\frac{\partial B(X,k^2)}{\partial X} - \frac{\partial G(X,k^2)}{\partial X}\right) \sim \delta\frac{\partial(B-G)}{\partial X} + O\left(\frac{1}{k^6}\right). \tag{A10}$$

The finite part of the logarithmically divergent integrals is defined as follows:

$$\int_f\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+X)^2[1-B(X,k^2)]} \equiv \int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{(k^2+X)^2[1-B(X,k^2)]} - \frac{1}{(k^2+\mu^2)^2[1-B(\mu^2,k^2)]}\right), \tag{A11}$$

$$\int_f \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+X)^2[1-B(X,k^2)]^2} \equiv \int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{(k^2+X)^2[1-B(X,k^2)]^2} - \frac{1}{(k^2+\mu^2)^2[1-B(\mu^2,k^2)]}\right), \tag{A12}$$

$$\int_f \frac{d^4k}{(2\pi)^4}\frac{\partial B(X,k^2)/\partial X}{(k^2+X)[1-B(X,k^2)]^2} \equiv \int \frac{d^4k}{(2\pi)^4}\left(\frac{\partial B(X,k^2)/\partial X}{(k^2+X)[1-B(X,k^2)]^2} - \frac{\delta\partial B/\partial X}{(k^2+\mu^2)[1-B(\mu^2,k^2)]^2}\right), \tag{A13}$$

where we have taken advantage of the relationship between $\overline{B}$ and B.

The two remaining definitions involve quadratically divergent quantities. We handle them by appealing to the result of the unshifted theory:

$$\tfrac{1}{3}\lambda_N \int \frac{d^4k}{(2\pi)^4}\frac{1}{[(k+p)^2+\mu^2][1-\overline{B}(\mu^2,k^2)]} = \delta M^2 + \tilde{z}p^2 + g(p^2)\,, \tag{A14}$$

where δM^2 and $\tilde{z}$ are infinite constants whose definition varies depending upon how the left-hand side is regulated, and $g(p^2)$ is a finite function satisfying $g(0)=0=\lim_{p^2\to 0} g(p^2)/p^2$. Then we define

$$\int_f \frac{d^4k}{(2\pi)^4}\frac{\frac{1}{3}\lambda_N}{(k^2+X)[1-\overline{B}(X,k^2)]}$$
$$\equiv \int \frac{d^4k}{(2\pi)^4}\left(\frac{\frac{1}{3}\lambda_N}{(k^2+X)[1-\overline{B}(X,k^2)]} - \delta M^2 + \frac{\lambda}{3}\frac{(X-\mu^2)}{(k^2+\mu^2)^2[1-B(\mu^2,k^2)]} - \frac{\lambda}{3}\frac{\delta B(X,k^2)}{(k^2+\mu^2)[1-B(\mu^2,k^2)]^2}\right). \tag{A15}$$

The last integral whose finite part we must define has a misleading compact formal form. First, we rewrite the expression so that the quadratically divergent part is immediately recognized as a self-energy:

$$\int \frac{d^4k}{(2\pi)^4}\frac{\partial\overline{B}(X,k^2)/\partial X}{1-\overline{B}(X,k^2)} = \int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+X)^2}\int\frac{d^4k}{(2\pi)^4}\frac{\frac{1}{3}\lambda_N}{[(p+k)^2+\mu^2][1-\overline{B}(\mu^2,k^2)]}$$
$$+\int \frac{d^4k}{(2\pi)^4}\frac{\partial B(X,k^2)}{\partial X}\frac{B(X,k^2)-B(\mu^2,k^2)}{[1-B(\mu^2,k^2)][1-B(X,k^2)]}$$
$$+\int \frac{d^4k}{(2\pi)^4}\left(\frac{\partial B(X,k^2)}{\partial X} - \frac{\partial G(X,k^2)}{\partial X}\right)\left(\frac{1}{1-B(\mu^2,k^2)}\right).$$

The finite part is defined as

$$\int_f \frac{d^4k}{(2\pi)^4}\frac{\partial\overline{B}(X,k^2)/\partial X}{1-\overline{B}(X,k^2)} \equiv \int \frac{d^4p}{(2\pi)^4}\left[\frac{1}{(p^2+X)^2}\left(\int\frac{d^4k}{(2\pi)^4}\frac{1}{[(p+k)^2+\mu^2][1-\overline{B}(\mu^2,k^2)]} - \delta M^2 - p^2\tilde{z}\right) - \frac{g(p^2)}{(p^2+\mu^2)^2}\right]$$
$$+\int \frac{d^4k}{(2\pi)^4}\frac{1}{1-B(\mu^2,k^2)}\left(\frac{\partial B(X,k^2)}{\partial X} - \frac{\partial G(X,k^2)}{\partial X} - \frac{\delta\partial(B-G)}{\partial X}\right)$$
$$+\int \frac{d^4k}{(2\pi)^4}\left(\frac{\partial B(X,k^2)}{\partial X}\frac{B(X,k^2)-B(\mu^2,k^2)}{[1-B(X,k^2)][1-B(\mu^2,k^2)]} - \frac{\delta B(X,k^2)\delta\partial B/\partial X}{[1-B(\mu^2,k^2)]^2}\right). \tag{A16}$$

By substituting expressions (A11)–(A13), (A15), and (A16) into the renormalized expression for the effective potential (3.9), we can derive Eq. (3.37) of the text provided that

$$aX+b = AX+D+C\int\frac{d^4k}{(2\pi)^4}\frac{k^2}{(k^2+X)^2} + \left(\frac{6}{\lambda_N}+\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+X)^2}\right)\delta M^2$$
$$+\left(2-\frac{\lambda}{48\pi^2}\ln\frac{X}{\mu^2}\right)\int\frac{d^4k}{(2\pi)^4}\left(\frac{\delta B(X,k^2)}{(k^2+\mu^2)[1-B(\mu^2,k^2)]^2} - \frac{(X-\mu^2)}{(k^2+\mu^2)^2[1-B(\mu^2,k^2)]}\right)$$
$$-\left(2-\frac{\lambda}{48\pi^2}\ln\frac{X}{\mu^2}\right)\tfrac{1}{3}\lambda\phi^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+\mu^2)^2[1-B(\mu^2,k^2)]^2}$$
$$-\int\frac{d^4k}{(2\pi)^4}\left(\frac{\delta M^2+\tilde{z}k^2}{(k^2+X)^2}+\frac{g(k^2)}{(k^2+\mu^2)^2}\right) - \int\frac{d^4k}{(2\pi)^4}\frac{\delta B(X,k^2)\delta\partial B/\partial X}{[1-B(\mu^2,k^2)]^2}$$
$$-\int\frac{d^4k}{(2\pi)^4}\frac{\delta\partial(B-G)/\partial X}{1-B(\mu^2,k^2)} + \tfrac{1}{3}\lambda\phi^2\int\frac{d^4k}{(2\pi)^4}\frac{\delta\partial B/\partial X}{(k^2+\mu^2)[1-B(\mu^2,k^2)]^2}$$
$$-\tfrac{1}{3}\lambda\phi^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+\mu^2)^2[1-B(\mu^2,k^2)]}. \tag{A17}$$

In the above expression A, D, and C are infinite constants of order $1/N$. Moreover, we have used the relationships

$$\left(\frac{\partial X}{\partial \phi^2}\right)^{-1} = \left(\frac{6}{\lambda_N} + \int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+X)^2}\right),$$
$$\frac{\lambda}{3}\left(\frac{\partial X}{\partial \phi^2}\right)^{-1} = 2\left(1 - \frac{\lambda}{48\pi^2}\ln\frac{X}{\mu^2}\right). \tag{A18}$$

Many of the terms in (A17) are readily absorbed into a redefinition of A and D. The choice $C = \bar{z}$ disposes of the wave function renormalization. The terms involving a more subtle cancellation are

$$A'X + C' + \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+\mu^2)^2}\frac{1}{[1-B(\mu^2,k^2)]}\left[\frac{\lambda}{48\pi^2}(X-\mu^2)\ln\frac{X}{\mu^2} - \frac{\lambda\phi^2}{3} - \left(\frac{\delta\partial(B-G)}{\partial X}\right)(k^2+\mu^2)^2\right]$$
$$+ \int \frac{d^4k}{(2\pi)^4}\frac{(k^2+\mu^2)\delta B(X,k^2) - \frac{1}{3}\lambda\phi^2}{(k^2+\mu^2)^2[1-B(\mu^2,k^2)]^2}\left(2 - \frac{\lambda}{48\pi^2}\ln\frac{X}{\mu^2} - (k^2+\mu^2)\delta\frac{\partial B}{\partial X}\right). \tag{A19}$$

However, from the definitions of δB, $\delta\partial(B-G)/\partial X$, $\delta\partial B/\partial X$, and $X(\phi)$ we find

$$(k^2+\mu^2)\delta B(X,k^2) - \tfrac{1}{3}\lambda\phi^2 = 2(X-\mu^2)\left(-1 + \frac{\lambda}{96\pi^2}\ln\frac{k^2}{\mu^2}\right),$$

$$(k^2+\mu^2)\delta\frac{\partial B}{\partial X} + \frac{\lambda}{48\pi^2}\ln\frac{X}{\mu^2} = \ln\frac{k^2}{\mu^2},$$

and that

$$\frac{\lambda}{48\pi^2}(X-\mu^2)\ln\frac{X}{\mu^2} - \frac{\lambda\phi^2}{3} - \delta\frac{\partial(B-G)}{\partial X}(k^2+\mu^2)^2$$

is proportional to $(X-\mu^2)$. Hence all the infinities in the two integrals in (A18) can be canceled by $A'X+C'$. Therefore, there exist counterterms A, D, and C, which are independent of X and ϕ, and which cancel all the infinities, leaving the renormalized result given in the text, Eq. (3.37).

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[5]It is inconvenient to sum the graphs in Fig. 1(a) directly. Rather, their derivative with respect to ϕ^2 can be related to a mass renormalization and to $2dV/d\phi^2$. The diagrams illustrated in Fig. 1(b) can be summed using a modified propagator which incorporates the order-one mass renormalization. The effective potential calculated by this procedure leads to Eq. (2.22) of the text.

[6]In this paper we use the combinatorical trick as it is described in Ref. 1. An alternative version of this trick which is suitable for investigating the vacuum expectation value of composite operators such as $\langle\Phi\Phi\rangle$ when $\langle\Phi\rangle = 0$ is discussed in Ref. 3.

[7]The integral in the equation defining $X(\phi)$ dictates the value of $\sqrt{X}$ for complex X. If the imaginary part of X is negative, the imaginary part of $\sqrt{X}$ must also be negative and no solution exists. If the imaginary part of X is positive then the counterterm no longer cancels the infinities and the equation is not renormalized. Hence the only solutions are real positive X.