the general R_ξ formalism, see Ref. 5, where the second-order scalar-particle self-energy with an arbitrary value of ξ is also evaluated.

[8]This problem has been investigated, with the use of conventional formalism for the vector field, by T. Appelquist and H. Quinn, Phys. Lett. 39B, 229 (1972).

PHYSICAL REVIEW D VOLUME 10, NUMBER 10 15 NOVEMBER 1974

Infrared bootstrap for the electron mass in finite quantum electrodynamics

Philip D. Mannheim*

Institute for Advanced Study, Princeton, New Jersey 08540

(Received 17 June 1974)

This paper is concerned with the infrared structure of Johnson-Baker-Willey finite quantum electrodynamics. In this theory the insertion of $\overline{\psi}\psi$, the composite mass operator, into the electron propagator satisfies a homogeneous Bethe-Salpeter equation whose solution is conformally invariant at short distances with an anomalous dimension, $\gamma_{\overline{\psi}\psi}(\alpha)$. To determine whether the theory is forced to actually choose a nontrivial solution to this homogeneous equation we calculate the effective potential, $V(\langle\overline{\psi}\psi\rangle)$, using the dressed scalar vertex as input. We find that the infrared divergences of the theory cause the effective potential to develop a degenerate minimum away from the origin in classical field space. Thus dynamical γ_5 symmetry breaking takes place with the electron acquiring a mass $m \sim \langle\overline{\psi}\psi\rangle$. It is thought that this may be a general mechanism for generating masses in an otherwise conformal-invariant theory.

I. INTRODUCTION

This paper is concerned with the possibility that dynamical γ_5 symmetry breaking is the agency for introducing a scale into particle physics. In discussing theories in which all of the mass is to be dynamical we must start off with an underlying massless theory with dimensionless couplings. Such theories will exhibit conformal invariance with anomalous dimensions at short distances provided there is a renormalization-group fixed point.[1-3] We restrict ourselves to such theories only in this paper. These theories are generally regarded as being off-shell theories only without a sensible mass-shell limit. Thus the good ultraviolet limit is accompanied by a bad infrared limit. In perturbation theory we avoid but do not solve the infrared problem by renormalizing off the mass shell. Eventually, however, we have to sum the perturbation series and go to the mass shell, at which point we then have to face the infrared problem. The main point of this paper is that this infrared problem is then solved by dynamical γ_5 symmetry breaking, so that the fermions in the theory acquire masses by translating to the new vacuum. In this approach Wilson's skeleton theory[4] will be an exactly conformal-invariant renormalizable theory with either anomalous or canonical dimensions at short distances, depending on whether the ultraviolet-stable fixed point is nontrivial or at the origin; and all of the breaking of conformal invariance is achieved through the γ_5 degeneracy of the vacuum with no soft operators (or dilatons) being needed in the theory. This is the realization of an idea we suggested in a recent publication.[5]

The theory we analyze in detail in this paper is Johnson-Baker-Willey quantum electrodynamics[6-11] (finite QED) which possesses an explicit dynamical-symmetry-breaking solution. These authors have considered the Bethe-Salpeter equation (see Fig. 1)

$$m\tilde{\Gamma}_S(p,p,0)=m_0Z_2+\int\frac{d^4k}{(2\pi)^4}\,m\tilde{\Gamma}_S(k,k,0)i\tilde{S}(k)\times\tilde{K}(p,k,0)i\tilde{S}(k)\qquad(1)$$

for the insertion of the renormalized scalar operator, $\theta=\overline{\psi}\psi$, carrying zero momentum into the electron propagator. In the generalized Landau gauge where Z_2 is finite the electron propagator is canonical and the above equation admits of a solution

$$m\tilde{\Gamma}_S(p,p,0)=C(\alpha)m\left(\frac{-p^2}{m^2}\right)^{\gamma_\theta(\alpha)/2}\qquad(2)$$

for asymptotic p.[12] Thus if γ_θ, the anomalous dimension of θ, is negative the theory has a zero bare mass m_0 (in the limit of infinite cutoff).[13] Equation (1) then becomes a homogeneous bootstrap equation for the renormalized mass operator and admits of a nonvanishing physical mass. This is a typical degenerate vacuum situation in which the solution to a Lagrangian field theory possesses less symmetry than the original zero-bare-mass Lagrangian. The question of whether the nontrivial solution of Eq. (1) is preferred over the trivial solution is a stability problem, so we are thus led to calculate the effective potential seen by $\langle\bar{\psi}\psi\rangle$ in order to investigate the structure of the vacuum. We shall adapt the method developed by Coleman and Weinberg.[14] Since their approach is not too well suited to the case where there are composite operators, we shall adapt their method to our problem by developing a physically motivated (but not as of yet formally justified) technique for handling composite-operator effective potentials. Our method will also admit of a loop expansion and we shall use the completely dressed vertex of Eq. (2) as input for the loop calculation of the effective potential. We find that the potential develops a minimum away from the origin so that the physical mass then bootstraps itself about this γ_5 degenerate vacuum. As has been previously stressed[9,15,16] the nontrivial solution to Eq. (1) corresponds to a nonconserved axial-vector current due to a nonperturbative renormalization anomaly, so that this spontaneous breakdown of a continuous symmetry needs no accompanying Goldstone boson. We thus note the one-way nature of the Goldstone theorem in theories in which all of the mass is dynamical. Though a conserved current implies a degenerate vacuum and a massless particle, there is no theorem which forces a degenerate vacuum to be accompanied by a conserved current. The second possibility is automatically realized in finite QED. The fact that we have obtained spontaneous breakdown in QED is perhaps surprising since, being infrared-stable, it is usually thought of as being well behaved in the massless limit. However, the infrared stability of QED is a statement about our ability to handle soft photons, and says nothing about the limit of vanishing electron mass. Thus in finite QED the photon stays massless, but the theory does not tolerate a massless electron.

II. CALCULATION OF THE EFFECTIVE POTENTIAL IN FINITE QED

In a recent publication Coleman and Weinberg[14] have developed a useful method for investigating whether radiative corrections can be the source of spontaneous symmetry breakdown. They start from Schwinger's generating functional, $W(J)$, in which the field of interest, ϕ, is coupled linearly to an external source J. Functional variations of $W(J)$ with respect to $J(x)$ then give the connected ϕ Green's functions, $G^{(n)}(x_1, \ldots, x_n)$. A classical field is then introduced through the definition

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} = \frac{\langle 0^+ | \phi(x) | 0^- \rangle}{\langle 0^+ | 0^- \rangle}\bigg|_J \,. \tag{3}$$

From $\phi_c(x)$ we next construct an effective action functional

$$\Gamma(\phi_c) = W(J) - \int d^4x\, J(x)\phi_c(x)\,, \tag{4}$$

which can be expanded in two ways. First, we expand in a functional Taylor series about the point $\phi_c = 0$ (i.e., about the normal vacuum), so that

$$\Gamma(\phi_c) = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n\, \Gamma^{(n)}(x_1, \ldots, x_n) \times \phi_c(x_1) \cdots \phi_c(x_n)\,. \tag{5}$$

A second expansion is in powers of momentum about the point where all external momenta vanish, which in coordinate space is equivalent to

$$\Gamma(\phi_c) = \int d^4x[-V(\phi_c) + \tfrac{1}{2}(\partial_\mu \phi_c)^2 Z(\phi_c) + \cdots]\,. \tag{6}$$

Here $V(\phi_c)$, an ordinary function, is known as the effective potential and satisfies

$$V(\phi_c) = \sum_n \frac{1}{n!} \Gamma^{(n)}(p_i = 0)\phi_c{}^n\,. \tag{7}$$

The minima of $V(\phi_c)$ give the true vacua of the theory. Note that up to this point no distinction has been made between fundamental fields and composite operators. The coefficients $\Gamma^{(n)}$ are defined through Eq. (5) and can be related functionally to the connected $G^{(n)}$. In the event that $\phi(x)$ is a fundamental field which can propagate from point to point internally in diagrams it then follows that the $\Gamma^{(n)}$ are amputated one-particle irreducible (1PI) graphs. Thus for fundamental fields Eq. (7) is a very convenient starting point, and lends itself directly to the loop expansion of Coleman and Weinberg which sums particular infinite classes of infrared-divergent 1PI graphs. For composite operators, however, the $\Gamma^{(n)}$ have no simple interpretation (if there is no kinetic energy associated with the composite operator then there is no meaning to amputation), and from a practical viewpoint Eq. (7) is not too useful.

Various authors[17] have noted that a great deal of computational simplicity is achieved by using

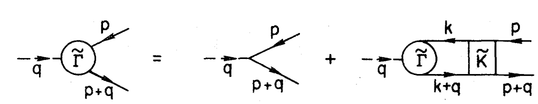

FIG. 1. The Bethe-Salpeter kinematics for the dressed vertex.

the Feynman path-integral method which uses functional integration instead of functional differentiation. This method expands the quantum action in powers of $\hbar$ about the classical action, and powers of $\hbar$ are then found to correspond to numbers of loops in the above-discussed loop expansion. In this approach the infinite class of graphs in a given order in the loop expansion corresponds to a single term in the $\hbar$ expansion. Though the functional-integration method has great practical value it also cannot be readily extended to composite operators, unless some meaningful classical action can be constructed for them. This difficulty is compounded by the fact that in the case in which we are interested the composite $\bar\psi\psi$ does not correspond to a physical particle state.

Since the available techniques are not readily adaptable to composite operators we shall now develop an alternative approach. We note first that a composite operator does not propagate but just brings two fields together and creates a pair at a point (i.e., it is an insertion into a Green's function) so that it acts more like a source than a field, with the most convenient objects for calculation being its connected Green's functions. These have a simple diagrammatic interpretation and are easily calculable. In order to utilize this fact we shall use $\langle\bar\psi\psi\rangle$ as the "source" of $\bar\psi\psi$,[18] by writing the fermion mass term in a disguised form. We introduce

$$e^{iW(\langle\bar\psi\psi\rangle)} = \left\langle 0 \left| T\exp\left\{ i\int d^4x[\mathcal{L}(x)+\bar\psi(x)\psi(x)\langle\bar\psi(x)\psi(x)\rangle]\right\}\right| 0\right\rangle . \tag{8}$$

Variations of W with respect to $\langle\bar\psi(x)\psi(x)\rangle$ give precisely the connected $\bar\psi\psi$ Green's functions defined by the theory with a chiral-invariant Lagrangian and with $\langle\bar\psi\psi\rangle = 0$. Thus

$$W(\langle\bar\psi\psi\rangle) = \sum_n \frac{1}{n!}\int d^4x_1\cdots d^4x_n\, G^{(n)}(x_1,\ldots,x_n)\langle\bar\psi(x_1)\psi(x_1)\rangle\cdots\langle\bar\psi(x_n)\psi(x_n)\rangle . \tag{9}$$

Having now obtained the coefficients we take $\langle\bar\psi(x)\psi(x)\rangle$ to be independent of x so that we may write

$$W(\langle\bar\psi\psi\rangle) = -\int d^4x\, V(\langle\bar\psi\psi\rangle), \tag{10}$$

where we obtain a composite-operator effective potential

$$V(\langle\bar\psi\psi\rangle) = \sum_n \frac{1}{n!} G^{(n)}(p_i = 0)\langle\bar\psi\psi\rangle^n . \tag{11}$$

Now though we described $\langle\bar\psi\psi\rangle$ as a source we only used that fact to obtain Eq. (11) by functional differentiation. It is not to be thought of as a conventional source since we do not intend to switch it off. In this approach spontaneous breakdown is the statement that we cannot switch off $\langle\bar\psi\psi\rangle$, i.e., that Eq. (11) has a minimum away from the origin. Thus our approach is to assume that the fermion gets a mass through the nonvanishing of $\langle\bar\psi\psi\rangle$, giving the mass term in Eq. (8). We then study Eq. (11) to see if it is stable about this assumed nonvanishing $\langle\bar\psi\psi\rangle$. This is thus a self-consistency approach like the mean-field approach in phase transitions. There is, however, one essential difference with mean-field theory. There the external field couples linearly to the spontaneous magnetization, but the induced internal mean field couples to the spontaneous magnetization in a nonlinear fashion and remains after the external field is removed. In our case if we are to induce a spontaneous mass term of the form $\bar\psi(x)\psi(x)\langle\bar\psi\psi\rangle$ at all then the coupling to the internal degrees of freedom will also be linear. Hence our approach can only work when there is an induced composite mass operator. It does not work when a fundamental field acquires an expectation value. It is as though mass acts like a bootstrapped $\langle\bar\psi\psi\rangle$ source.

It is important to note that the coefficients of Eq. (11) are the Green's functions of the massless theory ($\langle\bar\psi\psi\rangle = 0$) and they will typically be built out of the vertices $\tilde\Gamma_S(p,p,0)$ of Eq. (1), which itself was solved using massless fermion propagators to yield Eq. (2). It is this feature which makes our mass bootstrap self-consistent.

We must point out that our procedure is still only at the level of an ansatz. It is not at all clear whether the object defined by Eq. (11) is in fact an effective potential, and whether it bears any relation (or whether it even should bear any relation) to the more familiar potential of Eq. (7). Though we have given a physical argument above, and though we expect that the class of graphs of Fig. 2 has some fundamental significance, we

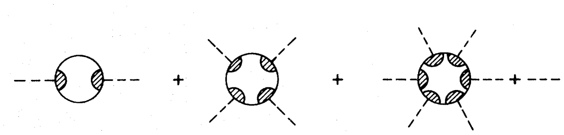

FIG. 2. The class of graphs used to calculate the composite-operator effective potential. The shaded blob represents the complete dressed scalar vertex.

have not yet found a more formal derivation of Eq. (11).

Now of course we do not know the exact Green's function required for Eq. (11) so we must now try to make a sensible approximation. We shall again proceed loopwise as in Ref. 14 (in fermion QED this has also been suggested as a general procedure by Adler[19]) and sum the infinite series of infrared-divergent connected graphs exhibited in Fig. 2. In these graphs the dressed vertices are given by Eq. (2) and the fermion is massless, so that we only have loops with an even number of external lines. This is not the complete set of one-fermion loop graphs since we are ignoring graphs like those of Fig. 3. We will indicate later why this may not be a severe restriction. We note that each graph in Fig. 2 has a conformal-invariant structure. Though the series looks like a perturbation expansion we recall that each vertex has already undergone a nonperturbative summation, so that our $V(\langle\bar\psi\psi\rangle)$ is highly nonperturbative.

Before we sum up the graphs of Fig. 2 we can immediately determine whether $\langle\bar\psi\psi\rangle = 0$ can remain a well-defined minimum. For this we need only look at $V''(\langle\bar\psi\psi\rangle = 0)$ and $V''''(\langle\bar\psi\psi\rangle = 0)$. Thus [from Eq. (11)]

$$V''(\langle\bar\psi\psi\rangle = 0) = -C^2(\alpha) m^{-2\gamma} \int \frac{p^2 dp^2}{8\pi^2} \frac{p^{2\gamma}}{p^2} = \frac{-C^2(\alpha) m^{-2\gamma}}{8\pi^2} \frac{(p^2)^{1+\gamma}}{(1+\gamma)} \Big|_0^\infty \tag{12}$$

and only has an infrared-divergent logarithm when $\gamma = -1$. (In $\gamma < -1$ we have an infrared power.) Also

$$V''''(\langle\bar\psi\psi\rangle = 0) = \frac{C^4(\alpha) m^{-4\gamma}}{16\pi^2} \frac{(p^2)^{2\gamma}}{2\gamma} \Big|_0^\infty , \tag{13}$$

which has an infrared-divergent logarithm at $\gamma = 0$. Now since it is infrared logarithms which give rise to spontaneous breakdown we see that we will not expect anything special to happen at $\langle\bar\psi\psi\rangle = 0$ unless γ is an integer. We now sum the series of Fig. 2 to obtain

$$V(\langle\bar\psi\psi\rangle) = i \int \frac{d^4p}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{(-1)}{2n} \left[\mathrm{Tr} \left\{ \frac{i}{(\not p + i\epsilon)} [-iC(\alpha)]^2 \left(\frac{-p^2}{m^2}\right)^\gamma \frac{i}{(\not p + i\epsilon)} \right\} \langle\bar\psi\psi\rangle^2 \right]^n . \tag{14}$$

Essentially the same formula is obtained by Coleman and Weinberg for the 1PI loop contribution to $\lambda\phi^4$ (apart from the $p^{2\gamma}$ factor). The crucial distinction is the one additional minus sign coming from the closing of the fermion loop. This extra factor will give us a stable potential instead of an unstable one in $\gamma < 0$. We now sum on n and make a Wick rotation to the Euclidean region to find

$$V(\langle\bar\psi\psi\rangle) = -\frac{1}{8\pi^2(1-\gamma)} [C^2(\alpha)\langle\bar\psi\psi\rangle^2 m^{-2\gamma}]^{2/(1-\gamma)} \times \int_0^A dx\, x^{(1+\gamma)/(1-\gamma)} \ln(1+1/x), \tag{15}$$

where

$$A = \frac{\Lambda^{2(1-\gamma)}}{C^2(\alpha)\langle\bar\psi\psi\rangle^2 m^{-2\gamma}} . \tag{16}$$

This integral is readily evaluated and we give the solution in various cases.

We consider first the case $\gamma = 0$. We find

$$V(\langle\bar\psi\psi\rangle, \gamma = 0) = -\langle\bar\psi\psi\rangle^4 \frac{C^4(\alpha)}{32\pi^2} \left[\frac{4\Lambda^2}{\langle\bar\psi\psi\rangle^2 C^2(\alpha)} - 1 - \ln\left(\frac{\Lambda^2}{\langle\bar\psi\psi\rangle^2 C^2(\alpha)}\right)^2 \right] . \tag{17}$$

We renormalize the theory by adding $\langle\bar\psi\psi\rangle^2$ and $\langle\bar\psi\psi\rangle^4$ counterterms to Eq. (8). Because of infrared divergences we cannot adjust these counterterms to specify the value of $G^{(4)}$ at $p_i = 0$, so we follow Coleman and Weinberg and renormalize instead away from $\langle\bar\psi\psi\rangle = 0$ by introducing an arbitrary point M in classical field space and renormalize so $V''(0) = 0$, $V''''(M) = -11C^4(\alpha)/2\pi^2$ [so $V'(M) = 0$]. This then leads to

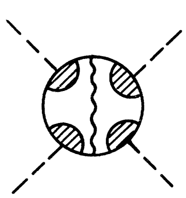

FIG. 3. A typical additional photon dressing to the class of graphs of Fig. 2.

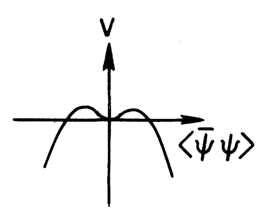

FIG. 4. The effective potential $V(\langle\bar\psi\psi\rangle)$ for a free Fermi theory.

$$V(\langle\bar\psi\psi\rangle,\gamma=0)=-C^4(\alpha)\frac{\langle\bar\psi\psi\rangle^4}{32\pi^2}\left[2\ln\left(\frac{\langle\bar\psi\psi\rangle^2}{M^2}\right)-1\right],\tag{18}$$

which is plotted in Fig. 4. It gives an upside-down double-well potential which possesses no stable absolute minimum. This is to be expected since we have changed one sign compared to the essentially equivalent infrared-stable $\lambda\phi^4$ calculation of Ref. 14. Now when $\gamma=0$ the vertex $\tilde\Gamma_S(p,p,0)$ is given entirely by the bare vertex term m_0Z_2 of Eq. (1). We are thus led to conclude that a free Fermi theory with a nonvanishing bare mass is completely unstable. [In a free theory the ultraviolet divergences of Eq. (17) are simply due to the fact that the ill-defined $\bar\psi(x)\psi(x)$ has a free field singularity.] We feel that this point should be investigated further since it seems to imply that conventional QED with infinite renormalization constants is a theory which is built on the local minimum at the origin in Fig. 4 as though it were the true minimum of the input theory (or possibly a theory built on the local maximum where $\langle\bar\psi\psi\rangle=0$). Thus the usual bare normal vacuum used for setting up perturbation theory [which buries the difficulties of Eq. (17) by normal ordering without knowing in which vacuum this is being done] may not be the ground state of the system. In order to stabilize the potential we will proceed below to consider $\gamma<0$, so that all of the scalar vertex comes from the kernel of Eq. (1). Thus the theory in which all of the mass of the electron is dynamical may be the only consistent theory of QED. This could then provide a physical rationale for the actual existence of an eigenvalue at all.

For $0>\gamma>-1$ Eq. (15) may be evaluated by integrating by parts to isolate the infinities as surface terms, leaving us with a standard product of Γ functions. Thus we obtain

$$V(\langle\bar\psi\psi\rangle,0>\gamma>-1)=-\frac{1}{16\pi}[C^2(\alpha)\langle\bar\psi\psi\rangle^2m^{-2\gamma}]^{2/(1-\gamma)}\csc\left(\frac{2\pi\gamma}{1-\gamma}\right)-\frac{1}{8\pi^2}C^2(\alpha)\langle\bar\psi\psi\rangle^2m^2\frac{1}{(1+\gamma)}\left(\frac{\Lambda^2}{m^2}\right)^{1+\gamma}.\tag{19}$$

This function is smooth as $\langle\bar\psi\psi\rangle\to0$ and possesses no infrared logarithm until we go to $\gamma=-1$. Thus we can renormalize at $\langle\bar\psi\psi\rangle=0$ and may then remove the second term of Eq. (19) by a standard $\langle\bar\psi\psi\rangle^2$ counterterm selected so $V''(\langle\bar\psi\psi\rangle=0)=0$, which then leaves only the first term of Eq. (19). The resulting potential is plotted in Fig. 5 and the origin is stable in this case, with the fermion staying massless. However, our renormalization scheme is ambiguous and we will discuss an alternative prescription below which leads to a double-well potential.

For $\gamma=-1$ we evaluate Eq. (15) directly to find

$$V(\langle\bar\psi\psi\rangle,\gamma=-1)=-\frac{C^2(\alpha)\langle\bar\psi\psi\rangle^2m^2}{16\pi^2}\times\left[\ln\left(\frac{\Lambda^4}{C^2(\alpha)\langle\bar\psi\psi\rangle^2m^2}\right)+1\right].\tag{20}$$

Now we cannot renormalize at $\langle\bar\psi\psi\rangle=0$, so we add a logarithmically divergent $\langle\bar\psi\psi\rangle^2$ counterterm and adjust its coefficient so $V''(M)=m^2C^2(\alpha)/4\pi^2$, so that $V'(M)=0$. With this choice we obtain

$$V(\langle\bar\psi\psi\rangle,\gamma=-1)=\frac{C^2(\alpha)\langle\bar\psi\psi\rangle^2m^2}{16\pi^2}\left[\ln\left(\frac{\langle\bar\psi\psi\rangle^2}{M^2}\right)-1\right],\tag{21}$$

which exhibits the stable double-well potential of Fig. 6. Thus if $\gamma=-1$ the infrared divergences force us away from the origin and we are obliged to translate to the new degenerate vacuum with the fermion acquiring a mass of order M. Hence we can self-consistently identify the parameter m of Eq. (2) with M of Eq. (21).[20]

For $\gamma<-1$ we obtain

$$V(\langle\bar\psi\psi\rangle,\gamma<-1)=\frac{1}{16\pi}[C^2(\alpha)\langle\bar\psi\psi\rangle^2m^{-2\gamma}]^{2/(1-\gamma)}\times\csc\left(\frac{\pi(1+\gamma)}{1-\gamma}\right),\tag{22}$$

which is completely finite and gives the unstable single-well potential of Fig. 7.

We have singled out the value $\gamma=-1$ as special

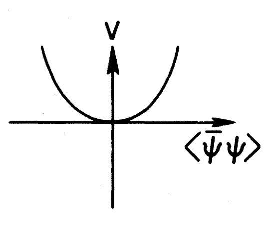

FIG. 5. The stable effective potential obtained in $0>\gamma>-1$.

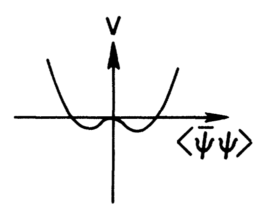

FIG. 6. The double-well potential obtained in $\gamma=-1$.

in the sense that only for this value are we necessarily forced away from the origin. However, the unrenormalized equations, Eqs. (19) and (20), are completely continuous as $\gamma \to -1$, though our renormalization procedure is discontinuous since there is no parameter M in $\gamma > -1$. Now we are free to renormalize the case $\gamma > -1$ also at $M \neq 0$, and this can be done in such a way as to give a double-well potential like that of Fig. 6, or to give a single-well potential again.[21] This is an ambiguity of the renormalization prescription, which we cannot for the moment guard against. Of course $\gamma(\alpha)$ has a unique value at the eigenvalue for α, so an independent determination of $\gamma(\alpha)$ would resolve this ambiguity, and tell us which case we are in.

We now discuss briefly the neglect of graphs like those of Fig. 3. The series of Fig. 2 sums out to a term of order $C^2(\alpha)$. The dressings like that of Fig. 3 to all of the graphs of Fig. 2 should, we hope, then give a term to the effective potential of order $\alpha C^2(\alpha)$ without altering the structure of Fig. 6 so that the dressing is a higher-order modification of the one-loop approximation.

Thus to sum up our findings we note that from the requirement of stability we have $0 > \gamma \geq -1$, so that QED with a vanishing bare mass only exists in this range. If $0 > \gamma > -1$ the electron may get a mass, and if $\gamma = -1$ the electron must get a mass. Thus the anomalous dimension of the composite mass operator acts as a signal for spontaneous breakdown, and in conformal theories $\beta(\alpha)$ bootstraps coupling constants and $\gamma_\theta(\alpha)$ bootstraps masses.

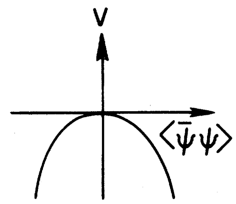

FIG. 7. The unstable potential obtained in $\gamma < -1$.

Note added in proof. Since submitting this paper we have clarified the status of Eq. (11). It is not in fact an effective potential, but rather it gives a vacuum energy difference between a massive and an underlying massless theory, viz.,

$$\epsilon(m) = \langle \Omega^{(m)} | H^{(m)} | \Omega^{(m)} \rangle - \langle \Omega^{(0)} | H^{(0)} | \Omega^{(0)} \rangle = \sum_n \frac{1}{n!} G^{(n)}(p_i = 0) m^n ,$$

so that when $\gamma_\theta(\alpha) = -1$ the massive theory is energetically favored. Moreover, we have also eliminated the ambiguity in $0 > \gamma_\theta(\alpha) > -1$, and these points will be discussed in future publications.

ACKNOWLEDGMENTS

I would like to thank Professor S. L. Adler for his interest and for his careful reading of the manuscript. I would also like to thank Dr. E. Weinberg and Dr. A. Patrascioiu for useful discussions. I would like to thank Dr. C. Kaysen for the hospitality of the Institute for Advanced Study.

---

*Research sponsored by the National Science Foundation under Grant No. GP-40768X.

[1] M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

[2] C. G. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

[3] K. G. Wilson, Phys. Rev. D 3, 1818 (1971).

[4] K. G. Wilson, Phys. Rev. 179, 1499 (1969).

[5] P. D. Mannheim, Phys. Rev. D 9, 3438 (1974).

[6] K. Johnson, M. Baker, and R. Willey, Phys. Rev. 136, B1111 (1964).

[7] K. Johnson, R. Willey, and M. Baker, Phys. Rev. 163, 1699 (1967).

[8] M. Baker and K. Johnson, Phys. Rev. 183, 1292 (1969).

[9] M. Baker and K. Johnson, Phys. Rev. D 3, 2516 (1971).

[10] M. Baker and K. Johnson, Phys. Rev. D 3, 2541 (1971).

[11] K. Johnson and M. Baker, Phys. Rev. D 8, 1110 (1973).

[12] This solution is readily obtained also using the Callan-Symanzik approach as discussed by S. L. Adler and W. A. Bardeen, Phys. Rev. D 4, 3045 (1971); 6, 734(E) (1972). In order to make contact with their notation we introduce $Z_\theta^{-1/2}$ which renormalizes $\bar{\psi}(x)\psi(x)$ so that $Z_S = Z_2 Z_\theta^{-1/2}$ renormalizes Γ_S. γ_θ is defined as $m(\partial/\partial m)[\ln Z_\theta^{1/2}(g_\Lambda, \Lambda/m)|_{g_\Lambda,\Lambda}]$ so that $\gamma_\theta = d_\theta - d_\theta^c$ (i.e., the conventional definition of γ_θ is through a Z_θ that would renormalize the inverse θ Green's function). With this definition of γ_θ Eq. (2) then follows. Note also that, as these authors have demonstrated, γ_θ is gauge-independent. Note also (Ref. 9) that Z_θ is not cutoff-independent in the finite gauge, though $m_0 Z_\theta^{1/2} = m$ is finite.

[13] This condition on γ_θ is not the same as the one recently discussed by S. Weinberg [Phys. Rev. D 8, 3497 (1973)] for the legitimacy of neglecting asymptotically the terms which order by order in perturbation theory are negligible. His condition using our definition of Z_θ in Ref. 12 is that $\gamma_\theta < 1$, i.e., that the mass operator, $m\bar{\psi}\psi$, remains soft asymptotically. Thus when $d_\theta < 4$ there is *a posteriori* a good Callan-Symanzik limit, which is to be expected intuitively. However, only if $d_\theta < 3$ will the bare mass vanish in the limit of infinite cutoff.

[14] S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888

(1973).

[15]M. Baker, K. Johnson, and B. W. Lee, Phys. Rev. 133, B209 (1964).

[16]K. Johnson, in *The Ninth Latin American School of Physics*, edited by I. Saavedra (Benjamin, New York, 1968).

[17]A. Salam and J. Strathdee, Phys. Rev. D 9, 1129 (1974); R. Jackiw, *ibid.* 9, 1686 (1974).

[18]Throughout it is understood that $\langle\bar{\psi}\psi\rangle$ has been divided by m^2 so as to keep ordinary dimensional analysis correct. Thus $\langle\bar{\psi}\psi\rangle$ has ordinary dimension 1, but dynamical dimension $3+\gamma_\theta$.

[19]S. L. Adler, Phys. Rev. D 5, 3021 (1972).

[20]We make here a clarifying remark about this introduction of a scale M for masses and about our use of the word "scale" in this paper. If we have a solution to Eq. (1) with an anomalous dimension at all, then on ordinary dimensional grounds we need some mass scale in the theory. In perturbation theory with a massless fermion this is introduced through an off-shell subtraction point, μ, which then parametrizes the ultraviolet behavior of the theory. However, even though there is now a scale this does not mean that the pole in the fermion propagator has moved away from zero. μ is just a dimensional scale, whereas our interest is in obtaining a physical scale by actually being obliged to move the pole to some nonzero, but of course arbitrary, point M, which then parametrizes the infrared behavior of the theory. For simplicity we parametrized Eq. (2) directly in terms of the physical mass m and then proceeded self-consistently. The alternative is to parametrize Eq. (2) in terms of $(-p^2/\mu^2)^{\gamma/2}$ and then study the theory in $p^2 \ll \mu^2$, the opposite of the usual ultraviolet limit. This again leads to a degenerate vacuum whose position then gives a physical scale, as opposed to the purely dimensional one given initially by μ.

[21]We note that the first term of Eq. (19) dominates for large $\langle\bar{\psi}\psi\rangle$, so that the potential will always be stable for any renormalization scheme. The coefficient of the $\langle\bar{\psi}\psi\rangle^2$ term can be made either positive or negative by renormalization, so that this potential can then be made to act like a wrong sign $(\mu^2<0)$ σ model.