Bound states and asymptotic scaling laws in field theory*

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We examine some of the problems associated with understanding wide-angle scaling laws in quantum field theory. We have in mind a model in which the hadrons are bound states of quarks. In this paper we examine elastic electromagnetic form factor behavior in a renormalizable scalar-field-theory model. We investigate the extent to which the asymptotic behavior is determined by the short-distance properties of the theory. The dependence on both the bound-state wave function and the strong radiative corrections is considered. The differences between the scalar theory and a more realistic quark-vector-gluon model are pointed out.

I. INTRODUCTION

Experiments on wide-angle elastic scattering[1] and elastic form-factor behavior[2] indicate that these processes are reasonably well described by a set of scaling laws of the form

$$\frac{d\sigma}{dt} \to \frac{1}{s^n} f(\theta_{\text{c.m.}}),$$

where n depends on the particular process.[3] It seems possible to understand the power laws in models in which the hadrons are bound states of quarks.[4-6] In the model of Matveev, Muradyan, and Tavkhelidze[4] and Brodsky and Farrar,[5] the power laws come from drawing simple connected Feynman diagrams with dimensionless coupling constants and counting powers.[7] The dominant power comes from graphs with the minimum number of quark constituents. The example of the nucleon form factor is shown in Fig. 1.

The apparent success of this naive approach makes it a very important problem to understand this behavior in quantum field theory. This will be shown to involve both the infrared and short-distance properties of field theory. To make clear what we mean by this distinction, it is useful to discuss briefly fixed-angle scattering and form-factor behavior when the external particles correspond to the elementary fields of a renormalizable theory. In some theories (such as pseudoscalar-meson theories) the infrared region is suppressed,[8] and it can be shown that apart from renormalization effects, these processes become insensitive to the mass (M) of the external particles. By renormalizing off the mass shell at some Euclidean point of scale $\lambda \gg M$, a renormalization-group[9] equation can be written which governs the momentum dependence. In such theories, wide-angle scattering is directly probing short distances. In Abelian vector-gluon theories, on the other hand, the infrared region is not suppressed and the short-distance effects are shrouded in a cloud of $\log(E/M)$ factors in each order of perturbation theory.[10] In the first case, the measured power-law falloff (of the proton elastic form factor, for example) can be understood only if the short-distance dynamics produces a rather large negative anomalous dimension. In the second case, while leading-log summations give results that are not unreasonable,[10] a convincing calculational method is still missing.

A constituent model for these processes is similar to the neutral-vector-exchange theories with elementary external particles in that both short-distance effects and infrared effects are important. We have in mind an underlying non-Abelian quark-gluon theory (perhaps with "color" as the gauge symmetry) with the physical hadrons as gauge singlets.[11] Then short-distance effects can be dealt with making use of asymptotic freedom,[12] and we are left with the infrared problem to deal with. In more physical terms we ask, how sensitive are these processes to the large-distance details of the hadron wave function? It is a different, and perhaps simpler, problem than with elementary external particles and neutral vector exchange. We do not have to dig the power-law behavior out of an infinite series of perturbation-theory infrared logarithms. It is already present on the most naive level, and we ask whether infrared effects can substantially modify it.

In this paper we will examine this problem by focusing on the elastic form-factor behavior. For reasons of simplicity, we will carry out our analysis by considering a scalar field model, namely the ϕ^3 theory in six dimensions (ϕ^3_6). Even though this theory is a nonphysical theory, it has great mathematical and graphical similarity with the more physical non-Abelian quark-gluon theories. In particular, it is renormalizable, it has a fundamental trilinear coupling, and furthermore it happens to be asymptotically free.[13] It is simpler since we do not have to worry about Dirac spinology and the complications of non-Abelian gauge invariance. We stress, though, that this model is not a completely satisfactory prototype for the non-Abelian theory in that it is not as infrared sensitive. The controlling of infrared logarithms in that theory is less straightforward, and we will try to stress, throughout the text of this paper, where these differences may come in and what kind of effects they might have.

We believe, though, that our choice of model will give us a well-defined general picture of the main features and possible difficulties to be encountered in this problem. The more physical theories will be taken up later.

In Sec. II we describe the model and outline the elements of the form-factor problem. Section III concerns the Bethe-Salpeter (BS) wave function.[14] Its behavior at short distances is discussed using the Wilson operator product expansion (OPE).[15] The connection of the OPE to both the homogeneous and inhomogeneous BS equation[14] is established. This connection makes use of a theorem about the infrared structure of the BS kernel which we prove in Appendix A. The entire discussion refers to our scalar field model, but can easily be generalized to other theories. Much of the material of Sec. III is interesting independently of the rest of the paper, and it contains, we believe, several new observations.

In Sec. IV we point out, by a simple example, that the form factor is not only sensitive to the short-distance properties of the BS wave function. An attempt to determine the wave function in other kinematic regions is discussed. This is, of course, a very important and yet to be solved problem. Given the behavior of the wave function, there is still the behavior of the strong radiative corrections [as shown in Fig. (3)] to deal with. We show in Sec. V that these corrections only influence the form-factor behavior through their short-distance behavior even if they are quite singular in the infrared. In an asymptotically free theory, they will give only a logarithmic modification to the form factor. In Sec. VI we summarize our results and discuss the extension to colored quark-gluon theories. The breakdown of the naive power-counting arguments in wide-angle scattering (Ref. 7) is discussed, and we speculate about a solution to this problem. A few remarks about the connection of this work to the quark-confinement problem are included.

The considerations in this paper are not necessarily tied to asymptotic freedom even though the model we use is asymptotically free. The main effort here is to separate infrared from short-distance effects. Once this separation is made, the renormalization group can be applied. An ultraviolet-stable fixed point with zero or small anomalous dimensions would lead to results similar to those in this paper.

II. THE FORM FACTOR

The problem to be examined in this paper is a scalar field (ϕ_6^3) prototype of the pion form factor. The more realistic model we have in mind is a colored quark-gluon model, with the bound state a color singlet.[16] The essential features there can be incorporated into the ϕ_6^3 model in a variety of ways. For example the interaction $g\vec{\phi}\cdot\psi^*\vec{\tau}\psi$ with ψ a complex scalar isodoublet and $\vec{\phi}$ a scalar isotriplet would suffice. There is a conserved charge corresponding to the symmetry $\psi\to e^{i\theta}\psi$. The force between two ψ constituents is attractive in the singlet channel and a doubly charged, isosinglet bound state would necessarily consist of at least two ψ constituents. Another model is the interaction $g\phi\psi^*\psi$ with ψ a complex scalar and ϕ a Hermitian scalar.[17] A doubly charged bound state (the force between two ψ's is again attractive) would again consist of at least two ψ constituents. Either model is asymptotically free, and the reader can imagine using either one throughout this paper.

The scalar quark ψ will be given a mass m (see Ref. 18) and the scalar gluon ϕ will be taken to be massless.[19] Wave-function renormalization subtractions will be performed at the Euclidean point $k^2=-\lambda^2$ and the vertex will be subtracted at the symmetric Euclidean point $k_1^2=k_2^2=k_3^2=-\lambda^2$. The corresponding coupling constant will be g_λ. We will take $m\ll\lambda<q$, with q the momentum transfer in the form factor.

Graphical notation for propagators and bound-state wave functions is shown in Fig. 2. The homogeneous BS equation for the two-component wave function will be examined in the next section. The calculation of the form factor can be organized as shown in Fig. 3. We are making use here of the conventional Bethe-Salpeter scattering formalism.[14] There are then, two basic parts to the problem.

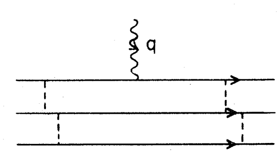

FIG. 1. A graph for the proton elastic form factor. Power counting with the assumption that q is the only dimensional quantity asymptotically leads to $F_1(q^2)\sim 1/q^4$. Invariant normalization $u^\dagger u=2E\sim 2q$ is used for each quark line.

First, one must determine the behavior of the two-component BS wave function, and then the strong radiative corrections must be included and controlled.

To get some perspective on the problem, we first recall the prediction one expects, following the mnemonics of Refs. 4 and 5 for our particular model. We consider a connected Feynman graph, such as the one in Fig. 4, and count powers. Any such graph with two constituent lines running through it will lead to the asymptotic behavior

$$F(q^2) \sim 1/q^4\,, \tag{2.1}$$

up to logarithmic powers which depend on the order of perturbation theory. Graphs with more constituent lines will be down by powers of q^2. Thus, within logarithmic corrections, Eq. (2.2) will represent the naive asymptotic behavior for our pion form factor in six dimensions. The problem is to understand how this power law might survive in a complete analysis.

An exact evaluation of the graph in Fig. 4 gives

$$F(q^2) \sim (1/q^4)\log^2(-q^2/m^2)\,, \tag{2.2}$$

showing a logarithmic sensitivity to m^2 the mass of the constituent. This already indicates that we are not dealing with a pure short-distance problem in the renormalization-group sense. Nearly everything in the following sections is essentially an effort to deal with this problem.

III. THE BS WAVE FUNCTION AT SHORT DISTANCES

In this section the connection of the homogeneous and inhomogeneous BS equations to the Wilson operator product expansion is established. It is a brief discussion completely in the context of asymptotically free theories, and serves as an introduction to the remainder of the paper which is more directly concerned with the form-factor calculation.

The inhomogeneous BS equation for the connected, truncated four-point function of four ψ fields is depicted in Fig. 5. The arrows indicate the direction of electrical charge flow. A bound state corresponds to the existence of a pole in the four-point function. Separation of the pole piece leads to the homogeneous BS equation for the bound-state wave function as shown in Fig. 6. Analytically, the equation is

$$\phi(k^2, (p+k)^2) = \frac{i}{(2\pi)^6}\int d^6q\, \phi(q^2, (p+q)^2)D(q^2) \times D((p+q)^2)K(q,p,k)\,. \tag{3.1}$$

$\phi(k^2, (p+k)^2)$ is the truncated wave function and $D(q^2) \equiv [1/(q^2-m^2)]\times d(q^2/\lambda^2, (m^2/\lambda^2), g_\lambda)$ is the ψ propagator. Dependence on the renormalization mass λ, the ψ mass m, and coupling constant will often be suppressed. $K(q,p,k)$ is the two-particle irreducible BS kernel.

The BS wave function $\phi(k^2, (p+k)^2)$ can be determined without reference to the BS equation in the limit $-k^2 \to \infty$ at fixed p. This is the limit $p\cdot k/k^2 \to 0$ or $k^2/(p+k)^2 \to 1$. We use the definition of the wave function

$$\phi(k^2, (p+k)^2)D(k^2)D((p+k)^2) = \int d^6x\, e^{ik\cdot x}\langle \pi(p)|T\psi(x)\psi(0)|0\rangle \tag{3.2}$$

and the Wilson operator product expansion[15] of the two ψ fields. In an asymptotically free theory, naive power counting can be used to determine the

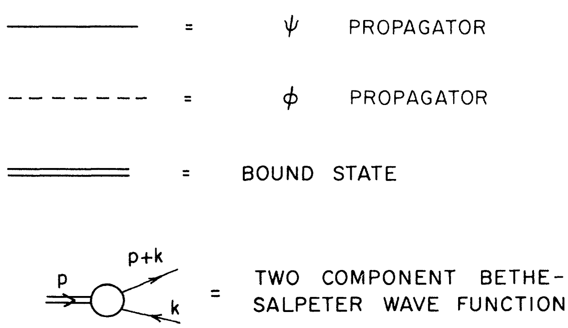

FIG. 2. Graphical notation for propagators and the bound-state wave function.

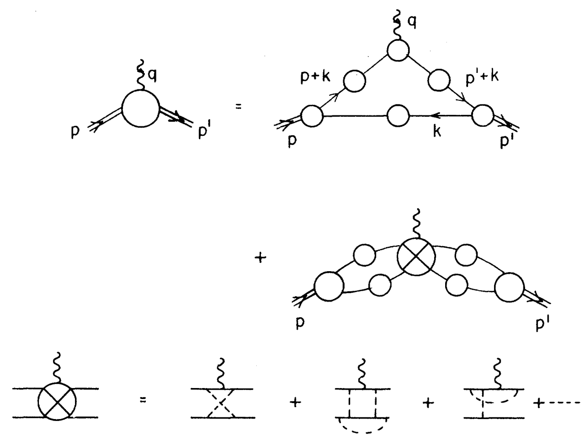

FIG. 3. The components of the form-factor calculation. The crossed circle represents corrections which are two-particle irreducible in the incoming and outgoing bound-state channels.

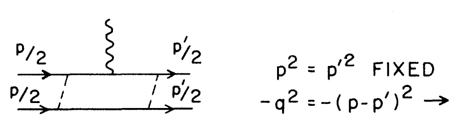

FIG. 4. A form-factor perturbation-theory graph for the ϕ_6^3 theory.

asymptotic behavior up to logarithmic corrections. Scalar fields have naive dimension m^2 in six dimensions, and the operator of lowest dimension in the OPE which can connect the vacuum to the bound state is $\psi^2(0)$. The asymptotic behavior of the right-hand side (RHS) of Eq. (3.2) is

$$\langle \pi(p)|\psi^2(0)|0\rangle (1/k^6)[\log(-k^2/\lambda^2)]^{-1+\gamma_{\psi\psi}}, \tag{3.3}$$

where the power $\gamma_{\psi\psi}$ is calculable in perturbation theory. It is the anomalous dimension of the operator $\psi^2(x)$. The asymptotic behavior of the propagator $D(k^2)$ is, apart from a multiplicative constant,

$$D(k^2) \sim (1/k^2)[\log(-k^2/\lambda^2)]^{\gamma}, \tag{3.4}$$

where γ is also calculable in perturbation theory. Thus

$$\phi(k^2, (k+p)^2) \sim G(p^2)(1/k^2)[\log(-k^2/\lambda^2)]^{-1+\gamma_{\psi\psi}-2\gamma}, \tag{3.5}$$

where $G(p^2)$ is, apart from a multiplicative constant, the matrix element in (3.3).

It is useful and instructive to see how this behavior arises from the inhomogeneous BS equation. We examine this and then return to the homogeneous Eq. (3.2). The OPE argument predicts the asymptotic behavior (3.5) for the connected four-point function $\Gamma(p,k)$ of Fig. 5 as well as the bound-state wave function, and we will show how it is built up through the iteration of the kernel. We use the momentum labeling of Fig. 7 with only two independent momenta for simplicity, and again consider the limit $-k^2 \to \infty$ with $p\cdot k/k^2 \to 0$. We first consider the behavior of the kernel itself, the first term in the iteration. Because of its two-particle irreducible structure, it can be shown that the kernel is infrared-convergent when p is scaled to zero and/or the mass parameters are scaled to zero at fixed k.

It is this property that allows the asymptotic behavior to be determined (or a least speculated about in nonasymptotically free theories) using renormalization-group methods.[9] A proof of this result is given in Appendix A. Its consequence is

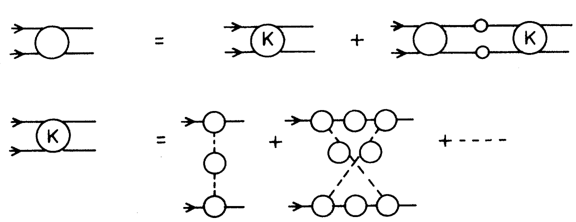

FIG. 5. The inhomogeneous Bethe-Salpeter equation and the skeleton expansion of the two-particle irreducible kernel.

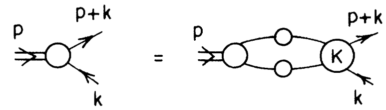

FIG. 6. The homogeneous Bethe-Salpeter equation.

$$K(p,k) \sim (1/k^2) f(k^2/\lambda^2, g_\lambda) \tag{3.6}$$

in the asymptotic limit we are considering.

The asymptotic behavior of $f(k^2/\lambda^2, g_\lambda)$ follows from the observation that the object

$$I(k^2/\lambda^2, g_\lambda) \equiv f(k^2/\lambda^2, g_\lambda) d^2(k^2/\lambda^2, 0, g_\lambda{}^2) \tag{3.7}$$

does not rescale under renormalization. We have taken the $m \to 0$ limit in the propagator. Then

$$I(k^2/\lambda^2, g_\lambda) = I(-1, g_k), \tag{3.8}$$

where, on the right-hand side, the symmetric Euclidean point $-k^2$ has been chosen as the renormalization point. The right-hand side is then an infinite series in $g_k{}^2$. The higher-order terms (beyond $g_k{}^2$) are present because the kernel is being evaluated at an asymmetric point. Since $g_k{}^2 \to [\log(-k^2/\lambda^2)]^{-1}$ asymptotically, the lowest order ($g_k{}^2$) term in $I(-1, g_k{}^2)$ gives the leading asymptotic behavior of $I(k^2/\lambda^2, g_\lambda)$. We have

$$I(k^2/\lambda^2, g_\lambda) \sim [\log(-k^2/\lambda^2)]^{-1}, \tag{3.9}$$

and using Eqs. (3.4) and (3.6),

$$K(p,k) \sim (1/k^2)[\log(-k^2/\lambda^2)]^{-1-2\gamma}. \tag{3.10}$$

This dominant term comes from just the first, single exchange, contribution in the skeleton expansion of the kernel (Fig. 5).

We next consider the second term in the iteration shown in Fig. 7. The leading term here (by a power of a logarithm) comes from inserting for each kernel the first term in its skeleton expansion. One then has a box graph with all vertex and self-energy insertions. This graph is logarithmically divergent in perturbation theory as $p^2 \to 0$ so that the renormalization group cannot be directly applied. Its asymptotic behavior is easily calculated by noting that the dominant region of integration is $q < k$. k is routed through the right hand rung of the graph so that for $q < k$, leading asymptotic behavior can be determined by setting p and q equal to zero in this part of the box. The existence of the limit follows, as before, from the theorem of Appendix A. In the rest of

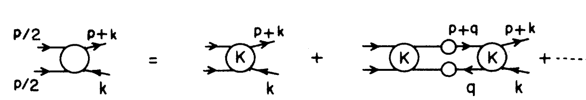

FIG. 7. The connected four-point function as an iteration of the kernel. The incoming momenta are each taken to be $p/2$ for simplicity.

the box, after performing the angular integration, q^2 is integrated up to k^2. The resulting expression is a product of factors for each part of the box:

$$\left[\int^{k^2}\frac{dq^2}{q^2}\frac{\gamma_{\psi\psi}}{\log(-q^2/\lambda^2)}+\cdots\right]\{(1/k^2)[\log(-k^2/\lambda^2)]^{-1-2\gamma}\}. \tag{3.11}$$

The second factor is the behavior of the right hand rung, the ladder term in the kernel. In the first factor, we exhibit only the upper limit and asymptotic form of the integrand. The inverse log arises from the fact that the combination of vertex functions and propagators to be integrated is exactly the square of the invariant charge of the renormalization group. The constant $\gamma_{\psi\psi}$ comes from the factors of π and the angular integration. Carrying out the integration, we have, through second order in the iteration,

$$\Gamma(p,k)\sim(1/k^2)[\log(-k^2/\lambda^2)]^{-1-2\gamma}\times(1+\{\gamma_{\psi\psi}\log[\log(-k^2/\lambda^2)]+f_1(p^2)\}+\cdots). \tag{3.12}$$

The function $f_1(p^2)$ comes from the lower end of the q^2 integration. All of the logarithmic sensitivity to p^2 as $p^2\to 0$ is buried in this function.

The next term in the iteration of the kernel gives the leading behavior

$$(1/k^2)[\log(-k^2/\lambda^2)]^{-1-2\gamma}\left\{\frac{(\gamma_{\psi\psi})^2}{2!}\log^2[\log(-k^2/\lambda^2)]+f_1(p^2)\gamma_{\psi\psi}\log[\log(-k^2/\lambda^2)]+f_2(p^2)\right\}, \tag{3.13}$$

and the complete sum exponentiates into the factored form

$$\Gamma(p,k)\sim(1/k^2)[\log(-k^2/\lambda^2)]^{-1+\gamma_{\psi\psi}-2\gamma}F(p^2), \tag{3.14}$$

where

$$F(p^2)=1+f_1(p^2)+f_2(p^2)+\cdots.$$

This result is to be compared with Eq. (3.5) for the wave function. The present discussion shows how the Wilson OPE comes about in perturbation theory for the asymptotically free theory being considered here. The k behavior and the p behavior factorize so that the large-k behavior at fixed p can be determined despite the infrared divergences in $F(p^2)$ when $p^2\to 0$. In both (3.5) and (3.14), the $1/k^2$ factor is present because a quantum-number assignment forces at least two ψ fields to be present. In a channel with the quantum numbers of the ϕ, for example, the leading power-law behavior would be $(k^2)^0$.

We now return to the homogeneous equation (3.1), still looking at the limit $k^2/(p+k)^2\to 1$. It is quite easy to find solutions in this limit by making use of the structure of the BS kernel proved in Appendix A. It is found that the equation admits solutions of the form[20]

$$\phi(k^2,(k+p)^2)\sim[\log(-k^2/\lambda^2)]^A \tag{3.15}$$

and

$$\phi(k^2,(k+p)^2)\sim(1/k^2)[\log(-k^2/\lambda^2)]^B, \tag{3.16}$$

with both A and B determined. The first of these solutions we have already disposed of in the OPE approach by the condition that there are no elementary fields with the quantum numbers of the bound state. This information is not in the BS equation, but armed with our OPE wisdom we forget (3.15) and concentrate on (3.16).

After performing the angular integration in (3.1) and inserting the solution (3.16), we have

$$(1/k^2)[\log(-k^2/\lambda^2)]^B=\gamma_{\psi\psi}(1/k^2)[\log(-k^2/\lambda^2)]^{-1-2\gamma}\int^{k^2}\frac{dq^2}{q^2}[\log(-q^2/\lambda^2)]^{B+2\gamma}+\gamma_{\psi\psi}\int_{k^2}^{\infty}\frac{dq^2}{(q^2)^2}[\log(-q^2/\lambda^2)]^{-1+B}. \tag{3.17}$$

In the lower part of the integral, we have set $p,q=0$ in the kernel (using Appendix A) and used its asymptotic form (3.10). Only the upper limit and asymptotic form ($k>\lambda\gg m$) of the integrand are displayed in the q^2 integral. In the upper part, we have taken $k\to 0$ in the kernel again using the theorem of Appendix A. Again the constant $\gamma_{\psi\psi}$ comes from the factors of π and the angular integral. It is exactly the same constant that appears in (3.11) and (3.12). The approximations in (3.17) are allowable since we only want to determine the leading behavior of the solution. The contribution from the upper part is clearly suppressed by one power of $\log(-k^2/\lambda^2)$, so we look at the lower part.

If $B+2\gamma>-1$, then the dominant contribution will come from the upper limit and B is fixed to be

$$B=-1-2\gamma+\gamma_{\psi\psi}; \tag{3.18}$$

this agrees exactly with the behavior found in

(3.5) and (3.14). The homogeneous BS equation is another framework for calculating the anomalous dimension of $\psi^2(x)$. It is more restrictive, however, since we must have $B+2\gamma>-1$, that is,

$$\gamma_{\psi\psi}>0\,, \tag{3.19}$$

or else (3.16) will not be a solution. The sign of $\gamma_{\psi\psi}$ is simply a question of the sign of the force between the two constituents. In either of the ϕ_6^3 models described in Sec. II or for the more realistic Yang-Mills theory, (3.19) is satisfied if the force between the constituents is attractive. We are, of course, always considering channels with attractive forces. It is quite reasonable that a solution discovered by OPE methods or through the inhomogeneous BS equation should only satisfy the bound-state equation if the force is attractive.

To summarize: The short-distance behavior of the BS wave function can be determined by the Wilson OPE or equivalently by a direct solution of the homogeneous BS equation. We emphasize that this is an *exact* asymptotic solution since the *complete* kernel can be controlled in the required limit.

IV. MORE ON THE BS WAVE FUNCTION

It has been emphasized in a long series of papers by Ciafaloni and Menotti[21] that form-factor behavior can in general depend upon more than just the short-distance properties of the bound-state wave function. This can be seen in the treatment of these authors, even though they neglect the effect of the strong radiative corrections of Fig. 3. We will make the same approximation in this section (Fig. 8) and turn to the full calculation in Sec. V. Our purpose here is to reemphasize the role of nonshort distances in the form-factor calculation[22] and to connect it to the discussion of the previous section.

In order to do this, it is convenient to introduce the Deser-Gilbert-Sudarshan-Ida (DGSI) spectral representation[23] of the wave function and perform the d^6k loop integration in Fig. 8. A convenient form of the spectral representation for the truncated wave function $\phi(k^2,(p+k)^2)$ is

$$\phi(k^2,(p+k)^2)=\int_{-1}^{+1}dz\int_0^\infty dt\,\frac{g(z,t)}{[\frac{1}{2}(1-z)k^2+\frac{1}{2}(1+z)(p+k)^2-m^2-t+i\epsilon]^2}\,. \tag{4.1}$$

The power 2 for the denominator is chosen with the known short-distance properties in mind. This behavior places conditions on the spectral function $g(z,t)$ in (4.1). We know that in the limit $-k^2\to\infty$ with $k^2/(k+p)^2\to 1$,

$$\phi(k^2,(k+p)^2)\sim G(p^2)(1/k^2)[\log(-k^2/\lambda^2)]\,, \tag{4.2}$$

where B is given by (3.18). This leads to a condition on the zeroth moment of $g(z,t)$:

$$g_0(t)\equiv\int_{-1}^{+1}dz\,g(z,t)\underset{t\to\infty}{\sim}[\log(t/\lambda^2)]^B\,. \tag{4.3}$$

The discussion of the previous section can be extended to the full tower of operators of leading twist in the light-cone expansion. (This happens to be twist 4 for the ϕ_6^3 model.) The result is a condition on the nth moments of the $g(z,t)$:

$$g_n(t)\equiv\int_{-1}^{+1}dz(1+z)^n g(z,t)\underset{t\to\infty}{\sim}[\log(t/\lambda^2)]^{B_n}\,, \tag{4.4}$$

with the B_n calculable in perturbation theory ($B_0\equiv B$).

Other asymptotic limits of (4.1) which do not correspond to short distances in coordinate space can have a different behavior from (4.2). For example, the limit $-(p+k)^2\to\infty$ at fixed k^2 (see Ref. 24) is sensitive (because of the $1+z$ in the denominator) to the $z\to-1$ behavior of $g(z,t)$. Examples of $g(z,t)$ can be written down[25] which satisfy (4.3) and (4.4) and yet produce a less rapidly falling form factor [less than (4.2)].

To see that the form factor is sensitive to the wave function in this limit, we calculate the triangle graph of Fig. 8 using (4.1) for the wave function. We introduce Feynman parameters with the labeling of Fig. 8 and perform the d^6k loop integration. The result is

$$F(q^2)=\text{const}\times\int_0^1 dx_1\cdots dx_5\,\delta(1-x_1-\cdots-x_5)x_3x_4$$

$$\times\int_0^\infty dt\,dt'\int_{-1}^{+1}dz\,dz'\,\frac{g(z,t)g(z',t')}{\{[x_1+x_3\frac{1}{2}(1+z)][x_2+x_4\frac{1}{2}(1+z')](-q^2)+m^2+2tx_3+2t'x_4\}^4}\,.$$

The mass of the bound state p^2 has been set equal to zero since the constituent mass m^2 serves as an infrared cutoff. Thus, we are dealing with an essentially Euclidean problem. The z or $z' \to 1$ sensitivity is obvious. The naive power law $1/q^4$ would emerge here if $g(z,t)$ would behave like t^0 for large t and like $(1+z)^\gamma$, $\gamma > 0$ for $z \to -1$. A more singular $z \to -1$ behavior [but compatible with (4.2) and (4.3)] could easily give a less rapidly falling form factor. Thus, the form-factor behavior can be quite sensitive to the $(k+p)^2/k^2 \to \infty$ behavior of the bound-state wave function if it is singular enough.

How singular is it? The solution to the homogeneous BS equation in this region is more difficult since it requires more information about the kernel than we have used in the short-distance case. We have not yet solved this problem, but at least in the scalar-field theory case, it seems possible to do this. To see why and to get some feeling for the possible behavior of the wave function, we will describe a closely related problem that we have looked at in detail. The truncated, connected four-point function of Fig. 7 (with just the two independent momenta as shown in that figure) is an object which can be analyzed iteratively. Keeping p^2 fixed, we can let $-k^2$ and/or $-(p+k)^2 \to \infty$ and explore its behavior. In the short-distance limit $k^2/(p+k)^2 \to 1$, it follows from the OPE that the bound-state wave function and the four-point function have exactly the same behavior. We will use the four-point function as a model of the wave function and examine its behavior in the other asymptotic limits as well. Whether or not this is a reasonable thing to do is unclear. There is certainly no guarantee that the wave function and connected four-point function have even similar behavior in any limit except the short-distance limit.

Each term in the iteration of Fig. 7 can be computed at least in leading-log approximation. Taking $R = k^2/(k+p)^2$, we find ($b > 0$ for asymptotically free theories)

$$\Gamma(p,k) \sim [1/(k+p)^2]\left[1 + bg_\lambda{}^2 \log\left(\frac{-(k+p)^2}{\lambda^2}\right)\right]^{-1-2\gamma} \left\{ 1 + g_1(R)\log\left[1 + bg_\lambda{}^2\log\left(\frac{-(k+p)^2}{\lambda^2}\right)\right] + g_2(R)\log^2\left[1 + bg_\lambda{}^2\log\left(\frac{-(k+p)^2}{\lambda^2}\right)\right] + \cdots \right\} . \qquad (4.6)$$

$R = 1$ corresponds to short distances and $R = 0$ corresponds to holding k^2 fixed. The first term in the sum is just the asymptotic form of the kernel. This is in fact the exact high-energy behavior of the kernel for any R, not just a sum of leading logs. This behavior can be determined exactly since the infrared properties of the kernel allow the renormalization group to be used for any R. This is a simple extension of the theorem of Appendix A to asymptotic limits other than short distances. The $R = 0$ case can be dealt with using the renormalization group in the scalar theory, but not in the Yang-Mills theory. This is because of logarithmic sensitivity to k^2 as $k^2 \to 0$ in this theory, which prevents the renormalization-group scaling arguments from being applied.[26] This is one of several ways in which the scalar theory is less infrared sensitive than the colored quark model.

The higher terms in (4.6) with coefficients $g_1(R)$, $g_2(R)$, etc., correspond to the higher terms in Fig. 7, and have been computed in leading-log approximations only. The leading terms come from the generalized ladder approximation to the kernel (see Fig. 5). The coefficients $g_m(R)$ can be computed from uncrossed m-loop ladder graphs without vertex and self-energy corrections. In the limit $R \to 1$,

$$g_n(R) \to \frac{\gamma_{\psi\psi}{}^n}{n!} , \qquad (4.7)$$

so that (4.6) agrees with (3.14). For general R we have not been able to sum (4.6), but the following fact is clear: For any value of R, the known short-distance behavior of the theory leads, at least in leading-log approximation, to a power series in $\log\{\log[-(k+p)^2/\lambda^2]\}$. For at least one value of $R (R=1)$ this series sums to a power of $\log[-(k+p)^2/\lambda^2]$, and it is reasonable to expect it not to be more singular than this for any R. Assuming this to be the case, then for any R,

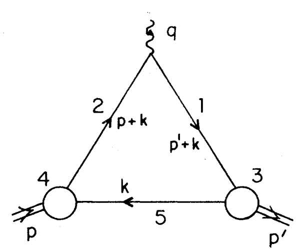

FIG. 8. The triangle approximation to the form factor.

$\Gamma(p, k)$ will behave like $1/(k+p)^2$ up to a finite power of a logarithm.

In the remainder of this paper we will assume that the bound-state wave function $\phi(k^2, (p+k)^2)$ has a similar behavior. We are presently investigating the homogeneous BS equation in an attempt to prove this. For the scalar theory, this seems to be feasible since the kernel can be controlled in all the necessary kinematic regions by using the renormalization group.

V. THE CALCULATION OF THE FORM FACTOR

We will examine successively the two contributions shown in Fig. 3. We will make the *assumption* that in any of the asymptotic regions discussed above, the wave function has the same power-law behavior that it has at short distances. For purposes of this section, we will neglect logarithmic factors in the wave function and make what we call the rung approximation[27]:

$$\phi(k^2, (p+k)^2) \sim \frac{1}{k^2+(p+k)^2} \sim \frac{1}{(k+p/2)^2} . \tag{5.1}$$

We have set $p^2=0$ anticipating that the constituent mass m will serve as an infrared cutoff. We emphasize that we have not proved (5.1) to be the correct power-law behavior of the wave function in all the kinematic regions that might be important in the form-factor calculation. It is because of this and the fact that we are only dealing with a model field theory here that it does not seem worthwhile to try to include logarithmic factors in Eq. (5.1).

The first contribution of Fig. 3 involves only propagator insertions and the photon-ψ-ψ vertex correction. We first neglect the propagator corrections. It is then easy to see that to any order in perturbation theory, there is no more than a double logarithmic sensitivity to the constituent mass m^2. This can be seen either by examining the Feynman parametric expression for an arbitrary term in the perturbation expansion[28] or by the following simpler argument.

The form of the vertex insertion is $(p+p'+2k)_\mu$, $V(q^2, (p+k)^2, (p'+k)^2)$. It can be shown[29] that the asymptotic q^2 behavior of V is insensitive to whether or not $(p+k)^2$ and $(p'+k)^2$ are being held fixed or scaled up with q^2. Thus the ϕ_6^3 theory is similar to the Yukawa theory in four dimensions in that elastic form-factor behavior (with elementary external particles) can be determined by renormalization-group methods. Consequently, the double logarithmic sensitivity to m^2, present without the vertex insertion [see Eq. (2.2)] is not changed by the inclusion of these corrections.

The large-q^2 behavior of the form factor is therefore

$$(1/q^4)\log^2(-q^2/m^2)F(q^2/\lambda^2, g_\lambda{}^2) \tag{5.2}$$

plus nonleading terms. With the m^2 sensitivity factored out explicitly, a renormalization-group equation can be written for $F(q^2/\lambda^2, g_\lambda{}^2)$. The short distance and infrared effects have been separated. Using the asymptotic freedom of this theory, the large-q^2 behavior of $F(q^2/\lambda^2, g_\lambda{}^2)$ is $[\log(-q^2/\lambda^2)]^{-\gamma}$, where γ is the anomalous dimension of the ψ field. This is a consequence of the Ward identity.

The above result will not be true in the quark-vector-gluon theory. In both the color theory and the Abelian theory,[10] the photon-quark-quark vertex is not insensitive to $(p+k)^2$ and $(p'+k)^2$ as q^2 is scaled up relative to them. We return to this problem in Sec. VI.

The propagator insertions can now be included. These corrections, though, increase the m^2 sensitivity. In perturbation theory, to order $(2n)$, each $D(k^2)$ behaves like

$$\frac{1}{k^2-m^2}\left[\log\frac{k^2-m^2}{\lambda^2}\right]^n$$

as $k^2 \to m^2$ due to the onset of a threshold for the production of massless particles, and thus, in the form-factor calculation one can exhibit arbitrarily high powers of $\log m^2$. However, the asymptotic q^2 behavior of the form factor can be shown to be unaffected by such infrared behavior. One way to do so is by introducing a spectral representation for each of the three ψ propagators shown in the first contribution of Fig. 3. Recall that

$$D(k^2) = \frac{1}{k^2-m^2}\, d(k^2/\lambda^2, m^2/\lambda^2, g_\lambda),$$

with $d(-1, m^2/\lambda^2, g_\lambda)=1$. With a subtraction at $k^2=-\lambda^2$, the spectral form is

$$D(k^2) = \left[\frac{1}{k^2-m^2} + \int_0^\infty \frac{dM^2\sigma(M^2)}{(k^2-M^2)(\lambda^2+M^2)}\right] + \frac{\lambda^2+m^2}{k^2-m^2}\int_0^\infty \frac{dM^2\sigma(M^2)}{(k^2-M^2)(\lambda^2+M^2)} . \tag{5.3}$$

This expression can incorporate the correct asymptotic behavior of $D(k^2)$ as well as singular threshold behavior. Asymptotically, $\sigma(M^2) \sim [\log(M^2/\lambda^2)]^{\gamma-1}$ with $\gamma>0$, so that $D(k^2)$ satisfies (3.4).

The analysis leading to (5.2) can be repeated. One must analyze Feynman parametric integrals together with three integrals over spectral weight functions. We outline the main steps. The contribution in square brackets in (5.3) is easy to treat. If this piece is used for all three propagators the asymptotic q^2 behavior is sensitive only to the high-M^2 behavior of each $\sigma(M^2)$. In place of (5.3) one now finds the leading behavior

$$\frac{1}{q^4}\,[\log(-q^2/\lambda^2)]^{2+3\gamma}\,G(q^2/\lambda^2, g_\lambda)\,, \qquad (5.4)$$

where G behaves like $[\log(-q^2/\lambda^2)]^{-\gamma}$ at high q^2. This is in fact the dominant q^2 behavior of the first contribution of Fig. 3.

Contributions involving the second term (5.3) are suppressed at least by factors of $[\log(-q^2/\lambda^2)]^{\gamma}$. It is this piece that contains the singular threshold behavior. In Feynman-parameter computations, an additional parameter must be introduced when this piece is used because of the two propagator-like factors in the denominator. It is simplest to think of performing the parameter integrations first, including those in the vertex correction. Then all orders in the vertex correction can be summed and the vertex produces an overall factor of $[\log(-q^2/\lambda^2)]^{-\gamma}$ as before. The high-M^2 part of the spectral integral will be suppressed by a power of q^2 when this second part of the propagator is used. The reason for this is that the two propagatorlike factors, or equivalently the additional Feynman parameter, lead to an additional factor of M^{-2} in the final M^2 integration.

The low-M^2 part of the integral is suppressed by a power of $[\log(-q^2/\lambda^2)]^{\gamma}$. The reason for this is that the integral is convergent in the threshold region ($M^2 \to m^2$) even with $\sigma(M^2)$ singular in this limit. Recall that in perturbation theory to order $2n$, $\sigma(M^2) \sim [\log(M^2-m^2)]^n$ in this limit. The integral in fact converges even with $\sigma(M^2) \sim \delta(M^2-m^2)$ corresponding to the behavior $D(k^2) \sim 1/(k^2-m^2)^2$ near threshold.

The above result, that the infrared singular structure of the propagators does not affect the q^2 behavior of the form factor, depends critically on the fact that it is the constituent mass and not the bound-state mass which is serving as the infrared cutoff. This can be seen already with no vertex insertions or propagator insertions except on the line carrying momentum k. Then with $m=0$, P^2, the mass of the bound state serves as the cutoff, and arbitrarily high powers of $\log(q^2/p^2)$ are found. Any such term can be bounded by using $1/k^4$ as the propagator. This corresponds to $\sigma(M^2) = \delta(M^2)$. This bounding integral, however, behaves like

$$(1/q^2)\,[1/(p^2)^2]\log(-q^2/p^2)\,,$$

suggesting that the logarithms could exponentiate to change the power law. The reason for this difference between using the constituent mass and the bound-state mass as the cutoff is made clear by an example in Appendix B.

The second contribution of Eq. (5.3) contains the two-particle irreducible corrections. It can be analyzed along the same lines. If we neglect the insertions on the four propagators leading into the crossed circle, a renormalization-group argument can be used to directly determine the q^2 dependence. The argument hinges on the structure of the graphs contributing to the crossed circle, namely that they are two-particle irreducible in the incoming and outgoing bound-state channels. Because of this, it can be shown that, just as in the case of the vertex correction, the double logarithmic sensitivity to m^2 of the bare graph is not increased in any order of perturbation theory. This involves a straightforward analysis of Feynman parametric integrals.[28] Corrections which are not two-particle irreducible in the above sense, have of course been summed into the definition of the Bethe-Salpeter wave function. This we are assuming to be of the form (5.1) up to a logarithm.

The m^2 sensitivity can be factored out as before. The leading term which contains the factor $[\log(-q^2/m^2)]^2$ is

$$\frac{1}{q^4}[\log(-q^2/m^2)]^2 H(q^2/\lambda^2, g_\lambda^2)\,, \qquad (5.5)$$

where a renormalization-group analysis can be used to determine the large-q^2 behavior of H. For our asymptotically free theory, it will be dominated by the lowest order graphs in the skeleton expansion of the crossed circle because these contain the least number of factors, namely four, of the invariant charge of the renormalization group. A simple analysis shows that the large-q^2 behavior of $H(q^2/\lambda^2, g_\lambda^2)$ is $[\log(-q^2/\lambda^2)]^{-2-2\gamma}$. Finally, the four propagator insertions on the lines leading into the crossed circle can be included. Again, only the high-k^2 part of the propagators will affect the q^2 dependence. Four factors of $[\log(-q^2/\lambda^2)]^{\gamma}$ are picked up so that the complete high-q^2 behavior of the second contribution of Fig. 3 can be written down. With the approximation (5.1) for the wave function, its behavior is $(1/q^4)[\log(-q^2/m^2)]^2 \times[\log(-q^2/\lambda^2)]^{-2+2\gamma}$. It is thus suppressed by a factor of $[\log(-q^2/m^2)]^{-2}$ relative to the first contribution, [see Eq. (5.4)].

To conclude, if the wave function is assumed to have the behavior (5.1) up to logarithmic corrections, then one can show that only the short-distance behavior of the strong radiative corrections will affect the q^2 dependence of the form factor. The above discussion can easily be modified to include logarithmic factors in the wave function.

VI. SUMMARY AND DISCUSSION

A. Summary

We have examined the behavior of the elastic form factor for a composite particle in a scalar field theory. This is a first step in an attempt to understand fixed-angle scattering scaling laws within a field-theoretical framework. We have used the ϕ_6^3 theory as a prototype of a more realistic quark-gluon theory. This theory, which is asymptotically free, is structurally simpler and less sensitive in the infrared. Our main results are as follows:

1. The form-factor behavior depends on the bound-state wave function and the strong radiative corrections as shown in Fig. 3.

2. The bound-state wave function can be determined at short distances using the Wilson operator product expansion or by directly solving the homogeneous BS equation.

3. The form factor is sensitive to the asymptotic behavior of the wave function in limits other than the short-distance limit. We have examined the four-point function in all such asymptotic limits in order to argue that the power-law behavior of the wave function is the same.

4. If it is assumed that this is true, then the effect of all the strong radiative corrections can be included, and they only modify the asymptotic q^2 dependence through their ultraviolet behavior. In an asymptotically free theory, this amounts only to a logarithmic variation.

B. Extension to the Yang-Mills theory

Nearly everything presented in this paper can be carried over to the Yang-Mills theory. Apart from a more complex notation and difficulties of non-Abelian gauge invariance, the only new problem is the additional infrared sensitivity which we have already mentioned several times. This problem, we recall, arises in at least two places. First, the photon-ψ-ψ vertex correction (Fig. 3) is not as straightforward to deal with as in the Yang-Mills theory where ψ is a quark field. When all three momenta are scaled up together, the renormalization group can be applied as in the ϕ_6^3 theory, but not so in the region of integration $-q^2 \gg -(p+k)^2$, $-(p'+k)^2$. In $2n$th-order perturbation theory, the dominant term in this region has a behavior

$$\{\log[q^2/(p+k)^2]\log[q^2/(p'+k)^2]\}^n .$$

This double-log behavior in each order was first calculated by Sudakov[30] in quantum electrodynamics. In that theory, the signs alternate in each order and the leading terms exponentiate to an expression which falls more rapidly than any power as $-q^2 \to \infty$. If this behavior can be trusted and if the Yang-Mills theory behaves similarly, then this region of integration will be suppressed and only the renormalization-group region will contribute. This problem is under investigation.

This distinction between ϕ_6^3 and Yang-Mills will also probably arise in the solution of the homogeneous BS equation. The perturbative result (4.6) for the connected four-point function $\Gamma(p,k)$ is certainly not valid for $R=0$ [fixed $-k^2$ with $-(p+k)^2 \to \infty$]. The very first term in the iteration, the kernel $K(p,k)\,(q=0)$, cannot be determined by renormalization-group methods in this limit. Just as in the vertex correction, it is logarithmically sensitive to k^2 and p^2 as $-(p+k)^2 \to \infty$.

C. Fixed-angle scattering

Landshoff has recently observed that there can be linear infrared sensitivity to the mass parameters in this process coming from pinch singularities. This causes a less rapidly falling s behavior than given by simple dimensional analysis. With two quark constituents in each of the two incoming and two outgoing bound states, this contribution arises from the possibility of two independent quark-quark scatterings. In each one the quark lines stay at some fixed distance from their mass shell relative to s and t. It is quite possible that the quark-quark scattering amplitude in this limit is similar to the photon-quark-quark vertex when $-q^2 \to \infty$ with $(p+k)^2$ and $(p'+k)^2$ fixed, namely, rapidly damped.[31] This would suppress the Landshoff mechanism and reinstate the simple power-counting result up to logarithmic corrections.

D. Quark confinement

It is quite possible that the infrared instabilities of the Yang-Mills theory could be responsible for the strong long-range forces that screen quark quantum numbers from physical states. In the scalar field theory model we have investigated, this might also happen since the theory is asymptotically free and therefore can become strongly coupled at low momenta. However, it seems that in this model form-factor and fixed-angle scattering behavior is not sensitive to the details of how this happens. The form-factor behavior is, for example, insensitive to the low-momentum behavior of the strong radiative corrections. The form factor does, however, depend upon $\phi(k^2,(p+k)^2)$ in non-short-distance regions. As discussed in Sec. V, a singular behavior here could destroy the naive scaling result. On the basis of a perturbative analysis we have guessed that this does not happen [Eq. (5.1)]. Even if (5.1) is wrong, say when $-(k+p)^2 \to \infty$ at fixed $-k^2$, it seems unlikely that it would be more singular, i.e., fall off less rapidly. Such a behavior would require large contributions from the large longitudinal distances[24] which contribute in this kinematic limit. We expect the coordinate-space wave function rather to be damped in this limit if the constituents are to be strongly bound. Because of the sensitivity to low-momentum structure of the strong radiative corrections in the Yang-Mills theory, the connection of form-factor behavior to quark confinement is not so clear.

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APPENDIX A: ASYMPTOTIC BEHAVIOR OF THE BS KERNEL

In this appendix we will show that the two-particle irreducible BS kernel $K(q,p,k)$ of the massless ϕ_6^3 theory[32] is free of infrared divergences in the limit where q and p are scaled to zero relative to k. This will justify the renormalization-group arguments used in Sec. III.

Some graphical elements of the kernel are shown in Fig. 5. All the graphs can be divided into two classes. One class includes the one-particle reducible, one ϕ exchange graphs, the other includes the rest. To show that the one meson exchange graphs are infrared finite we just have to show that the vertex function with one external momentum scaled to zero with respect to the other two is infrared finite. That this is in fact true can be seen by explicitly examining some vertex function graphs. We will now give a general, simple proof of this statement. Any vertex graph can be represented as shown in Fig. 9. The four-point function denoted by $T(s,k)$ is the sum of connected Feynman graphs which are one-particle irreducible in the zero momentum channel. When all its external momenta are fixed, Kinoshita's theorem tells us that it is free of infrared divergences.[33] By power-counting arguments $T(s,k)$ behaves like an inverse power of squared momentum, such that as $s \to 0$,

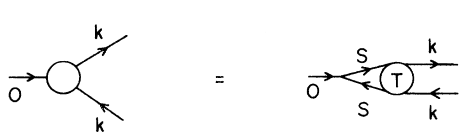

FIG. 9. General decomposition of a vertex function.

$$T(s,k) \sim \frac{1}{k^2}\, t(s^2/k^2) \ . \tag{A1}$$

$t(s^2/k^2)$ cannot have worse than a logarithmic divergence as $s \to 0$. Therefore, the final integration over $\int (d^6 s/s^4) T(s,k)$ is infrared finite.

To show that the remaining kernel graphs are infrared finite we will use a reduction technique together with simple power-counting arguments. Suppose first that we choose a routing such that the momentum k goes through the least number of internal lines of the kernel. We then shrink every such line to a point, so that one obtains the reduced graphs shown in Fig. 10(a). The reduced kernel K_R can be disconnected. Simple power counting will tell us whether the reduced graph is finite in the infrared, namely the Dyson degree of divergence must be positive. Let K_R have N external lines joining at the point P. Then the degree of divergence of the reduced graph is

$$D_N = 2N - 4 \ . \tag{A2}$$

This formula is valid even when K_R is disconnected [Fig. 10(b)]. Then D_N is simply the sum of the degree of divergence of each piece. Because the kernel is two-particle irreducible, $N>2$ and therefore $D_N > 0$.[34]

Up to now, we have only shown that the reduced graphs discussed above are overall infrared finite. The reduced graph may still contain some divergent subintegrations. Suppose now that we route the momentum k through the kernel graph in any other possible way and, as before, we obtain a corresponding reduced graph by shrinking all internal lines which carry k to a point. If we can show that all such reduced graphs are themselves overall infrared convergent then our proof will follow since, by choosing all the possible routings, we

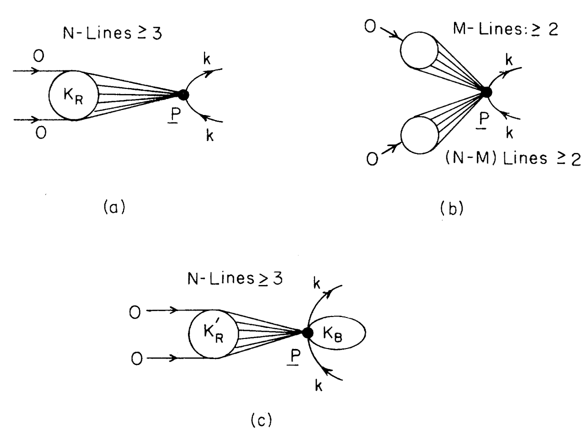

FIG. 10. Reduced graphs obtained by various possible routings of the momentum k through the kernel graphs, as explained in Appendix A.

are effectively exposing all the possible subintegrations which may be potentially infrared divergent. The general form of the reduced graphs obtained by the various choices of nonminimal routings is given in Fig. 10(c). Again K_R' may be disconnected. Carrying out the power-counting arguments for every K_R' and K_B resulting from each distinct routing, we find that if K_B is infrared finite, every reduced graph must be finite because, again, two-particle irreducibility forces $N>2$. The following argument shows that K_B is infrared finite. Let L be the number of internal lines carrying the momentum k that have been shrunk. There is then an overall factor of $(k^2)^{-L}$. Since the kernel must behave as $(k^2)^{-1}$, if D_B is the degree of divergence of K_B and D_n is as given in (A1) then

$$D_n + D_B - 2L = -2, \tag{A3}$$

and therefore K_B will be infrared convergent ($D_B > 0$) if

$$L > (N-1)\ . \tag{A4}$$

It is not hard to see that the case $L=(N-1)$ corresponds to just its minimal case, and thus it corresponds to the case previously discussed. Therefore (A4) must be necessarily true.

APPENDIX B

In this appendix, we look at an example to clarify some of the statements made in Sec. V. We claimed there that the threshold singularities of $D(k^2)$ (see the first graph of Fig. 3) could strongly affect the q^2 behavior of the form factor if the constituent mass is zero. We illustrate this by using the form (5.1) for the wave function and neglecting all the radiative corrections except the propagator insertion on the bottom rung. As an extreme model of singular threshold behavior we take $D(k^2) = K/(k^2-m^2)^2$ and then compare the $m \neq 0$ and $m=0$ cases.

We obtain the following parametric integral:

$$\int \frac{dx_1 dx_2 dx_3 dx_4 dx_5\, \theta(1-x_1-x_2-x_3-x_4-x_5)}{[-q^2(x_1+x_3/2)(x_2+x_4/2) - p^2(x_1+x_2+x_3/2+x_4/2)(1-x_1-x_2-x_3/2-x_4/2) + m^2(1-x_3-x_4)]^3}\ ; \tag{B1}$$

as $-q^2 \to \infty$, we observe that when $m \neq 0$ the leading contribution to (B1) is in the region x_1, x_2, x_3 and $x_4 \to (m^2/-q^2)^{1/2}$, so that m^2 will serve as the infrared cutoff. Then, the large-q^2 behavior of (B1) can be seen to be of the form

$$\frac{1}{q^6}\left[\left(\frac{-q^2}{m^2}\right)^{1/2}\right]^2 \log(-q^2/m^2)\ ,$$

which has the canonical $1/q^4$ behavior.

If the mass were zero, $-p^2(x_1+x_2+x_3/2+x_4/2)$ would serve as the infrared cutoff, and the important region would then be $x_1, x_2, x_3, x_4 \to (p^2/q^2)$. The large-$q^2$ behavior of (B1) in this case would be

$$(1/q^6)(q^2/p^2)^2\ ,$$

which behaves like $1/q^2$ and would dominate the canonical result. It is only the propagator carrying momentum k (the bottom rung) whose threshold singularities can affect the q^2 behavior in this way.

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[13]This does not violate the general result of S. Coleman and D. Gross [Phys. Rev. Lett. 31, 851 (1973)] since they only considered theories with quartic scalar couplings. The well-known problems of ϕ^3 theories do not appear in perturbation theory, so that they do not affect the considerations of this paper.

[14]For a review of the BS formalism and its application to scattering of bound states, see S. Mandlestam, Proc. R. Soc. A233, 248 (1953).

[15]K. G. Wilson, Phys. Rev. 179, 1499 (1969), and unpublished work; see also W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser *et al.* (MIT Press, Cambridge, Mass., 1970).

[16]See Ref. 11 for a description. With a gauge symmetry unbroken, the infrared instability of this theory might well be the source of the strong long-range forces necessary to form bound states in color singlet channels. The infrared problems in the form-factor calculation will be shown to be less difficult to deal with than this yet to be understood dynamical problem. See Sec. VI for further discussions of this point.

[17]The model as it stands is a two-coupling-constant theory since a ϕ^3 interaction is necessarily induced through renormalization. This can be avoided by introducing another complex field χ and demanding invariance under the symmetry operation $\phi \to -\phi$ and $\psi \to \chi$. The χ plays no other role in the rest of the paper so it is suppressed. This problem does not arise in the SU(2) model.

[18]It will turn out to be important to give the ψ a mass at least on the order of magnitude of the bound-state mass. It will be shown in Sec. V that it is very important that the constituent mass rather than the bound-state mass provides the infrared cutoff in the form-factor calculation. Otherwise, a singular infrared behavior of the strong radiative corrections can substantially modify the form-factor behavior.

[19]For our purposes, it really is not important whether the ϕ is massless or not. The ψ mass will provide an IR cutoff. In the quark-gluon theory, it is necessary to keep the gluon massless in order to preserve renormalizability and asymptotic freedom.

[20]A solution to the BS equation of the form (3.15) was found by E. Poggio, S.-H. H. Tye, and E. Tomboulis, MIT Report No. CPT 404, 1974 (unpublished). These authors were looking at the question of dynamical symmetry breaking in non-Abelian gauge theories.

[21]M. Ciafaloni, Phys. Rev. 176, 1898 (1968); M. Ciafaloni and P. Menotti, *ibid.* 173, 1575 (1968); M. Ciafaloni and P. Menotti, Nuovo Cimento Lett 6, 545 (1973); P. Menotti, Phys. Rev. D 9, 2767 (1974).

[22]This is in contrast to the situtation in potential scattering where the behavior of the bound-state wave function at the origin determines the asymptotic behavior of the form factor. See, for example, S. D. Drell, A. C. Finn, and M. H. Goldhaber, Phys. Rev. 157, 1402 (1967). This feature of potential scattering can also emerge in a simple relativistic bound-state treatment. See S. D. Drell and K. Johnson, Phys. Rev. D 6, 3248 (1972).

[23]S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. 115, 731 (1959); M. Ida, Prog. Theor. Phys. (Kyoto) 23, 1151 (1960). We will take the angular momentum of the bound state to be zero. The DGS representation is easily modified to describe a non-s-wave bound state. See, for example, Ref. 22.

[24]Just as in electroproduction, this corresponds in coordinate space to $x^2 \approx 1/k^2$ and $p \cdot x \to \infty$ (large longitudinal distances and finite transverse distances from the light cone).

[25]As a simple example of a $\phi(k^2, (k+p)^2)$ corresponding to such a $g(z, t)$, we could take $\phi(k^2, (p+k)^2) = [1/(k^2 - m^2)]\, F(k^2, (p+k)^2)$, where $F(k^2, (p+k)^2)$ is only logarithmically varying in k^2 and $(p+k)^2$. This would lead to $F(q^2) \sim (1/q^2)$ (mod log).

[26]This is discussed in more detail in Sec. VI.

[27]A word about orders of magnitude is perhaps worthwhile here. In addition to the assumption that (5.1) is the correct asymptotic *power* behavior of the wave function as $(k+p)^2 = \infty$ either with or without k^2 fixed, we are also assuming that this asymptotic form is relevant for the form-factor calculation. Since $-(k+p)^2$ ranges only up to $-q^2$, this amounts to an assumption that q is large enough so that $g_q \ll 1$, where g_q is the running coupling constant of the renormalization group.

[28]We have done this using the formalism of N. Nakanishi, Prog. Theor. Phys. Suppl. 18, 1 (1961).

[29]A proof of this fact has been given by J. Borenstein (private communication).

[30]V. Sudakov, Zh. Eksp. Teor. Fiz. 30, 87 (1956) [Sov. Phys.—JETP 3, 65 (1956)].

[31]This possiblity has been suggested by J. C. Polkinghorne, Phys. Lett. 49B, 277 (1974).

[32]This theorem can be shown to hold also for the colored quark-gluon theory, with the proper modifications to account for the more complex structure of such a theory.

In a different context, analogous proofs of this theorem have been given for massless QED and the massless $g\phi^4$ theory by K. Johnson, R. Willey, and M. Baker, Phys. Rev. 163, 1699 (1967); E. C. Poggio, MIT Ph.D. thesis, 1971 (unpublished).

[33]T. Kinoshita, J. Math. Phys. 3, 650 (1962).

[34]For the case when K_R is disconnected, the degree of divergence of each piece is separately positive. A typical disconnected reduced graph is shown in Fig. 10(b). The degree of divergence for each separate piece is $[2M-2]$ and $[2(N-M)-2]$, respectively, which are necessarily always greater than zero. Their sum, obviously, adds up to D_N as in Eq. (A1).