Dynamical symmetry breaking in asymptotically free field theories*

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Two-dimensional massless fermion field theories with quartic interactions are analyzed. These models are asymptotically free. The models are expanded in powers of $1/N$, where N is the number of components of the fermion field. In such an expansion one can explicitly sum to all orders in the coupling constants. It is found that dynamical symmetry breaking occurs in these models for any value of the coupling constant. The resulting theories produce a fermion mass dynamically, in addition to a scalar bound state and, if the broken symmetry is continuous, a Goldstone boson. The resulting theories contain no adjustable parameters. The search for symmetry breaking is performed using functional techniques, the new feature here being that a composite field, say, $\bar{\psi}\psi$, develops a nonvanishing vacuum expectation value. The "potential" of composite fields is discussed and constructed. General results are derived for arbitrary theories in which all masses are generated dynamically. It is proved that in asymptotically free theories the dynamical masses must depend on the coupling constants in a nonanalytic fashion, vanishing exponentially when these vanish. It is argued that infrared-stable theories, such as massless-fermion quantum electrodynamics, cannot produce masses dynamically. Four-dimensional scalar field theories with quartic interactions are analyzed in the large-N limit and are shown to yield unphysical results. The models are extended to include gauge fields. It is then found that the gauge vector mesons acquire a mass through a dynamical Higgs mechanism. The higher-order corrections, of order $1/N$, to the models are analyzed. Essential singularities, of the Borel-summable type, are discovered at zero coupling constant. The origin of the singularities is the ultraviolet behavior of the theory.

I. INTRODUCTION

The usual method of generating spontaneous symmetry breaking in quantum field theory is to introduce an elementary scalar field which develops a nonvanishing vacuum expectation value. This mechanism is, of course, not necessary. The general features of spontaneous symmetry breaking, such as the Goldstone theorem itself,[1,2] are independent of whether the Goldstone particle is associated with an elementary or composite field. Indeed it was stressed by Nambu and Jona-Lasinio,[3] in one of the pioneering papers on the subject, that the origin of the spontaneous chiral-symmetry breaking might be dynamical, as it is in the analogous phenomenon of superconductivity. In fact these authors analyzed a specific field-theoretic model, which indicated dynamical symmetry breaking. Unfortunately this model, involving a quartic fermion interaction in four dimensions, was unrenormalizable. Thus it was necessary to introduce a cutoff and the validity of the approximations made to solve the model was very unclear.

In recent years specific field-theoretic models, which employ the Goldstone mechanism to break various symmetries, have increased in importance. To a large extent this is due to the advent of the non-Abelian gauge theories of the weak and electromagnetic interactions.[4] Such theories must break the gauge symmetry spontaneously, generating masses for the gauge vector mesons by means of the Higgs mechanism,[5] in order to preserve their renormalizability. Although one can always achieve this by explicitly introducing the requisite number of scalar fields, one pays the price of having many additional parameters in the resulting theory. Although it is clear that the Higgs mechanism will work whether or not the symmetry breaking is put in by hand or dynamically produced,[6] no realistic models of dynamical symmetry breaking have been constructed.

Non-Abelian gauge theories of the strong interactions have also been proposed recently,[7] in order to explain Bjorken scaling. In such theories it appears impossible to break the gauge symmetry by explicitly introducing Higgs particles, without destroying the asymptotic freedom which is the reason for their existence. Instead it has been suggested[7,9] that the symmetry remains unbroken, yet the infrared singularities of the theory prevent the appearance of the charged gauge mesons and quarks in physical states.

Alternatively the gauge symmetry of asymptotically free theories could be broken dynamically. It was pointed out, in Ref. 7, that asymptotic freedom itself suggests a mechanism for this occurrence. Namely, the strength of the interaction at long ranges, or at small momenta, in such theories can become very large, irrespective of the value of the "physical" couplings. Given an attractive channel such forces might inevitably produce bound states which could act as dynamical Goldstone (or Higgs) particles. Such a mechanism, while not nearly as attractive for the strong interactions as containment, might be useful for theories of the weak interactions.

In this paper we examine some two-dimensional model field theories, involving fermions with quartic interactions. These models are essentially equivalent to the Nambu–Jona-Lasinio models, save the fact that in two dimensions they are renormalizable. Our reason for choosing them is that they are, with the exception of Yang-Mills theories in four dimensions, the only known physical asymptotically free theories. Thus the idea is to throw some light on the above problems in a theory in which one can hope to make reasonable approximations.

In order to perform explicit calculations we consider N-component fermion fields in the limit of large N. In such an expansion one can sum, in each order of $1/N$, to all orders in the coupling constant. This is an extremely nice expansion, since even to lowest order it provides us with a very nontrivial theory. Furthermore, as far as we can see (based partially on higher-order calculations), there is no reason to expect that nonleading corrections can materially affect our results if N is large enough. In four-dimensional gauge theories, on the other hand, no small expansion parameter appears to exist in the small-momentum region.[10]

We employ functional methods developed by Jona-Lasinio[11] to search for spontaneous symmetry breaking. The utility of these methods for treating dynamical symmetry breaking has been emphasized recently by Coleman and Weinberg.[12] In studying massless scalar quantum electrodynamics they found a broken-symmetry solution generated by higher-order corrections to the semiclassical potential. In addition they discovered the phenomenon of "dimensional transmutation," namely, the conversion of a dimensionless coupling constant into a mass scale parameter that occurs when a massless theory acquires masses dynamically.

We employ these methods to study our models in the limit of large N where the calculations can be explicitly performed. In our case we must construct the "potential" of a composite field, $\bar{\psi}\psi$. Such potentials are, in general, difficult to construct and possess quite unusual properties. [Thus, for example, they vanish in the semiclassical (tree) approximation.] However, due to the quartic nature of the interaction it is possible to formally introduce an elementary field, σ, which is essentially equal to $\bar{\psi}\psi$, thus greatly facilitating the construction of this potential.

We find that the increasing attractive interaction at long distances invariably produces bound states and dynamical symmetry breaking. The resulting theories produce a fermion mass, a scalar bound state, and if the broken symmetry is continuous, a bound-state Goldstone boson. All dimensionless parameters are calculable, and the theory ends up involving no adjustable parameters. This is in accord with general arguments that we make regarding asymptotically free theories. Conversely we argue that infrared stable theories (in which the effective couplings vanish for small momenta) cannot produce all masses dynamically. This is substantiated by an examination of a four-dimensional ϕ^4 theory, in the large-N limit. Here we find no spontaneous symmetry breaking.

We also extend our models to include gauge fields. We find that the gauge mesons acquire a mass through a dynamical Higgs mechanism, as one might expect. We also extend the models by continuing the space-time dimension above two, i.e., into the "postcritical" region.

In order to check the validity of the large-N expansion we have calculated some of the higher-order corrections to various Green's functions. In doing this we have discovered the existence of essential singularities at zero coupling constant. These arise, not from the summation of an increasing number of Feynman graphs, but rather from the ultraviolet divergences of a renormalizable field theory. They provide a concrete example of the type of essential singularities one might expect in quantum field theory. They are Borel-summable and do not invalidate the use of the perturbation series as an asymptotic expansion.

The outline of the paper is as follows. In Sec. II we present the two-dimensional models to be discussed. Section III introduces the large-N approximation, which is used to solve the theory to lowest order. It is shown that the Green's functions contain a tachyon pole, which, it is argued, is a manifestation of perturbing about an unstable vacuum. This is substantiated in Sec. IV where the σ potential is constructed and shown to have an asymmetric minimum. The theory constructed about this point generates a fermion mass. The broken-symmetry solution is discussed in detail, including the calculation of many physical parameters and an illustration of how "partial conservation of axial-vector current" (PCAC) works here. Section V is devoted to an analysis of the potential of $\bar{\psi}\psi$ and the subtleties involved in its construction. Here we justify the somewhat simplified discussion given in the previous section. In Sec. VI we present a general analysis of dynamical symmetry breaking in massless theories using the renormalization group. Section VII includes gauge mesons in our models and illustrates how the Higgs mechanism occurs dynamically. Section VIII describes the continuation to dimension $2+\epsilon$ of these models. In Sec. IX we calculate the fermion propagator to order $1/N$, and exhibit the resulting essential singularities in the coupling. Finally Sec. X consists of some concluding remarks.

II. THE MODEL

The models that we shall consider in this paper will contain N-component fermion fields with quartic interactions in two space-time dimensions. The salient feature of these models is that they are the simplest, and indeed, with the exception of non-Abelian gauge theories in four dimensions, the only physically sensible asymptotically free theories. Furthermore, if we let N be very large these theories can be solved in an expansion in powers of $1/N$. This expansion, which for Ising-type systems yields the spherical model, has been employed to investigate critical behavior.[13] Here we shall use it to study spontaneous symmetry breaking.

The simplest of the models to be discussed is described by the Lagrangian density,[14]

$$\mathcal{L}_\psi = \bar\psi(i\partial\!\!\!/)\psi + \tfrac{1}{2}g^2(\bar\psi\psi)^2\,, \tag{2.1}$$

where ψ is the N-component, massless fermion field. This Lagrangian is invariant under the discrete γ_5 transformation

$$\psi \to \gamma_5\psi\,, \tag{2.2}$$

which ensures the masslessness of the fermion to any order of perturbation theory. In the large-N limit, as we shall see, g^2 will vanish like $1/N$ so that we define

$$\lambda = g^2 N\,. \tag{2.3}$$

This theory is renormalizable (in two dimensions) and will require only wave-function and coupling-constant renormalization. It is easy to verify that no new interactions, such as $(\bar\psi\gamma^\mu\psi)\times(\bar\psi\gamma_\mu\psi)$ are generated, at least to order $g^8 = \lambda^4(1/N)^4$. The resulting theory is then characterized by a single dimensionless parameter, g^2 (and N of course).

One might wonder whether we have not chosen the "wrong" sign for g^2. Indeed with the choice in Eq. (2.1) the interaction Hamiltonian is given by

$$H_I = -\tfrac{1}{2}g^2(\bar\psi\psi)^2\,, \tag{2.4}$$

and one might very well question whether such a theory is stable (i.e., possesses a ground state). If ψ were a scalar field, this would certainly be the case, at least for the classical field theory. However, for fermion fields the situation is much less clear. Indeed we argue that the sign exhibited in Eq. (2.1) is, for a quartic fermion coupling, the "right" sign. To see this we shall consider the theory generated by a Yukawa interaction, in which the mass of the scalar and its coupling become infinite in such a way as to reproduce in the limit our local quartic coupling.

Consider the Yukawa Lagrangian

$$\mathcal{L}' = \bar\psi(i\partial\!\!\!/)\psi + \tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2 + gm\bar\psi\psi\phi\,. \tag{2.5}$$

Here one would expect positivity for any real value of g. This has been rigorously proved for a massive fermion.[15] Now when one lets m become infinite the above Yukawa interaction becomes equivalent (order by order in perturbation theory) to our original interaction, Eq. (2.1), the combination of vertices $(igm)^2$ and scalar propagator $i/(p^2-m^2)$ yielding for infinite m the local coupling $+ig^2$. Since we expect the positivity of the theory to survive in the large-m limit, especially because the resulting theory is asymptotically free, the sign in Eq. (2.1) is the physical one. Indeed the Lagrangian $\mathcal{L} = -g^2(\bar\psi\psi)^2$ corresponds to the local limit of a Yukawa theory with imaginary coupling.

The $m\to\infty$ limit can be taken in $\mathcal{L}'$ by rescaling the scalar field

$$\sigma = m\phi \tag{2.6}$$

and letting $m\to\infty$. The resulting Lagrangian

$$\mathcal{L}_\sigma = \bar\psi(i\partial\!\!\!/)\psi - \tfrac{1}{2}\sigma^2 - g\bar\psi\psi\sigma \tag{2.7}$$

yields identical fermion Green's functions as does $\mathcal{L}_\psi$. This can also be seen by examining the generating functional for these Green's functions in the path-integral formulation:

$$\begin{aligned} Z(\eta,\bar\eta) &= \text{const}\times\int d\psi\, d\bar\psi \exp\{i[\, i\bar\psi\partial\!\!\!/\psi + \tfrac{1}{2}g^2(\bar\psi\psi)^2 + \bar\eta\psi + \bar\psi\eta]\} \\ &= \text{const}'\times\int d\psi\, d\bar\psi\, d\sigma \exp[\, i(i\bar\psi\partial\!\!\!/\psi - \tfrac{1}{2}\sigma^2 - g\bar\psi\psi\sigma + \bar\eta\psi + \bar\psi\eta)] \end{aligned} \tag{2.8}$$

and performing the σ integration in the latter expression.

We shall often, for simplicity, consider the theory generated by $\mathcal{L}_\sigma$. The bare σ propagator, in momentum space, is simply $-i$. The discrete symmetry which prevents ψ from acquiring a mass in perturbation theory is

$$\begin{aligned} \psi &\to \gamma_5\psi\,, \\ \sigma &\to -\sigma\,. \end{aligned} \tag{2.9}$$

We shall also be interested in theories possessing an internal symmetry beyond the U(N) symmetry of $\mathcal{L}_\psi$. An example of such a theory is

$$\mathcal{L}_\psi = \bar{\psi}(i\partial\!\!\!/)\psi + \tfrac{1}{2}g^2[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] \tag{2.10}$$

or equivalently

$$\mathcal{L}_\sigma = \bar{\psi}(i\partial\!\!\!/)\psi - \tfrac{1}{2}(\sigma^2 + \pi^2) + g(\sigma\bar{\psi}\psi + i\pi\bar{\psi}\gamma_5\psi) . \tag{2.11}$$

(We use the Bjorken-Drell metric; γ_5 is Hermitian.) This theory is invariant under the Abelian chiral group

$$\psi \to e^{i\theta\gamma_5}\psi ,\qquad \begin{pmatrix}\sigma\\ \pi\end{pmatrix} \to \begin{pmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{pmatrix}\begin{pmatrix}\sigma\\ \pi\end{pmatrix} . \tag{2.12}$$

Another example is the 4-fermion version of the σ model with

$$\mathcal{L}_\psi = \bar{\psi}(i\partial\!\!\!/)\psi + \tfrac{1}{2}g^2[(\bar{\psi}\psi)^2 - (\bar{\psi}\,\vec{\tau}\gamma_5\psi)^2] \tag{2.13}$$

or equivalently

$$\mathcal{L}_\sigma = \bar{\psi}(i\partial\!\!\!/)\psi - \tfrac{1}{2}(\sigma^2 + \vec{\pi}^2) + g(\sigma\bar{\psi}\psi + i\vec{\pi}\cdot\bar{\psi}\,\vec{\tau}\gamma_5\psi) , \tag{2.14}$$

where the τ's are the matrices of the fundamental representation of SU(M) (normalized so that $\mathrm{Tr}[\tau^a\tau^b] = M\delta_{ab}$). The fermion field then consists of N separate M-dimensional representations of SU(M), with the obvious summation convention

$$\vec{\pi}\cdot\bar{\psi}\,\vec{\tau}\gamma_5\psi = \sum_{a=1}^{M^2-1}\sum_{i=1}^{N}\sum_{k,j=1}^{M}\pi^a\bar{\psi}_i^k\tau^a_{kj}\psi_i^j . \tag{2.15}$$

The resulting theory is invariant under chiral SU(M)×SU(M), with $(\sigma, \vec{\pi})$ or $(\bar{\psi}\psi, \bar{\psi}\,\vec{\tau}\gamma_5\psi)$ transforming according to the (M,M) representation and ψ [$\bar{\psi}$] transforming according to the (M, 1) [(1,M)] representation.

Some of these models will require other interactions as counterterms. Thus, for example, the fermion σ model will generate in the one-loop expansion divergences which require a counterterm of the form $(\bar{\psi}\gamma^\mu\vec{\tau}\psi)^2$. The term will, however, only appear to order $g^4 = \lambda^2/N^2$ and thus be suppressed by one factor of $1/N$. We shall therefore be able, at least to leading order in $1/N$, to ignore such complications. They must be taken into account in higher orders but will not affect our considerations.

III. THE 1/N LIMIT

We now proceed to solve the model in the large-N limit. The dominant graphs in this limit will be those containing the maximal number of fermion loops—since each of these yields a factor of g^2N. Keeping λ fixed, as $N\to\infty$, it is easy to see, in the σ-meson formulation [Eq. (2.7)], that the only radiative corrections of order 1 to the 4-point function are to the σ propagator. The lowest-order σ self-energy graph (Fig. 1) is simply

$$\Pi(P) = -(g)^2N\int\frac{d^2k}{(2\pi)^2}\frac{\mathrm{Tr}[k\!\!\!/(k\!\!\!/ - P\!\!\!/)]}{k^2(k-P)^2} = \frac{+i\lambda}{2\pi}\int_0^1 d\alpha\left[\ln\left(\frac{-\Lambda^2}{\alpha(1-\alpha)P^2}\right) - 2\right] , \tag{3.1}$$

where Λ is an ultraviolet cutoff. We renormalize by requiring that the σ propagator, $D(P) = -i/[1 + i\Pi(P)]$, satisfy

$$D_R(P^2) = -i \text{ at } P^2 = -\mu^2 . \tag{3.2}$$

This means that we must subtract $\Pi(P^2)$ at $P^2 = -\mu^2$:

$$\Pi_R(P^2, \mu^2) = -\frac{i\lambda}{2\pi}\ln(-P^2/\mu^2) ,\qquad D_R(P^2, \mu^2) = \frac{-i}{1 + (\lambda/2\pi)\ln(-P^2/\mu^2)} . \tag{3.3}$$

All other radiative corrections are of order $1/N$. Thus, the 4-point function is given by the graphs in Fig. 2 (which are equivalent to the graphs of Fig. 3) and is equal to

$$G(P_1P_2;P_3P_4) = ig^2\left[\frac{1}{1 + (\lambda/2\pi)\ln(s/\mu^2)} + \frac{1}{1 + (\lambda/2\pi)\ln(u/\mu^2)}\right] , \tag{3.4}$$

where $s = -(P_3 - P_1)^2$, $u = -(P_4 - P_1)^2$ [we are employing the standard Bjorken-Drell metric, $g_{00} = -g_{11} = 1$, so that positive s and u mean spacelike energy squared and momentum transfer squared].

The dependence on the arbitrary subtraction parameter μ is, of course, spurious. A change in μ can be compensated for by an appropriate change of λ and the scale of the fields, as dictated by the renormalization group. To evaluate the renormalization-group parameters we note that to order $1/N$ there is no wave-function renormalization of the ψ field, nor is the vertex $\bar{\psi}\psi\sigma$ renormalized. Therefore the renormalized coupling g_R is related to the bare coupling g_0 by $g_R = g_0\sqrt{Z_\sigma}$, where Z_σ is the wave-function renormalization constant of the

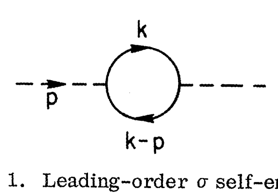

FIG. 1. Leading-order σ self-energy.

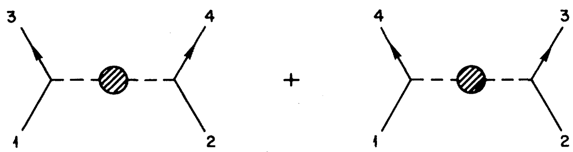

FIG. 2. Leading-order, fermion 4-point function.

σ field. Thus the β function and the anomalous dimension of σ (γ_σ) are related by

$$\beta(g) = \mu \frac{\partial}{\partial\mu} g_R \bigg|_{g_0, \Lambda \text{fixed}} = g\mu\frac{\partial}{\partial\mu}\sqrt{Z_\sigma} = g\gamma_\sigma(g) . \quad (3.5)$$

To evaluate $\beta(g)$ we note that $D_R(P, \mu)$ should obey

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + 2\gamma_\sigma(g)\right] D_R(P, \mu) = 0 , \quad (3.6)$$

from which we deduce

$$\beta(g) = -\frac{\lambda g}{2\pi}\gamma_\sigma(g) = -\frac{\lambda}{2\pi} . \quad (3.7)$$

The negative sign of $\beta(g)$ means that the theory is asymptotically free.[7,8,16] The "effective coupling constant," i.e., that which would be defined by subtracting at $P^2 = -\mu^2 e^{2t}$, satisfies

$$\frac{d\bar{g}(g,t)}{dt} = \beta(\bar{g}) , \quad \bar{g}(g, 0) = g . \quad (3.8)$$

Thus it is given by

$$\bar{g}^2(g, t) = \frac{g^2}{1 + (\lambda/\pi)t} . \quad (3.9)$$

The similarity between the effective coupling constant and the fermion 4-point function is not coincidental. In fact $G(P_1 \cdots P_4)$ satisfies the renormalization-group equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} \right) G(P_1 \cdots P_4; g, \mu) = 0 \quad (3.10)$$

and thus

$$G(\lambda P_1^0, \lambda P_2^0, \lambda P_3^0, \lambda P_4^0; g, \mu) = G(P_1^0, P_2^0, P_3^0, P_4^0; \bar{g}(\ln\lambda, g), \mu) . \quad (3.11)$$

If we choose the reference momenta P_i^0 so that $P_i^0 P_j^0 = (\delta_{ij} - \frac{1}{4})\mu^2$, then the right-hand side of Eq.

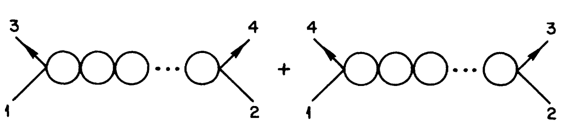

FIG. 3. Leading-order graphs which are equivalent to those in Fig. 2.

(3.11) is evaluated at the subtraction point $[(P_1 - P_3)^2 = (P_1 - P_4)^2 = -\mu^2]$. Thus

$$G(\lambda P_1^0, \lambda P_2^0, \lambda P_3^0, \lambda P_4^0; g, \mu) = 2i\bar{g}^2(\ln\lambda, g) . \quad (3.12)$$

The effective coupling, $\bar{g}^2$, vanishes for large Euclidean momenta ($t \to \infty$), logarithmically (as $1/t$). This is common to *all* asymptotically free theories. What is unusual in the $1/N$ limit is that Eq. (3.9) holds for all t. Thus we can also explore the small-momentum behavior of the theory. Since the only zero of β is the ultraviolet attractive one at the origin $\bar{g}^2$ will necessarily diverge as $t \to -\infty$ (zero momentum). In Ref. 7 it was shown that if β decreases sufficiently fast (faster than $-g$) then $\bar{g}^2$ will become infinite at a finite value of the momentum. This is the case here, where $\bar{g}^2$ develops a pole at

$$t = -\frac{\pi}{\lambda} , \quad P^2 = -\mu^2 e^{-2\pi/\lambda} . \quad (3.13)$$

This pole is present for any value of λ, approaching zero when $\lambda \to 0$.

The meaning of the pole in $\bar{g}^2$ is apparent from Eq. (3.12) [or for that matter directly from Eq. (3.4)], since the effective coupling constant is simply the fermion 4-point function: Namely, there appears in this amplitude a pole at spacelike momenta, a tachyon, with mass squared given by Eq. (3.13).

The existence of this tachyon pole could mean one of two things. First, the theory could be simply nonsense at least in the leading $1/N$ approximation. This indeed is the case for a ϕ^4 theory in four dimensions. Alternatively we could simply be constructing the theory about the "wrong" vacuum state. This is reasonable since the 4-point function [Eq. (3.4)] in position space does not satisfy cluster decomposition. When we separate the fermion-antifermion pairs by a large spacelike separation, the Green's function does not fall off exponentially. This can be explained if the vacuum about which we have been perturbing, the normal vacuum which is invariant under $\psi \to \gamma_5\psi$, $\sigma \to -\sigma$, is not the ground state. In the following we shall show that this indeed is the case—and the pole in $\bar{g}^2$ is simply the signal for spontaneous symmetry breaking. The symmetry breaking will generate a fermion mass and prevent us from concluding from the pole in $\bar{g}^2$ at some small spacelike momenta that the fermion amplitudes develop tachyon poles.

Finally let us note that identical results hold for the other models described in Sec. II. The only difference is that in the case of the SU(M) σ model one should replace $\lambda = g^2N$ by $\lambda = g^2NM$.

IV. SPONTANEOUS SYMMETRY BREAKING

In the previous section we have seen that in the large-N limit our models necessarily develop tachyon poles for any value of the coupling. If the theory is to be consistent in this approximation it must be that the normal symmetric vacuum is not in fact the ground state. If this is the case we would expect that in the true ground state $\bar{\psi}\psi$ has a nonvanishing vacuum expectation value. This can be verified either by studying the integral equations for the Green's function of the theory and looking for symmetry-breaking solutions or by examining the "potential" of $\bar{\psi}\psi$.

Let us first consider our theories with the addition of a constant external source coupled to $\bar{\psi}\psi$. This is analogous to adding, say in an Ising model, a constant external magnetic field and then examining whether $\langle\bar{\psi}\psi\rangle \neq 0$ when the external field vanishes. In our case such an external source is simply a mass term. Thus we consider the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - M)\psi + \tfrac{1}{2}g^2(\bar{\psi}\psi)^2 \tag{4.1}$$

or

$$\mathcal{L}_\sigma = \bar{\psi}(i\not{\partial} - M)\psi - \tfrac{1}{2}\sigma^2 - g\sigma\bar{\psi}\psi .$$

As before we construct the 4-point fermion Green's function which is now given by [see Eq. (3.4)]

$$G(P_1P_2;P_3P_4) = ig^2\left\{\frac{1}{1+(\lambda/2\pi)[B(s,M^2)-B(\mu^2,M^2)]} + (s \leftrightarrow u)\right\} , \tag{4.2}$$

where B is essentially the massive fermion loop of Fig. 1:

$$B(s,M^2) = \left(\frac{s+4M^2}{s}\right)^{1/2} \ln\left(\frac{(s+4M^2)^{1/2}+\sqrt{s}}{(s+4M^2)^{1/2}-\sqrt{s}}\right) . \tag{4.3}$$

$B(s,M^2)$ is a monotonically increasing function of s whose minimum value is $B(0,M^2)=2$. Therefore, as long as $M>0$, there will be no pole for spacelike momentum ($s>0$) for small enough values of λ. One requires that

$$\frac{\lambda}{2\pi} < \frac{1}{B(\mu^2,M^2)-B(0,M^2)} . \tag{4.4}$$

However, when M is decreased, for fixed λ and μ, a bound-state pole develops [as soon as $B(\mu^2,M^2) \geq 2\pi/\lambda$], whose mass decreases as M decreases. At the point at which the inequality in Eq. (4.4) is just violated the bound-state mass is zero. This occurs, for small λ, when

$$M^2 \approx \mu^2 e^{-2\pi/\lambda} . \tag{4.5}$$

Thus for any value of the coupling a zero-mass fermion-antifermion bound state is formed when the mass is reduced below this critical value. When the mass is decreased even further, the bound state would appear to become a tachyon. However, at the point at which its mass vanishes the vacuum might very well be unstable — due to the emission of zero-momentum bound states, and the tachyon present in Eq. (3.4) merely a consequence of constructing the amplitude by perturbing about an unstable vacuum.

One could try to construct the amplitudes of the broken-symmetry theory by solving the Schwinger-Dyson equations for the Green's functions in a nonperturbative fashion. Instead, we shall use functional techniques. It is convenient to employ the Lagrangian as given by Eq. (2.7), since σ is essentially equal to $g\bar{\psi}\psi$, and to investigate the "potential" as a function of the classical σ field.

This potential *is not* exactly equal to the potential of the composite operator $\bar{\psi}\psi$. The two, however, are closely related. In Sec. V we discuss this relationship and show how one can construct the $\bar{\psi}\psi$ potential from the σ potential calculated below. Furthermore, we shall show that for purposes of probing for symmetry breaking, it is necessary to investigate the σ potential, for the ground state must occur at a minimum of this potential, at which point $\sigma_c = \langle 0|\sigma|0\rangle = \langle 0|g\bar{\psi}\psi|0\rangle$.

Thus we consider the vacuum-to-vacuum amplitude in the presence of an external source coupled to σ:

$$e^{iW(J)} \equiv \int d\psi\, d\bar{\psi}\, d\sigma \exp\{i[\mathcal{L}_\sigma(\sigma,\bar{\psi},\psi)+J\sigma]\} . \tag{4.6}$$

$W(J)$ is the generator of the connected Green's functions of the σ field. The classical σ field, σ_c, is defined by

$$\sigma_c(x) = \frac{\delta W}{\delta J(x)} = \langle 0|\sigma(x)|0\rangle_J . \tag{4.7}$$

The Legendre transform of $W(J)$,

$$\Gamma(\sigma_c) = \int d^4x\, \sigma_c(x) J(x) - W(J) , \tag{4.8}$$

is equal to the expectation value of the Hamiltonian, in the state which the vacuum expectation value of $\sigma(x)$ is $\sigma_c(x)$.[17] Translational invariance dictates that σ_c be independent of space-time. In that case

$$\Gamma = \int d^4x\, V(\sigma_c) . \tag{4.9}$$

The condition that the energy be minimal is

$$J = \partial V/\partial\sigma_c = 0$$

and

$$\partial^2 V/\partial \sigma_c^{\,2} > 0\,. \qquad (4.10)$$

Spontaneous symmetry breaking will occur if this transpires for a nonzero value of σ_c.

Now Γ is easy to construct, since it is the generating functional of the one-particle irreducible (1PI) n-point functions of the σ field. Thus $V(\sigma_c)$ is given by

$$V(\sigma_c) = \sum \frac{1}{n!}(\sigma_c)^n \Gamma_n(0, \ldots, 0)\,, \qquad (4.11)$$

where $\Gamma_n(0, \ldots, 0)$ is the sum of all 1PI Green's functions with n external σ lines carrying zero four-momentum.

In the tree approximation V is simply the negative of the nonderivative terms in $\mathcal{L}_\sigma$ involving the σ field, i.e.,

$$V(\sigma_c)_{\rm tree} = \tfrac{1}{2}\sigma_c^{\,2}\,.$$

In this approximation, order $(1/N)^0$, the minimum is at $\sigma_c = \langle 0|\sigma|0\rangle = 0$. This is the normal symmetric vacuum.

There are clearly other contributions to V of order $(1/N)^0$, for example, the graph depicted in Fig. 1. In fact the leading terms in V for large N are given by the tree graphs plus all one-loop graphs (Fig. 4). At first sight it would seem that the higher-order graphs are of order $g^{2k}N = \lambda^k/N^{k-1}$, however, these graphs separately are highly infrared-divergent and must be summed to yield a finite result (up to ultraviolet divergences). We therefore sum all the one-loop graphs with an ultraviolet cutoff Λ:

$$V = \tfrac{1}{2}\sigma_c^{\,2} - Ni\sum_{n=1}^{\infty}\int^{\Lambda}\frac{d^2k}{(2\pi)^2}\,\frac{1}{2n}\,\frac{(g^2\sigma_c^{\,2})^n}{(k^2)}$$

$$= \tfrac{1}{2}\sigma_c^{\,2} - \frac{\lambda}{4\pi}\sigma_c^{\,2}[\ln\Lambda^2 + 1 - \ln(g^2\sigma_c^{\,2})]\,. \qquad (4.12)$$

The potential V requires renormalization, which can be performed (following Coleman and Weinberg) by subtracting (4.12) at some value, σ_0, of the classical field. This is related to our previous renormalization of the σ propagator. In fact if we were to define $(\partial^2 V/\partial\sigma^2)|_{\sigma=0} = 1$ this would be equivalent to subtracting the σ propagator (in the normal theory) at zero momentum. However, this is impossible due to the infrared divergences at zero momentum, or zero field. Therefore we renormalize, following Coleman and Weinberg[12] by demanding that

$$\left.\frac{\partial^2 V}{\partial \sigma_c^{\,2}}\right|_{\sigma=\sigma_0} = 1\,. \qquad (4.13)$$

Then we have

$$V(\sigma_c, \sigma_0, g) = \tfrac{1}{2}\sigma_c^{\,2} + \frac{\lambda}{4\pi}\sigma_c^{\,2}\left[\ln\left(\frac{\sigma_c}{\sigma_0}\right)^2 - 3\right]. \qquad (4.14)$$

The subtraction parameter is completely arbitrary, just as the subtraction parameter in momentum space μ was. A change in σ_0 is equivalent to a change in g and the scale of the σ field, so that V obeys the renormalization-group equation

$$\left[\sigma_0\frac{\partial}{\partial\sigma_0} + \tilde\beta(g)\frac{\partial}{\partial g} - \tilde\gamma(g)\sigma_c\frac{\partial}{\partial\sigma_c}\right]V(\sigma_c, \sigma_0, g) = 0\,. \qquad (4.15)$$

Since our renormalization procedure here is different from the previous prescription $\tilde\beta(g)$ and $\tilde\gamma(g)$ might be different from their previous values [Eq. (3.7)]. Indeed if the potential in Eq. (4.14) is inserted into Eq. (4.15) we find

$$\beta(g) = g\tilde\gamma(g) = -\frac{\lambda g/2\pi}{1+\lambda/2\pi}\,. \qquad (4.16)$$

These functions differ from those calculated in Sec. III beyond the lowest order. This is not unexpected since only the lowest-order terms in the expansion of β and γ are independent of the specific renormalization procedure.

It is now seen that the symmetric point, $\sigma_c = 0$, is never a minimum of the potential (see Fig. 5). The one-loop corrections give rise to a negative term which dominates, for small σ_c, the tree approximation no matter how small g is. For large σ_c the potential is positive and increasing, and thus the theory is stable. The minimum of the potential occurs at $\sigma_c = \sigma_M$, where

$$J = V'(\sigma_M, \sigma_0, g) = \sigma_M\left\{1 + \frac{\lambda}{2\pi}\left[\ln\left(\frac{\sigma_M}{\sigma_0}\right)^2 - 2\right]\right\} = 0\,, \qquad (4.17)$$

$$V''(\sigma_M, \sigma_0, g) = 1 + \frac{\lambda}{2\pi}\ln\left(\frac{\sigma_M}{\sigma_0}\right)^2 = \frac{\lambda}{\pi}\,,$$

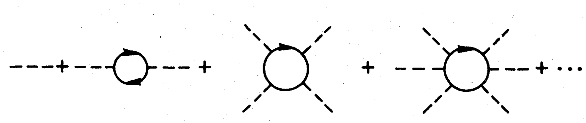

FIG. 4. Feynman graphs which contribute to $V(\sigma)$ in leading order in $1/N$.

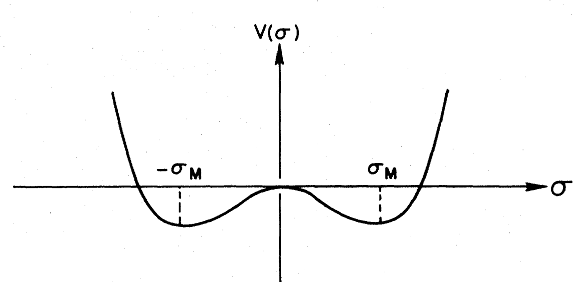

FIG. 5. Form of $V(\sigma)$ to leading order in $1/N$.

i.e., when

$$|\sigma_M| = \sigma_0 \exp\left(1 - \frac{\pi}{\lambda}\right) . \tag{4.18}$$

(Note that this result is essentially nonperturbative, since σ_M has zero asymptotic expansion in powers of λ.)

Thus we see that the reason that we found a tachyon pole previously was that we were perturbing about a maximum of the potential. In the true ground state σ, or $g\bar{\psi}\psi$, has a nonvanishing vacuum expectation value. If we shift the σ field by this amount we can then carry out perturbation theory about the asymmetric vacuum. Choosing, say, $\sigma_M = \sigma_0 \exp(1 - \pi/\lambda)$, the discrete symmetry $\sigma \to \sigma$, $\psi \to \gamma_5 \psi$ is broken and the fermion acquires a mass:

$$M_F = g\sigma_0 \exp(1 - \pi/\lambda) . \tag{4.19}$$

The fermion mass is of course a physical parameter and therefore cannot depend on the choice of renormalization point. This means that $M_F(\sigma_0, g)$ must obey the homogeneous renormalization-group equation

$$\left[\sigma_0 \frac{\partial}{\partial \sigma_0} + \tilde{\beta}(g) \frac{\partial}{\partial g}\right] M_F(\sigma_0, g) = 0 . \tag{4.20}$$

Since M_F must be proportional to σ_0 it follows that the solution of (4.20) is

$$M_F(\sigma_0, g) = \sigma_0 \exp\left[-\int^g \frac{dx}{\tilde{\beta}(x)}\right] . \tag{4.21}$$

If we insert the value of $\tilde{\beta}(g)$, Eq. (4.16), into Eq. (4.21), we recover the previously derived expression for M_F (up to a multiplicative constant).

In order to calculate the Green's functions of the broken-symmetry theory it is necessary to relate the subtraction procedure carried out for $V(\sigma)$ to that required for the σ propagator. We recall that $V''(\sigma)$ evaluated at $\sigma = \sigma_M$ is equal to $-i$ times the inverse σ propagator evaluated at zero four-momentum. Thus the subtraction procedure used above for $V(\sigma)$ requires that

$$D_\sigma^{-1}(P^2 = 0) = iV''(\sigma_M) = \lambda/\pi . \tag{4.22}$$

The value of σ_0 is, however, completely arbitrary. One can eliminate the explicit dependence on this parameter by choosing σ_0 to be any particular value. One convenient choice is to choose σ_0 to coincide with the value of σ_M. However, to do that one must fix the coupling constant to be π:

$$\sigma_0 = \sigma_M \to \lambda = \pi \tag{4.23}$$

and then

$$M_F = g\sigma_0 = \left(\frac{\pi}{N}\right)^{1/2} \sigma_0 . \tag{4.24}$$

With this choice of σ_0 it is manifestly apparent that our theory contains only one free parameter, say M_F or σ_0. This is altogether reasonable since our starting point was a theory of massless fermions which was determined by one dimensionless coupling constant g. We end up (with the above choice of σ_0) with a theory determined by one-dimensional coupling constant σ_0 (or M_F). This is an example of the phenomenon of "dimensional transmutation" discovered, in scalar QED, by Coleman and Weinberg.[12] In our case, however, we arrive at *a theory in which, aside from the over-all mass scale (characterized by M_F) there are no free adjustable parameters. All dimensionless quantities in the theory are calculable numbers.*

If σ_0 is left arbitrary ($\sigma_0 \neq \sigma_M$) the same result will emerge. All Green's functions which appear to depend on the two parameters σ_0 and g will in fact only depend on these parameters through their dependence on $M_F(g, \sigma_0)$.

We can now calculate the Green's functions of the theory to order $1/N$. As before they are given by the tree graphs generated by $\mathcal{L}_\sigma$, except that the fermion has acquired the mass M_F and we must subtract at zero momentum. Therefore the σ propagator is

$$\begin{aligned} D_R^{-1}(P^2) &= +i\left\{\frac{\lambda}{\pi} + \frac{\lambda}{2\pi}\left[\left(\frac{-P^2 + 4M_F^2}{-P^2}\right)^{1/2} \ln\frac{(-P^2 + 4M_F^2)^{1/2} + (-P^2)^{1/2}}{(-P^2 + 4M_F^2)^{1/2} - (-P^2)^{1/2}} - 2\right]\right\} \\ &= \frac{+i\lambda}{2\pi}\left(\frac{-P^2 + 4M_F^2}{-P^2}\right)^{1/2} \ln\frac{(-P^2 + 4M_F^2)^{1/2} + (-P^2)^{1/2}}{(-P^2 + 4M_F^2)^{1/2} - (-P^2)^{1/2}} , \end{aligned} \tag{4.25}$$

where we have used Eq. (4.24) and the fact that $B(0, M_F^2) = 2$. The fermion 4-point function is then given by

$$G(P_1P_2; P_3P_4) = \left[\frac{1}{B(t, M_F^2)} + \frac{1}{B(u, M_F^2)}\right] \frac{2\pi i}{N} . \tag{4.26}$$

Note that the dependence of G on the coupling constant (except via M_F) has disappeared. The "physical" coupling constant is therefore a pure number. We could define this coupling by, say, the value of G when all momenta are set equal to zero (divided by $2i$). In that case the 4-fermion coupling constant is equal to π/N.

There still exists in the theory a fermion-antifermion bound state (the σ particle). However, it now occurs at threshold, where $B(4M_F^2, M_F^2)=0$, so

$$M_\sigma = 2M_F . \tag{4.27}$$

In the leading-$1/N$ approximation the fact that the binding energy of σ is zero is reasonable. After all M_σ must, like M_F, obey the homogeneous renormalization-group equation. Therefore the ratio $M_\sigma/2M_F$ must be a constant independent of g. When g is taken to zero, however, we would expect the σ binding energy to vanish, and therefore Eq. (4.27) must be true in the leading-$1/N$ approximation. In higher order we might, however, discover that

$$M_\sigma = 2M_F\left[1 \pm O\left(\frac{1}{N}\right)\right].$$

If we evaluate the residue of the σ pole in Eq. (4.26) we learn that the coupling of the σ to the fermions is given by

$$g_{\sigma F\bar{F}}{}^2 = \frac{4\pi M_F{}^2}{N} . \tag{4.28}$$

Note that $g_{\sigma F\bar{F}}{}^2$ is a *physical* coupling constant and therefore is independent of the renormalization procedure by satisfying the homogeneous renormalization-group equation. Since it has dimensions of mass squared it must be, and is, equal to $M_F{}^2$ up to a multiplicative constant.

Let us now consider the class of models which possess a continuous symmetry. Consider the theory described by Eq. (2.11), which is invariant under an Abelian U(1) chiral group. One now calculates the potential as a function of $\sigma_c = \langle 0|\sigma(x)|0\rangle$ and $\pi_c = \langle 0|\pi(x)|0\rangle$. Due to the symmetry the potential is a function of $s^2 = \sigma_c{}^2 + \pi_c{}^2$. It is easy to see that $V(s^2)$ is given by the same expression as before, Eq. (4.14), except that we must replace $\sigma_c{}^2$ by s^2. We again renormalize V by demanding that

$$\left.\frac{\partial^2 V}{\partial s^2}\right|_{s=s_0} = 1 ; \tag{4.29}$$

no further renormalization is required due to the U(1) symmetry. Thus

$$V(s^2 = \sigma_c{}^2 + \pi_c{}^2, s_0, g) = \tfrac{1}{2} s^2 + \frac{\lambda}{4\pi} s^2 [\ln(s/s_0)^2 - 3] . \tag{4.30}$$

Again the symmetric vacuum $\sigma_c = \pi_c = 0$ is not stable, and the true ground state is given by

$$s^2 = \sigma_c{}^2 + \pi_c{}^2 = s_0{}^2 \exp(2 - 2\pi/\lambda) = s_M{}^2 . \tag{4.31}$$

In this case we have a continuum of vacua, related by the U(1) transformations ($\sigma_c = s_M \cos\theta$, $\pi_c = s_M \sin\theta$). Choosing the π field to remain with vanishing vacuum expectation value, the ground state is characterized by

$$\sigma_c = s_M = s_0 \exp(1 - \pi/\lambda) , \quad \pi_c = 0 \tag{4.32}$$

and perturbation theory can be derived by shifting the σ field in Eq. (2.11) by this amount, the fermion acquiring a mass M_F given by Eq. (4.19).

As before we can construct the σ and π propagators by noting that the above subtraction procedure is equivalent to normalizing these propagators at zero four-momentum. Indeed

$$D_\sigma{}^{-1}(P^2=0) = i \left.\frac{\partial^2}{\partial \sigma_c{}^2} V(s)\right|_{\sigma_c = s_M;\ \pi_c=0} = i\,\frac{\lambda}{\pi} \tag{4.33}$$

and

$$D_\pi{}^{-1}(P^2=0) = i \left.\frac{\partial^2}{\partial \pi_c{}^2} V(s)\right|_{\sigma_c = s_M;\ \pi_c=0} = 0 . \tag{4.34}$$

There is no mixing since $\partial^2 V/\partial\sigma_c \partial\pi_c$ vanishes when $\pi_c = 0$.

Therefore the full σ propagator will be identical to that evaluated above [Eq. (4.25)] and the theory will possess a σ bound state. In addition there will exist a bound-state π meson with zero mass. This is of course the Goldstone boson associated with the conserved axial-vector current $A_\mu = \bar{\psi}\gamma_5\gamma_\mu\psi$. In fact the "pion" propagator is easily evaluated to be

$$D_\pi{}^{-1}(P^2) = \frac{i\lambda}{2\pi}\left(\frac{P^2}{P^2 + 4M_F{}^2}\right) B(-P^2, M_F{}^2) \tag{4.35}$$

and thus the fermion-antifermion 4-point function has a zero-mass pseudoscalar pole, whose residue gives the pion-fermion coupling constant

$$g_{\pi\bar{F}F}{}^2 = \frac{4\pi M_F{}^2}{N} . \tag{4.36}$$

As expected, due to the U(1) symmetry the pion-fermion coupling equals the σ-fermion coupling.

The educated reader will probably object to our assertion of the existence of the Goldstone phenomenon in two dimensions. In fact Coleman has shown[18] that due to the untameable infrared divergences associated with massless particles in two dimensions, a sensible theory cannot possess Goldstone bosons. This in fact is true and would be evident if we were to calculate in the above theory to higher orders in $1/N$. The existence, in leading order, of zero-mass bound states will give rise to infrared infinities arising from virtual π states of the form $\int d^2k/k^2$. This of course means that the lowest-order approximation is meaningless, and to investigate the stability of the theory one would have to work to all orders in $1/N$ (or at least to evaluate the most highly infrared-singular contributions).

Nonetheless, we believe that our model exhibits features of dynamical symmetry breaking that would be present in four-dimensional models (particularly asymptotically free ones). If one desires to eliminate the infrared divergences that arise in higher orders, one could continue the dimension of space-time to $2+\epsilon$. Alternatively, one can introduce a chiral U(1) gauge group, in which case the dynamical Goldstone boson is eliminated by the Higgs mechanism. This is interesting in its own right and will be considered in Sec. VI.

It is amusing to see how the analog of PCAC works in the above model. In the symmetric theory the axial-vector current $A_\mu = \bar\psi\gamma_5\gamma_\mu\psi$ is conserved, the vacuum is symmetric, and the fermion mass is zero. In the asymmetric theory A_μ is still conserved, the vacuum is not symmetric, a massless Goldstone boson is formed, and the fermion acquires a mass. Let us examine the fermion matrix element of A_μ

$$\langle P'|A_\mu(0)|P\rangle_{q=P-P'} = \bar u(P)[g_A(q^2)\gamma_5\gamma_\mu + g_P(q^2)q_\mu\gamma_5 + g_S(q^2)\epsilon_{\mu\nu}q^\nu]u(P)\,. \tag{4.37}$$

Note that in two dimensions the axial-vector current has both an induced pseudoscalar form factor (g_P) and an induced scalar form factor (g_S).

The contributions to this matrix element in the leading-$1/N$ approximation arise from the graphs of Fig. 6. The axial-vector form factor gets contribution from Fig. 6(a) alone so that $g_A=1$. The pseudoscalar form factor arises from the graphs summarized by Fig. 6(b), whose contribution yields

$$g_P(q^2)q_\mu = g^2N\int\frac{d^2k}{(2\pi)^2}\,\mathrm{Tr}\left[\gamma_5\gamma_\mu\frac{1}{\not k - M_F}\gamma_5\frac{1}{\not k+\not q - M_F}\right]\times D_\pi(q^2)$$
$$= -2M_F\,q_\mu/q^2 \tag{4.38}$$

upon using Eq. (4.35). The induced scalar form factor vanishes. (Note that the cut due to the fermion-antifermion intermediate state cancels in the above expression.)

It is easily seen that the axial-vector current is indeed conserved, since $g_A 2M_F + q^2 g_P = 0$. The "pion decay constant" is easily evaluated to be

$$\langle 0|A_\mu(0)|\pi(q)\rangle = -iq_\mu f_\pi = -iq_\mu\left(\frac{N}{\pi}\right)^{1/2} \tag{4.39}$$

consistent with the analog of the Goldberger-Treiman relation: $f_\pi = 2M_F g_A/g_{\pi\bar F F}$ [using Eq. (4.36)] and with the fact that *a dimensionless physical parameter must be a pure number*.

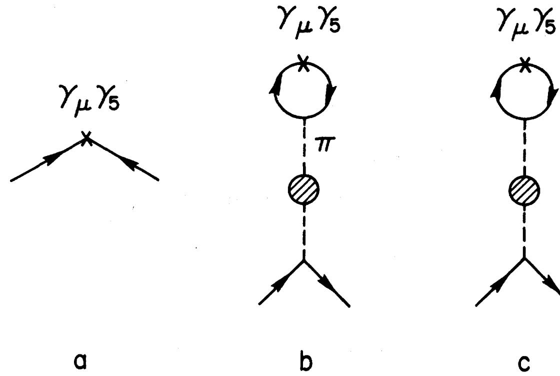

FIG. 6. Axial-vector fermion form factors to leading order in $1/N$.

The generalization of the above to the models which possess an SU(M)×SU(M) chiral symmetry, described by Eq. (2.14) is clear. One need only replace $\lambda = g^2N$ by $\lambda = g^2NM$.

Finally we should emphasize that, in the case of the simple model (Eq. 2.1) we do not expect higher-order corrections to qualitatively change the picture as long as N is sufficiently large. The new vacuum that we have found occurs in a region of finite σ, where we expect the $1/N$ expansion to be a valid asymptotic expansion. This is not the case, as previously remarked, for our continuous symmetry models which develop in leading-order zero-mass bound-state Goldstone bosons, since higher orders will contain infrared divergences. But even here one can by the Higgs phenomenon (Sec. VII) or by continuing to $2+\epsilon$ dimensions, ensure that the dynamical symmetry will survive to all orders in $1/N$.

V. THE POTENTIAL OF COMPOSITE FIELDS

The potential that we have constructed in Sec. IV is not the same as the potential of the composite field $\bar\psi\psi$ for an arbitrary external source. In this section we shall show how one can construct the latter in terms of the σ potential previously derived.

To construct the "potential" of the composite field $\Sigma \equiv g\bar\psi\psi$ (for notational convenience it is useful to include g in the definition of Σ) we first define the generating function $\mathcal{W}(J)$ of the connected n-point functions of Σ:

$$e^{i\mathcal{W}(J)} = \mathrm{const}\times\int d\psi\, d\bar\psi \exp\{i[\bar\psi\, i\not\partial\psi + \tfrac{1}{2}g^2(\bar\psi\psi)^2 + Jg\bar\psi\psi]\}\,. \tag{5.1}$$

We then define a c-number function Σ_c to be

$$\Sigma_c = \frac{\delta \mathcal{W}(J)}{\delta J} = \langle 0 | g \bar{\psi}\psi | 0 \rangle_J \,. \tag{5.2}$$

The Σ potential is then defined by the Legendre transform of $\mathcal{W}(J)$. For constant J and Σ_c:

$$\mathcal{V}(\Sigma_c) = \Sigma_c J - \mathcal{W}(J)\,. \tag{5.3}$$

So far these definitions are completely analogous to those used for noncomposite fields. However, there are important differences. If Σ were an elementary field operator then $\mathcal{V}(\Sigma_c)$ (for constant Σ_c) would be the generator of the 1PI Green's functions of the Σ field with zero external momentum. This is not the case here; the graphical interpretation of $\mathcal{V}(\Sigma_c)$ is much more complicated. On the other hand, the physical interpretation of this functional is unchanged. In the standard fashion one can show that $\mathcal{V}(\Sigma_c)$ is equal to the energy density in the state where Σ is constrained to have Σ_c as its expectation value. Again in the ground state $\mathcal{V}(\Sigma_c)$ must be stationary with respect to variations of Σ, among the various extrema we must pick the absolute minimum. Thus the vacuum is determined by

$$J \equiv \frac{\partial \mathcal{V}_c(\Sigma_c)}{\partial \Sigma_c} = 0\,, \quad \frac{\partial^2 \mathcal{V}(\Sigma_c)}{\partial {\Sigma_c}^2} > 0\,. \tag{5.4}$$

In most theories it is extremely difficult to construct the potential of a composite operator. Here, however, it is relatively simple since we can easily relate $\mathcal{V}(\Sigma_c)$ to $V(\sigma)$, which we calculated in Sec. IV. This can be achieved by introducing into the definition of $\mathcal{W}(J)$ the J-independent constant

$$\int d\sigma \exp[-\tfrac{1}{2}(\sigma - J - g\bar{\psi}\psi)^2]\,,$$

which allows us to replace Eq. (5.1) by

$$e^{i\mathcal{W}(J)} = \text{const} \times \int d\psi\, d\bar{\psi}\, d\sigma \exp[i(\bar{\psi} i\partial\!\!\!/ \psi - \tfrac{1}{2}\sigma^2 + g\sigma\bar{\psi}\psi + \sigma J - \tfrac{1}{2}J^2)]\,. \tag{5.5}$$

This, however, is simply equal to $\exp(-\frac{1}{2} iJ^2)$ multiplied by the path-integral expression for $\exp[iW(J)]$ [Eq. (4.6)]. Thus

$$\mathcal{W}(J) = W(J) - \tfrac{1}{2} J^2\,. \tag{5.6}$$

We note that in the tree approximation $V(\sigma) = \frac{1}{2}\sigma^2$ and therefore $W(J) = \frac{1}{2}J^2$. Therefore $\mathcal{W}(J)$ vanishes in the tree approximation. This must be the case for a composite field which does not, of course, appear explicitly in the Lagrangian.

The "classical" Σ_c operator can then be expressed as

$$\Sigma_c = \frac{\partial \mathcal{W}(J)}{\partial J} = \frac{\partial W(J)}{\partial J} - J = \sigma_c - J\,. \tag{5.7}$$

Similarly the Σ potential can be evaluated in terms of the σ potential:

$$\begin{aligned}\mathcal{V}(\Sigma_c) &= \Sigma_c J - \mathcal{W} \\ &= (\sigma_c - J)J - (W - \tfrac{1}{2}J^2) \\ &= V(\sigma_c) - \tfrac{1}{2}J^2\,.\end{aligned} \tag{5.8}$$

Until now, we have ignored all renormalization counterterms. In order to derive an expression for the renormalized potential, we must replace the Lagrangian in Eq. (5.1) by

$$\bar{\psi} i\partial\!\!\!/ \psi + \tfrac{1}{2} g^2 {Z_\sigma}^{-1} (\bar{\psi}\psi)^2 + JgZ_\sigma^{-1}\bar{\psi}\psi\,, \tag{5.9}$$

where ${Z_\sigma}^{-1}$ renormalizes the coupling constant:

$$g_R = g_0 (Z_\sigma)^{1/2}\,. \tag{5.10}$$

This guarantees that $\mathcal{W}(J)$ will generate the renormalized connected Green's functions of $\Sigma_c = g\bar{\psi}\psi$. Similarly, the renormalized $W(J)$ is defined with $\mathcal{L}_\sigma$ replaced by

$$-\tfrac{1}{2} {Z_\sigma}^{-1}\sigma^2 + gZ_\sigma^{-1}\sigma\bar{\psi}\psi + {Z_\sigma}^{-1}\sigma J\,.$$

Repeating the previous manipulations, we now derive for the renormalized potentials

$$\mathcal{V}(\Sigma_c) = V(\sigma_c) - \tfrac{1}{2} {Z_\sigma}^{-1} J^2\,, \tag{5.11}$$

where

$$\Sigma_c = \sigma_c - {Z_\sigma}^{-1} J\,. \tag{5.12}$$

In an asymptotically free theory, the behavior of Z_σ as the ultraviolet cutoff Λ is taken to infinity can be calculated.[19] Indeed, in our model

$${Z_\sigma}^{-1} = \frac{1}{1 + (\lambda/\pi) g^2 \ln(\Lambda/\mu)} \xrightarrow[\Lambda\to\infty]{} 0\,. \tag{5.13}$$

Therefore, in the infinite cutoff limit, Σ_c and σ_c are equal, and $\mathcal{V}(\Sigma_c)$ coincides with $V(\sigma_c)$.

The above considerations show that the broken-symmetry solution constructed in Sec. IV using the σ Lagrangian is indeed the correct one for our original four fermion theory. In addition they illustrate some of the complexities one encounters when probing for symmetry breaking via composite operators. A further example is given in the Appendix where we discuss $\lambda\phi^4$ models in four dimensions.

VI. DYNAMICAL SYMMETRY BREAKING AND ASYMPTOTIC FREEDOM

In the preceding sections we have developed in detail a particular two-dimensional model which exhibits dynamical symmetry breaking. In this section we shall discuss some of the general features of theories which contain no mass parameters but which generate masses dynamically. We shall argue that dynamical symmetry breaking is likely to be a common occurrence in asymptotically free theories. In addition we shall determine the dependence of the dynamically generated masses on the coupling constants for small values of the latter. Conversely we shall argue that theories for which the origin of coupling constant space is infrared stable cannot generate masses dynamically. To illustrate this a ϕ^4 theory in four space-time dimensions is studied in the Appendix employing the $1/N$ limit as before. The only significant difference between this theory and our two-dimensional models is that the latter are asymptotically free and the former are not. Consequently spontaneous symmetry breaking does not occur.

Let us consider a class of renormalizable field theories which contain no dimensional parameters. In other words, all masses and superrenormalizable couplings vanish. Such theories are characterized by dimensionless couplings g_i, and a renormalization parameter, μ, introduced to get the scale of the momentum at which one subtracts the divergent Green's functions. In such a theory an arbitrary Green's function satisfies the renormalization-group equation

$$\left[\mu \frac{\partial}{\partial \mu}+\sum_i \beta_i(g_1,g_2,\ldots)\frac{\partial}{\partial g_i}+\sum_a \gamma_a(g_1,g_2,\ldots)\right] \times G(P_a,g_i,\mu)=0, \quad (6.1)$$

where $\beta_i(\gamma_a)$ describes the change in the renormalized coupling constant g_i (the scale of the field labeled a) when one changes the renormalization scale μ keeping the ultraviolet cutoff and bare couplings fixed.

For simplicity let us consider theories characterized by one coupling constant, and define

$$\beta(g)\approx \tfrac{1}{2} b_0 g^3+O(g^5), \quad g\simeq 0. \quad (6.2)$$

An asymptotically free theory is one for which $b_0<0$, in which case the "effective coupling constant," defined by

$$\frac{d\bar{g}(g,t)}{dt}=\beta(\bar{g}), \quad \bar{g}(g,0)=g, \quad (6.3)$$

approaches zero when $t\to\infty$. In such theories the large-Euclidean-momentum asymptotic behavior of the Green's functions will be essentially that of free field theory, up to calculable logarithmic deviations (see Ref. 7). The only known physically sensible theories of this type are the two-dimensional fermion models considered in this paper and non-Abelian gauge thoeries in four dimensions.[20]

Conversely if $b_0>0$ the theory exhibits free field theory behavior for small values of the momenta. The effective coupling vanishes when $t=\ln P^2\to-\infty$:

$$\bar{g}^2(t) \underset{t\to-\infty}{\sim} \frac{-1}{b_0 t}. \quad (6.4)$$

Examples of such theories are QED of massless fermions and $g^2\phi^4$ in four dimensions.

In all such theories it is conceivable that masses are generated dynamically; indeed the models considered in this paper provide a concrete example of such a phenomenon. If that is the case we can deduce some general features of the resulting theory which follow from the use of the renormalization group.

Theorem I. The physical masses must have the following dependence on the coupling constant:

$$\mathfrak{M}(g,\mu)=\mu \exp\left[-\int^g \frac{dx}{\beta(x)}\right]. \quad (6.5)$$

This follows trivially from the dimension of $\mathfrak{M}$ and the fact that, being a physical parameter it must satisfy the homogeneous renormalization-group equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]\mathfrak{M}(g,\mu)=0.$$

The same equation must be satisfied by all other physical parameters.

Theorem II. Any physical parameter (for example, a coupling constant) $P(g,\mu)$ which has physical dimension $(\text{mass})^{d_p}$ must be equal to $[\mathfrak{M}(g,\mu)]^{d_p}$ up to a calculable number:

$$P(g,\mu)=\text{constant}\times[\mathfrak{M}(g,\mu)]^{d_p}. \quad (6.6)$$

Thus all dimensional physical parameters are pure numbers which are, in principle, calculable. In this sense such theories contain no free parameters.

Furthermore the dependence of an arbitrary Green's function, satisfying Eq. (6.1), on the renormalization scale parameter μ can be eliminated. It is clear that $G(P_a,g,\mu)$ can be written as

$$G(P_a,g,\mu)=[\mathfrak{M}(g,\mu)]^D\prod_a \exp\left[-\int^g\frac{\gamma_a(x)\,dx}{\beta(x)}\right] \times F(P_a/m), \quad (6.7)$$

where D is the dimension of G and F a calculable function involving no adjustable parameters.

As a consequence of Theorem I we can determine how the dynamically produced masses change when we vary the coupling constant. It is particularly interesting to consider the limit of vanishing coupling, keeping the renormalization procedure and the subtraction point μ fixed. Using Eqs. (6.3) and (6.4) we derive from Eq. (6.5)

$$\mathfrak{M}(g,\mu)\underset{g\to0}{\sim}\mu\exp\left(\frac{1}{b_0 g^2}\right). \quad (6.8)$$

The behavior of the physical masses as the coupling is turned off depends crucially on whether the theory is asymptotically free or not. For an infrared-stable theory ($b_0>0$) the masses diverge exponentially as the coupling vanishes. This, to our mind, is physically unreasonable and leads us to conclude that *infrared-stable theories cannot produce masses dynamically*.[21] This argument can easily be generalized to the case where the theory involves many coupling constants. For an infrared-stable theory the physical masses must satisfy

$$\mathfrak{M}(\mu, g_1, \ldots, g_N) = \mu f(g_1, \ldots, g_N) = \mu' f\left(\bar{g}_1\left(\ln\frac{\mu'}{\mu}\right), \ldots, \bar{g}_N\left(\ln\frac{\mu'}{\mu}\right)\right), \tag{6.9}$$

where $d\bar{g}_i^2(t)/dt = \beta_i(\bar{g}_1, \ldots, \bar{g}_N)$. If the theory is infrared-stable then $\bar{g}_i^2(t) \approx -1/t$ when $t \to -\infty$. Therefore if we let $\mu' \to 0$ in Eq. (6.9) we see that $f(g_i)$ must diverge as e^{+1/g_i^2} when all the coupling constants g_i^2 vanish at the same rate.

In asymptotically free theories, on the other hand, the masses vanish exponentially [$\exp(-1/g^2)$] when the coupling is turned off. This is altogether reasonable. In fact one might expect that bound states will be produced in an asymptotically free theory no matter how small the "physical" coupling g is, as long as the first nontrivial zero of $\beta(g)$ (g_1) is large enough. Indeed irrespective of the value of $\bar{g}(t=0)=g$ one can make the effective coupling $\bar{g}$ increase to g_1 for sufficiently small momenta. If g_1 is large enough for the theory to produce masses this will then occur independent of the value of g. Certainly in a theory in which β is negative everywhere (so that $g_1 \to \infty$), as is the case in the models considered above, bound states should necessarily be produced.

VII. DYNAMICAL HIGGS MECHANISM

In Sec. IV we constructed models in which a continuous symmetry group is dynamically broken, and bound-state Goldstone bosons were formed. Here we shall show that if one makes the theory invariant under a gauge group this dynamical symmetry breaking gives the gauge meson a mass. We shall only discuss the simple case of chiral symmetry [corresponding to Eq. (2.11)]:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ + eB\!\!\!/\gamma_5)\psi + \tfrac{1}{2}g^2[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] + \tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 \tag{7.1}$$

or equivalently

$$\mathcal{L}' = \bar{\psi}(i\partial\!\!\!/ + eB\!\!\!/\gamma_5)\psi + \tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \tfrac{1}{2}(\sigma^2+\pi^2) + g\bar{\psi}(\sigma + i\gamma_5\pi)\psi . \tag{7.2}$$

This Lagrangian is invariant under the local gauge transformations

$$\begin{aligned} \delta B_\mu &= \partial_\mu \eta, \\ \delta\psi &= -\tfrac{1}{2} ie\gamma_5\eta\psi, \\ \delta\sigma &= -\eta\pi e, \\ \delta_\pi &= \eta\sigma e . \end{aligned} \tag{7.3}$$

In two dimensions the coupling constant e has dimension of mass corresponding to a superrenormalizable coupling. We define e to be the finite bare coupling constant and thus it will not have any effect on the asymptotic freedom and the renormalization-group properties of the theory. If we were to introduce into Eq. (7.1) scalar fields, so as to generate spontaneous symmetry breaking in the tree graph approximation, we would necessarily have at least one additional parameter. As we shall see this is unnecessary, for the model automatically generates a Higgs scalar dynamically.

The limit studied is now $N\to\infty$ for fixed $\alpha = e^2N$ and $\lambda = g^2N$. To leading order in N, the σ potential is unaffected by the new interaction. The symmetry breaking develops as above: σ acquires a vacuum expectation value and the fermion mass given by Eq. (4.19). The σ bound state occurs, as before, with mass equal to $2M_F$. As for the B_μ and π fields, the one fermion loop graphs of Fig. 7 have the potential of generating a vector-meson mass. It is straightforward to compute these graphs. The result can be expressed as the bilinear part of the effective Lagrangian in the limit $N\to\infty$ (in the momentum representation):

$$\mathcal{L}_{\text{eff}}(B_\mu, \pi) = -\tfrac{1}{2}B_\mu P^2 B_\mu + \tfrac{1}{2}B_\mu P_\mu P_\nu B_\nu - \frac{\alpha}{2\pi}B_\mu^2 + \frac{\alpha}{2\pi}B_\mu\frac{P_\mu P_\nu}{P^2}B_\nu + \alpha B_\mu M^2 U\frac{P_\mu P_\nu}{P^2}B_\nu + igemN\pi U P_\mu B_\mu + \tfrac{1}{4}\alpha\pi P^2 U\pi , \tag{7.4}$$

where

$$U = \frac{1}{\pi[P^2(P^2-4M_F^2)]^{1/2}}\ln\frac{(-P^2+4M_F^2)^{1/2}-(-P^2)^{1/2}}{(-P^2+4M_F^2)^{1/2}+(-P^2)^{1/2}} . \tag{7.5}$$

Note that this effective Lagrangian, which contains only bilinear forms of the fields, is not chiral-invariant by itself. This is because in the broken-symmetry theory, the Ward identities relate Green's functions involving different numbers of external lines.

It is still necessary to introduce a gauge-fixing term. We make the convenient choice

$$\mathcal{L}_c = \frac{1}{2}\left[(P^\mu B_\mu)\left(1 - \frac{\alpha}{\pi P^2} - \frac{2\alpha M^2 U}{P^2}\right)^{1/2} - \frac{iegNU}{(1-\alpha/\pi P^2 - 2\alpha M^2 U/P^2)^{1/2}}\pi\right]^2, \tag{7.6}$$

which diagonalizes the B_μ and π propagators. With this choice the B_μ propagator is simply $-ig_{\mu\nu}/(P^2+\alpha/\pi)$ and the π propagator is

$$4i/U(-P^2+\alpha/\pi)(1-\alpha/\pi P^2 - \alpha 2UM_F{}^2/P^2)\,.$$

Hence the vector meson has acquired a mass α/π. The gauge-dependent pole (at $P^2=+\alpha/\pi$) in the π propagator only serves to cancel the unwanted polarizations of the vector-meson field B_μ. There are no more zero-mass states in the theory—the bound-state Goldstone π has become a bound-state Higgs particle.

We also note that the mass of the vector meson, α/π, is independent of the value of g. In fact it is identical to its value in two-dimensional quantum electrodynamics.[22] That this must be so is obvious from the renormalization group, since the ratio of M_B to α/π can only be a pure number, independent of g.

VIII. CONTINUATION TO $d=2+\epsilon$ DIMENSIONS

We now consider how dynamical spontaneous symmetry breaking can occur when the models of the previous sections are continued to $2+\epsilon$ dimensions ($\epsilon>0$). Continuation of the theory described by the Lagrangian (2.1) in $2+\epsilon$ dimensions has already been considered by Wilson.[13] Among other results, he finds that the theory exhibits a nontrivial Gell-Mann–Low eigenvalue, of order ϵ, at which point scaling occurs with anomalous dimension. For values of the coupling constant below the fixed point, the infrared behavior is governed by the origin, the ultraviolet behavior by the fixed point.[23] The fact that a $(\bar\psi\psi)^2$ coupling for dimension greater than two is nonrenormalizable by the usual power-counting arguments raises serious questions as to the meaning of this section. However, we shall find that the nonrenormalizability certainly does not appear in the leading $1/N$ approximation. It turns out that the next order in

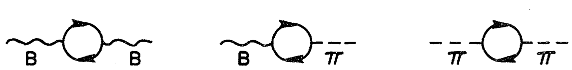

FIG. 7. The Feynman graphs that contribute to the vector-meson and pseudoscalar-meson self-energies.

$1/N$ is also renormalizable. This depends crucially on the presence of the nontrivial eigenvalue and the value of the anomalous dimensions of the fields. Hence, renormalizability is still an open question. Of course, only the value $\epsilon=1$ is physically meaningful.[24]

We show that spontaneous symmetry breaking occurs for values of the coupling constant greater than the nontrivial fixed point. This is again related to the presence of an infrared tachyon in the symmetric theory.

In order to have a dimensionless coupling constant, we use the Lagrangian

$$\mathcal{L} = \mu^\epsilon[i\bar\psi\partial\!\!\!/\psi + g(\bar\psi\psi)\sigma - \tfrac{1}{2}\sigma^2], \tag{8.1}$$

where μ is an arbitrary mass parameter. With this choice, the Maxwellian dimension of ψ in units of mass is 1, and g^2 is dimensionless.

The renormalized σ propagator is in the $N\to\infty$ limit[25]:

$$\frac{-i\mu^{-\epsilon}}{1+(g^2N/\pi\epsilon)c(\epsilon)(k^2/\mu^2)^{\epsilon/2}-(g^2N/2\pi\epsilon)c(\epsilon)}, \tag{8.2}$$

with

$$C(\epsilon) = \pi^{1-d/2}2^{2-d}\frac{[\Gamma(\frac{1}{2}d)]^2\Gamma(2-d/2)}{\Gamma(d-1)} \xrightarrow[\text{for } \epsilon\to 0]{} 1\,.$$

This propagator exhibits many of the features already discussed by Wilson in Ref. (13). There is a nontrivial eigenvalue $\lambda_0 \equiv g_0{}^2N = 2\pi\epsilon/c(\epsilon)$, where the σ propagator scales with an anomalous dimension and is independent of μ.[26] The theories with $\lambda<\lambda_0$ are those considered by Wilson.[13] For $\lambda>\lambda_0$, one sees that the σ propagator develops a spacelike pole (a tachyon) with mass equal to

$$i\mu\left[1-\frac{\epsilon}{\lambda c(\epsilon)}\right]^{1/\epsilon}.$$

Spontaneous symmetry breaking then occurs in the same fashion as in Sec. II. For completeness, we give the renormalized σ potential in the $N\to\infty$ limit:

$$V(\sigma)=\tfrac{1}{2}\sigma^2\left[1+\frac{\lambda}{2^{d-4}\pi^{d/2}}\,\frac{\Gamma(2-d/2)}{(d-2)}\,\frac{1}{d}\left(\frac{g_\sigma}{\mu}\right)^{d-2}-\frac{\lambda(d-1)}{2^{d-3}\pi^{d/2}}\,\frac{\Gamma(2-d/2)}{(d-2)}\right]. \tag{8.3}$$

Here, we have chosen $V''(\sigma)=1$ at $g\sigma=\mu$. For $d\neq 2$, this leads to a simpler formula than the renormalization procedure of Sec. II. It is also more sensible, since $g\sigma$, and not σ, is the real physical quantity, related to the fermion mass. Thus $g\sigma_M$, where σ_M is the position of the minimum of the potential, is equal to the fermion mass and is an invariant of the renormalization group.

This potential has the same general shape as in two dimensions, Fig. 5, provided that

$$\lambda>\frac{(d-2)\pi^{d/2}2^{d-3}}{(d-1)\Gamma(2-d/2)} \tag{8.4}$$

and spontaneous symmetry breaking occurs. As for the renormalization of the theory we note that the σ propagator, Eq. (8.2) behaves for large k^2 like $(k^2)^{-(1/2)\epsilon}$, whereas each term of its expansion in g^2N actually blows up. This asymptotic behavior is of course directly related to the ultraviolet stable eigenvalue. The summation to all orders in g^2N has improved the convergence properties of the theory. It is then trivial to check that in next order in $1/N$ no more than one subtraction in the $\sigma\bar{\psi}\psi$ vertex, in the σ propagator and in the ψ propagator is required to render them finite. Thus the renormalizability of the theory is the same as in two dimensions. We have not investigated higher orders.

Hence, for dimensions greater than two, two qualitatively different theories emerge from the same Lagrangian [Eq. (2.1)], for different values of the physical coupling constant. These different theories are distinguished by the sign of the Callan-Symanzik function $\beta(g)$ (Fig. 8):

$$\beta(g)=-\frac{g\lambda}{2\pi}+\epsilon g\,,\qquad g_0{}^2N=2\pi\epsilon\,. \tag{8.5}$$

For $\beta(g)>0$, i.e., $0<g<g_0$, an ordinary theory with a massless fermion emerges. For $\beta(g)=0$, i.e., $g=g_0$, the fermion is still massless, and the theory possesses exact scale invariance, albeit with anomalous dimensions. Finally, in the, so-called "postcritical theory," $\beta(g)<0$, i.e., $g>g_0$, spontaneous symmetry breaking occurs, the fermion acquires a mass, and there appears a massive fermion-antifermion bound state. In all three cases, the ultraviolet behavior of the theory is governed by the fixed point g_0.

For the same reasons as given in the Appendix, it seems impossible to define a similar postcritical theory for ϕ^4 in $4-\epsilon$ dimensions.

IX. HIGHER ORDERS

We shall consider, in this section, higher-order ($1/N$) corrections to our model. These corrections can be systematically computed regarding the one-loop σ propagator Eq. (4.25) as the "bare" σ propagator. One then sums ordinary perturbation theory graphs with this replacement. For example, the $1/N$ corrections to the fermion self-energy, the vertex function, and the σ self-energy are given by the graphs of Fig. 9. Of course the σ propagator is itself a sum of an infinite set of Feynman graphs so the resulting terms in the $1/N$ expansion can be quite complicated functions of λ.

We find that the leading corrections exhibit unexpected features. Namely, the Green's functions, as well as the renormalization-group parameters, develop an essential singularity in the coupling constant λ, at $\lambda=0$. This of course means that perturbation theory in λ does not yield a convergent series expansion for any value of λ. This divergence does not arise from the large number of Feynman graphs encountered in high orders in λ. Rather it is a consequence of the ultraviolet behavior of a renormalizable theory (thus it is absent in an analogous superrenormalizable theory).

Such an essential singularity does not mean that perturbation theory is useless, as long as the formal perturbation series yields an asymptotic expansion of the Green's functions for small coupling. We find, not surprisingly since our Green's functions have been perturbatively constructed, that this is the case. More important we find that the perturbation series is Borel-summable. This appears to be connected with the asymptotic freedom of the theory. A nonasymptotically free theory, such as ϕ^4, would exhibit a non-Borel-summable essential singularity.

Another reason for studying the higher-order

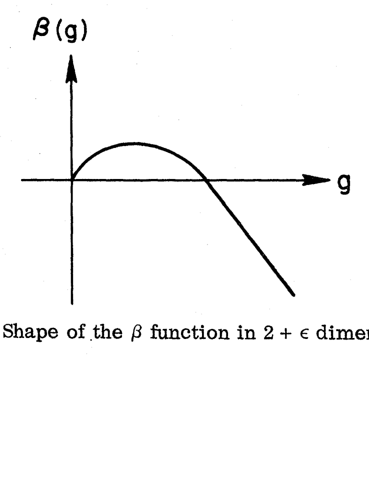

FIG. 8. Shape of the β function in $2+\epsilon$ dimensions.

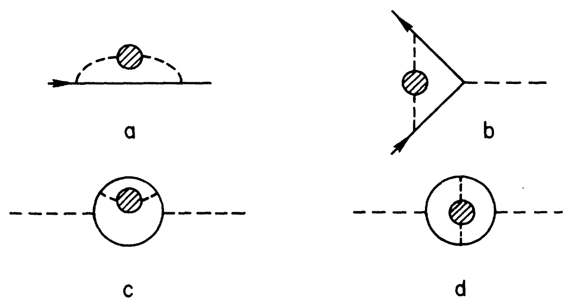

FIG. 9. $1/N$ corrections to (a) fermion self-energy; (b) σ vertex; (c), and (d) σ self-energy.

corrections in $1/N$ is the possibility of singularities at finite λ. Each order in $1/N$ has, in general, a very nontrivial $\lambda = g^2 N$ dependence. In principle singularities could arise of the type, say $1/(\lambda - \lambda_0)$, $\lambda_0 > 0$. In such a case the $1/N$ expansion would be valid for any $\lambda \neq \lambda_0$, but would break down for $\lambda = \lambda_0$. We shall argue that such is not the case, and that the only singularity is at the origin.[27]

In the theory where the fermion is initially massless [Eq. (2.1)], the only dependence of the coupling constant g is ultimately in the fermion mass M_F via Eq. (4.19). This means that the theory in fact has no free dimensionless parameters. Hence, we rather study the massive theory, where there is no symmetry breaking, and it is meaningful to discuss the singularity structure in g. Also, the presence of a mass term in the Lagrangian cannot alter the structure of the renormalization-group parameters.

The essential features of the $1/N$ corrections are already displayed in the fermion propagator $\Sigma(P)$ [Fig. 9(a)]

$$i\Sigma^{-1}(P) = \not{P}[1 + (1/N)A(P^2)] + m[1 + (1/N)c(P^2)]. \tag{9.1}$$

Let us evaluate the wave-function renormalization constant A. To perform this calculation we note that in each of the infinite set of graphs summarized by Fig. 9(a) the only subtractions required are in the σ propagator, i.e., the apparent over-all logarithmic divergence is spurious. This of course means that A will not require additional subtractions, so that we can easily evaluate it[28]:

$$A(P^2) = \frac{\lambda}{8\pi P^2}\int_0^\infty d(k^2)\left[1 - \frac{k^2 + P^2 + m^2}{[(k^2 + P^2 + m^2)^2 - 4P^2 m^2]^{1/2}}\right]\frac{1}{1 + (\lambda/2\pi)[B(k^2, m^2) - B(\mu^2, m^2)]}, \tag{9.2}$$

where B is given in Eq. (4.3).

The fact that the fermion requires only a finite wave-function renormalization to order $1/N$ means that to this order the fermion anomalous dimension vanishes. $A(P^2)$ satisfies the homogeneous renormalization-group equation, $[\mu\partial/\partial\mu + \beta(g)\partial/\partial g]A = 0$, where β is evaluated to leading order in $1/N$, as a consequence of the fact that $\lambda D_\sigma(k^2)$ obeys this equation. In calculating the radiative corrections of order $1/N$ to the vertex [Fig. 9(b)] or to the σ self-energy [Figs. 9(c) and 9(d)] we would find additional divergences arising from the over-all loop integration. The ensuing calculation is therefore much more difficult, for one cannot simply use the renormalized σ propagator inside the graphs of Figs. 9(c) and 9(d). One must expand these in a power series in λ, subtract term by term, and then resum to get the renormalized amplitudes.

Returning now to the fermion self-energy we see that the absence of the tachyon pole in the σ propagator ensures that the integrand in Eq. (9.2) is well defined throughout the whole integration region. Consequently the only singularities in λ can arise at $\lambda = 0$ (or $\lambda = \infty$). For positive λ the integral can be expanded in a power series about $\lambda = 0$. However, it is evident that this series is only an asymptotic expansion, and has zero radius of convergence. This fact is best seen by trying to compute $A(P^2)$ for small negative λ. The integrand then develops a pole for large $k^2 \approx \mu^2 e^{1/|\lambda|}$ and in that region the integral has the form

$$\int_{k_{\min}}^\infty \frac{d(k^2)}{k^4(1 + \lambda \ln k^2)}.$$

If we change variables $u = \ln k^2$, we can rewrite this as

$$F(\lambda, u_{\min}) = \int_{u_{\min}}^\infty du \frac{e^{-u}}{1 + \lambda u}.$$

This is easily recognized as the classic example of a Borel-summable essential singularity. As a result one sees that the power series in λ for $A(P^2)$ has the form

$$A(P^2) \approx \frac{\lambda}{2\pi}\sum_{n=0}^\infty \left(\frac{-\lambda}{2\pi}\right)^n n!\,. \tag{9.3}$$

What is the origin of this essential singularity? The growth of the coefficients of the Taylor series expansion about $\lambda = 0$ resembles that estimated in rigorous treatments of quantum field theory.[29] However, in these estimates the $n!$ arises due to the contribution of $n!$ Feynman graphs to nth-order perturbation theory. Here, in contrast, there is only one Feynman graph contributing to each term. The origin of the $n!$ is rather the ultraviolet behavior of individual Feynman graphs. Thus the λ^n contribution to $A(P^2)$ involves the λ^n term in the σ propagator, which grows for large momentum like $(\ln k^2)^n$. The factor of $n!$ then arises from an integral which behaves like

$$\lambda^n \int_{k_{\min}}^{\infty} \frac{dk^2}{k^4} (\ln k^2)^n \sim \lambda^n n! \,.$$

Indeed if one were to introduce an ultraviolet cutoff, Λ, consistently in the theory, the above integral would be replaced by

$$\lambda^n \int_{k_{\min}}^{\infty} \frac{dk^2}{k^4} (\Lambda^2)^n \approx (\lambda\Lambda^2)^n \,,$$

and the perturbation series would be absolutely convergent for $|\lambda| \leq 1/\Lambda^2$. The essential singularity would occur at $\lambda \cong -1/\Lambda^2$.

Alternatively we can understand the origin of the essential singularity at $\lambda = 0$ as arising from the fact that the theory, yields nonsense when λ is negative, no matter how small. For when λ is negative the σ propagator develops a tachyon pole (independent of symmetry breaking or the value of the fermion mass), and it is the presence of this pole in the integration region in Eq. (9.2) that gives rise to the singularity. Thus the same calculation performed for the ϕ^4 theory in four dimensions leads to an integral of the form

$$\int_{u_{\min}}^{\infty} \frac{e^{-u} du}{1 - \lambda u} \,,$$

which is ill defined for any value of λ. One could define this integral in some fashion or another, however, the essential singularity at $\lambda = 0$ is not Borel-summable. Thus it is not clear how the $1/N$ corrections in the ϕ^4 theory can help to cure the problems which are manifest in the leading order (see the Appendix), for these very problems appear to prevent one from using perturbation theory to higher orders.

Returning to our asymptotically free four fermion theories, it is clear that, to any order in $1/N$, the only singularities in λ of renormalized Green's functions will occur at $\lambda = 0$. Indeed after Wick rotation a renormalized Green's function is defined by an integral over Euclidean momenta that converges at infinity and is perfectly well defined everywhere in the integration region for any positive value of λ. The prototype of such an integral is of course Eq. (9.2). It also follows that all the renormalization-group parameters, β, γ, etc., have similar analyticity properties as functions of λ. Thus we see no reason why the $1/N$ expansion should not be a valid asymptotic expansion of the theory.

X. CONCLUSIONS

The models analyzed in this paper, formulated in two space-time dimensions, are clearly unrealistic. However, we believe that the phenomenon exhibited by these models is indicative of what one would expect in more realistic models. In fact the restriction to two dimensions is only in order to have an asymptotically free theory in which one has an explicit expansion parameter (N). The only asymptotically free theory in four dimensions necessarily involves gauge fields[7,8] and does not lend itself to any simple approximation. Beyond that the role of two dimensions is minimal. Our models are neither trivial nor soluble. On the contrary, we have found quite complicated behavior for the theory already in second order in $1/N$; i.e., bound states, dynamical symmetry breaking, and essential singularities in the coupling constants. These phenomena are also stable under small changes in the parameters of the theory, including the bare masses and the dimension of space-time. This is to be contrasted with other well-known two-dimensional models, which do not exhibit such stability, and which have a trivial S matrix.[30]

There are many other possible uses for our models, which we have not explored in detail. One can explicitly perform calculations to study how asymptotic forms are approached, to study the properties of amplitudes involving bound states, to study patterns of symmetry breaking, etc. In addition it is not unlikely that further exploration will lead to new and, perhaps surprising, results such as the essential singularities that we discovered in higher orders in $1/N$.

ACKNOWLEDGMENTS

The authors would like to thank Roger Dashen and Arthur Wightman for useful conversations. One of the authors (D. J. Gross) would like to express his appreciation to Dr. C. Kaysen for the hospitality shown him at the Institute for Advanced Study.

APPENDIX: ϕ^4 MODELS

Four-dimensional scalar field theories with quartic interactions bear a striking resemblance to the two-dimensional fermion theories considered above. Here we shall investigate for such theories whether one can generate dynamical symmetry breaking in the $N \to \infty$ limit.

Let us consider the theory given by the Lagrangian density

$$\mathcal{L} = (\partial_\mu \vec{\phi}) \cdot (\partial^\mu \vec{\phi}^*) - \tfrac{1}{2} g^2 (\vec{\phi} \cdot \vec{\phi}^*)^2 , \qquad \text{(A1)}$$

where $\vec{\phi}$ is a complex N-dimensional vector. Contrary to the two-dimensional four-fermion theories considered previously we have chosen the opposite sign for the interaction. That is because in the theory in which $\mathcal{L}_I = \frac{1}{2} g^2 (\vec{\phi} \cdot \vec{\phi}^*)^2$ the energy has no lower bound. This is obvious on a classical level and can be forcefully argued in relativistic quantum mechanics.[31] Indeed if one were to choose this sign then the theory would be asymptotically free. One can then show, with the aid of the renormalization group, that the potential of $\vec{\phi}_c$ approaches, for large values of $\vec{\phi}_c$, its classical value $[-(\vec{\phi} \cdot \vec{\phi}^*)^2]$ up to logarithms. Thus the energy density per unit volume is arbitrarily negative for large classical fields, and the energy spectrum has no lower bound. We note that this argument cannot be given for our fermion models since ψ is an anticommuting field.

We again solve the model for large N, and examine whether in the true ground state spontaneous symmetry breaking occurs and $\vec{\phi} \cdot \vec{\phi}^*$ develops a nonzero vacuum expectation value. As before this model is the local limit of a $\vec{\phi} \cdot \vec{\phi}^* \sigma$ coupling and can be generated by

$$\mathcal{L}_c = \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi}^* + \tfrac{1}{2} \sigma^2 - g\sigma(\vec{\phi} \cdot \vec{\phi}^*) . \qquad \text{(A2)}$$

In the large N limit the Green's functions are given by the tree graphs of $\mathcal{L}_\sigma$ with the bare σ propagator, $+i$, replaced by the sum of the one-loop graphs, renormalized at $P^2 = -\mu^2 (\lambda = g^2 N)$:

$$D_\sigma(P^2) = \frac{i}{1 - (\lambda/16\pi^2) \ln(-P^2/\mu^2)} . \qquad \text{(A3)}$$

This theory is not asymptotically free. In fact one easily sees that

$$\beta(g) = g\gamma_\sigma(g) = \frac{g\lambda}{16\pi^2} . \qquad \text{(A4)}$$

Thus the effective coupling increases for increasing momentum, and vanishes for small momentum $[2t = \ln(-P^2/\mu^2)]$:

$$\bar{g}^2(t) = \frac{g^2}{1 - (\lambda/8\pi^2)t} . \qquad \text{(A5)}$$

In fact the effective coupling develops a pole at a finite value of t (as before) and simultaneously the σ propagator, and thus the scalar n-point functions, will contain a tachyon pole. This occurs at

$$P^2 = -\mu^2 e^{16\pi^2/\lambda} . \qquad \text{(A6)}$$

The difference between this and the previous case is that as the coupling is turned off the tachyon pole approaches infinity instead of zero. Because of this the tachyon will not be removed by having $\vec{\phi}$ acquire a mass. Indeed if $\vec{\phi}$ were to have mass m, then the σ propagator would be

$$D_\sigma(P^2) = \frac{i}{1 - (\lambda/16\pi^2)[B(-P^2, m^2) - B(\mu^2, m^2)]} , \qquad \text{(A7)}$$

where B is given by Eq. (4.3). Since B is an increasing function of $-P^2$ a pole develops when $B(-P^2, m^2) = B(\mu^2, m^2) + 16\pi^2/\lambda$, which corresponds to spacelike P^2 for any value of λ. As $\lambda \to 0$ the pole approaches infinity exponentially.

We therefore do not expect the theory to rid itself of the tachyon by spontaneous symmetry breaking. Instead if this theory is at all physically sensible the tachyon must disappear when one calculates to higher order in $1/N$. This can happen if $\beta(g)$ develops a zero at some finite g in higher orders.

If we construct the σ potential for this theory we find that

$$V(\sigma) = -\tfrac{1}{2}\sigma^2 + \frac{\lambda}{32\pi}\sigma^2 \left[\ln\left(\frac{\sigma}{\sigma_0}\right)^2 - 3\right] . \qquad \text{(A8)}$$

Although the sign of the tree graph contribution has opposite sign compared with σ potential for the fermion models, the over-all shape of the potentials is similar. $V(\sigma)$ again has the form depicted in Fig. 5. Thus at first sight it appears that the normal vacuum is unstable and that spontaneous symmetry breaking occurs. This is not what is expected on the basis of the arguments presented in Sec. VI for this infrared free theory. We note, however, that the position of the minimum (σ_M) and thereby the scalar mass generated are given by

$$M\phi^2 = g|\sigma_M| = g\sigma_0 \exp\left(1 + \frac{16\pi^2}{\lambda}\right) , \qquad \text{(A9)}$$

which, as expected, blows up when the coupling is turned off.

This apparent contradiction is removed when we construct, according to the methods of Sec. V, the renormalized potential of $\Sigma_c = \langle 0 | g\vec{\phi} \cdot \vec{\phi} | 0 \rangle$. As before $\mathcal{V}(\Sigma_c)$ can be constructed in terms of $V(\sigma)$, and one derives

$$\Sigma_c = \sigma + Z^{-1} J , \qquad \text{(A10)}$$

$$\mathcal{V}(\Sigma_c) = V(\sigma) + \tfrac{1}{2} Z^{-1} J^2 , \qquad \text{(A11)}$$

$$J = \frac{\partial \mathcal{V}(\Sigma_c)}{\partial \Sigma_c} = \frac{\partial V(\sigma)}{\partial \sigma} . \qquad \text{(A12)}$$

In this case, however, the coupling constant renormalization Z^{-1} diverges as the cutoff Λ is taken to infinity. In fact Z^{-1} blows up at a finite Λ:

$$Z^{-1}=\frac{1}{1-(\lambda/8\pi^2)\ln(\Lambda/\mu)}\,. \tag{A13}$$

Thus our methods yield a meaningless expression for $\mathcal{V}(\Sigma_c)$, and the apparent symmetry breaking in $V(\sigma)$ has no physical relevance.

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[1]J. Goldstone, Nuovo Cimento 19, 154 (1961).

[2]J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1961).

[3]Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).

[4]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8)*, edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367.

[5]P. Higgs, Phys. Lett. 12, 132 (1964); F. Englert and R. Brout, Phys. Rev. Lett. 13, 321 (1964); G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, *ibid.* 13, 585 (1964); P. Higgs, Phys. Rev. 145, 1156 (1966); T. W. B. Kibble, *ibid.* 155, 1554 (1967); G. 't Hooft, Nucl. Phys. B35, 167 (1971).

[6]R. Jackiw and K. Johnson, Phys. Rev. D 8, 2386 (1973); J. Cornwall and R. Norton, *ibid.* 8, 3338 (1973).

[7]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); Phys. Rev. D 8, 3633 (1973); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).

[8]G. 't Hooft (unpublished).

[9]S. Weinberg, Phys. Rev. Lett. 31, 494 (1973).

[10]G. 't Hooft has recently considered large N expansions for gauge theories. There, however, the leading approximation is extremely complicated, and some progress has been made. [Nucl. Phys. B72, 461 (1974)].

[11]G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).

[12]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

[13]K. Wilson, Phys. Rev. D 7, 2911 (1973).

[14]When $N=1$ this is, of course, the soluble Thirring model. For any $N>1$, however, the model is neither trivial nor exactly soluble.

[15]J. Glimm, Commun. Math. Phys. 5, 343 (1967); 6, 120 (1967); R. Schrader, Ann. Phys. (N.Y.) 70, 457 (1972); 70, 612 (1972).

[16]Incidentally, these models provide concrete examples of asymptotically free theories which, as far as one can tell, are sensible in all regards.

[17]S. Coleman, lectures given at the 1973 International Summer School of Physics "Ettore Majorana," Erice, Italy (unpublished).

[18]S. Coleman, Commun. Math. Phys. 31, 259 (1973).

[19]Note that in a non-asymptotically-free theory, Z^{-1} either diverges or approaches a nonvanishing constant, in which case the composite potential would be either meaningless or more complicated. We thank S. Coleman and J. Schoenfeld for pointing out a mistake in an earlier version of this section.

[20]S. Coleman and D. J. Gross, Phys. Rev. Lett. 31, 851 (1973).

[21]Such infrared stable theories include, of course, Abelian massless gauge theories such as quantum electrodynamics. Our arguments do not apply, however, to situations where one demands that the physical coupling constant be fixed at a nontrivial fixed point of the renormalization group, as in Baker-Johnson-Willey-Adler quantum electrodynamics.

[22]J. Schwinger, Phys. Rev. 125, 397 (1962); 128, 2425 (1962).

[23]In this respect the situation is opposite to that of ϕ^4 theory in $4-\epsilon$ dimensions, which has been more extensively studied.

[24]At $\epsilon\to 2$, i.e., $d\to 4$, the theory resembles a Yukawa theory with a coupling constant of order $4-d$ (see Ref. 13).

[25]For the trace over γ matrices, we choose the convention $\mathrm{Tr}I=2$ for any dimension away from 2.

[26]We have chosen in (8.2) the renormalization point to be at $k^2=\mu^2$, for simplification. This is not necessary. Using the renormalization group another choice would lead to the same physics after appropriate redefinitions of μ and g.

[27]The connection of our conclusions with the recent work of Dashen and Frishman on a scale-invariant Thirring model with isospin is very unclear, since their model should, by Fierz transformation, be essentially equivalent with ours. See R. Dashen and Y. Frishman, Phys. Lett. 46B, 439 (1973); P. K. Mitter and P. H. Weisz, Phys. Rev. D 8, 4410 (1973).

[28]The lowest-order contribution to A (of order λ) is superficially linearly divergent, however, symmetric integration renders it equal to a finite constant, which can be absorbed in a finite renormalization of Σ.

[29]A. Jaffe, Commun. Math. Phys. 1, 127 (1965).

[30]These include the Thirring model and two-dimensional quantum electrodynamics (Ref. 22).

[31]S. Coleman (private communication).