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Quantization of a coupled Fermi field and Robertson-Walker metric

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The combined Dirac field and Robertson-Walker metric system is quantized in a Heisenberg picture. One of the Einstein equations is implemented as a constraint on the state vectors and leads to a spectrum for the allowed values of the mass of the spinor field.

I. INTRODUCTION

In recent years, physics has seen many advances in understanding the problem of quantizing the gravitational field.[1,2] It is no great wonder that this subject should command so much attention, considering the intellectual and technical challenges it presents. In particular, the approach known as "quantum cosmology" or "quantum models" has been the object of heightened interest following the original work of DeWitt[3] and Misner.[4] This approach involves freezing out a large number (frequently almost all) of degrees of freedom and quantizing the remaining ones. The main advantage of this approach is that problems which are peculiar to quantum gravity can be discussed in at least partial isolation from those which beset *any* quantum field theory. Specifically, questions concerning gravitational collapse and the quantum effects therein can be explored in depth.

Most work has been concerned with quantizing the gravitational field alone. Attempts have also been made[5] to include matter in the form of classical dust or classical perfect fluids. Some recent work[6] considered the problem of a scalar matter field coupled to a Robertson-Walker metric, in which the *combined* system was quantized in the true canonical sense. This work was one of our motivations for considering a fermion matter field in the same background. One can anticipate that the change in field statistics will lead to a radically different technical structure.

This work is also of interest in relation to that of Davis and Ray[7] who discuss the classical, unquantized, massless Einstein-Dirac equations in a static, plane-symmetric spacetime, in which a zero energy-momentum tensor is obtained.

The plan of the paper is as follows. In Sec. II the Einstein equations for the *vierbein* field $L_{\mu a}$ and the Dirac equations for the spinor field ψ are obtained by variation of the Lagrangian

$$\mathfrak{L}_{\text{Einstein}} + \mathfrak{L}_{\text{matter}} = (\det L)\left[\frac{R(L)}{\kappa^2} + \tfrac{1}{2} i L^{\mu a}\bar{\psi}\gamma_a \overleftrightarrow{\nabla}_\mu \psi - \frac{mc}{\hbar}\bar{\psi}\psi\right]. \qquad (1.1)$$

It is shown that for an assumed form for the spinor field $\psi = \psi(t)$ coupled to a Robertson-Walker metric, only the $K=0$ metric

$$ds^2 = N^2(t)\,dt^2 - R^2(t)(dx^2 + dy^2 + dz^2) \qquad (1.2)$$

is compatible with the Einstein and Dirac equations. The only nontrivial Einstein equation which involves the spinor field is the G_{00} equation, which in the present case reduces to

$$m\bar{\psi}\psi = \text{constant} \times \left(\frac{\dot{R}}{R}\right)^2. \qquad (1.3)$$

This constraint has to be imposed on the easily obtained solutions of the Dirac equations.

In Sec. III the classical model of Sec. II is quantized in the Heisenberg picture, and the (assumed) equal-time canonical anticommutation relations (hereafter referred to as CAR's) are found to be consistent with the operator Dirac equations of motion for the spinor field ψ. The G_{00} constraint equation is incorporated in the quantum theory by imposing it as a constraint which projects out the physically allowed state vectors from the (finite-dimensional) Hilbert space which carries the CAR representation. The major result of this is that only certain values of the parameter m in the Lagrangian are found to be consistent with the quantum scheme, thus leading to a "mass" spectrum.

In Sec. IV the effect of adding a cosmological term $\Lambda g_{\mu\nu}$, where Λ is the cosmological constant and $g_{\mu\nu}$ is the metric, to Einstein's equations is investigated. This is equivalent to adding a term $\Lambda(\det L)$, where $L_{\mu a}$ is the *vierbein* basis (see Sec. II for a full explanation) to the Lagrangian (1.1). The classical solutions to this model are presented, and it is found that the presence of this extra term in no great way affects the results of Sec. II (with $\Lambda = 0$).

The work of Trautman[8] motivated us to consider the addition of a nonzero torsion term to the Lagrangian (1.1), in the hope that, as previously suggested for the classical case by Trautman,[8] gravitational collapse of the model may be avoided after quantization. The effect of this term on the

Dirac equations and Einstein's equations, which appear in the form of (differential) constraints on the spinor fields, is investigated, although it was not possible to solve these equations explicitly.

The final section puts forward an interpretation of these results, and discusses their significance. In addition, we indicate further areas which this work suggests to be of possible interest.

II. THE CLASSICAL MODEL

We consider the coupled Einstein and Dirac fields, using a *vierbein* basis $L_{\mu a}$, satisfying

$$L_\mu{}^a L_{\nu a} = g_{\mu\nu}\,, \qquad L_{\mu a} L^\mu{}_b = \eta_{ab} = (1,-1,-1,-1)\,, \tag{2.1}$$

with $\mu, \nu, a, b = 0, 1, 2, 3$, where $g_{\mu\nu}$ is the metric. The generally covariant and SL(2, C) gauge-invariant Einstein + Dirac Lagrangian is

$$\mathcal{L} = (\det L)\left[\frac{R(L)}{\kappa^2} + \tfrac{1}{2} i L^{\mu a}\bar{\psi}\gamma_a \overleftrightarrow{\nabla}_\mu \psi - \frac{mc}{\hbar}\bar{\psi}\psi\right], \tag{2.2}$$

where $A\overleftrightarrow{\nabla}B = A(\nabla B) - (\nabla A)B$, ψ is the fermion field, $\bar{\psi} = \psi^\dagger\gamma_0$, $\nabla_\mu\psi = (\partial_\mu + iB_\mu)\psi$, $\nabla_\mu\bar{\psi} = \partial_\mu\bar{\psi} - i\bar{\psi}B_\mu$, and the spinor connection B_μ is defined by

$$B_\mu = \tfrac{1}{4} L_{\beta c} L^\beta{}_{a|\mu}\sigma^{ac} = \tfrac{1}{2} B_{\mu ac}\sigma^{ac}\,, \tag{2.3}$$

where

$$L^\beta{}_{a|\mu} = L^\beta{}_{a,\mu} + \Gamma^\beta{}_{\mu\alpha} L^\alpha{}_a \tag{2.4}$$

and $\Gamma^\beta{}_{\mu\alpha}$ is the affine connection in some coordinate basis. For the sake of convenience we will define the rest of our notation now. $R(L)$ is the Ricci scalar given by $R = R^{\mu\nu}g_{\mu\nu}$, where

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\alpha,\nu} - \Gamma^\alpha{}_{\mu\nu,\alpha} + \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha} - \Gamma^\alpha{}_{\mu\nu}\Gamma^\beta{}_{\alpha\beta}\,. \tag{2.5}$$

κ^2 is defined by $\kappa^2 = 16\pi G/c$, where G is the Newtonian constant and c is the speed of light.

The Pauli representation of γ matrices is used,

$$\{\gamma_a, \gamma_b\} = \eta_{ab}\,, \tag{2.6}$$

where

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \tag{2.7}$$

and the σ_k are the usual Pauli spin matrices. γ_5 is defined by

$$\gamma_5 = \epsilon^{abcd}\gamma_a\gamma_b\gamma_c\gamma_d = \gamma_0\gamma_1\gamma_2\gamma_3 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{2.8}$$

where

$$\epsilon^{0123} = -\epsilon_{0123} = +1\,.$$

We also have

$$\{\gamma_a, \sigma_{cd}\} = -2i\,\epsilon_{cda}{}^b\gamma_b\gamma_5\,,$$

where σ_{cd} is defined by

$$[\gamma_c, \gamma_d] = -2i\sigma_{cd}\,. \tag{2.9}$$

The field equations obtained by varying the Lagrangian (2.2) with respect to $L^{\nu b}$ are

$$\frac{G_{\mu\nu}}{\kappa^2} = -\tfrac{1}{4} i L_{(\nu}{}^b\bar{\psi}\gamma_b\overleftrightarrow{\nabla}_{\mu)}\psi\,, \tag{2.10}$$

where

$$P_{(ab)} = \tfrac{1}{2}(P_{ab} + P_{ba})\,.$$

We have used the fact, following from the spinor equations of motion, that

$$\mathcal{L}_{\text{matter}} = 0\,. \tag{2.11}$$

Since the basic idea of quantum-model calculations is to remove all but a finite number of degrees of freedom, our first step is to assume a specific, simple form for the metric/*vierbein* and then check that classically there do exist modes of the spinor field compatible with this choice.

The geometries of interest (as in Ref. 6) are those provided by the usual homogeneous and isotropic Robertson-Walker metrics

$$ds^2 = N^2(t)\,dt^2 - R^2(t)\,\mathcal{S}_{ij}\,dx^i\,dx^j\,, \tag{2.12}$$

where $\mathcal{S}_{ij}$ is the metric for a three-space of constant curvature K. The case $K = +1$ is that of a three-sphere, while $K = 0$ and $K = -1$ correspond to the flat and hyperbolic cases, respectively. We note here the fact (for later use) that this three-space is a Lie group space and therefore the appropriate techniques of differential geometry can be used upon it.

We have

$$g_{\mu\nu} = \begin{pmatrix} N^2(t) & 0 \\ 0 & -R^2(t)\,\mathcal{S}_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} \dfrac{1}{N^2(t)} & 0 \\ 0 & -\dfrac{\mathcal{S}^{ij}}{R^2(t)} \end{pmatrix}. \tag{2.13}$$

A convenient choice for a corresponding *vierbein* field is

$$L_{\mu a}=\begin{pmatrix} N(t) & 0 \\ 0 & R(t)\,\mathcal{S}_{i(j)}\end{pmatrix},$$

$$L^{\mu a}=\begin{pmatrix} \dfrac{1}{N(t)} & 0 \\ 0 & -\dfrac{\mathcal{S}^{i(j)}}{R(t)}\end{pmatrix}, \tag{2.14}$$

where $\mathcal{S}_{i(j)}$ is the "*dreibein*" field for $\mathcal{S}_{ij}$ and δ_{ij}, satisfying

$$\mathcal{S}_{i(j)}\,\mathcal{S}_k{}^{(j)}=\mathcal{S}_{ik}$$

and

$$\mathcal{S}_{j(i)}\,\mathcal{S}^j{}_{(k)}=\delta_{ik}, \quad i,j,k=1,2,3\ . \tag{2.15}$$

Greek indices are raised and lowered by $g_{\mu\nu}$, whereas Latin indices are raised and lowered by η_{ab}.

The "G_{00} equation" becomes (with this choice of *vierbein* field)

$$\left(-3\frac{\dot R^2}{R^2}+\frac{N^2}{2R^2}K\right)=-\tfrac{1}{4}i\kappa^2 N(\bar\psi\gamma_0\dot\psi-\dot{\bar\psi}\gamma_0\psi)\ , \tag{2.16}$$

where

$$\dot R=\frac{dR}{dt}\,,\quad \dot\psi=\frac{d\psi}{dt}\,,\quad \dot{\bar\psi}=\frac{d\bar\psi}{dt}\ .$$

This is sufficient to imply that the choice of a homogeneous matter field is natural. That is, we take $\psi=\psi(t)$, $\bar\psi=\bar\psi(t)$ (with no dependence on the spatial coordinates) in order to maintain the Robertson-Walker form of the metric. It remains to be shown that our choice is consistent with the remaining equations of motion.

The "G_{ij} equation" becomes

$$\frac{G_{ij}}{\kappa^2}=-\tfrac{1}{16}iR\bar\psi[\mathcal{S}_i{}^{(k)}(\mathcal{S}_{m(p)}\mathcal{S}^m{}_{(l),j}+\mathcal{S}_{m(p)}\mathcal{S}^n{}_{(l)}\Gamma^m{}_{jn})+\mathcal{S}_j{}^{(k)}(\mathcal{S}_{m(p)}\mathcal{S}^m{}_{(l),i}+\mathcal{S}_{m(p)}\mathcal{S}^n{}_{(l)}\Gamma^m{}_{in})]\epsilon_k{}^{pl0}\gamma_0\gamma_5\psi\ , \tag{2.17}$$

while the $G_{0i}=0$ "constraint" equation becomes

$$\frac{G_{0i}}{\kappa^2}=0=-\tfrac{1}{8}i\big[R\mathcal{S}_i{}^{(j)}(\bar\psi\gamma_j\dot\psi-\dot{\bar\psi}\gamma_j\psi)+\tfrac{1}{2}N\bar\psi(\mathcal{S}_{m(j)}\mathcal{S}^m{}_{(k),i}+\mathcal{S}_{m(j)}\mathcal{S}^n{}_{(k)}\Gamma^m{}_{in})\epsilon_0{}^{jkl}\gamma_l\gamma_5\psi\big]\ . \tag{2.18}$$

Now the quantity

$$(\mathcal{S}_{m(p)}\mathcal{S}^m{}_{(l),i}+\mathcal{S}_{m(p)}\mathcal{S}^n{}_{(l)}\Gamma^m{}_{in})\epsilon_0{}^{pls}\gamma_s\gamma_5 \tag{2.19}$$

occurs in both the G_{0i} and G_{ij} equations. $\Gamma^m{}_{in}$ is given by

$$\Gamma^m{}_{in}=\tfrac{1}{2}(\Lambda^m{}_{in}+\Lambda^m{}_{ni})\ , \tag{2.20}$$

where $\Lambda^m{}_{in}=\mathcal{S}^{m(k)}\mathcal{S}_{n(k),i}$.

Also,

$$\mathcal{S}_{n(j),i}-\mathcal{S}_{i(j),n}=C_{pqj}\mathcal{S}_n{}^{(p)}\mathcal{S}_i{}^{(q)}\ , \tag{2.21}$$

where the group structure constants C_{ijk} are given (for $K>0$) by

$$C_{ijk}=(\tfrac{1}{2})^{1/2}\epsilon_{ijk} \tag{2.22}$$

and for $K=0$ by

$$C_{ijk}=0\ . \tag{2.23}$$

(We will not discuss the $K<0$ case since, like $K>0$, it is ultimately inconsistent.) The above expression then reduces (when $K>0$) to

$$-\frac{1}{2\sqrt2}\epsilon_{ljp}\epsilon_0{}^{pls}\mathcal{S}_i{}^{(j)}\gamma_s\gamma_5\ . \tag{2.24}$$

The field equations then become [with this form substituted in (2.18) and (2.17)]

$$G_{0i}=0=-\tfrac{1}{8}iN\mathcal{S}_i{}^{(j)}\times\left[\frac{R}{N}(\bar\psi\gamma_j\dot\psi-\dot{\bar\psi}\gamma_j\psi)-\frac{1}{2\sqrt2}\bar\psi\gamma_j\gamma_5\psi\right], \tag{2.25}$$

$$\frac{G_{ij}}{\kappa^2}=\frac{\mathcal{S}_{ij}}{\kappa^2}\left[\frac{2R}{N}\frac{d}{dt}\left(\frac{\dot R}{N}\right)+\left(\frac{\dot R}{N}\right)^2-\frac{K}{6}\right]=-\frac{i}{8\sqrt2}\mathcal{S}_{ij}R\bar\psi\gamma_0\gamma_5\psi\ , \tag{2.26}$$

whereas the Dirac equations obtained by varying $\mathcal{L}_{\text{matter}}$ with respect to $\psi,\bar\psi$ become

$$\frac{R}{N}\dot\psi=-\frac{3}{4\sqrt2}\gamma_5\psi-\frac{imc}{\hbar}R\gamma_0\psi-\tfrac{3}{2}\frac{\dot R}{N}\psi\ ,$$
$$\frac{R}{N}\dot{\bar\psi}=-\frac{3}{4\sqrt2}\bar\psi\gamma_5+\frac{imc}{\hbar}R\bar\psi\gamma_0-\tfrac{3}{2}\frac{\dot R}{N}\bar\psi\ . \tag{2.27}$$

These two Dirac equations tell us that

$$\frac{R}{N}(\bar\psi\gamma_j\dot\psi-\dot{\bar\psi}\gamma_j\psi)+\frac{3}{2\sqrt2}\bar\psi\gamma_j\gamma_5\psi=0\ , \tag{2.28}$$

which, to be compatible with the $G_{0i}=0$ equation above, requires that

$$\bar\psi\gamma_j\gamma_5\psi=0\ , \tag{2.29}$$

which is essentially the spin part of the interaction. A check of the Bianchi identities, taking into account the Dirac equations, also shows the $G_{0i}=0$

equation (2.25) to be correct. If we let

$$\psi=\begin{pmatrix} f \\ g \end{pmatrix},$$

where f, g are each 2-component spinors, then

$$\bar{\psi}\gamma_j\gamma_5\psi=0 \Rightarrow f^\dagger\sigma_j f+g^\dagger\sigma_j g=0\,, \tag{2.30}$$

which must imply that

$$f=g=0\,, \tag{2.31}$$

from which our chosen metric may be deduced to be incompatible with the field equations. It may be that this incompatibility is closely linked to our choice of a homogeneous matter field, but we suggest that the inclusion of some spatial dependence of the matter fields would be unsuitable for our choice of background.

However, this does work in flat space ($K=0$), since then

$$g_{ij}=\delta_{ij}, \quad C_{ijk}=0 \tag{2.32}$$

and the expression (2.19) reduces to zero. The Dirac equations become

$$\begin{aligned} \frac{R}{N}\dot{\psi} &= -\frac{imc}{\hbar}R\gamma_0\psi-\tfrac{3}{2}\frac{\dot{R}}{N}\psi\,, \\ \frac{R}{N}\dot{\bar{\psi}} &= \frac{imc}{\hbar}R\bar{\psi}\gamma_0-\tfrac{3}{2}\frac{\dot{R}}{N}\bar{\psi}\,, \end{aligned} \tag{2.33}$$

whereas the *vierbein* equations become

$$G_{0i}=0=-\tfrac{1}{8}iR(\bar{\psi}\gamma_i\dot{\psi}-\dot{\bar{\psi}}\gamma_i\psi)\,, \tag{2.34}$$

which is now *automatically satisfied* by virtue of the the Dirac equations,

$$G_{ij}=0=\frac{2R}{N}\frac{d}{dt}\left(\frac{\dot{R}}{N}\right)+\left(\frac{\dot{R}}{N}\right)^2 \tag{2.35}$$

and

$$\begin{aligned} \frac{G_{00}}{\kappa^2} &= -\frac{3\dot{R}^2}{\kappa^2R^2} \\ &= -\frac{iN}{4}(\bar{\psi}\gamma_0\dot{\psi}-\dot{\bar{\psi}}\gamma_0\psi)\,. \end{aligned} \tag{2.36}$$

The equation (2.36) can be reduced, using the Dirac equations, to

$$\frac{mc}{\hbar}\bar{\psi}\psi=\frac{6\dot{R}^2}{\kappa^2N^2R^2}\,. \tag{2.37}$$

The above equations are of course underdetermined, and in order for them to be solved classically a choice of time must be made. This can be done in any of the usual ways within the context of these second-order equations. As an example, consider the case when the time variable is provided implicitly by the choice $N=1$ (cosmic time). Then equations (2.35) and (2.33) can be solved readily to give

$$R=R_0(a+t)^{2/3}\,, \tag{2.38}$$

$$\psi=\frac{1}{(a+t)}\begin{pmatrix} \exp\left(\dfrac{-imct}{\hbar}\right)d \\ \exp\left(\dfrac{imct}{\hbar}\right)e \end{pmatrix}, \tag{2.39}$$

$$\bar{\psi}=\frac{1}{(a+t)}\left(\exp\left(\frac{imct}{\hbar}\right)d^\dagger,\ -\exp\left(\frac{-imct}{\hbar}\right)e^\dagger\right), \tag{2.40}$$

where a, R_0 are constants and d, e are constant two-component spinors. This shows clearly that the system does experience gravitational collapse at a constant value of t, $t=-a$, at which ψ and $\bar{\psi}$ diverge, and also that the radius parameter has no maximum value.

The G_{00} equation tells us that the spinors d and e must satisfy the simple constraint

$$\frac{mc}{\hbar}(d^\dagger d-e^\dagger e)=\frac{8}{3\kappa^2}\,. \tag{2.41}$$

The spinor fields can be conveniently redefined by

$$\chi=(a+t)\psi, \quad \bar{\chi}=(a+t)\bar{\psi} \tag{2.42}$$

so that $\chi, \bar{\chi}$ satisfy the free-field Dirac equations

$$\begin{aligned} &\dot{\chi}+\frac{imc}{\hbar}\gamma_0\chi=0\,, \\ &\dot{\bar{\chi}}-\frac{imc}{\hbar}\bar{\chi}\gamma_0=0\,, \end{aligned} \tag{2.43}$$

but are subject, however, to the constraint equation

$$\frac{mc}{\hbar}\bar{\chi}\chi=\frac{8}{3\kappa^2}\,. \tag{2.44}$$

The parameter t can be eliminated from Eqs. (2.38)–(2.40) (choosing $a=0$ for convenience) to give

$$\psi=\left(\frac{R_0}{R}\right)^{3/2}\begin{pmatrix} \exp\left[-\dfrac{imc}{\hbar}\left(\dfrac{R}{R_0}\right)^{3/2}\right]d \\ \exp\left[\dfrac{imc}{\hbar}\left(\dfrac{R}{R_0}\right)^{3/2}\right]e \end{pmatrix} \tag{2.45}$$

and

$$\bar{\psi}=\left(\frac{R_0}{R}\right)^{3/2}\left(\exp\left[\frac{imc}{\hbar}\left(\frac{R}{R_0}\right)^{3/2}\right]d^\dagger\,,\ -\exp\left[\frac{-imc}{\hbar}\left(\frac{R}{R_0}\right)^{3/2}\right]e^\dagger\right), \tag{2.46}$$

or equivalently

$$R = R_0\left(\frac{d^\dagger d - e^\dagger e}{\bar\psi\psi}\right)^{1/3} = \frac{R_0}{\bar\psi\psi}\,\frac{8}{3\kappa^2}\,\frac{\hbar}{mc}\ . \qquad (2.47)$$

These equations describe the intrinsic dynamics of the system expressed as a correlation between ψ, $\bar\psi$, and R and are of course independent of the choice of time. Other suitable choices of time which yield equivalent solutions are $R(t) = t$, $R(t) = N(t)$.

For the purposes of conventional canonical quantization it is necessary to construct an action principle from which the equations of motion can be derived. Actually this does not play a dominant role in the quantum scheme used in Sec. III, but for the sake of completeness let us state that the above system of equations, (2.33), (2.35), and (2.36), can be derived from the Lagrangian

$$\mathcal{L}(t) = 6\dot R^2\,\frac{R}{N\kappa^2} + NR^3\left[\frac{i}{2N}(\bar\psi\gamma_0\dot\psi - \dot{\bar\psi}\gamma_0\psi) - \frac{mc}{\hbar}\bar\psi\psi\right]\ , \qquad (2.48)$$

in which R, N, ψ, and $\bar\psi$ are all to be varied independently. This Lagrangian is simply the usual one for general relativity (minus a time divergence) with the matter Lagrangian added on and the form of the metric given by (1.2) substituted. We have checked explicitly that the resulting equations are precisely Eqs. (2.33), (2.35), and (2.36)—a step that is necessary, as one cannot always guarantee that a reduced action principle of this type will reproduce the original equations of motion in which the reduced field is substituted.

This does work, however, in the flat-space case since the $G_{0i} = 0$ equation is given identically by the Dirac equations, whereas in curved space ($K \neq 0$), to obtain the $G_{0i} = 0$ equation it would be necessary to choose a metric of the form

$$g_{\mu\nu} = \begin{pmatrix} N^2(t) - N^iN_i & N_j \\ N_i & {}^3g_{ij}\end{pmatrix}\ ,$$

$$\mu, \nu = 0, 1, 2, 3, \quad i, j = 1, 2, 3 \qquad (2.49)$$

and the vary $\mathcal{L}(K \neq 0)$ with respect to N_i. This implies that a different choice of metric, possibly one of the Bianchi types, with $N_i \neq 0$, may yield positive results in the curved-space case. This problem is currently under investigation.

III. THE QUANTIZED MODEL

In Ref. 6 the combined Robertson-Walker–scalar-field system was quantized using a strict canonical approach. In the present case the first-order nature of the Dirac equations and the specific constraint form of the G_{00} equation result in this no longer being the natural approach.

Instead, in this section the flat-space model is quantized in a Heisenberg picture. The classical equations of motion are treated as operator equations [by inserting carets: $\psi_\alpha(t) \to \hat\psi_\alpha(t)$, etc.] and their consistency with the (assumed) equal-time CAR's is checked.

We shall consider the case

$$N = 1, \quad R = R_0 t^{2/3} \qquad (3.1)$$

throughout. The other choices of time mentioned in Sec. II can be dealt with, using similar techniques.

Since

$$\frac{\partial\mathcal{L}}{\partial\dot{\hat\psi}_\alpha(t)} = \tfrac{1}{2}iR^3(t)\,\hat\psi^{\dagger\,\alpha}(t)\ , \qquad (3.2)$$

where $\mathcal{L}$ is the flat-space Lagrangian given by (2.48), we expect that $R^3(t)\hat\psi^{\dagger\,\alpha}(t)$ is conjugate to $\hat\psi_\beta(t)$. The corresponding equal time CAR's are

$$\{\hat\psi^{\dagger\,\alpha}(t), \hat\psi_\beta(t)\} = \frac{\hbar\delta^\alpha{}_\beta}{R^3(t)} \qquad (3.3)$$

and

$$\{\hat\psi_\alpha(t), \hat\psi_\beta(t)\} = \{\hat\psi^{\dagger\,\alpha}(t), \hat\psi^{\dagger\,\beta}(t)\} = 0\ , \qquad (3.4)$$

bearing in mind that, from Eq. (2.39), "equal-time" means "equal-$R(t)$."

In terms of the $\hat\chi$ fields, where

$$\hat\chi_\alpha(t) = t\hat\psi_\alpha(t)\ , \quad \hat\chi^{\dagger\,\beta}(t) = t\hat\psi^{\dagger\,\beta}(t)\ , \qquad (3.5)$$

the CAR's become

$$\{\hat\chi^{\dagger\,\alpha}(t), \hat\chi_\beta(t)\} = \frac{\hbar}{R_0^{\,3}}\delta^\alpha{}_\beta\ , \qquad (3.6)$$

$$\{\hat\chi_\alpha(t), \hat\chi_\beta(t)\} = \{\hat\chi^{\dagger\,\alpha}(t), \hat\chi^{\dagger\,\beta}(t)\} = 0\ . \qquad (3.7)$$

We shall work with the redefined fields $\hat\chi_\alpha(t)$, $\hat\chi^{\dagger\,\beta}(t)$ wherever it is convenient to do so.

If the $\hat\chi_\alpha(t)$, $\hat\chi^{\dagger\,\beta}(t)$ fields are expanded according to

$$\hat\chi_\alpha(t) = \sum_{\beta=1}^{4}\hat a_\beta(u_\beta)_\alpha\ , \quad \hat\chi^{\dagger\,\beta}(t) = \sum_{\alpha=1}^{4}(u_\alpha^\dagger)^\beta\hat a_\alpha^\dagger\ , \qquad (3.8)$$

where

$$u_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad u_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad u_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix},\quad u_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \qquad (3.9)$$

then it is found that the operators $\hat{a}_\alpha$, $\hat{a}_\alpha^\dagger$ satisfy

$$\{\hat{a}_\beta, \hat{a}_\alpha^\dagger\} = \frac{\hbar}{R_0^3}\,\delta_{\alpha\beta} \tag{3.10}$$

and

$$\{\hat{a}_\alpha, \hat{a}_\beta\} = \{\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger\} = 0 \ . \tag{3.11}$$

There is, by virtue of the Jordan-Wigner theorem, a unique (up to unitary equivalence) representation of the CAR's on a *finite-dimensional* (actually of dimension $2^4 = 16$) vector space $\mathcal{H}$. This is in marked contrast to the infinite-dimensional Hilbert space of the boson field in Ref. 6.

Now the G_{00} constraint equation is

$$\frac{mc}{\hbar}\,\hat{\bar{\chi}}^\alpha \hat{\chi}_\alpha = \frac{8}{3\kappa^2} \ , \tag{3.12}$$

which becomes, with the use of (3.8),

$$\frac{mc}{\hbar}\,(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_3^\dagger \hat{a}_3 - \hat{a}_4^\dagger \hat{a}_4) = \frac{8}{3\kappa^2} \ . \tag{3.13}$$

There are various ways in which this equation might be understood. The most natural one in the present context is to regard it as a constraint equation on $\mathcal{H}$ in the form

$$\left[(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_3^\dagger \hat{a}_3 - \hat{a}_4^\dagger \hat{a}_4) - \frac{8\hbar}{3\,mc\,\kappa^2}\right] |\ \rangle = 0 \tag{3.14}$$

which projects out those states $|\ \rangle$ in $\mathcal{H}$ which are physically allowed. Such a quantum scheme is not a truly canonical one (in which constraints must be solved *before* quantization), but is rather more like the "superspace" quantization of the gravitational field.[1]

We have the relations

$$\hat{a}_\alpha^\dagger \hat{a}_\alpha + \hat{a}_\alpha \hat{a}_\alpha^\dagger = \frac{\hbar}{R_0^3} \ , \tag{3.15}$$

$$(\hat{a}_\alpha)^2 = (\hat{a}_\alpha^\dagger)^2 = 0 \ . \tag{3.16}$$

It follows, using (3.15) and (3.16), that

$$(\hat{a}_\alpha^\dagger \hat{a}_\alpha)^2 = \frac{\hbar}{R_0^3}\,(\hat{a}_\alpha^\dagger \hat{a}_\alpha) \tag{3.17}$$

so that the operator $\hat{a}_\alpha^\dagger \hat{a}_\alpha$ (no sum on α) has eigenvalues 0, $\hbar/R_0^3$. Therefore the operator

$$\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_3^\dagger \hat{a}_3 - \hat{a}_4^\dagger \hat{a}_4$$

has eigenvalues $(-2, -1, 0, 1, 2)\ \hbar/R_0^3$ and in effect Eq. (3.14) can be regarded as an *eigenvalue equation* for $1/m$. Thus our quantum scheme leads to the mass spectrum

$$m = \frac{8R_0^3}{3c\,\kappa^2 n} \ , \quad n = -2, -1, 0, 1, 2 \tag{3.18}$$

which relates the allowed mass m to the "rate-of-growth" parameter R_0. (The parameter m does indeed have the units of mass, since R_0^3 has units L^3T^{-2}, κ^2 has units $M^{-1}L^2T^{-1}$, and c has units LT^{-1}.)

Since the operator $\hat{a}_\alpha^\dagger \hat{a}_\alpha$, $\alpha = 1, 2, 3, 4$, has two spin states (up or down), it follows that the operator

$$\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_3^\dagger \hat{a}_3 - \hat{a}_4^\dagger \hat{a}_4$$

has $2^4 = 16$ spin states as mentioned above. It can easily be seen, therefore, that the eigenvalues ± 2 are 1-fold degenerate, and the eigenvalues ± 1 are 4-fold degenerate. Thus the dimensions of the final space of physically allowed state vectors are 1 or 4. The eigenvalue zero (which gives $m = \infty$ and so must be discarded, but is stated here for completeness) is 6-fold degenerate.

There are several points in this discussion which require further comment. First, although it is quite clear that the emergence of a mass spectrum is certainly a quantum effect (for example, we would have obtained an *infinite* series had we considered a boson field), and does give credence to our quantum scheme, it is interesting to note that Planck's constant itself does not appear in the final answer.

Second, we must attach an interpretation to the presence of negative (and infinite) values of mass in its spectrum. Clearly on physical grounds (and in order to have a positive-definite energy) one would be inclined to choose the positive, noninfinite values of mass.

Third, the rate of growth parameter R_0 can be related to the Hubble constant H_0 at some reference time t_0 by

$$H_0 = \tfrac{2}{3}\left(\frac{R_0}{R(t_0)}\right)^{3/2} \ ,$$

in terms of which the mass spectrum becomes

$$m = \frac{3}{8n\pi G} R(t_0)^3 H_0^2 \ , \quad n = 1, 2 \ . \tag{3.19}$$

It is interesting to note that, if in the mass spectrum (3.19) we take $R(t_0)$ to be the radius of the known universe at the present time and insert the correct values for κ^2 and c, we see that

$$m \approx 10^{56} \text{ grams} \ ,$$

which is indeed the mass of the known universe. It is well known that the mass of the universe, $R(t_0)$, and H are related in an order-of-magnitude sense, but it is intriguing to see the relation emerge in the above way.

The Heisenberg-picture time development of the system should be expressible in the form $[H, \hat{\chi}_\alpha(t)] = -i\hbar\dot{\chi}_\alpha(t)$. Indeed, one can check that the appropriate Hamiltonian H is given by

$$H=\frac{mc}{\hbar}R^3\hat{\bar{\psi}}^{\alpha}\hat{\psi}_{\alpha}$$
$$=\frac{mc}{\hbar}R_0{}^3\hat{\bar{\chi}}^{\alpha}\hat{\chi}_{\alpha}\ . \qquad (3.20)$$

That is, the $\hat{\chi}$ fields develop in time according to

$$\hat{\chi}_{\alpha}(t)=e^{(iH/\hbar)(t-t')}\hat{\chi}_{\alpha}(t')e^{(-iH/\hbar)(t-t')}\ , \qquad (3.21)$$

and similarly for $\hat{\chi}^{\dagger\,\beta}(t)$, where

$$\hat{\chi}_{\alpha}(t)=t\hat{\psi}_{\alpha}(t),\quad \hat{\chi}^{\dagger\,\beta}(t)=t\hat{\psi}^{\dagger\,\beta}(t)\ .$$

It can be checked explicitly that the CAR's are preserved in time, and are thus consistent with the equations of motion, that is

$$\{\hat{\chi}_{\alpha}(t),\hat{\chi}^{\dagger\,\beta}(t)\}=\{\hat{\chi}_{\alpha}(t'),\hat{\chi}^{\dagger\,\beta}(t')\}$$

or

$$\frac{d}{dt}\{\hat{\chi}_{\alpha}(t),\hat{\chi}^{\dagger\,\beta}(t)\}=0\ .$$

The time evolution of the $\hat{\psi}$ fields must be derived from these relations using the defining equation (2.43).

IV. FURTHER DEVELOPMENTS

The effect of the addition of a cosmological term $\Lambda g_{\mu\nu}$ to the field equations, where Λ is the cosmological constant, can be readily discussed. This is equivalent to adding on a term $\Lambda(\det L)$ to the Lagrangian (2.2), and yields the Einstein equations

$$\frac{G_{\mu\nu}}{\kappa^2}+\Lambda g_{\mu\nu}=T_{\mu\nu}\ . \qquad (4.1)$$

The $G_{0i}=0$ equation and the Dirac equations remain as in Sec. II, so we find the same inconsistency as previously. The flat-space ($K=0$) case is therefore the only one considered. The field equations become

$$\frac{2R}{N}\frac{d}{dt}\left(\frac{\dot{R}}{N}\right)+\left(\frac{\dot{R}}{N}\right)^2-\Lambda R^2=0 \qquad (4.2)$$

from the G_{ij} equation,

$$\kappa^2\frac{mc}{\hbar}\bar{\psi}\psi=\frac{6\dot{R}^2}{N^2R^2}-2\Lambda \qquad (4.3)$$

from the G_{00} equation, and the $G_{0i}=0$ equation is automatically satisfied by virtue of the Dirac equations, as before.

When we consider the choice of time given by $N=1$, solutions to (4.2) and (2.33) are given by

$$R=R_0\left\{\frac{\sinh[\frac{1}{2}\sqrt{3}\Lambda(b+t)]}{\sinh[\frac{1}{2}\sqrt{3}\Lambda b]}\right\}^{2/3}\ , \qquad (4.4)$$

$$\psi=\tfrac{1}{2}\sqrt{3}\Lambda\,\mathrm{csch}[\tfrac{1}{2}\sqrt{3}\Lambda(b+t)]\begin{pmatrix}\exp\left(-\frac{imct}{\hbar}\right)\alpha\\ \exp\left(\frac{imct}{\hbar}\right)\beta\end{pmatrix}, \qquad (4.5)$$

$$\bar{\psi}=\tfrac{1}{2}\sqrt{3}\Lambda\,\mathrm{csch}[\tfrac{1}{2}\sqrt{3}\Lambda(b+t)]\times\left(\exp\left(\frac{imct}{\hbar}\right)\alpha^{\dagger},-\exp\left(-\frac{imct}{\hbar}\right)\beta^{\dagger}\right), \qquad (4.6)$$

where b, R_0 are constants and α, β are constant two-component spinors. The G_{00} equation becomes

$$\frac{mc}{\hbar}(\alpha^{\dagger}\alpha-\beta^{\dagger}\beta)=\frac{8}{3\kappa^2}\ . \qquad (4.7)$$

It is to be noticed that this system of equations (and their solutions) does in fact reduce to the system in Sec. II (with no cosmological term present) when the limit $\Lambda\to 0$ is taken. As in Sec. II, the time parameter t can be eliminated from this system of equations to give a correlation between R, ψ, and $\bar{\psi}$ which describes the intrinsic dynamics of the system. The equations (4.4)–(4.6) show that the model experiences gravitational collapse at the constant-time value $t=-b$, at which ψ and $\bar{\psi}$ diverge as previously, and also show that the radius parameter has no maximum value.

As mentioned in Sec. I, the work of Trautman[8] prompted us to consider the addition of a nonzero torsion term to the Lagrangian (2.2), keeping in mind that, as suggested by Trautman,[8] this contribution may help us to avoid the classical singularity of the model at $R(t)=0$. The effect of the first-order variational formalism is to provide a torsional contribution to the affine connection and hence the additional interaction

$$\mathcal{L}'=\tfrac{3}{32}\kappa^2(\det L)(i\bar{\psi}\gamma_a\gamma_5\psi)(i\bar{\psi}\gamma^a\gamma_5\psi)\ , \qquad (4.8)$$

as has been noted by Weyl, Sciama, and Kibble.[9]

As a consequence of the presence of the term $\mathcal{L}'$ in the Lagrangian (2.2), the field equations are modified according to

$$\frac{G_{\mu\nu}}{\kappa^2}=\tfrac{1}{4}g_{\mu\nu}\left(\tfrac{1}{2}iL^{\alpha a}\bar{\psi}\gamma_a\overleftrightarrow{\nabla}_{\alpha}\psi-\frac{mc}{\hbar}\bar{\psi}\psi\right)\kappa^2-\tfrac{1}{4}iL_{(\nu}{}^{b}\bar{\psi}\gamma_b\overleftrightarrow{\nabla}_{\mu)}\psi\ , \qquad (4.9)$$

where we use the fact, following from the spinor equations of motion, that

$$\mathcal{L}_{\mathrm{matter}}=-\tfrac{3}{32}\kappa^2(\det L)(i\bar{\psi}\gamma_a\gamma_5\psi)(i\bar{\psi}\gamma^a\gamma_5\psi)\ . \qquad (4.10)$$

The Dirac equations become

$$\frac{R}{N}\dot{\psi}=-\frac{3\gamma_5\psi}{4\sqrt{2}}-\frac{imc}{\hbar}R\gamma_0\psi-\frac{3}{2}\frac{\dot{R}}{N}\psi$$
$$+\tfrac{3}{16}i\kappa^2R\gamma_0(i\gamma_a\gamma_5\psi)(i\bar{\psi}\gamma^a\gamma_5\psi)\,,$$
$$\frac{R}{N}\dot{\bar{\psi}}=-\frac{3\bar{\psi}\gamma_5}{4\sqrt{2}}+\frac{imc}{\hbar}R\bar{\psi}\gamma_0-\frac{3}{2}\frac{\dot{R}}{N}\bar{\psi}$$
$$-\tfrac{3}{16}i\kappa^2R(i\bar{\psi}\gamma_a\gamma_5\psi)(i\bar{\psi}\gamma^a\gamma_5)\gamma_0\,. \tag{4.11}$$

The $G_{0i}=0$ equation is (as in Sec. II) found to be

$$\frac{R}{N}(\bar{\psi}\gamma_i\dot{\psi}-\dot{\bar{\psi}}\gamma_i\psi)-\frac{1}{2\sqrt{2}}\bar{\psi}\gamma_i\gamma_5\psi=0\,, \tag{4.12}$$

which once again is incompatible with the Dirac equations, so we consider the flat-space ($K=0$) case only.

The flat-space Dirac equations become

$$\frac{R}{N}\dot{\psi}=-\frac{imc}{\hbar}R\gamma_0\psi-\frac{3}{2}\frac{\dot{R}}{N}\psi$$
$$+\tfrac{3}{16}iR\gamma_0(i\gamma_a\gamma_5\psi)(i\bar{\psi}\gamma^a\gamma_5\psi)\kappa^2\,,$$
$$\frac{R}{N}\dot{\bar{\psi}}=\frac{imc}{\hbar}R\bar{\psi}\gamma_0-\frac{3}{2}\frac{\dot{R}}{N}\bar{\psi}$$
$$-\tfrac{3}{16}iR(i\bar{\psi}\gamma_a\gamma_5\psi)(i\bar{\psi}\gamma^a\gamma_5)\gamma_0\kappa^2\,, \tag{4.13}$$

and the $G_{0i}=0$ equation,

$$\frac{R}{N}(\bar{\psi}\gamma_i\dot{\psi}-\dot{\bar{\psi}}\gamma_i\psi)=0\,, \tag{4.14}$$

is automatically satisfied by the Dirac equations.

The G_{00} equation is

$$-\frac{3\dot{R}^2}{\kappa^2R^2}=-\tfrac{1}{8}iN(\bar{\psi}\gamma_0\dot{\psi}-\dot{\bar{\psi}}\gamma_0\psi)-\tfrac{1}{4}N^2\frac{mc}{\hbar}\bar{\psi}\psi\,. \tag{4.15}$$

The G_{ij} equation is

$$\frac{2R}{N}\frac{d}{dt}\left(\frac{\dot{R}}{N}\right)+\left(\frac{\dot{R}}{N}\right)^2$$
$$=-\kappa^2\frac{R^2}{N^2}\left[\tfrac{1}{8}iN(\bar{\psi}\gamma_0\dot{\psi}-\dot{\bar{\psi}}\gamma_0\psi)-\tfrac{1}{4}N^2\frac{mc}{\hbar}\bar{\psi}\psi\right]. \tag{4.16}$$

If we subtract Eqs. (4.15) and (4.16), we find that

$$\frac{2}{\kappa^2}\left[\frac{N}{R}\frac{d}{dt}\left(\frac{\dot{R}}{N}\right)-\left(\frac{\dot{R}}{R}\right)^2\right]=-\tfrac{1}{4}iN[\bar{\psi}\gamma_0\dot{\psi}-\dot{\bar{\psi}}\gamma_0\psi]\,, \tag{4.17}$$

whereas if we add Eqs. (4.15) and (4.16), then

$$\frac{2}{\kappa^2}\left[\frac{N}{R}\frac{d}{dt}\left(\frac{\dot{R}}{N}\right)+2\left(\frac{\dot{R}}{R}\right)^2\right]=\tfrac{1}{2}N^2\frac{mc}{\hbar}\bar{\psi}\psi\,. \tag{4.18}$$

With the choice of time given by $N=1$, Eq. (4.17) tells us that

$$\frac{\dot{R}}{R}=-\tfrac{1}{8}i\kappa^2\int_0^t(\bar{\psi}\gamma_0\dot{\psi}-\dot{\bar{\psi}}\gamma_0\psi)\,dt'\,, \tag{4.19}$$

so that the Dirac equations become

$$\dot{\psi}=-\frac{imc}{\hbar}\gamma_0\psi+\tfrac{3}{16}i\kappa^2\int_0^t(\bar{\psi}\gamma_0\dot{\psi}-\dot{\bar{\psi}}\gamma_0\psi)\,dt'\,\psi$$
$$+\tfrac{3}{16}i\kappa^2\gamma_0(i\gamma_a\gamma_5\psi)(i\bar{\psi}\gamma^a\gamma_5\psi)\,,$$
$$\dot{\bar{\psi}}=\frac{imc}{\hbar}\bar{\psi}\gamma_0+\tfrac{3}{16}i\kappa^2\int_0^t(\bar{\psi}\gamma_0\dot{\psi}-\dot{\bar{\psi}}\gamma_0\psi)\,dt'\,\bar{\psi}$$
$$-\tfrac{3}{16}i\kappa^2(i\bar{\psi}\gamma_a\gamma_5\psi)(i\bar{\psi}\gamma^a\gamma_5)\gamma_0\,. \tag{4.20}$$

Alternatively, if the case $m=0$ is taken, and then we put $N=1$, Eq. (4.18) tells us that

$$R=R_0(a+t)^{2/3}\,, \tag{4.21}$$

so that the Dirac equations become

$$\dot{\psi}=-\frac{\psi}{(a+t)}+\tfrac{3}{16}i\gamma_0(i\gamma_a\gamma_5\psi)(i\bar{\psi}\gamma^a\gamma_5\psi)\kappa^2\,,$$
$$\dot{\bar{\psi}}=-\frac{\bar{\psi}}{a+t}-\tfrac{3}{16}i(i\bar{\psi}\gamma_a\gamma_5\psi)(i\bar{\psi}\gamma^a\gamma_5)\gamma_0\kappa^2\,, \tag{4.22}$$

and Eq. (4.19) becomes

$$(\bar{\psi}\gamma_0\dot{\psi}-\dot{\bar{\psi}}\gamma_0\psi)=-\frac{16i}{3\kappa^2(a+t)^2}\,.$$

Although it seems obvious that these coupled equations must yield solutions for ψ, $\bar{\psi}$, it is not at all clear how to solve them classically, and even if one could, no obvious quantization scheme springs to mind.

V. CONCLUSIONS

We have discussed the coupled Dirac field and Robertson-Walker metric, and quantized the combined system in the Heisenberg sense. The spinor field Dirac equations are linear and can easily be solved classically. This enables the corresponding quantum field equations to be readily solved and shown to be compatible with the canonical anticommutation relations.

The G_{ij} Einstein equation in both the classical and quantum theories simply leads to an explicit relation between the Robertson-Walker radius $R(t)$ and the chosen time coordinate t. The crucial G_{00} equation imposes a constraint in the classical theory between $\psi(t)$, $\bar{\psi}(t)$, and $R(t)$. We have chosen to interpret the associated quantum equation as a constraint on the allowed state vectors. This immediately leads to an eigenvalue spectrum for the mass m in terms of the rate-of-growth parameter R_0 in $R(t)=R_0t^{2/3}$. That is, the theory can only be quantized for any given value of R_0, if m has one of a finite number of discrete values. We note that if this rate-of-growth parameter is expressed in terms of the radius and Hubble constant of the physical universe, then the resulting possible masses predicted are all of the order of magnitude of that of the physical universe. The significance of this result is, however, unclear.

One of the main reasons for studying quantum gravity is the phenomenon of gravitational collapse. It is difficult to say whether or not a quantum model of the type above exhibits gravitational collapse since no complete set of properties characterizing this situation in the quantum case has yet been formulated. However, insofar as the spinor field operator equations are linear and exactly the same as the classical ones it is difficult to see how the singularity at $R(t)=0$ could be avoided. This, of course, is not the case for the model including torsion but unfortunately we have not been able to make much progress in this direction.

There are several points worthy of further discussion. The most obvious is the consideration of a different metric, possibly one of the more complicated Bianchi-type universes, with the shift functions N_i included, which would introduce some anisotropic degrees of freedom into the system and which we hope would lead to a consistent curved-space interaction.

It would also be interesting to investigate the model using $\bar{\psi}\psi$ as the choice of time. This is the analog of the natural choice $t=\phi(t)$ used in the scalar-field case.[6]

Finally, there is the question of the general coordinate invariance of the model and the not dissimilar problem of the invariance under the local SL(2, C) gauge group of *vierbein* transformations. The latter should not be too difficult because of the linear nature of the spinor field equations and the SL(2, C) covariant form of the operator G_{00} constraint equation. We hope to return to this question in a later publication.

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