PHYSICAL REVIEW D VOLUME 10, NUMBER 10 15 NOVEMBER 1974

Presence of a particle in a given space-time region and the continuous action of a particle detector

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(Received 12 April 1973)

We attempt to answer several questions relating to the presence of a particle, with known initial state, in a given region of space during a given period of time. In the process, we try to extend the quantum theory of observation to the case in which the observation goes on continuously. Although this mode of observation is common in practice, the theory does not seem to describe it adequately. In general, questions like those that we are asking do not have simple, unambiguous answers in quantum theory. In order to approach such a question, we usually have to assume that the particle interacts with something in the space-time region of interest, and then to infer the probability of the particle's presence from the probability of its interacting with the other object. This fact suggests either that we are asking the wrong questions or that perhaps quantum mechanics should be regarded as a theory of interaction events rather than a theory of systems.

I. INTRODUCTION

Quantum mechanics, including relativistic theories of particles and of fields, enables one to calculate dynamical variables and states of systems at instants of time—more generally, on three-dimensional spacelike surfaces. The theory gives straightforward answers to questions such as "What is the probability distribution of the dynamical variable α at time t_0 (or on spacelike surfaces s_0)?" Time is not normally treated as a dynamical variable itself but rather as a numerical parameter incapable of having an indefinite value.[1]

Correspondingly, the quantum theory of observation consists of statements about observations at a precise time; it is supposed that a system interacts very strongly and briefly with a measuring instrument, by means of a time-dependent interaction Hamiltonian proportional to $\delta(t-t_0)$, and that this interaction establishes a correlation between later states of particle and apparatus sufficient to permit the particle state to be inferred from observation of the apparatus state.[2]

The quantum theory of observation has often been questioned, but not, so far as we know, on account of the "precise-time" feature mentioned above. We consider this feature unrealistic because experimenters do not have very good control over the time at which a particle interacts with apparatus and, indeed, customarily leave particle detectors turned on for times long compared with transit times of particles passing through the detectors. If an experimenter wishes to know the time at which a counter detects a particle, he does not (except very roughly) predetermine the time by his procedures; instead, he waits for a count and then somehow measures the time at which the count occurred. In this sense times of observations play the role of observables, but the theory treats them as numerical parameters that are determined *a priori*.[1]

It has occurred to us, also, that some of the troubles perennially besetting relativistic quantum theory may be related to the asymmetric treatment of space and time coordinates (even in field theory), so we have thought it would be interesting to look into the relationship between variables and states defined in the usual way at a precise time, and the same quantities somehow defined over a time interval or at an imprecise time.

We have chosen what seems a reasonably simple case of this type, and we treat it here nonrelativistically; this treatment has been troublesome enough to discourage us from any present attempt to carry it through relativistically. The question that we attempt to answer is "What is the probability that a particle in a given initial state will be present in a given volume at least once during a given time interval?"[3]

II. THEORY OF OBSERVATION

Although this question has a clear meaning (and often a simple answer) in classical mechanics, its meaning is not very plain in the quantum context—for reasons related to our remarks in the first paragraph. But apparently it can be given a clear operational meaning as a question in the theory of observation, and in that form seems to us a more realistic question than those about instantaneous observations that the theory usually deals with. So we (tentatively) restate our question: "What is the probability that a 100%-efficient counter, occupying a volume V and turned on from t_1 till $t_1+\tau$, will detect a particle that has a given initial state (if the counter cannot detect the particle twice)?" If the time of detecting a particle is regarded as an observable, this second question is of a rather common type: "What is the probability that this observable has a value between t_1 and $t_1 + \tau$?"

Now our task is to apply the orthodox theory of instantaneous observations to one that extends over a time interval. There are two main ways of approaching this problem: (1) One can suppose that the volume V is occupied by a counter, capable of irreversible counting, that is turned on from t_1 until $t_1+\tau$ and is itself observed for the first time at $t_1+\tau$; (2) one can suppose that the volume is occupied by a counter which is turned on from t_1 until $t_1+\tau$ and is itself observed continuously during this interval. Each of these approaches corresponds more or less closely to procedures that are in actual use. The first is orthodox in that it involves a single instantaneous observation, but the continuous interaction of particle with detector prevents there being a simple, model-independent relationship between the particle's initial state and the probability of detection. We shall again comment briefly on this approach, in a later section.

The second approach is unorthodox by virtue of the assumed continuous observation. So far as we know, it has not been used before.[4] Therefore, aside from its presumed relevance to our question about the particle in the space-time volume, this treatment of particle detection appears to us interesting in its own right and worthy of investigation.

It seems reasonable to attack the problem of continuous observation during a given time interval by looking first at the case of n instantaneous observations at n times that span the interval and cover its end points, and then letting n go to infinity in such a way that in the limit there are no gaps between observations. Accordingly, we review some features of the theory of instantaneous observations as that theory applies to measurements with two possible outcomes.

According to this theory, the composite supersystem of apparatus plus microscopic system is initially in a pure state which is a simple (uncorrelated) product of system state and apparatus state. The strong instantaneous interaction of system and apparatus converts this product into a sum of products, each term of which sum consists of an eigenvector of the system variable being measured, multiplied by a distinctive apparatus state whose later recognition by an observer will permit him to infer that the system is in the pure state that accompanies the observed apparatus state in the sum of products. The norm of the product state in each term of the sum is the probability that that term will be singled out by later observation of the state of the apparatus.

At each stage of this process the composite supersystem is in a pure state: a product, then a sum of products, then a single product again. In the intermediate state, however, after the interaction but before the later observation of the apparatus, the state of either the observed system alone or the apparatus alone is a mixture, i.e., a superposition in which the coefficient of each term is not a number but a vector in a different space. Various authors[5] have discussed the time evolution of the apparatus's density matrix, seeking to describe how this matrix ultimately becomes diagonal with respect to a set of persistent equilibrium states which might correspond to alternative permanent records of the measurement. The final reduction, whereby the mixture representing the state of the apparatus gets reduced to a single component, seems to us still to be obscure. The mixture which was originally introduced as an objective description of a subsystem coupled to another subsystem suddenly gets interpreted as a description of subjective uncertainty, as if only one of its components were correct and an observation would reveal which.

However, this obscurity does not concern us in the present study. We shall simply summarize the effect of the observation on the state of the observed system as being equivalent to the application of a linear operator which converts a pure state into another pure state. In the case of an ideal measurement this linear operator is the projector onto an eigenstate of the measured quantity—in our case, the geometric projection operator into or out of the volume of the detector. However, in order to take account of instantaneous measurements that are less than ideal, we shall initially leave these operators unspecified. It will appear later that the effect of each instantaneous observation in a sequence is not the same as if that observation were isolated in time, so it is appropriate to represent that effect by means of an operator that contains some adjustable parameters. We shall look at the consequences of various particular choices of the operator, and in Sec. II B we shall find in the theory of irreversible processes a justification of some of our results.

So if the system being observed has a Schrödinger state vector $|\psi\rangle$ immediately before the observation at t_k, there are two operators Γ and $\overline{\Gamma}$, corresponding respectively to detection and nondetection at t_k, such that

$$\langle\psi|\psi\rangle P(t_k) = \langle\psi|\Gamma^\dagger\Gamma|\psi\rangle$$

and (1)

$$\langle\psi|\psi\rangle \overline{P}(t_k) = \langle\psi|\overline{\Gamma}^\dagger\overline{\Gamma}|\psi\rangle ,$$

where P and $\overline{P}=1-P$ are respectively the probabilities of detection and nondetection for the initial state $|\psi\rangle$. We shall call Γ and $\overline{\Gamma}$ "observation" operators; they must satisfy

$$\Gamma^\dagger\Gamma+\overline{\Gamma}^\dagger\overline{\Gamma}=1 . \tag{2}$$

In the ideal case which is usually treated Γ and $\overline{\Gamma}$ are complementary projection operators.

We shall take the state vectors immediately after the observation as $\Gamma|\psi\rangle$ corresponding to detection and $\overline{\Gamma}|\psi\rangle$ corresponding to nondetection. We see that the norm of each vector equals the norm of the initial state $|\psi\rangle$ multiplied by the probability of the outcome that corresponds to that vector. Thus if $|\psi\rangle$ already had a norm equal to that *a priori* probability of some earlier sequence of experimental results, the norm of either of the vectors that can follow the observation t_k will be the *a priori* probability of the earlier sequence of results followed by the appropriate result at t_k.

Therefore each of our state vectors will have a norm equal to the *a priori* probability of some specified sequence of observational results. We shall generate the vectors by applying to an initial vector of norm 1 a sequence of alternating observation operators and time-development operators, representing the effect on the vector of an observation, followed by propagation to the time of the next observation, followed by the effect of that observation, etc.

In particular, we shall consider the *a priori* probability $\overline{Q}_n$ of n unsuccessful searches for a particle by a counter. The final state vector in this case will be

$$|\psi(t_n+)\rangle=\overline{\Gamma}_nK(\Delta t_{n-1})\overline{\Gamma}_{n-1}K(\Delta t_{n-2})\cdots\overline{\Gamma}_2K(\Delta t_1)\overline{\Gamma}_1|\psi(t_1-)\rangle \equiv \overline{X}_n|\psi(t_1-)\rangle , \tag{3}$$

where $K(\Delta t)=\exp(-iH\Delta t/\hbar)$, and $|\psi(t_1-)\rangle$ has norm 1.

The operator $\overline{X}_n$ defined in Eq. (3) is thus the effective time-development operator; it plays, for n searches, the same role that $\overline{\Gamma}$ plays for one search. The probability of n unsuccessful searches is

$$\overline{Q}_n=\langle\psi(t_n+)|\psi(t_n+)\rangle =\langle\psi(t_1-)|\overline{X}_n^\dagger\overline{X}_n|\psi(t_1-)\rangle . \tag{4}$$

Similarly one can define an operator and calculate a probability for n failures followed by a success:

$$X_{n+1}=\Gamma_{n+1}K(\Delta t_n)\overline{X}_n$$

and (5)

$$R_{n+1}=\langle\psi(t_1-)|X_{n+1}^\dagger X_{n+1}|\psi(t_1-)\rangle .$$

The probability of a success on one (any) search in a total number n is

$$Q_n=1-\overline{Q}_n=\sum_{j=1}^{n}R_j . \tag{6}$$

We shall concentrate our attention on $\overline{X}_n$ and especially on

$$\overline{X}(\tau)=\lim_{n\to\infty}\overline{X}_n \text{ with} \sum_{1}^{n-1}\Delta t_j=\tau . \tag{7}$$

In the limiting case, the state vector resulting from nondetection between t_1 and $t_1+\tau$ is

$$|\psi(t_1+\tau)\rangle=\overline{X}(\tau)|\psi(t_1-)\rangle . \tag{8}$$

We shall require $|\psi(t_1+\tau)\rangle$ and hence $\overline{X}(\tau)$ to be differentiable with respect to τ. Thus we form

$$\frac{\Delta\overline{X}_n}{\Delta t_n}=\frac{\overline{X}_{n+1}-\overline{X}_n}{\Delta t_n}=\frac{[\overline{\Gamma}_{n+1}K(\Delta t_n)-1]\overline{X}_n}{\Delta t_n} . \tag{9}$$

We can write $K(\Delta t_n)=1+U_n\Delta t_n$, where U_n contains all non-negative powers of Δt_n, starting with $-iH/\hbar$, which is independent of Δt_n. Then

$$\frac{\Delta\overline{X}_n}{\Delta t_n}=\frac{[\overline{\Gamma}_{n+1}-1-i\overline{\Gamma}_{n+1}H\Delta t_n/\hbar+O(\Delta t_n{}^2)]\overline{X}_n}{\Delta t_n} \tag{10}$$

and we require the limit of this ratio to exist as $\Delta t_k\to 0$ for all $k=1,\ldots,n$, in such a way that all ratios $\Delta t_k/\Delta t_j$ are bounded. The trouble, if any, will come from the first terms of the numerator, which must yield a quantity of at least first order in Δt. The first factor in $\overline{X}_n$ is $\overline{\Gamma}_n$, and the next, $K(\Delta t_{n-1})$, is of order 1 in Δt. So we must require

$$(\overline{\Gamma}_{n+1}-1)\overline{\Gamma}_n=A\Delta t+O(\Delta t^2), \tag{11}$$

where A is a linear operator independent of Δt.[6] In general, then, $\overline{\Gamma}_n$ can depend on the Δt, most plausibly on Δt_{n-1}. Let us write

$$\begin{aligned}\overline{\Gamma}_n&\equiv\overline{\Gamma}(\Delta t_{n-1})\\ &=\overline{\gamma}+\beta(\Delta t_{n-1})\\ &=\overline{\gamma}+\beta_1\Delta t_{n-1}+\beta_2\Delta t_{n-1}{}^2+\cdots ,\end{aligned} \tag{12}$$

where $\overline{\gamma}$ and the β_k are linear operators independent of Δt_{n-1}. Now Eq. (11) implies that

$$(\overline{\gamma}-1)\overline{\gamma}+(\overline{\gamma}-1)\beta_1\Delta t_{n-1}+\beta_1\overline{\gamma}\Delta t_n=A\Delta t_n , \tag{13}$$

whence we find that

$$\overline{\gamma}^2=\overline{\gamma} \tag{14}$$

or $\overline{\gamma}$ is a projection operator. This is a necessary and sufficient condition for differentiability of $\overline{X}(\tau)$, and will be taken as a restriction on acceptable operators $\overline{\Gamma}$. In addition, Eqs. (1) and (2) require that, for all values of Δt from 0 to ∞, $\overline{\Gamma}(\Delta t)$ must not increase the norm of any vector on which it acts. The differentiability condition guarantees that this restriction is satisfied for $\Delta t=0$.

Now let us evaluate $\bar{X}(\tau)$ explicitly. Using Eq. (3) we write

$$\bar{X}_n = (\bar{\gamma} + \beta_1 \Delta t_{n-1})\left(1 - \frac{i}{\hbar} H \Delta t_{n-1}\right)(\bar{\gamma} + \beta_1 \Delta t_{n-2}) \cdots \times (\bar{\gamma} + \beta_1 \Delta t_1)\left(1 - \frac{i}{\hbar} H \Delta t_1\right) \bar{\Gamma}_1 , \quad (15)$$

where we have omitted powers of the Δt_j higher than the first everywhere except in $\bar{\Gamma}_1$ (which acts after a very long Δt_0). Now, multiplying adjacent factors in (15), using (14), and again dropping terms quadratic in the Δt_j, we obtain

$$\bar{X}_n = \left[\bar{\gamma} + \left(\beta_1 - \frac{i}{\hbar}H\right)\Delta t_{n-1}\right]\left[\bar{\gamma} + \left(\beta_1 - \frac{i}{\hbar}H\right)\Delta t_{n-2}\right] \cdots \left[\bar{\gamma} + \left(\beta_1 - \frac{i}{\hbar}H\right)\Delta t_1\right]\bar{\Gamma}_1$$
$$= \left[\bar{\gamma} + \bar{\gamma}\left(\beta_1 - \frac{i}{\hbar}H\right)\bar{\gamma}\sum_{i=1}^{n-1}\Delta t_i + \bar{\gamma}\left(\beta_1 - \frac{i}{\hbar}H\right)\bar{\gamma}\left(\beta_1 - \frac{i}{\hbar}H\right)\bar{\gamma}\sum_{i>j}\Delta t_i \Delta t_j + \cdots\right]\bar{\Gamma}_1 . \quad (16)$$

The last expression contains errors proportional to Δt_{n-1} and to Δt_1, but these errors will vanish as we let all the Δt_j go to zero with $\sum_{j=1}^{n-1}\Delta t_j = \tau$:

$$\bar{X}(\tau) = \left\{\bar{\gamma} + \bar{\gamma}\left(\beta_1 - \frac{i}{\hbar}H\right)\bar{\gamma}\tau + \left[\bar{\gamma}\left(\beta_1 - \frac{i}{\hbar}H\right)\bar{\gamma}\right]^2 \tau^2/2 + \cdots + \left[\bar{\gamma}\left(\beta_1 - \frac{i}{\hbar}H\right)\bar{\gamma}\right]^k \tau^k/k! + \cdots\right\}\bar{\Gamma}_1$$
$$= \bar{\gamma}\exp\left[\bar{\gamma}\left(\beta_1 - \frac{i}{\hbar}H\right)\bar{\gamma}\tau\right]\bar{\Gamma}_1$$
$$= \exp\left[\bar{\gamma}\left(\beta_1 - \frac{i}{\hbar}H\right)\bar{\gamma}\tau\right]\bar{\gamma}\bar{\Gamma}_1 . \quad (17)$$

This is the time-development operator from t_1 – to $t_1 + \tau$ for the case in which a continuous search by the counter fails to detect the particle during that period; its initial expectation value over a state of unit norm is the *a priori* probability of nondetection between t_1 and $t_1+\tau$. It can be interpreted as the action at time t_1 of $\bar{\gamma}\bar{\Gamma}_1 = \bar{\gamma}(1+\beta(\infty))$,[7] followed by action over a period of duration τ of an effective Hamiltonian

$$H_e = \bar{\gamma}(H + i\hbar\beta_1)\bar{\gamma} . \quad (18)$$

After this analysis one may wonder whether we have been misguided in our attempt to give meaning to our first question by restating it as a question about a continuing observation. The particle's wave function will certainly be changed by the attempt at detection, so whatever probability one calculates by studying the action of a counter will not be the probability that a free particle would have entered the space-time region. On the other hand it seems likely that, if one had a calculated value of this latter probability, it would be unverifiable. Regardless of this problem, we consider the detection problem worth studying for its own sake.

A. A particle observed by a counter

According to the theory of observation, one has considerable freedom in defining the boundary between the (classical) observer and the quantum system being observed. In this section we shall take the *system* to be a single particle, and the observer to be the counter with all its associated circuitry and a human observer.

It is tempting to investigate a sequence of observations by a 100%-efficient counter that does not give spurious counts. The observation operators of such a counter would be, respectively, the projection operators for particle states into and out of the counter volume. Unfortunately the latter operator, combined with H in Eq. (18), yields a pathological effective Hamiltonian which converts well-behaved wave functions to ill-behaved ones. Accordingly we define

$$\Lambda(n_0, V_0) = \sum_{n=n_0}^{N} |n\rangle\langle n|$$

and (19)

$$\bar{\Lambda}(n_0, V_0) = 1 - \Lambda(n_0, V_0) ,$$

where the $|n\rangle$ are bound energy eigenstates of a particle subject to a potential that vanishes in the counter volume and equals $V_0 > 0$ outside. The sum extends from some lowest state $|n_0\rangle$ (perhaps the ground state) to a state $|N\rangle$ whose energy is as close as possible to some specified fraction of V_0 (e.g., $E_N = \frac{1}{2}V_0$). Thus, in the limit $V_0 = \infty$, Λ is simply the projection operator into the counter volume, provided $|n_0\rangle$ is taken to be the ground state.

A higher energy for $|n_0\rangle$ gives us a counter that (realistically) cannot readily detect a particle of too-low energy. A large but finite V_0 avoids the problem of the pathological H_e, though at the cost of making the counter insensitive to extremely energetic particles. But by taking V_0 high enough one can to any desired degree reduce the effect of its finiteness on any given initial state. The deviation of Λ and $\bar{\Lambda}$ from geometric projection operators constitutes a special sort of nonlocality.

Accordingly, suppressing their dependence on n_0 and V_0, we shall use these nonlocal, energy-dependent projection operators as our observation operators:

$$\Gamma = \Lambda(n_0, V_0) \equiv \Lambda = \Lambda^2;$$
$$\bar{\Gamma} = 1 - \Lambda(n_0, V_0) \equiv \bar{\Lambda} = \bar{\gamma} , \tag{20}$$
$$\beta_1 = 0 .$$

Then the effective Hamiltonian is

$$H_e = \bar{\Lambda} H \bar{\Lambda} , \tag{21}$$

and the time-development operator is

$$\bar{X}(\tau) = \exp(-i\bar{\Lambda} H \bar{\Lambda}\tau/\hbar)\bar{\Lambda} , \tag{22}$$

which is unitary after t_1. Thus the probability of nondetection from t_1 to $t_1+\tau$ is the initial expectation value of

$$\bar{X}^\dagger \bar{X} = \bar{\Lambda} \exp(i\bar{\Lambda} H \bar{\Lambda}\tau/\hbar)\exp(-i\bar{\Lambda} H \bar{\Lambda}\tau/\hbar)\bar{\Lambda}$$
$$= \bar{\Lambda} . \tag{23}$$

This probability is equal to the probability of nondetection on the first search,[7] as if the later searches had not taken place.[8] After the first instant of being turned on the counter has been a perfect reflector.

If, as we intend, $|n_0\rangle$ is the ground state of the hypothetical particle in the counter volume and V_0 is very large compared to the energies that are abundant in the spectrum of the actual incident particle, the effect of an unsuccessful search by the counter is practically to obliterate the part of the particle's wave function that is inside the counter. Thus it is obvious that a failure to detect the particle reduces the probability that it will be detected soon afterwards, and that this probability approaches zero as the time interval between successive searches approaches zero. Working against this effect is the fact that the closer together in time the searches are, the more of them take place in a given fixed period.

Our calculation shows that the first effect overcomes the second. The wavelengths that $\bar{\Lambda}$ excludes from the counter cannot repenetrate the counter in time to be detected, while the repeated "chopping" of the wave function by the action of $\bar{\Lambda}$ introduces more and more undetectable high-energy components into the wave function. Thus, although its norm is unaffected by continuing nondetection, the wave function becomes drastically distorted in the process. These are discouraging results, inasmuch as the observation operators being used seem to be the best ones available for approximating the effect of a perfect counter.

Although we might avoid these difficulties by stopping short of the limit $\Delta t = 0$, we shall not investigate this possibility; there is no obvious criterion for choosing the correct Δt, and in any case we wish to study continuous rather than intermittent action of a counter.

If our counter is to have a chance of detecting the particle in continuous operation, its instantaneous efficiency must depend on the lapse of time since its last action. We must choose $\beta_1 \neq 0$, while continuing to make $\bar{\gamma}$ a projection operator. An example that seems reasonable is

$$\bar{\gamma} = 1; \quad \beta(\Delta t) = (e^{-\alpha\Delta t} - 1)\Lambda, \ \text{ or } \ \beta_1 = -\alpha\Lambda , \tag{24}$$

where α is a complex constant with positive real part:

$$\alpha = a + ib, \quad a > 0 .$$

Then $\bar{\Gamma}_n = \bar{\Lambda} + \Lambda e^{-\alpha\Delta t_{n-1}}$ and $\Gamma_n = \Lambda(1 - e^{-2a\Delta t_{n-1}})^{1/2}$ for $n > 1$ and

$$\bar{\Gamma}_1 = \bar{\Gamma}(\infty) = \bar{\Lambda} \text{ and } \Gamma_1 = \Lambda . \tag{25}$$

The instantaneous action of this counter approaches the previous case ($\bar{\Gamma} = \bar{\Lambda}$) as $\Delta t \to \infty$. In the limit $\Delta t \to 0$ the counting probability approaches zero and nondetection has no effect on the particle's wave function. By giving our counter an instantaneous efficiency that approaches zero in the limit of a continuous search, we have succeeded in giving it nonzero efficiency in its continuous action:

$$\bar{X}(\tau) = \exp(-iH_e\tau/\hbar)\bar{\Lambda} , \tag{26}$$

where

$$H_e = H + \hbar(b - ia)\Lambda , \tag{27}$$

which resembles the Hamiltonian of the nuclear optical model and (by virtue of the Hermitian part of β_1) produces a time-development operator after t_1 that is nonunitary.

So we have made progress; we have found a counter that *can* count. But we are far from having the perfect continuously acting counter that we seek. We do not believe that any optical-model potential produces complete absorption; in particular, $a = \infty$ produces complete reflection with no absorption, and renders $\bar{X}$ effectively unitary.

In his study of arrival time, Allcock[1,4] has considered the continuous action of a detector that fills the half-space $x > 0$. Without detailed analysis, he represents the effect on a particle wave function of a mechanism which periodically sweeps all amplitude out of the region $x>0$, by means of a constant negative imaginary potential filling the region of positive x. This procedure corresponds to our Eq. (27) with $b=0$. Our limit of 100%-efficient instantaneous detection corresponds to Allcock's periodic sweeps having infinite repetition rate; in this case, his imaginary potential becomes $-i\infty$ and produces complete reflection, or zero probability of detection. Thus, insofar as his results and ours can be compared, they lead to similar conclusions.

B. A counter observed by something else

Having found the "perfect" counter totally ineffective, we have been led to detract from its perfection in the hope that it may sometimes detect a particle. In the process, we have strayed rather far from our original question about the presence of a particle in a given space-time region. The detector that was supposed to give meaning to that question has now become our primary object of study. This being the case, it may still be worthwhile to give a little more attention to counters before returning to the original question.

The argument of Eqs. (1) to (18) would appear general enough to be applied in the case in which the observed system is taken to be particle plus counter and the observer is everything that follows, up to and including a human observer. Let us briefly examine this case.

We shall take the particle's Hamiltonian to be its kinetic energy, T, and shall suppose that the counter has a large number of discrete energy levels E_n, of which E_0 corresponds to the counter's quiescent state and the others to states in which it has counted. For simplicity we shall (unrealistically) take the E_n to be nondegenerate. Thus, for particle plus counter,

$$\begin{aligned} H &= T + H_c + f(t)\Lambda M \\ &= T + \sum_n |E_n\rangle\, E_n \left\langle E_n \right| + f(t)\Lambda \sum_{m,n} \left| E_m \right\rangle M_{mn} \langle E_n| \ . \end{aligned} \tag{28}$$

Here M is Hermitian. Λ has its previous meaning; it approximately projects particle states into the counter volume. Its presence in the interaction term of H guarantees that transitions can occur in the counter only when some of the particle's wave function is inside the counter. The inclusion of $f(t)$ makes possible an explicit time dependence of the interaction.

Since it is now the counter that is to be observed directly, the observation operators should act on counter states. The simplest choice, corresponding to that made in Eq. (20), is

$$\begin{aligned} \overline{\Gamma} &= |E_0\rangle\langle E_0| \ , \\ \Gamma &= 1 - \overline{\Gamma} = \sum_{n\neq 0} |E_n\rangle\langle E_n| \ . \end{aligned} \tag{29}$$

It is characteristic of the orthodox theory of observations that system and apparatus have an instantaneous interaction. In the present context, then, the observation operators must act instantaneously, as they did in the calculations of the preceding section, but on counter states instead of particle states. There is some ambiguity in the particle-counter interaction: Do particle and counter still interact instantaneously, or do they have a time-dependent interaction with the only instantaneous interaction being the one (for which there is no Hamiltonian) between counter and observer?

Let us investigate the first alternative by taking $f(t)=\delta(t-t_1)$ in Eq. (28), and supposing that the counter is observed instantaneously at some time after t_1. The assumed interaction Hamiltonian leads to a time-development operator from t_1- to t_1+ that can be put in the following forms:

$$\begin{aligned} K(t_1) &= \exp\left(-i\int_{t_1-}^{t_1+} H dt/\hbar\right) \\ &= \exp(-i\Lambda M/\hbar) \\ &= \Lambda \exp(-iM/\hbar) + \overline{\Lambda} \\ &= 1 + \Lambda[\exp(-iM/\hbar)-1] \, . \end{aligned} \tag{30}$$

If now the counter is observed immediately after t_1 and is found not to have counted, the operator that has acted on the state vector of particle plus counter at t_1 is

$$\overline{\Delta} = \overline{\Gamma} K(t_1) = |E_0\rangle\langle E_0|[\overline{\Lambda} + \Lambda \exp(-iM/\hbar)] , \tag{31}$$

which is thus the effective observation operator in this case. Although this may be a reasonable observation operator for a single instantaneous observation, we wish to use it as an ingredient in $\overline{X}(\tau)$ and so must require it to yield differentiable state vectors. Since it has not been made dependent on Δt, according to Eqs. (12) and (14) it must be a projection operator. So we ask whether the following operator vanishes:

$$\begin{aligned} \overline{\Delta}^2 - \overline{\Delta} &= |E_0\rangle\langle E_0|\Lambda[G|E_0\rangle\langle E_0|G - G] \\ &= |E_0\rangle\Lambda(G_{00}-1)\langle E_0|G \ . \end{aligned} \tag{32}$$

Here $\exp(-iM/\hbar)$ has been abbreviated as G. We see that the expression vanishes if and only if $G_{00}=1$. G is unitary, so if one of its elements equals 1, all its other elements in that row and in that column must vanish. Thus, if $G_{00}=1$, G cannot produce transitions from $|E_0\rangle$ to other states of the counter: The counter cannot count. This is a common affliction of our counters and can be brought about by simpler procedures. Thus an instantaneous Hamiltonian interaction of particle with counter, followed by an instantaneous non-Hamiltonian observation of the counter, cannot yield acceptable results in a continuous search, and we should let the particle and the counter interact continuously.

So we take $f(t) = 1$ in Eq. (28). Now the actual $\overline{\Gamma}$ is also the effective $\overline{\Gamma}$, and the earlier analysis can be applied unchanged. It tells us that the counting probability vanishes after t_1 if (as we assumed) $\overline{\Gamma}$ is a projection operator. We could look at more complicated $\overline{\Gamma}$'s which depend on Δt, but there seems to be little profit in doing so.

We seem to have discovered that, in order to make a particle detector work at all, one needs to make it "imperfect"; i.e., its counting efficiency in a given search depends on the lapse of time since the last previous search. Consideration of the detector's approach to equilibrium suggests that, in fact, every actual detector must have observation operators that depend on Δt, like those in Eqs. (24) and (25). If one treats the detector as part of the quantal system and supposes that the "observation" is an observation of the detector, one has to realize that the latter approaches its state of equilibrium in a nonzero time. Immediately after an observation that reveals the detector to be in its quiescent state(s) and the particle thus to be (more or less) outside the detector, the density matrix of the combined system begins evolving in the way discussed by George *et al.*[5] from this initial state. Whatever might be discovered by an observation made after a long time, it is likely that after a *short* time Δt the quiescent state(s) of the counter will appear with too high a probability, i.e., with a probability greater than would occur at equilibrium in the case in which the counter is turned off at the instant of the second observation. Thus it is reasonable to take Γ to be "less than" Λ, as in Eq. (25), and to approach zero as Δt approaches zero. The imaginary part $\hbar a$ of the complex effective potential of Eq. (27) is, therefore, approximately the reciprocal of the time constant of the detector in its approach to equilibrium. Allcock[1] assumes a similar relationship. There does not seem to be a simple interpretation of the real part $\hbar b$, which does not enter Γ.

The foregoing results apply only to the case of continuous observation. The other case (1), in which the counter is observed only at the end of the interval $(t_1, t_1+\tau)$, should be analyzable by the methods of George *et al.* by almost the same arguments that they apply to an instantaneous observation; the only difference would be that the counter is not an isolated system during any part of the time interval, so the approach to equilibrium would have to involve the density matrix of particle plus detector, rather than that of the detector alone. Such a change probably would not give rise to any puzzles.

Although the analysis of continuous interaction of a particle with a detector leads to reasonable conclusions, its results depend on details of the Hamiltonian and provide no criterion of 100% (or optimum) efficiency. There is no simple prescription of the sort that applies to a single instantaneous interaction of counter with particle and instantaneous observation of the counter (viz., integrate $|\psi|^2$ over the counter volume at the appropriate time), and a detailed treatment seems unlikely to yield 100% efficiency. This limitation of the theory of observation apparently applies to all observations made on particles, for all such observations are really noninstantaneous, and all depend on approximate localization of particles and therefore use detectors.

We have remarked previously that an attempt to detect a particle in a given space-time region by a process of continuous observation will so alter the particle's wave function that one is no longer dealing with the original question about the particle's entering the (empty) space-time region; instead, one is dealing with a question about the particle's entering a space-time region *occupied by an operating counter*. The use of an irreversible counter that is itself observed only once has the same shortcoming, for the particle's state becomes correlated with that of the counter and thus loses coherence even before the observation; the presence of the counter changes the state of the particle whether or not the ultimate observation reveals that the particle has been detected.

In the conventional theory of observation, it is recognized that instantaneous interaction plus observation changes the state of the observed system. But if the observation is instantaneous, the probabilities of various outcomes are determined quite simply by the initial state of the system; the collapse of the wave function will influence later observations but not the one that produced the collapse. On the other hand, if one is seeking a particle continuously throughout a time interval, the collapse of the wave function (or its preparation for collapse) in the earlier parts of the interval inevitably affects detection probabilities in the later parts.

However interesting or realistic it may be to inquire about the probability that a particle enters a given space-time region at least once (whether or not there is a counter there), such questions are not related to the postulates of quantum mechanics in the same way as are the usual questions about observables' instantaneously having values in specified ranges. At a given instant the different eigenvalues of an observable represent mutually exclusive alternatives, so it is appropriate to calculate a probability for some range of eigenvalues by summing or integrating the probabilities of the individual eigenvalues. But the probability of a particle's entering a space-time region cannot be similarly expressed as an integral of a density over the region, for its presence in different parts of the region does not correspond to a set of mutually exclusive alternatives. Even though in practice the time of detection of a particle acts like an observable, it clearly is not an observable in the usual sense,[1] and a theory in which it could be so treated would presumably differ fundamentally from the present theory.

So the probabilities that we seek to calculate are not related to the postulates of the theory in the same way as are instantaneous probabilities. We do not even have a method for calculating noninstantaneous counting probabilities, and, if we had a method, it would not after all give us the probability that we originally sought, for the particle's entering a space-time region *not* occupied by a counter. As we have remarked, if we had a formula for determining this last probability its predictions would be unverifiable since by the terms of the problem there would be no detector in the region.

III. OTHER APPROACHES

In view of the foregoing comments, one may wonder how anyone ever manages to predict counting rates or to use counters intelligently. In fact, physicists who use particle detectors have not been noticeably hampered by defects in the theory of observation. They simply assume that the counting rate of a given counter will be proportional to the particle flux through the counter window. Such an assumption is accurate in most cases commonly encountered because the particles' wave function at the counter window contains only waves that are ingoing[9] with respect to the counter. Then the flux is all ingoing, *and* arises solely from ingoing waves. In any situation in which ingoing and outgoing waves never coexist at any point on the counter surface and waves cannot leave the counter and then be reflected back into it, the probability that a particle enters the counter at least once during $(t_1, t_1+\tau)$ should be given by

$$-Q(\tau)=\int_{t_1}^{t_1+\tau} dt \oint_S \vec{J}_- \cdot d\vec{S} \,, \tag{33}$$

where the integral extends over the surface of the counter and over the appropriate time interval, and $\vec{J}_-$ is the current that is ingoing at the surface. This expression is equivalent to those actually used in the interpretation of (for example) collision experiments; it seems to answer our question about the space-time volume, and thus, indirectly, our question about continuous observation, for a class of special cases.

If Eq. (33) is to give the probability originally sought, for the particle's being in the space-time region at least once, there must be an additional term on the right equal to the probability that the particle is in the volume at t_1. The surface integral can be converted to a volume integral, so the entire probability can be written as an integral over the space-time region:

$$Q(\tau)=-\int_{t_1-}^{t_1+\tau} dt \int_V dV[\nabla\cdot\vec{J}_- - |\psi|^2\delta(t-t_1)] \,. \tag{34}$$

This possibility seems to contradict our earlier assertion that Q cannot be expressed as the integral of a density over the region. In fact, however, the integrand of Eq. (34) cannot be interpreted as a space-time density, for it depends on the choice of end times and on the geometry of the volume; the probability Q for a part of the volume V will not, in general, be the same integrand integrated over the smaller volume and over the time interval.

In any case, these expressions for Q can be correct and unambiguous only in special cases in which ingoing and outgoing waves do not coexist at the surface S, and particles cannot be backscattered into V. In order to achieve greater generality in these matters, let us specialize the geometry; let us take V to be a sphere of radius r_0 and the origin to be at the center of the sphere. Now we can write a general wave function for the particle:

$$\psi(\vec{r},t)=\sum_{LM}\int_0^\infty dk\, A_{LM}(k)Y_{LM}(\theta,\phi)g_L(kr)e^{-i\omega t} \,, \tag{35}$$

where $\omega=\hbar k^2/2m$, and the radial functions g_L are linear combinations of spherical Bessel and Neumann functions, or of ingoing and outgoing spherical Hankel functions:

$$\begin{aligned} g_L(kr) &= \tfrac{1}{2}[\eta_L(k)+1]j_L(kr)+i\tfrac{1}{2}[\eta_L(k)-1]n_L(kr) \\ &= \tfrac{1}{2}[h_L^{(2)}(kr)+\eta_L(k)h_L^{(1)}(kr)] \,. \end{aligned} \tag{36}$$

This wave function satisfies the time-dependent Schrödinger equation for a free particle, but it is well behaved at the origin only if all $\eta_L(k)=1$. We shall assume that ψ is normalized to 1 with this choice of the $\eta_L(k)$. Then we discover that

$$(\psi, \psi) = 1 = \frac{\pi}{2} \sum_{LM} \int_0^\infty dk |A_{LM}(k)|^2 / k^2 \ . \tag{37}$$

The radial current density is

$$J_r = \frac{\hbar}{2im} \left(\psi^* \frac{\partial \psi}{\partial r} - \psi \frac{\partial \psi^*}{\partial r} \right) . \tag{38}$$

Since we are interested in currents entering the sphere from all directions, we integrate J_r over the sphere, taking the Y_{LM} to be normalized to 1 over a unit sphere, thus getting the (outward) flux F through the spherical surface of radius r_0:

$$F(r_0, t) = \frac{\hbar r_0^2}{8im} \sum_{LM} \int dk dk' A^*_{LM}(k') A_{LM}(k) e^{i(\omega' - \omega)t} \times \left\{ [h_L^{(1)}(k'r_0) + \eta_L^*(k') h_L^{(2)}(k'r_0)] \frac{\partial}{\partial r_0} [h_L^{(2)}(kr_0) + \eta_L(k) h_L^{(1)}(kr_0)] - [h_L^{(2)}(kr_0) + \eta_L(k) h_L^{(1)}(kr_0)] \frac{\partial}{\partial r_0} [h_L^{(1)}(k'r_0) + \eta_L^*(k') h_L^{(2)}(k'r_0)] \right\} . \tag{39}$$

Now if we are willing to specialize the problem by taking the time interval of interest to include the whole period during which any part of the wave packet is inside the sphere, we expect Q to be given by

$$Q = -\int_{-\infty}^{\infty} F dt = \frac{\pi}{2} \sum_{LM} \int dk |A_{LM}(k)|^2 [1 - |\eta_L(k)|^2] / k^2 \ . \tag{40}$$

If ψ is interpreted as the complete wave function for $r \geq r_0$, Eq. (40) gives the net particle flow inward through the surface $r = r_0$. The $\eta_L(k)$ are sometimes written as

$$\eta_L = e^{2i\delta_L} , \tag{41}$$

where the δ_L are the partial-wave phase shifts. If the Hamiltonian of the particle is Hermitian not only outside the sphere but also inside, the δ_L are real, so $|\eta_L| = 1$, and the expression in Eq. (40) vanishes; the S matrix is unitary, or all the current that enters the sphere later leaves it.

In order to calculate a net inward flow of particles, one must either assume something inside the sphere that is equivalent to a non-Hermitian potential (e.g., an optical-model potential) or one must interpret ψ to be only part of the complete wave function. The argument that leads to Eq. (33) suggests that ψ should include only the waves that are ingoing at r_0, i.e., the $h_L^{(2)}$. This choice of ψ implies that all $\eta_L = 0$; then comparison with Eq. (40) indicates that

$$-\int_{-\infty}^{\infty} F dt = (\psi, \psi) = 1. \tag{42}$$

So if the probability Q is identified with the total inward particle flow through the surface $r = r_0$ during an infinite time interval, due to all ingoing waves at the surface, we find that $Q = 1$, *independent of the choice of* r_0. The whole packet seems to enter even a very small sphere, at some time. This result, like the others in this section, is not new. It does show that the argument involving flux due to ingoing waves at the counter window cannot be applied unchanged to the problem of flux through the sphere with arbitrary initial state. If the particle is represented by a wave packet much of which would miss the sphere in free propagation (i.e., if it has partial waves whose classical impact parameters greatly exceed r_0), these outlying parts of the packet nevertheless produce large ingoing waves at r_0. In the entire wave function these ingoing waves combine with outgoing waves at r_0 to produce the familiar r^L dependence of j_L, but the ingoing waves alone do not behave in this way.

So this method of calculating Q is unacceptable, and one might even wonder about the venerable argument that leads to Eq. (33). In this case, however, all is well. If one considers a plane packet going in the z direction and impinging on a small circular hole (window) in the sphere, one can readily calculate the flux F as a convergent double series involving j_L functions. But at the position of the hole (assumed symmetric about the $-z$ axis) the packet contains practically no outgoing Hankel functions, so the total particle flow Q through the hole, proportional to the hole's area, in fact arises almost solely from ingoing Hankel functions. In this case the higher partial waves, corresponding classically to particles that miss the hole, make contributions to Q that rapidly go to zero as L increases.

But the ingoing waves do not have this property when integrated over the whole sphere, and therefore have to be cut off somehow at a maximum value of L. The simplest criterion for cutting them off is to sum over L from zero only to $L = kr_0$, the angular momentum which classically would graze the sphere. One can interpret this cutoff in two ways: (1) The wave function really contains only j_L functions, but one calculates ingoing current at r_0 by using only ingoing Hankel functions for $L \leqslant kr_0$ and Bessel functions for higher L. (2) The sphere is really "black"; it totally absorbs the lower partial waves and leaves the higher ones unchanged, so the true wave function for $r \geqslant r_0$ contains exactly the radial functions that give the desired ingoing flux. These two interpretations yield the same value of Q, the one given by Eq. (40) with

$$\eta_L = 0 \text{ for } L \leqslant kr_0, \text{ and } \eta_L = 1 \text{ for } L > kr_0 , \qquad (43)$$

viz.,

$$Q = \frac{\pi}{2} \int_0^\infty dk \sum_{L=0}^{kr_0} \sum_M |A_{LM}|^2/k^2$$

$$= \frac{\pi}{2} \sum_{L=0}^{\infty} \sum_M \int_{L/r_0}^{\infty} dk |A_{LM}|^2/k^2 . \qquad (44)$$

The two interpretations of this result, suggested above, correspond respectively to the absence and the presence of a counter that occupies the sphere and fails to detect the particle (thus obliterating the wave function inside the sphere). We seem to have calculated the nondetection probability of the counter (for the special case of a spherical counter and $\tau = \infty$) without using the theory of observation, and also to have obtained the wave function of the particle that the counter has failed to detect. Although this wave function is different (in lacking low partial waves) from that of the free particle, the probability that the particle enters the space-time volume is the same in both cases. This fact alone shows that the usual theory of observation cannot yield the result obtained in this section.

The restriction to a spherical counter and infinite time interval can in principle be removed, though a more general calculation would be much more difficult. A more severe limitation of the argument comes from the use of semiclassical reasoning in support of the cutoff procedure, and in the ambiguity of the procedure itself. The sharp cutoff at $L = kr_0$ seems unlikely to be quite accurate in quantum mechanics, but once one begins using a more gradual cutoff there is no obvious criterion for deciding which to use. Presumably η_L should depend somehow on how much of the radial function j_L is inside the sphere, but we have not discovered such a criterion which seems clearly correct. Although convincing, the whole argument is rather contrived, and does *not* follow from the postulates of quantum mechanics.

Feynman's first paper on path integrals[11] seems to offer an expression for the probability that a particle enters a given space-time region. In Sec. 3 of that paper Feynman says, in effect, that the propagator from (x_1, t_1) to (x_2, t_2) including only paths lying wholly within a space-time region R is given by his usual integral over paths of $\exp(i\int L dt/\hbar)$, provided the integral over paths is confined to R. Integrating only over R imposes complicated and awkward limits on the space integrals by which Feynman investigates the properties of propagators; he calculates such integrals only over all space. However, one can produce the effect of these finite limits while still integrating over all space if one replaces the Lagrangian L by

$$L' = T - V - U , \qquad (45)$$

where T and V are the usual kinetic and potential energies, and

$$U = 0 \text{ in } R , \qquad U = -i\infty \text{ outside of } R . \qquad (46)$$

Thus, applied to our problem of the nondetection probability, Feynman's argument replaces the counter by an optical-model potential with infinite negative imaginary part. As we have stated, such a potential does not absorb but reflects completely; it produces a unitary propagator. So Feynman's suggestion does not solve our problem.

IV. OTHER QUESTIONS

The *ad hoc* argument involving the black sphere is the best that we have been able to do in attempting to answer the question, "What is the probability that a particle in a given initial state will be present in a given volume at least once during a given time interval?" The question about the continuously acting counter which we substituted for the first question seems to have been (approximately) answered indirectly, without the help of the theory of observation. Now we shall briefly consider a few other questions relating to the presence of a particle in a given space-time region.

The preceding investigation can be generalized to a study of the observable (?) $N(V, t_1, t_2)$, defined as the number of times a given particle enters volume V between times t_1 and t_2. We have been asking for the probability that $N = 0$; more complicated properties of N, if they have any meaning at all, are beyond our present powers of investigation. Nor do we know of any experiments in which N has been found to exceed 1.

A different type of experiment, readily analyzable by conventional quantum mechanics, involves an ensemble of particle-plus-counter systems with common Hamiltonian and initial state, in which the counters perform single instantaneous searches for their respective particles at times uniformly distributed between t_1 and t_2. If $\psi(\vec{r}, t)$ is the common wave function of the particles before the counters act, the probability of detection at a given time t in (t_1, t_2) is the integral over the counter volume of $|\psi(\vec{r}, t)|^2$. Then, search times being uniformly distributed over the interval, the probability of detection of a particle chosen at random from the ensemble is

$$\mathcal{P} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} dt \int_V |\psi|^2 dV . \tag{47}$$

This expression looks like a space-time integral of a density, but in fact the presumed density depends on $t_2 - t_1$ and will therefore, for example, be doubled if the interval is halved and the "density" computed separately for each half. Nor will $\mathcal{P}$ be equivalent to the Q to which most of this paper is devoted; for example, if a wave packet passes completely through V during a small fraction of the interval $t_2 - t_1$, Q will be 1 whereas $\mathcal{P}$ will be close to zero.

It is more plausible to interpret $\mathcal{P}$ as the mean fraction of the interval $t_2 - t_1$ that a particle spends in V, or the fourfold integral itself as the mean time during the interval that a particle is in V. This interpretation may be doubted because the experiment that measures $\mathcal{P}$ does not yield any information about the duration of any particle's stay in the counter. On the other hand, if each particle in a classical ensemble spent some time in the counter, the mean of these times for all the particles would be measured by exactly the experiment that we have described.

McClure[12] has considered the scattering of a packet of norm 1 from a local short-range potential, for the purpose of defining a space-time density of the scattering process. To this end he has taken the norm of the scattered packet at $t=\infty$ to be the total probability of scattering, and has expressed this quantity as an integral over space-time:

$$\mathcal{R} \equiv \int d^3x |\psi_{sc}(\vec{r}, \infty)|^2 = \frac{i}{\hbar} \int d^4x [\psi_0^* W \psi - \psi_0 W \psi^*] . \tag{48}$$

Here (in our notation) W is the scattering potential, ψ_0 is an unperturbed wave function representing the motion of the packet in the absence of a potential, and ψ is the actual wave function as affected by W; there is assumed to be a time t_1 such that, before t_1, $\psi = \psi_0$. McClure tentatively identifies the integrand as the interaction density that he seeks, although he cannot prove that it is positive-definite; possibly it can be made so by addition of the four-divergence of a suitable current which goes to zero fast enough at infinity. However, regardless of this question, McClure's expression for the total probability of scattering is correct and precise. His $\mathcal{R}$ should bear some resemblance to our $Q = 1 - \overline{Q}$. Accordingly, we express his result as

$$-i\hbar\mathcal{R} = \left\langle \psi_0(t_1) \right| \int_{t_1}^{\infty} dt [K_0^\dagger(t-t_1) W(t) K(t-t_1) - K^\dagger(t-t_1) W(t) K_0(t-t_1)] \left| \psi_0(t_1) \right\rangle , \tag{49}$$

where $\psi_0(t_1) = \psi(t_1)$ is assumed not yet to have interacted with W, and $K_0(t-t_1)$ and $K(t-t_1)$ are respectively the free-space and perturbed time-development operators from t_1 to t (K_0 is what we have previously called K).

After writing K_0 and K explicitly as exponentials we have compared the expression (49) with our expression for the initial expectation value of $1 - \overline{X}^\dagger(\infty)\overline{X}(\infty)$ with $\overline{\Gamma}_1 = 1$.[7] Although there are suggestive similarities between the two expressions, we have not been able to show their equivalence. This fact is perhaps not surprising, for our earlier arguments were pertinent to the probability that a particle is present at all in a region, regardless of how often it enters, how long it stays, or what happens to it there, whereas McClure's probability clearly does depend on how long the particle interacts with W or, more generally, on how strongly W affects it. Aside from issues in the theory of observation, the latter type of probability appears more fundamental and natural.

But McClure's argument does not give the probability that a particle *is in* a given space-time region; it gives the probability that it *interacts* with some assumed potential in the region. Our earlier efforts to deal with a particle's presence in a region led us, also, to assume the presence there of a counter or a black sphere, and when we were able to say anything about a free particle we found that we were calculating unverifiable quantities or were making *ad hoc* assumptions that did not follow from the postulates of quantum mechanics.

It is tempting to try to adapt McClure's result to our earlier concerns by somehow eliminating W and thus referring only to a space-time region and not simultaneously to a potential in that region. Thus one can assume that

$$W = \epsilon\Lambda \quad \text{for } t_1 \leq t \leq t_1 + \tau , \qquad W = 0 \quad \text{otherwise,} \tag{50}$$

so that it is constant in the desired region and zero elsewhere. Here Λ is the projection operator into the volume of interaction. Although the space and time dependences of W now seem appropriate, the result will still depend on the parameter ϵ. We are really interested in the case $\epsilon = 0$, but this case has $\mathcal{R} = 0$ also. Therefore we calculate $d\mathcal{R}/d\epsilon$ at $\epsilon = 0$, recalling that $\mathcal{R}$ depends on ϵ both explicitly through the dependence of W and implicitly through that of ψ. This first derivative vanishes also. The lowest nonvanishing derivative of $\mathcal{R}$ with respect to ϵ at $\epsilon = 0$ is the second:

$$\frac{d^2\mathcal{R}}{d\epsilon^2}\bigg|_0 = \frac{2i}{\hbar}\int d^4x \left(\psi_0^* \Lambda \frac{d\psi}{d\epsilon}\bigg|_0 - \frac{d\psi^*}{d\epsilon}\bigg|_0 \Lambda\psi_0\right) . \tag{51}$$

This expression contains the first derivatives of ψ and ψ^* at $\epsilon = 0$, which are proportional to the first-order corrections to ψ and ψ^* according to perturbation theory. These quantities can thus be explicitly substituted into (51):

$$\frac{d^2\mathcal{R}}{d\epsilon^2}\bigg|_0 = -\frac{2}{\hbar^2}\left\langle\psi_0(t_1)\middle|\int_{t_1}^{t_1+\tau} dt_3 \int_{t_1}^{t_3} dt_2 [K_0^\dagger(t_3-t_1)\Lambda K_0(t_3-t_2)\Lambda K_0(t_2-t_1) - K_0^\dagger(t_2-t_1)\Lambda K_0^\dagger(t_3-t_2)\Lambda K_0(t_3-t_1)]\middle|\psi_0(t_1)\right\rangle . \tag{52}$$

This expression is even harder than that in Eq. (49) to compare with those that we have derived earlier. It seems to be related to the term of order ϵ^2 in first-order perturbation theory whereby the norm of the perturbed state differs from 1. In the context of perturbation theory, it represents the only effect, on the norm, of single interactions of the particle with the potential. In perturbation theory, one would regard this lowest-order result as a first approximation which could be refined by inclusion of higher-order effects, but in our present venture we have no clue to the relative magnitudes of different terms in the perturbation series and so are not able to refine our result. If this quasiperturbative approach has any relevance to the problem of the free particle entering the space-time volume, its significance must lie in Eq. (52), which presumably yields a number that is proportional to the desired probability. But we are skeptical of such an expression, for it seems to give only the contribution of single-interaction amplitudes to the total probability of scattering, and thus seems incomplete and excessively specialized. On the other hand, if the potential really is zero, higher-order corrections should vanish. So we do not have a definitive comment on the significance of Eq. (52).

V. CONCLUDING REMARKS

We have now looked at several approaches to various problems relating to the presence of a particle in a given space-time region. Our impression after contemplating these arguments is that quantum mechanics is not well adapted to answering such questions. The trouble seems to come from the fact that the theory predicts states or dynamical variables as functions of a numerical parameter t, whereas the questions being asked in this paper tend to treat t as if it were an observable. If one is calculating the time evolution of a system, t does not make sense as an observable, for it is the free parameter whereby one distinguishes one state of the system from another. On the other hand, McClure's work and some of our earlier arguments suggest that t makes good sense as an observable not of a *system* but of an *event*, such as a collision or an interaction of a particle with something else. We can take either of two attitudes at this point: We can dismiss as unimportant or meaningless the questions discussed above which quantum mechanics seems ill suited to answer, or we can seek to reinterpret quantum mechanics as a theory of events rather than of systems.

ACKNOWLEDGMENTS

We take pleasure in acknowledging many helpful and interesting discussions with colleagues at Vanderbilt. We are especially indebted to Dr. Peter Iano, who suggested the black sphere to us and frequently made other helpful suggestions.

[1]Time has been given a status like that ofa dynamical variable in proper-time theories of particles, in studies of the time-energy uncertainty relation, and in attempts to define the time at which a particle arrives at a specified point. The subject of arrival time has been very fully discussed by G. R. Allcock [Ann. Phys. (N.Y.) 53, 253 (1969); 53, 286 (1969); 53, 311 (1969)]. His discussion reveals that, if arrival time is a dynamical variable, it is quite different from other such variables, for it has nonorthogonal eigenfunctions which, even in a nonrelativistic treatment, cannot be expressed in terms of positive-energy states alone. Also see comments by E. P. Wigner, in *Aspects of Quantum Theory,* Festschrift for Dirac's seventieth birthday (Cambridge University Press, Cambridge, 1972), pp. 237-247.

[2]See, for instance, D. Bohm, *Quantum Theory* (Prentice-Hall, New York, 1951), Chap. 22; E. P. Wigner, Am. J. Phys. 31, 6 (1963); I. Bloch, Phys. Rev. 156, 1377 (1967).

[3]Part of the ensuing discussion can be found in D. A. Burba's Ph.D. thesis, Vanderbilt University, 1970 (unpublished) (obtainable from University Microfilms, Ann Arbor, Mich.).

[4]In the second paper cited in Ref. 1, Allcock considers this problem as a side issue. In a later section we shall compare his treatment with ours.

[5]See, for example, A. Daneri, A. Loinger, and G. M. Prosperi, Nucl. Phys. 33, 297 (1962); K. Hepp, Helv. Phys. Acta 45, 237 (1972); C. George, I. Prigogine, and L. Rosenfeld, Nature 240, 25 (1972); and K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 38, 12 (1972).

[6]It might appear that one could derive a weaker condition on the $\overline{\Gamma}_n$ by requiring a product of *more than* two successive $\overline{\Gamma}_n$ to be of at least first order in the Δt. In fact, such a procedure leads to $\gamma^{-k+1} = \gamma^{-k}$, which is equivalent to Eq. (14).

[7]If the particle's wave function is far from the counter when the counter is turned on at t_1, the operator $\overline{\gamma}\overline{\Gamma}_1$ that acts at t_1 will be equivalent to the identity operator.

[8]E. P. Wigner (private communication) has encountered a similar phenomenon in studying a problem like the present one.

[9]We intend the adjectives "ingoing" and "outgoing" to be applied in a gauge in which the Schrödinger momentum operator is $\vec{p} = -i\hbar\nabla$.

[10]If the volume is a sphere and there are external forces lacking spherical symmetry (which might scatter the particle back into the sphere with altered L) the two interpretations are not equivalent. But in such a case the partial-wave expansion would be useful only for certain special initial states anyway.

[11]R. P. Feynman, Rev. Mod. Phys. 20, 367 (1948).

[12]J. A. McClure, Phys. Rev. 144, 1316 (1966); J. A. McClure, Ph.D. thesis, Vanderbilt University, 1963 (unpublished) (obtainable from University Microfilms, Ann Arbor, Mich.).