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Second quantization in the Kerr metric

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The second quantization of scalar and neutrino fields in the Kerr metric is discussed, and an instability to spontaneous emission is found in the scalar and neutrino cases. The dependence of these results on assumptions about the vacuum state is discussed, as is the semiclassical origin of this creation.

Of interest recently has been the problem of defining a consistent second quantization procedure on a given nonflat spacetime.[1-5] One of the key difficulties in any such procedure is the physical interpretation of such a procedure, or in other words, of interpreting the results in terms of particles or antiparticles. In the case of a time-varying background spacetime, one does not expect the concept of a single particle to be an invariant concept; the changing background gravitational field could be expected to create particles. Such could be the case even in a stationary gravitational field. We know that a stationary electromagnetic field will, if strong enough, produce (charged) particles out of the vacuum. Similarly, a stationary gravitational field might be expected to produce particles out of the vacuum.

Furthermore, we know from quantum electrodynamics that a background charge distribution can polarize the vacuum, leading to nonzero "vacuum" expectation values. One could expect a gravitational field to also induce vacuum polarization effects,[6] leading to nonzero "vacuum" expectation values. In particular, renormalization by normal ordering is no longer a valid procedure, as it is equivalent to the assumption that all field expectation values in the vacuum state are zero. We wish to calculate exactly these vacuum expectation values. Any renormalization procedure in general relativity must be covariant, and be based on physical assumptions (rather than just a desire to make various quantities finite). In homogeneous cosmologies Parker and Fulling[4] and Zel'dovich and Starobinskiĭ[3] have defined a procedure for renormalizing the energy-momentum tensor. The physical assumptions leading to their results are, however, obscure to this author.

We will therefore make no attempt at renormalization, instead examining quantities which give already finite results. In particular we shall examine the vacuum expectation value of the radial flow of energy and angular momentum in the Kerr vacuum for massless scalar and neutrino field.

The third section will discuss the limitation of our procedure, especially in light of the difficulty in defining a physically reasonable vacuum state.

I. NULL SCALAR FIELDS

The formal second quantization of the scalar field has been done in many places before (e.g., Fulling[2]), and we will just sketch the procedure.

The scalar Lagrangian is given by[7]

$$\mathcal{L}=\int_{x^0=\text{const}}\sqrt{-g}\,(\Phi^*_{,\mu}\Phi_{,\nu}g^{\mu\nu}-\tfrac{1}{6}R\Phi^*\Phi)\,d^3x\,, \tag{1.1}$$

where R is the Ricci scalar $g^{\alpha\beta}R_{\alpha\beta}$.

(This Lagrangian is the conformal Lagrangian.[8] Our results would be essentially the same if the term proportional to R were omitted.)

The momenta π, π^* conjugate to the fields Φ, Φ^* are defined by

$$\begin{aligned}\pi&=\frac{\delta\mathcal{L}}{\delta(\partial\Phi/\partial x^0)}=\sqrt{-g}\,g^{0\mu}\Phi^*_{,\mu}\,,\\ \pi^*&=\frac{\delta\mathcal{L}}{\delta(\partial\Phi/\partial x^0)}=\sqrt{-g}\,g^{0\mu}\Phi_{,\mu}\,.\end{aligned} \tag{1.2}$$

The Hamiltonian is

$$\begin{aligned}\mathcal{H}&=\int\left(\pi\frac{\partial\Phi}{\partial x^0}+\pi^*\frac{\partial\Phi^*}{\partial x^0}\right)d^3x-\mathcal{L}\\ &=\int\Bigg[\frac{\pi\pi^*}{\sqrt{-g}\,g^{00}}-\pi\frac{g^{0k}}{g^{00}}\Phi_{,k}-\pi^*\frac{g^{0k}}{g^{00}}\Phi^*_{,k}\\ &\qquad+\sqrt{-g}\left(\frac{g^{0k}}{g^{00}}\Phi_{,k}\frac{g^{0l}}{g^{00}}\Phi^*_{,l}-g^{ij}\Phi_{,i}\Phi^*_{,j}+\tfrac{1}{6}R\Phi^*\Phi\right)\Bigg]d^3x.\end{aligned} \tag{1.3}$$

Hamilton's equations become

$$\begin{aligned}\frac{\partial\pi}{\partial x^0}&=-\left(\frac{\pi g^{0k}}{g^{00}}\right)_{,k}+\left(\frac{g^{0l}}{(g^{00})^2}\Phi^*_{,l}g^{0k}\right)_{,k}\\ &\quad-(\sqrt{-g}\,g^{kj}\Phi^*_{,j})_{,k}-\tfrac{1}{6}R\Phi^*\,,\\ \frac{\partial\Phi}{\partial x^0}&=\frac{\pi^*-\sqrt{-g}\,g^{0k}\Phi_{,k}}{\sqrt{-g}\,g^{00}}\end{aligned} \tag{1.4}$$

plus the complex conjugate of these. These are equivalent to the second-order differential equation for Φ

$$(\Phi_{,\mu}g^{\mu\nu})_{;\nu}+\tfrac{1}{6}R\Phi=0\,. \tag{1.5}$$

Defining the state functional $V(\Phi)=\binom{\Phi}{\pi^*}$ we can define an inner product

$$\begin{aligned}\langle V(\Phi_1),\,V(\Phi_2)\rangle &\equiv \langle \Phi_1,\,\Phi_2\rangle\\ &=\tfrac{1}{2}i\int_{x^0=\text{const}}(\pi_2\Phi_1-\Phi_2^*\pi_1^*)d^3x\\ &=\tfrac{1}{2}i\int\sqrt{-g}\,g^{0\mu}(\Phi^*_{2,\mu}\Phi_1-\Phi_2^*\Phi_{1,\mu})d^3x\,.\end{aligned} \tag{1.6}$$

The Hamiltonian operator defined via (1.4) by

$$HV=i\frac{\partial V}{\partial x^0} \tag{1.7}$$

can be shown to be Hermitian with respect to this inner product (1.6).

The symmetry of (1.1) under the multiplication Φ by $e^{i\beta}$ and Φ^* by $e^{-i\beta}$ leads by Noether's theorem to the conserved current,

$$J_\mu=\tfrac{1}{2}i(\Phi^*_{,\mu}\Phi-\Phi^*\Phi_{,\mu})\,. \tag{1.8a}$$

If Φ is a solution to (1.5), then

$$\langle\Phi,\,\Phi\rangle=\int_{x^0=\text{const}}\sqrt{-g}\,J^0d^3x\,. \tag{1.8b}$$

The energy-momentum tensor is defined as[8]

$$\begin{aligned}T_{\mu\nu}&=\frac{\delta I}{\delta g^{\mu\nu}}\\ &=\tfrac{1}{3}[\Phi^*_{,\mu}\Phi_{,\nu}+\Phi^*_{,\nu}\Phi_{,\mu}-\tfrac{1}{2}g_{\mu\nu}\Phi^*_{,\rho}\Phi_{,}{}^{\rho}\\ &\quad-\tfrac{1}{2}(\Phi^*\Phi_{,\mu;\nu}+\Phi\Phi^*_{,\mu;\nu})\\ &\quad-\tfrac{1}{2}(R_{\mu\nu}-\tfrac{1}{6}g_{\mu\nu}R)\Phi^*\Phi],\\ &\qquad I=\int\mathcal{L}\,dx^0\,.\end{aligned} \tag{1.9}$$

$T_{\mu\nu}$ is trace-free, symmetric, and divergence-free.

We now specialize to the Kerr metric, which, in Boyer-Lindquist coordinates,[9] is

$$\begin{aligned}ds^2=&-\frac{\Sigma}{\Delta}dr^2-\Sigma d\theta^2-\frac{\sin^2\theta}{\Sigma}[a\,dt-(r^2+a^2)d\phi]^2\\ &+\frac{\Delta}{\Sigma}(dt-a\,d\phi)^2\,,\\ &\Delta=r^2-2Mr+a^2\,,\quad \Sigma=r^2+a^2\cos^2\theta\,.\end{aligned} \tag{1.10}$$

This metric has two Killing vectors, given by

$$\xi_t^\mu=\delta_t^\mu\,,\quad \xi_\phi^\mu=\delta_\phi^\mu$$

(we here use t, ϕ as indices rather than as coordinate values).

By definition, the Lie derivative of the metric with respect to these two vectors is zero. Noether's theorem now gives the conserved currents

$$\begin{aligned}\hat S_E^\mu=&\tfrac{1}{2}g^{\mu\nu}[\Phi^*_{,\nu}\mathcal{L}_{\xi_t}\Phi+(\mathcal{L}_{\xi_t}\Phi^*)\Phi_{,\nu}]\\ &-\xi_t^\mu(\Phi^*_{,\rho}\Phi_{,}{}^{\rho}-\tfrac{1}{6}R\Phi^*\Phi)\,,\\ \hat S_L^\mu=&\tfrac{1}{2}g^{\mu\nu}[\Phi^*_{,\nu}\mathcal{L}_{\xi_\phi}\Phi+(\mathcal{L}_{\xi_\phi}\Phi^*)\Phi_{,\nu}]\\ &-\xi_t^\mu(\Phi^*_{,\rho}\Phi_{,}{}^{\rho}+\tfrac{1}{6}R\Phi^*\Phi)\,.\end{aligned} \tag{1.11}$$

(These are obtained by taking the Lie derivative of the Lagrangian density, and using the Euler-Lagrange equations to eliminate terms containing $\delta\mathcal{L}/\delta\Phi$ and $\delta\mathcal{L}/\delta\Phi^*$.)

From the energy-momentum tensor (1.9) we obtain an additional set of conserved currents:

$$S_E^\mu=T^\mu{}_\nu\,\xi_t^\nu\,,\quad S_L^\mu=T^\mu{}_\nu\,\xi_\phi^\nu\,. \tag{1.12}$$

The two currents S^μ and $\hat S^\mu$ will differ by a divergence of a twoform, namely

$$S^\mu-\hat S^\mu=\tfrac{1}{6}[\xi^{[\mu}g^{\nu]\alpha}(\Phi^*_{,\alpha}\Phi+\Phi_{,\alpha}\Phi^*)]_{;\nu}\,. \tag{1.13}$$

(We use $R_{\mu\nu}=0$, and square bracket superscripts indicate antisymmetrization.)

The Killing vector ξ_t^μ is timelike far from the black hole. However, near the black hole $[r<M+(M^2-a^2\cos^2\theta)^{1/2}]$ this Killing vector becomes spacelike. On the other hand, we find that near the black hole, $(\xi_t^\mu+\omega_H\xi_\phi^\mu)$ is timelike but becomes spacelike as $r\to\infty$, $[\omega_H=a/2Mr_+,\ r_+=M+(M^2-a^2)^{1/2}]$.

We will define our normal modes by the requirement that

$$\mathcal{L}_{\xi_t^\mu}\varphi=-i\omega\varphi\,. \tag{1.14a}$$

By (1.2) and (1.7), and noting that in Boyer-Lindquist coordinates $\mathcal{L}_{\xi_t^\mu}\varphi=\partial\varphi/\partial t$, this equation becomes

$$HV(\varphi)=\omega V(\varphi)\,. \tag{1.14b}$$

(This holds only because in this coordinate system the metric is time-independent.) The normal modes are here the eigenmodes of the operator H. Given two normal modes, φ_1, φ_2, we find

$$\langle V(\varphi_1),\,HV(\varphi_2)\rangle=\langle HV(\varphi_1),\,V(\varphi_2)\rangle$$

or

$$(\omega_2-\omega_1^*)\langle V(\varphi_2),\,V(\varphi_1)\rangle=0\,,$$

i.e., the inner product is zero unless $\omega_1=\omega_2^*$. As the inner product is not positive definite, we cannot therefore conclude that ω must be real. In particular, it is possible that there exist (bounded) solutions to (1.14b) with $\text{Im}(\omega)>0$, corresponding to unstable growing modes. Such modes would have to have zero norm under our inner product.

No analytic proof exists at present that such unstable modes are impossible in the Kerr metric (except for the axisymmetric case). Detweiler and Ipser[9] have done a computer search, however, and have found no such unstable modes for any value of a. We shall hereafter assume this result. (A similar result for electromagnetic and gravitational perturbations has been found by Press and Teukolsky[10] and Hawking.[11])

Since the two Killing vectors commute, we can simultaneously define angular modes by

$$\pounds_{\xi^\mu_\phi}\varphi = -im\varphi \quad (m \text{ integer}).$$

Equation (1.5) now becomes a completely separable equation.[12] Letting

$$\varphi(\omega, m, x) = R(r)S(\theta)e^{-im\phi}e^{-i\omega t},$$

we find

$$\left[\frac{1}{\Delta}\frac{d}{dr}\Delta\frac{d}{dr} + \left(\frac{\omega(r^2+a^2)+am}{\Delta}\right)^2 + \frac{k^2}{\Delta}\right]R(r) = 0,$$
$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \left(\omega a\sin\theta + \frac{m}{\sin\theta}\right)^2\right]S(\theta) = k^2S(\theta). \tag{1.15}$$

k^2 is a separation constant, a function of ω, m, and some positive integer, determined by the requirement that $S(\theta)$ be regular at $\theta = 0, \pi$.

$S(\theta)$ is in fact just a spheroidal[13] harmonic which we normalize so that $S(\theta)$ is real and

$$\int S^2(\theta)\sin\theta\, d\theta = (2\pi)^{-1}.$$

Upon defining a new radial parameter $\hat{r}$ by

$$\frac{d\hat{r}}{dr} = \frac{r^2+a^2}{\Delta}$$

the radial equation becomes

$$\left[\frac{1}{r^2+a^2}\frac{d}{d\hat{r}}(r^2+a^2)\frac{d}{d\hat{r}} + \left(\omega + \frac{am}{r^2+a^2}\right)^2 - \frac{k^2\Delta}{(r^2+a^2)^2}\right]R(\hat{r}) = 0. \tag{1.16}$$

This equation has two linearly independent solutions, which we choose to be defined asymptotically by

$$R_+(\hat{r}) \sim \frac{1}{[2\pi\omega(r^2+a^2)]^{1/2}} \times \begin{cases} e^{-i\omega\hat{r}} + A_+(\omega m k)e^{i\omega\hat{r}}, & \hat{r}\to\infty \\ B_+(\omega, m, k)e^{-i\tilde{\omega}\hat{r}}, & \hat{r}\to-\infty, \end{cases}$$
$$R_-(\hat{r}) \sim \frac{1}{[2\pi\tilde{\omega}(r^2+a^2)]^{1/2}} \times \begin{cases} B_-(\omega, m, k)e^{i\omega\hat{r}}, & \hat{r}\to\infty \\ e^{i\tilde{\omega}\hat{r}} + A_-(\omega m k)e^{-i\tilde{\omega}\hat{r}}, & \hat{r}\to-\infty \end{cases}$$
$$\tilde{\omega} = \omega + m\omega_H. \tag{1.17}$$

The R_+ solution represents a wave originating at infinity, with no component coming out of the black hole, whereas R_- is a solution originating purely from out of the black hole. Another way of phrasing this is that at the horizon R_+ is purely ingoing [as can be seen by making a wave packet, and realizing that the velocity of this wave packet near the black hole is given by $-\partial\tilde{\omega}/\partial\omega$, or by noticing that R_+ is regular on the future horizon (see Misner[14])], whereas R_- is purely outgoing at infinity.

If R_1, R_2 are solutions of (1.15), then

$$\frac{d}{d\hat{r}}\left[(r^2+a^2)^{1/2}\left(\frac{d}{d\hat{r}}(r^2+a^2)^{1/2}R_1(\hat{r})\right)R_2(\hat{r}) - \left(\frac{d}{d\hat{r}}(r^2+a^2)^{1/2}R_2(\hat{r})\right)R_1(\hat{r})\right] = 0. \tag{1.18}$$

Using this and noting that if R is a solution then so is its complex conjugate R^*, we find

$$1 - |A_+|^2 = \frac{\omega + m\omega_H}{\omega}|B_+|^2, \tag{1.19a}$$

$$1 - |A_-|^2 = \frac{\omega}{\omega + m\omega_H}|B_-|^2, \tag{1.19b}$$

$$(\omega + m\omega_H)B_+A_-^* = -\omega B_-^* A_+, \tag{1.19c}$$

$$(\omega + m\omega_H)B_+ = \omega B_-. \tag{1.19d}$$

Equation (1.19a) implies that if $\omega(\omega + m\omega_H) < 0$, $|A_+|^2 > 1$. In other words, the amplitude of the reflected wave is larger than that of the incident wave. This is the phenomenon of superradiance, first noticed by Misner[14] and by Zeldovich[14] and discussed by Bekenstein,[15] Press and Teukolsky,[16] and others.

Defining $\lambda = +, -,$

$$\kappa(\omega, m, \lambda) = \begin{cases} +1, & (\lambda = +, \omega > 0) \text{ or } (\lambda = -, \tilde{\omega} > 0) \\ -1, & \text{otherwise} \end{cases} \tag{1.20}$$

and

$$\varphi(\omega, m, k, \lambda, x) = e^{-i\omega t} e^{-im\phi} S(\omega, m, k, \theta) R(\lambda, \omega, m, k, r)$$

we find that

$$\langle \varphi(\omega, m, k, \lambda), \varphi(\omega', m', k', \lambda') \rangle = \kappa(\omega, m, \lambda)\delta(\omega - \omega') \times \delta_{mm'}\delta_{kk'}\delta_{\lambda\lambda'} \tag{1.21}$$

[the $\delta(\omega - \omega')$ term arises formally from the Hermiticity of H, while the $\kappa(\omega, m, \lambda)\delta_{\lambda\lambda'}$ term is determined by examining the norm for wave packets in the limit as $t \to -\infty$].

We now formally second quantize this scalar field. The fields $\Phi(x), \Pi(x)$ become operators, the complex conjugation being replaced by Hermitian conjugation. $\Phi(x), \Pi(x)$ obey the commutation relations (with $x^0 = y^0$)

$$[\Phi(x), \Phi^\dagger(y)] = [\Pi(x), \Pi^\dagger(y)] = [\Phi(x), \Pi^\dagger(y)] = 0, \quad [\Phi(x), \Pi(y)] = i\delta^{(3)}(x, y), \tag{1.22}$$

and similarly for the Hermitian conjugates. $\delta^{(3)}(x, y)$ is the three dimensional Dirac δ function

$$\int_{x^0 = y^0 = \text{const}} \delta^3(x, y) f(y)\, d^3y = f(x).$$

These relations are consistent with the equations of motion (1.4) (see Fulling[2]).

Annihilation operators are defined by

$$a(\omega, m, \lambda, k) = \langle \Phi, \varphi(\omega, m, \lambda, k) \rangle, \quad \kappa(\omega, m, \lambda) > 0 \tag{1.23a}$$

$$b^\dagger(\omega, m, \lambda, k) = \langle \Phi, \varphi(-\omega, -m, \lambda, k) \rangle, \quad \kappa(\omega, m, \lambda) < 0. \tag{1.23b}$$

Using (1.22) and (1.21) we find that these obey the commutation relation

$$[a(\omega, m, \lambda, k), a^\dagger(\omega', m', \lambda', k')] = \delta(\omega - \omega')\delta_{\lambda\lambda'}\delta_{mm'}\delta_{kk'}, \quad [b(\omega, m, \lambda, k), b^\dagger(\omega', m', \lambda', k')] = \delta(\omega - \omega')\delta_{\lambda\lambda'}\delta_{mm'}\delta_{kk'}. \tag{1.24}$$

Had we defined a or b differently in (1.23) [i.e., defined a by (1.23a) for $\omega > 0$ rather than $\kappa(\omega, m, \lambda) > 0$], we would have found the wrong commutation relations here for $a, a^\dagger$. The normalization of the normal modes leads us naturally to this definition for the creation and annihilation operators.

We can now define the "vacuum" state $|0\rangle$ by the requirement that $a|0\rangle = b|0\rangle = 0$ for all a, b, and investigate the properties of this state.

Using Eq. (1.12b) we define an energy-current operator,

$$\mathfrak{S}_E^\mu(x) = \tfrac{1}{6} g^{\mu\nu} \xi_t^\rho (\{\Phi^\dagger_{,\nu}, \Phi_{,\rho}\} + \{\Phi^\dagger_{,\rho}, \Phi_{,\nu}\} - \tfrac{1}{2} g_{\nu\rho} \{\Phi^\dagger_{,\alpha}, \Phi^{,\alpha}\} - \tfrac{1}{2}\{\Phi^\dagger, \Phi_{,\nu;\rho}\} - \tfrac{1}{2}\{\Phi^\dagger_{,\nu;\rho}, \Phi\}); \tag{1.25}$$

$\{\,,\,\}$ indicates the anticommutator. This factor ordering is chosen so as to make the equations symmetric in Φ and $\Phi^\dagger$. Had we chosen the commutator, $\mathfrak{S}_E^\mu$ would have been a trivial infinite constant.

Now examine the vacuum expectation value of this current. The time component will in general be infinite (just as in flat spacetime). We will, however, examine the component $\langle 0|\mathfrak{S}_E^r|0\rangle$ which, far from the black hole, will represent an energy flow out of the black hole. In particular, the total energy gain by of the black hole is

$$-\frac{dE}{dt} = \int_{r,t\,\text{const}} \sqrt{-g}\, \langle 0|\mathfrak{S}_E^r|0\rangle\, d\theta\, d\phi. \tag{1.26}$$

The field Φ can be expanded in normal modes and creation and annihilation operators:

$$\Phi(x) = \sum_m \int_{\kappa(\omega m \lambda) > 0} d\omega \sum_{k\lambda} [a(\omega, m, k)\varphi(\omega, m, k, \lambda, x) + b^\dagger(\omega, m, k, \lambda) \times \varphi(-\omega; -m, k, \lambda, x)]. \tag{1.27}$$

Substituting into (1.25) and using (1.24), the normalization of the angular functions, and the asymptotic form of the normal modes (1.17), we find

$$-\frac{dE}{dt} = \sum_m \int d\omega \sum_k \omega^2 \left(\frac{|A_+(\omega, k, m)|^2 - 1}{2\pi|\omega|} - \frac{|B_-(\omega, k, m)|^2}{2\pi|\omega + m\omega_H|} \right). \tag{1.28a}$$

From (1.19) the integrand is zero unless $\omega(\omega + m\omega_H) < 0$:

$$\frac{dE}{dt} = \frac{\sum_m}{\pi} \int_{(\omega + m\omega_H)\omega < 0} d\omega\, |\omega| \sum_k [1 - |A_+(\omega, m, k)|^2]. \tag{1.28b}$$

This is a negative quantity, indicating a constant outflow of energy from the black hole.

A detailed examination of the radial equation shows that for the superradiant waves, $(\omega + m\omega_H)\omega < 0$, there always exists a potential barrier with height $\propto k^2$ implying that $1 - |A_+|^2 \sim O(e^{-\alpha k^2})$ for some α. As $k \gtrsim |m/2|$ for the superradiant waves, the integral (1.28) will converge. The dominant contribution to the energy flow will come from the low-order modes ($\omega \sim \omega_H$, $m \sim 1$). The integral will therefore be of order $\omega_H{}^2$.

A similar calculation for the radial flow of angular momentum gives

$$\frac{dL}{dt} = \frac{1}{\pi} \sum_m \int_{(\omega + m\omega_H)\omega < 0} d\omega\, m \left(\frac{\omega}{|\omega|}\right)(1 - |A_+|^2), \tag{1.29}$$

which will give a result of order $+\omega_H$. The angular momentum of the black hole is $-Ma$. This term therefore represents a decrease in the absolute value of the angular momentum of the black hole.

The area of the black hole, A, is equal to

$$8\pi[M^2 + (M^4 - M^2a^2)^{1/2}] = 4\pi a/\omega_H = 4\pi L/M\omega_H$$

(see Ref. 17). The half-life of the black hole for radiating away its angular momentum is therefore of order MA. As the age of the universe is only $\sim 10^{62}$ ($\approx 10^{10}$ years),[7] MA must be $\sim 10^{62}$ or $M \sim 10^{21}$ ($\sim 10^{16}$ g) (see Ref. 19) for this process to be significant on time scales of the age of the universe. Any astrophysical black hole is much larger than this ($M \sim 10^{33}$ g), indicating that this quantum instability would have a negligible effect. However, for the tiny black holes postulated by Hawking,[18] this effect could be significant, and such small black holes would probably not be spinning.

Let us now examine the energy and angular momentum current for a single particle sent into the black hole from infinity. The state corresponding to this particle (which we will assume has a fixed value of m and of k; see Ref. 20) is

$$|p\rangle = \int_{\omega > 0} d\omega\, \alpha(\omega)\, a^\dagger(\omega, m, k)|0\rangle$$
$$= \mathfrak{A}^\dagger(m, k)|0\rangle, \tag{1.30}$$

where

$$\int |\alpha(\omega)|^2 d\omega = 1 .$$

The energy current into the black hole in this state is given by

$$\left.\frac{dE}{dt}\right|_{|p\rangle} = \int \sqrt{-g}\, \langle p|\mathfrak{S}_E^r|p\rangle d\theta d\phi$$
$$= \int \sqrt{-g}\, (\langle 0|\, [\mathfrak{A}, \mathfrak{S}_E^r]\mathfrak{A}^\dagger|0\rangle + \langle 0|\mathfrak{S}_E^r[\mathfrak{A}, \mathfrak{A}^\dagger]\, |0\rangle) d\theta d\phi$$
$$= \int \sqrt{-g}\, \langle 0|\, [[\mathfrak{A}, \mathfrak{S}_E^r], \mathfrak{A}^\dagger]\, |0\rangle d\theta d\phi - \left.\frac{dE}{dt}\right|_{|0\rangle} . \tag{1.31}$$

Using the definition of $\mathfrak{S}_E^r$ (1.25) and the expansion of Φ in normal modes [Eq. (1.27)], the first term becomes

$$\int \sqrt{-g}\;\; S_E^r\left(\int \alpha(\omega)\varphi(\omega, m, k) d\omega\right) d\theta d\phi$$
$$= \int \omega[1 - |A_+(\omega, m, k)|^2]|\,\alpha(\omega)|^2 d\omega , \tag{1.32}$$

where $S_E^r(\varphi)$ is defined by Eqs. (1.12) and (1.9), i.e., this term is just the current one would associate with a nonquantized field $\int \alpha(\omega)\varphi(\omega, m, k)d\omega$. The total energy carried into the black hole by this particle is

$$\left.\frac{dE}{dt}\right|_{|p\rangle} - \left.\frac{dE}{dt}\right|_{|0\rangle} = \int \omega[1 - |A_+(\omega, m, k)|^2]|\,\alpha(\omega)|^2 d\omega . \tag{1.33}$$

If $\alpha(\omega)$ is appreciable only for ω such that $\omega(\omega + m\omega_H) < 0$, this expression will be negative, i.e., this state will cause a net decrease in energy of the black hole. [If we had chosen an antiparticle state $\int \beta(\omega) b^\dagger(\omega m k)|0\rangle$ instead, the same results would have been obtained.]

We can define a charge current vector operator:

$$\mathfrak{J}_\mu = \tfrac{1}{2} i\{\Phi^\dagger_{,\mu}, \Phi\} - \{\Phi^\dagger, \Phi_{,\mu}\}. \tag{1.34}$$

The net charge carried into the black hole by the state p is given by

$$\frac{dQ}{dt} = \int \sqrt{-g}\, \langle p|\mathfrak{J}^r|p\rangle d\theta d\phi$$
$$= \int \sqrt{-g}\, (\langle 0|\, [[\mathfrak{A}, \mathfrak{J}^r], \mathfrak{A}^\dagger]|\, 0\rangle + \langle 0|\mathfrak{J}^r|0\rangle) d\theta d\phi . \tag{1.35}$$

We find

$$\left.\frac{dQ}{dt}\right|_{|0\rangle} = \int \sqrt{-g}\, \langle 0|\mathfrak{J}^r|0\rangle d\theta d\phi$$
$$= -\int d\omega \sum_{m,k} \omega \left(\frac{1 - |A_+(\omega, m, k)|^2}{|\omega|} + \frac{|B_-(\omega, k, m)|^2}{|\omega + m\omega_H|}\right). \tag{1.36}$$

But from the structure of the radial equation (1.15), $A_+(\omega, m, k) = A_+(-\omega, -m, k)$, $B_-(\omega, m, k) = B_-(-\omega, -m, k)$, and this integral is zero. [In fact, because of the reality of Eq. (1.5) the vacuum expectation value of the current can be shown to be zero everywhere.] The first term in (1.35) becomes

$$\int \frac{d\omega\, d\omega'}{2\pi}\, \alpha^*(\omega)\alpha(\omega') \int d\theta d\phi \left[\left(\frac{d}{dr}\varphi^*(\omega, m, k)\right)\varphi(\omega', m, k) - \varphi^*(\omega, m, k)\frac{d}{dr}\varphi(\omega', m, k)\right]. \tag{1.37}$$

The net charge carried down the black hole by this particle is

$$\Delta Q = \int \frac{dQ}{dt} dt$$
$$= \int_{\omega>0} d\omega\, |\alpha(\omega)|^2 \frac{\omega}{|\omega|}[1 - |A_+(\omega, m, k)|^2]. \tag{1.38}$$

If $\alpha(\omega)$ is again dominant only in the superradiant modes, this charge current will represent a net charge flow out of the black holes. Had we used a state of the form $\int d\omega\, \beta(\omega) b^\dagger(\omega, m, k)|0\rangle$, we would have found

$$\Delta Q = \int_{\omega>0} d\omega |\beta(\omega)|^2 \left(\frac{-\omega}{|\omega|}\right)[1 - |A_+(-\omega, -m, k)|^2], \tag{1.39}$$

i.e., if $\beta(\omega) = \alpha(\omega)$ this would have been just the negative of (1.38). (The particles whose creation operators are $b^\dagger$ carry opposite charge to those with creation operator $a^\dagger$.)

The total charge q of one of these solutions is

$$q = \int \sqrt{-g}\, \langle p|\, \mathfrak{J}^0 |p\rangle d^3x = +1$$

for the states $|p\rangle = \int \alpha(\omega) a^\dagger(\omega, m, k)|0\rangle$ and $q = -1$ for $|p\rangle = \int \beta(\omega) b^\dagger(\omega, m, k)|0\rangle$.

Sending a positively charged particle into the black hole in a superradiant mode *decreases* the charge of the black hole.

One can define "out" creation and annihilation operators in a similar manner to the "in" operators already defined. The basic "out" normal modes are now defined by their behavior at future infinity and at the future horizon. Positive frequency is defined for the outgoing states (no flow into the future horizon) by demanding that they behave as $e^{-i\omega t}(e^{i\omega r^*} + A_+ e^{-i\omega r^*})$ near infinity with $\omega > 0$. Similarly, the ingoing positive frequency states (no flow out to infinity) behave as $e^{i\tilde{\omega} t}(e^{-i\tilde{\omega} r^*} + A_- e^{+i\tilde{\omega} r^*})$ near the horizon with $\tilde{\omega} > 0$. Expanding Φ in these normal modes will give the associated "out" creation and annihilation operators. The "in" vacuum $|0\rangle$ will not be annihilated by the "out" annihilation operators, but will rather correspond to a coherent superposition of "out" particle states. This can be interpreted as particle creation by the black hole.

The second quantization procedure for electromagnetic and gravitational perturbations will probably differ very little from those for scalar waves except for the additional problem of gauge. In both, an inner product for solutions of the field equations is defined [i.e., the generalizations of (1.6); see Ref. 21]. For the electromagnetic equations, with two solutions for the vector potential, A_1^μ, A_2^μ, it is

$$\langle A_1, A_2\rangle = i \int_{x^0=\text{const}} \sqrt{-g}\, g^{0\mu}(A_1^{*\nu} A_{2[\mu,\nu]} - A^*_{1[\mu,\nu]} A_2^\nu)\, d^3x. \tag{1.40}$$

For gravitational perturbations $h_{1\mu\nu}, h_{2\mu\nu}$ of the background metric $g_{\mu\nu}$, the inner product is

$$\langle h_1, h_2\rangle = \int \sqrt{-g}\, g^{0\mu}[k_1^{*\alpha\beta}(2k_{2\alpha\mu;\beta} - k_{2\alpha\beta;\mu} + \tfrac{1}{2} g_{\alpha\beta} k^\gamma_{2\gamma;\mu}) - k_2^{\alpha\beta}(2k^*_{1\alpha\mu;\beta} - k^*_{1\alpha\beta;\mu} + \tfrac{1}{2} g_{\alpha\beta} k^\gamma_{1\gamma;\mu})]d^3x,$$
$$k_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} h^\alpha{}_\alpha. \tag{1.41}$$

Both the electromagnetic and the gravitational inner products are gauge-independent in source-free spacetimes, i.e.,

$$\langle A_{1\mu} + \lambda_{,\mu}, A_{2\alpha}\rangle = \langle A_{1\mu}, A_{2\alpha}\rangle,$$
$$\langle h_{1\mu\nu} + \xi_{(\mu;\nu)}, h_{2\alpha\beta}\rangle = \langle h_{1\mu\nu}, h_{2\alpha\beta}\rangle$$

(parentheses in subscript denote symmetrization).

The normalization of the normal modes will again indicate a natural definition of creation and annihilation operators. Both fields also superradiate, which will lead to particle creation and an energy flow out to infinity.

From where does this particle creation arise? In the case of a static electromagnetic field, the positive and negative virtual charges in the vacuum can be thought of as being pulled apart by the electrostatic field. But for particles in the gravitational field, all masses are attracted by a gravitational field. In the case of the Kerr metric, however, there is a strong gravitational spin-orbit coupling between the orbiting particles and the spin of the black hole. Near the black hole, this spin orbit coupling becomes strong enough to create negative energy (as seen from infinity) orbits (see, for example, Carter[22]). It is the possibility of decay into such orbits that leads to particle creation in the Kerr metric.

We shall now proceed to investigate neutrinos in the Kerr metric. As has been pointed out by the author,[23] neutrinos do not superradiate, and the stress-energy tensor can have negative energy within the ergosphere. We will show, however, that particle creation can still occur, and that the "negative energy" has a simple interpretation in hole theory.

II. NEUTRINOS

We proceed as for the scalar fields. The neutrino Lagrangian is given by

$$\mathcal{L}=\int_{x^0=\text{const}}\sqrt{-g}\,\overline{\psi}\gamma^\mu\nabla_\mu\psi\, d^3x\,. \tag{2.1}$$

The Dirac matrices, γ^μ, obey the relationships

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}. \tag{2.2}$$

The operators ∇_μ are the spinor covariant derivatives (Bargmann,[24] Brill and Wheeler[25]) with spinor affine connections Γ_μ such that

$$\nabla_\mu\psi=\left(\frac{\partial}{\partial x^\mu}-\Gamma_\mu\right)\psi,\quad \nabla_\mu\overline{\psi}=\frac{\partial}{\partial x^\mu}\overline{\psi}+\overline{\psi}\Gamma_\mu,$$

$$\gamma^\mu{}_{;\nu}-\Gamma_\nu\gamma^\mu+\gamma^\mu\Gamma_\nu=0\,.$$

$\overline{\psi}$ is defined as

$$\psi^\dagger\alpha\,,$$

where α obeys

$$\alpha\gamma^\mu-\gamma^{\mu\dagger}\alpha=0 \tag{2.3}$$

and

$$\alpha_{,\mu}+\Gamma_\mu^\dagger\alpha-\alpha\Gamma_\mu=0\,.$$

We will further assume that $\mathrm{Im}(\mathrm{Tr}\,\Gamma_\mu)=\lambda_{,\mu}$. By an appropriate choice of spinor basis (or rather by an appropriate transformation $\psi\to S\psi$, $\overline{\psi}\to\overline{\psi}S^{-1}$, $\gamma^\mu\to S\gamma^\mu S^{-1}$, etc.) we can choose Γ^μ such that $\mathrm{Tr}\,\Gamma^\mu=0$.

The momentum conjugate to the field ψ is

$$\pi=\frac{\delta\mathcal{L}}{\delta(\partial\psi/\partial x^0)}=i\sqrt{-g}\,\overline{\psi}\gamma^0\,. \tag{2.4}$$

The Hamiltonian becomes

$$\mathcal{H}=\int\left[-\pi(\Gamma_0+\gamma^0\gamma^k\nabla_k\psi)\right]d^3x\,, \tag{2.5}$$

and Hamilton's equations become

$$\frac{\partial\pi}{\partial x^0}=-\pi\Gamma_0-\frac{\partial}{\partial x^k}(\pi\gamma^0\gamma^k)-\pi\gamma^0\gamma^k\Gamma_k\,, \tag{2.6a}$$

$$\frac{\partial\psi}{\partial x^0}=-(-\Gamma_0+\gamma^0\gamma^k\nabla_k)\psi\,. \tag{2.6b}$$

These equations can be summarized by the Euler-Lagrange equation from (2.1):

$$\gamma^\mu\nabla_\mu\psi=0\ ,\quad (\nabla_\mu\overline{\psi})\gamma^\mu=0\,. \tag{2.7a}$$

It is known that neutrinos are purely "left-handed,"[26] a condition which can be expressed by an additional constraint:

$$(1+i\gamma^5)\psi=0\,,\qquad \gamma^5=\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\,. \tag{2.7b}$$

For any two solutions of Eqs. (2.7), the current

$$J^\mu(\psi_1,\psi_2)=\overline{\psi}_1\gamma^\mu\psi_2 \tag{2.8}$$

is conserved (i.e., divergence-free) and defines an inner product on the solutions

$$\langle\psi_1,\psi_2\rangle=\int_{x^0=\text{const}}\sqrt{-g}\,\overline{\psi}_1\gamma^0\psi_2 d^3x\,. \tag{2.9}$$

The energy-momentum tensor is given by Brill and Wheeler[25] as

$$T_{\mu\nu}=\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}=\tfrac{1}{2}i\left[\overline{\psi}\gamma_{(\mu}\nabla_{\nu)}\psi-(\nabla_{(\mu}\overline{\psi})\gamma_{\nu)}\psi\right] \tag{2.10}$$

[the parentheses ($\mu\ \nu$) about the indices indicate symmetrization].

Specializing to the Kerr metric (1.10) we choose our γ matrices so that

$$\begin{aligned}\gamma^t&=\frac{r^2+a^2}{(\Sigma\Delta)^{1/2}}\tilde{\gamma}^0+\frac{a\sin\theta}{\Sigma^{1/2}}\tilde{\gamma}^2\,,\\ \gamma^\phi&=\frac{a}{(\Sigma\Delta)^{1/2}}\tilde{\gamma}^0+\frac{1}{\Sigma^{1/2}\sin\theta}\tilde{\gamma}^2\,,\\ \gamma^r&=\left(\frac{\Delta}{\Sigma}\right)^{1/2}\tilde{\gamma}^3\,,\quad \gamma^\theta=\frac{1}{\Sigma^{1/2}}\tilde{\gamma}^1\,,\\ \tilde{\gamma}^0&=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \tilde{\gamma}^i=\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix},\end{aligned} \tag{2.11}$$

where σ_i are the Pauli matrices.

The Killing vectors again give (by Noether's theorem) two conserved currents:

$$\hat{S}_E^\mu=i\overline{\psi}\gamma^\mu\pounds_{\xi_t^\mu}\psi+\text{comp. conj.}\,,\qquad \hat{S}_L^\mu=i\overline{\psi}\gamma^\mu\pounds_{\xi_\phi^\mu}\psi+\text{comp. conj.} \tag{2.12}$$

[As spinors are not directly geometrical objects, there is a degree of arbitrariness in defining a Lie derivative for them. The γ matrices define a correspondence between spinors and tensors. There exist essentially two separate ways of obtaining a co-spinor from a spinor ψ, i.e., $\overline{\psi}$ the Dirac adjoint, and $\tilde{\psi}=\psi^t C$ (where C obeys $C^t=-C$, $(C\gamma^\mu)^t=-C\psi^\mu$).[27] If we now demand that the Lie derivatives of the matrices C and α be zero, that the Lie derivative be defined covariantly, and that for a Killing vector $\mathcal{L}_\xi \gamma^\mu = 0$, we obtain

$$\mathcal{L}_\xi \psi = \xi^\mu \nabla_\mu \psi + \tfrac{1}{4}\gamma^{[\mu}\gamma^{\nu]}\psi \xi_{\mu;\nu} ,$$
$$\mathcal{L}_\xi \gamma^\mu = \tfrac{1}{2}\gamma^\alpha(\xi^\mu{}_{;\alpha} + \xi_{\alpha;}{}^\mu) . \tag{2.13}$$

This corresponds to a generalization of the naive procedure where the Lie derivative of a spinor in a coordinate system tied to the Killing vector is just the ordinary coordinate derivative.]

The Killing vectors also give us conserved currents via the energy-momentum tensor

$$S_E^\mu = T^\mu{}_\nu \xi_t^\nu$$
$$= \tfrac{1}{4} i[\overline{\psi}\gamma^\mu(\xi^\nu{}_t \nabla_\nu)\psi + \overline{\psi}(\xi_t^\nu \gamma_\nu)\nabla^\mu \psi] + \text{comp. conj.} , \tag{2.14}$$

and similarly for S_L^μ. The currents in (2.12) and (2.14) differ by an absolute divergence

$$S^\mu - \hat{S}^\mu = \tfrac{1}{4} i(\overline{\psi}\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\psi\xi_\rho)_{;\nu} . \tag{2.15}$$

The normal mode solutions are defined by

$$\mathcal{L}_{\xi_t^\mu}\varphi(\omega) = -i\omega\varphi(\omega) . \tag{2.16}$$

These normal modes with real ω form a complete set of solutions for the neutrino equations in the Kerr metric. Defining the operator

$$H = i[-\Gamma_t + (\gamma^t)^{-1}\gamma^k \nabla_k] , \tag{2.17}$$

we find that for the normal modes [by (2.18), (2.15), (2.7)]

$$H\psi = \omega\psi .$$

The inner product (2.9) is positive definite in the Kerr metric[23] and the operator H is Hermitian with respect to this inner product,

$$\langle H\psi_1, \psi_2\rangle - \langle \psi_1, H\psi_2\rangle = i\frac{\partial}{\partial t}\int_{t=\text{const}} \sqrt{-g}\,\overline{\psi}_1\gamma^t\psi_2 d^3x - i\int \sqrt{-g}\,\overline{\psi}\frac{\partial}{\partial t}(\sqrt{-g}\,\gamma^t)\psi d^3x = 0 ,$$

as the inner product is independent of t [by the conservation of (2.8)] and $\sqrt{-g}\,\gamma^t$ is independent of time.

This implies $(\omega_1^* - \omega_2)\langle\psi_1, \psi_2\rangle = 0$, from which we must have $\omega_1^* = \omega_1$ for $\psi_1 = \psi_2$; and for $\omega_1 \neq \omega_2$, $\langle\psi_1, \psi_2\rangle = 0$, i.e., no unstable modes exist for neutrinos.

For these normal mode solutions, $\hat{S}_E^\mu$ is just ωJ^μ. The energy current calculated from the energy-momentum tensor S_E^u has an additional term given by (2.17). This extra term seems to correspond to an internal energy density for the particle, which could be interpreted as a sort of internal Zitterbewegung energy for the particle.

Again, we also define the angular modes by

$$\mathcal{L}_{\xi_\phi^\mu}\varphi = -im\varphi . \tag{2.18}$$

Using (2.18), (2.16), (2.17), (2.11), and (2.7) we find[23] that the normal mode solution for the neutrino equations are separable. Setting

$$\psi = \frac{e^{-i\omega t}e^{-im\phi}}{[\Delta \sin^2\theta(r + ia\cos\theta)]^{1/4}}\begin{pmatrix}\eta\\ \eta\end{pmatrix} ,$$
$$\eta = \begin{pmatrix} R_1(r)S_1(\theta)\\ R_2(r)S_2(\theta)\end{pmatrix} , \tag{2.19}$$

it can be shown that the angular and radial functions obey the equations

$$\left(\frac{d}{dr} - \frac{i}{\Delta}[\omega(r^2 + a^2) + ma]\right)R_1(r) = \frac{k}{\Delta^{1/2}}R_2(r) ,$$
$$\left(\frac{d}{dr} + \frac{i}{\Delta}[\omega(r^2 + a^2) + ma]\right)R_2(r) = \frac{k}{\Delta^{1/2}}R_1(r) , \tag{2.20}$$

$$\left(\frac{d}{d\theta} + \omega a\sin\theta + \frac{m}{\sin\theta}\right)S_1(\theta) = kS_2(\theta) ,$$
$$\left(\frac{d}{d\theta} - \omega a\sin\theta - \frac{m}{\sin\theta}\right)S_2(\theta) = -kS_1(\theta) . \tag{2.21}$$

The angular functions are normalized so the $S_1(\theta)$ is real and (for any ω, m, k)

$$\int S_1{}^2(\theta)d\theta = \int S_2{}^2(\theta)d\theta = \frac{1}{4\pi} . \tag{2.22a}$$

The equality of these two integrals follows from Eq. (2.21) which implies $S_1(-\theta) = \pm S_2(\theta)$, $S_2(-\theta) = \mp S_1(\theta)$. This also implies

$$\int S_1(\theta)S_2(\theta)d\theta = 0 . \tag{2.22b}$$

The two linearly independent solutions to (2.20) are defined by

$$(R_1^+(r), R_2^+(r)) \sim \frac{1}{\sqrt{2\pi}} \times \begin{cases} (A_+(\omega, m, k)e^{i\omega\hat{r}}, e^{-i\omega\hat{r}}), & \hat{r} \to \infty\\ (0, B_+(\omega, m, k)e^{-i(\omega + m\omega_H)\hat{r}}), & \hat{r} \to -\infty ;\end{cases}$$
$$(R_1^-(r), R_2^-(r)) \sim \frac{1}{\sqrt{2\pi}} \times \begin{cases} (B_-(\omega, m, k)e^{i\omega\hat{r}}, 0), & \hat{r} \to \infty\\ (e^{i(\omega + m\omega_H)\hat{r}}, A_-(\omega, m, k)e^{-i(\omega + m\omega_H)\hat{r}}), & \hat{r} \to -\infty .\end{cases} \tag{2.23}$$

Again the + solutions represent particles originating from infinity, whereas the − solutions represent particles originating from the past horizon of the black hole. (The boundary conditions at the horizon are chosen so that a wave packet made of the + solutions has a velocity into the black hole near the horizon.)

For two solutions to Eq. (2.20), (R_1,R_2) and $(\tilde{R}_1,\tilde{R}_2)$, the quantities $\tilde{R}_1R_2-\tilde{R}_2R_1$ and $\tilde{R}_1^*R_1+\tilde{R}_2^*R_2$ are independent of r. We thus find

$$1-|A_\pm(\omega,m,k)|^2=|B_\pm(\omega,m,k)|^2\,,$$
$$A_-(\omega,m,k)B_+(\omega,m,k)=A_+(\omega,m,k)B_-(\omega,m,k)\,,$$
$$B_-^*(\omega,m,k)=-B_+(\omega,m,k)\,. \qquad (2.24)$$

Furthermore, using (2.22) and (2.23) we find that

$$\langle\varphi(\lambda,\omega,m,k),\varphi(\lambda',\omega',m',k')\rangle=\delta_{\omega\omega'}\delta_{mm'}\delta_{kk'}\delta_{\lambda\lambda'}\,. \qquad (2.25)$$

We now show that a normal mode solution can have positive energy density near infinity and negative energy density (in an orthonormal frame) near the black hole. The energy density with respect to some timelike vector n^μ is given by $T_{\mu\nu}n^\mu n^\nu$. Choose the vector $n_\mu=\nabla_\mu t$. This vector is timelike everywhere [in the Kerr metric $g^{tt}=(r^2+a^2)^2/\Delta\Sigma-a^2\sin^2\theta/\Sigma>0$ everywhere outside the horizon]. In a $\lambda=+$ normal mode solution with $\omega>0$, one finds that near infinity

$$T_{\mu\nu}n^\mu n^\nu=\frac{\omega[S_1{}^2(\theta)+|A_+|^2S_2{}^2(\theta)]}{2\pi(\Delta\Sigma)^{1/2}\sin\theta}>0\,. \qquad (2.26a)$$

Near the horizon, on the other hand,

$$T_{\mu\nu}n^\mu n^\nu=\frac{\omega+m\omega_H}{2\pi(\Delta\Sigma)^{1/2}\sin\theta}|B_+|^2S_1{}^2(\theta)\,, \qquad (2.26b)$$

which is negative if $\omega+m\omega_H<0$.

To quantize the neutrino fields, impose the canonical anticommutation relations (with $x^0=y^0$)

$$\{\Pi(x),\Psi(y)\}=i\delta^{(3)}(x,y) \qquad (2.27a)$$

or

$$\sqrt{-g}\{\overline{\Psi}(x)\gamma^t(x),\Psi(y)\}=\delta^{(3)}(x,y)\,. \qquad (2.27b)$$

Define

$$a(\lambda,\omega,m,k)=\langle\varphi(\lambda,\omega,m,k),\Psi\rangle;\quad \kappa(\omega,m,\lambda)>0\,, \qquad (2.28)$$
$$b^\dagger(\lambda,\omega,m,k)=\langle\varphi(\lambda,\omega,m,k),\Psi\rangle;\quad \kappa(\omega,m,\lambda)<0\,.$$

From (2.27) and (2.25) we find[28]

$$\{a(\lambda,\omega,m,k),a^\dagger(\lambda',\omega',m',k')\}=\delta_{\lambda\lambda'}\delta_{\omega\omega'}\delta_{mm'}\delta_{kk'} \qquad (2.29)$$

and similarly for b, $b^\dagger$. Furthermore we find

$$\{a,b\}=\{a^\dagger,b\}=0 \text{ for all } a,b\,.$$

The vacuum state is now defined by

$$a|0\rangle=b|0\rangle=0 \text{ for all } a,b\,.$$

Note that for neutrinos, no special choice of annihilation operators is forced on us. We can equally well call what in terms of this vacuum is a one-particle state [i.e., $a^\dagger(\omega,m,k,\lambda)|0\rangle$] our new vacuum state, and redefine our annihilation operators so that for this particular state $\tilde{a}=a^\dagger$, while maintaining all the rest of the annihilation operators.

We have chosen our vacuum to correspond to our choice for the scalar vacuum—namely, an observer near the past horizon will see no particles coming out of the past horizon of the black hole (i.e., for him, all states with negative frequency, $\omega+m\omega_H<0$ are filled). This state is also in some sense a minimum in energy (i.e., the energy flowing out of the black hole is minimal in the $|0\rangle$ state when compared to any other state of the form $a_1^\dagger a_2^\dagger\cdots a_n^\dagger b_1^\dagger b_2^\dagger\cdots b_m^\dagger|0\rangle$).

We investigate the energy and angular momentum flow through a surface near infinity. The energy-momentum tensor operator is defined by

$$T_{\mu\nu}=\tfrac{1}{2}i[\overline{\Psi},(\gamma_{(\mu}\nabla_{\nu)}\Psi)]+\text{Herm. conj.} \qquad (2.30)$$

([,] is the commutator).

The net inflow of energy and angular momentum through a surface $r=$ const as $r\to\infty$ is given by

$$\frac{dE}{dt}=-\int\sqrt{-g}\,\xi_t^\mu\langle 0|T_\mu{}^r|0\rangle d\theta d\phi\,,$$
$$\frac{dL}{dt}=-\int\sqrt{-g}\,\xi_\phi^\mu\langle 0|T_\mu{}^r|0\rangle d\theta d\phi\,. \qquad (2.31)$$

Expand the fields in terms of the creation and annihilation operators (2.28)

$$\Psi(x)=\sum_m\int_{\kappa(\omega m\lambda)>0}d\omega\sum_{k\lambda}[a(\omega mk\lambda)\varphi(\omega,m,k,\lambda,)+b^\dagger(\omega,m,k,\lambda,)\times\varphi(-\omega,-m,k,\lambda)]\,.$$

Equations (2.31) become [using (2.23)]

$$\frac{dE}{dt}=\frac{1}{2\pi}\int d\omega\,\omega\sum_{mk}[\kappa(\omega,m,+)(|A_+(\omega,m,k)|^2-1)+\kappa(\omega,m,-)|B_-(\omega,m,k)|^2]\,,$$
$$\frac{dL}{dt}=\frac{1}{2\pi}\int d\omega\sum_{mk}m[\kappa(\omega,m,+)(|A_+(\omega mk)|^2-1)+\kappa(\omega,m,-)|B_-(\omega mk)|^2]\,. \qquad (2.32)$$

If $\kappa(\omega,m,+)\kappa(\omega,m,-)=+1$, the integrands are zero [by (2.24)]. However, when $\kappa(\omega,m,+)\kappa(\omega,m,-)=-1$ [which corresponds to $\omega(\omega+m\omega_H)<0$ by the definition of $\kappa(\omega, m, \lambda)$ in (1.20)] they do not cancel and the integrals become

$$\frac{dE}{dt} = \frac{4}{2\pi} \sum_m \int_{\substack{\omega(\omega+m\omega_H)<0;\\ \omega>0}} d\omega\, \omega \sum_k (|A_+(\omega, m, k)|^2 - 1)\,. \tag{2.33}$$

This expression is negative, indicating an energy flow out of the black hole. Similarly, there exists a corresponding flow of angular momentum out of the black hole:

$$\frac{dL}{dt} = \frac{4}{2\pi} \sum_m \int_{\substack{\omega>0;\\ \omega(\omega+m\omega_H)<0}} d\omega\, m \sum_k (|A_+(\omega, m, k)|^2 - 1)\,.$$

Our conclusions are very similar to those for the scalar case. There will be a net spin down of the black hole on a time period of order $1/M^3$ because of neutrino emission.

The result in Eq. (2.26) can now be interpreted. Near the black hole, we find a net flow of neutrinos and antineutrinos with $\omega(\omega + m\omega_H) < 0$ into the black hole. If we send in a neutrino whose frequency is such that $\omega + m\omega_H < 0$ toward the black hole, we find that a part of this flux is suppressed. To the observer near the horizon, this neutrino looks like the absence of an antineutrino (i.e., the filling of a hole in the Dirac sea). The fact that the energy density is negative (with respect to the original $|0\rangle$ state) is not surprising (i.e., a state with no antineutrino has less energy than one with an antineutrino in it).

The absence of superradiance in neutrinos cannot be used to violate the area theorems of Hawking.[29] The spontaneous excitation of neutrinos in the vacuum state serves to constantly increase the area of the black hole. Because of the exclusion principle, one can at best suppress this constant area increase (this possibility was suggested by Feynman[30]).

III. CRITIQUE

We will now make explicit the assumptions which have led to our results, and indicate possible problems with these assumptions.

The basic assumption we have made is in the manner in which we have defined our vacuum state $|0\rangle$. For the $\lambda = +$ states, we are fairly confident of our procedure and definition of creation and annihilation operators. These states represent particles coming from infinity where spacetime becomes flat. The procedure we have followed for these states corresponds to the definition of the vacuum in elementary particle theory.[31]

However, we are much more uncertain about our procedure for the $\lambda = -$ states, states originating in the black hole itself. We have demanded that an observer near the black hole see no particles coming out of the black hole. On the other hand, the region behind the past horizon is highly dynamic, and includes a naked singularity. Our assumption about the definition of creation and annihilation operators [Eqs. (1.23) and (2.28)] is a very special assumption about the fields in the dynamic region.[34] Furthermore, work by Fulling[2,32] and the author[33] on Fock representations of scalar particles and neutrinos in accelerated coordinates in flat spacetime indicates that we have here also made a very special assumption about the representation of the field operators. (This question is further discussed in a paper under preparation in which a different state from that given here is suggested as the vacuum state.)

Furthermore, our transition from Eqs. (1.27), or (2.32) to (1.28), or (2.33) has performed the summation in a very special way, namely, we have summed first over λ, and then over the remaining variables ω, m, k. As this summation is formally indefinite [like the sum $\sum_{m=0}^{\infty}(-1)^m$], the result one obtains depends critically on the order of summation.

A realistic black hole would be expected to have been formed in the collapse of some matter distribution. For such a black hole, the past horizon (plus the naked singularity) does not exist. Furthermore, all particle states must originate at infinity. Consider now some material body, assumed to be stationary before some time $t = 0$. Before its collapse, any wave packet state originating at infinity must eventually return to infinity (the matter does not absorb the wave packet). During the collapse process, however, the wave packets inside the body may eventually reemerged during the collapse, but will be very strongly redshifted by the increasing gravitational field; i.e., their energy will be very much reduced. Furthermore, states originating at infinity could now disappear through the future horizon of the black hole. If one calculates mode by mode, one finds that the energy current through a surface near infinity becomes infinite in the vacuum state.

The formal sign of this infinity is different in the case of bosons and fermions. Naively, the zero-point oscillations for bosons have infinite positive energy, whereas the filled negative energy sea for fermions will have an infinite negative energy. The black hole will gain an infinite amount of energy from bosons and lose an infinite amount due to fermions per united time.

To make any physical sense these infinities must somehow be renormalized. The question is whether a physically reasonable renormalization procedure will indicate particle creation by the field of a collapsing star. Hawking[35] has recently examined this question and has come to the conclusion that such a star will lose energy at an asymptotic rate proportional to M^{-2}, where M is its mass. He, however, also does not treat the problems of renormalization except at infinity.

The most naive method of renormalization would be to normal order the energy-momentum tensor. This is unsatisfactory as it simply assumes that vacuum fluctuations will have no observable gravitational effects.

Alternately, one can argue that when the energy of a photon, etc. becomes high enough ($E \sim 1$ in units with $G = c = \hbar = 1$), quantum gravitational cutoffs will come into play. This energy is, however, $\sim 10^{22}$ MeV, leading to a vacuum energy density of $\sim 10^{120}$ MeV/cm^3 (Planck density) which is of no help.

Another possibility is to use the fact that the expectation values for fermions and bosons differ in the sign of their infinities to cancel each other. For massless particles the four massless fermions (electron and muon neutrino and antineutrino) could cancel the four states (two helicity states for each of photons and gravitons) of bosons. Where the massive particles fit in would be a rather serious problem to this viewpoint.

As yet no acceptable renormalization procedure, to my knowledge, has been found. However the problem of renormalizing the energy-momentum tensor is finally solved, some features of the present investigation would probably still remain. One would expect that in a stationary metric such as the Kerr metric, quantum vacuum instabilities could arise, leading to a spin down of the black hole. Again these effects would probably be significant only for small black holes (but still appreciably larger than the Planck mass black holes, i.e., 10^{-5} g). The ultimate resolution may be possible only when some theory is found in which both quantum mechanics and the principles of general relativity can be united.

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[6]The vacuum as used throughout this paper is not what is referred to as the vacuum in elementary particle physics (i.e., the basic state invariant under the full Poincaré group) which is the state with nothing in it. Here, there exists some given background, either a nonzero gravitational field or some classical distribution of sources. The vacuum state is in some sense the ground state for that theory having in some, as yet unknown, sense a minimum number of quanta of the quantized field present. For example, in quantum electromagnetics, if a given nonquantized source for the field is given, the vacuum would be the state with no incoming photons, rather than a state with no electromagnetic field at all (i.e., the nonquantized charge would be expected to have a $1/r^2$ electromagnetic field about it in the "vacuum" state).

[7]Our convention is to set $\hbar = c = 8\pi G = 1$; the metric $g^{\mu\nu}$ has signature $(+\,-\,-\,-)$ and the Riemann tensor is defined by

$$\xi_{\mu;\nu;\rho} - \xi_{\mu;\rho;\nu} = R^{\alpha}{}_{\mu\nu\rho}\,\xi_{\alpha}.$$

[8]See, for example, L. Parker, Phys. Rev. D 7, 976 (1973).

[9]S. L. Detwiler and J. R. Ipser, Astrophys. J. 185, 675 (1973).

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[11]S. Hawking, Commun. Math. Phys. 33, 323 (1973).

[12]B. Carter, Phys. Rev. 174, 1154 (1968).

[13]*Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards Applied Mathematics Series, No. 55 (U.S.G.P.O., Washington, D.C., 1964), Chap. 21.

[14]C. W. Misner (private communication to W. Press and S. Teukolsky, 1972); Ya. B. Zeldovich, Zh. Eksp. Teor. Fiz. Pis'ma Red. 14, 270 (1971) [JETP Lett. 14, 180 (1971)].

[15]J. Bekenstein, Phys. Rev. D 7, 949 (1973).

[16]W. Press and S. Teukolsky, Nature 238, 211 (1972).

[17]See, for example, S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, New York, 1973).

[18]S. W. Hawking, Mon. Not. R. Astron. Soc. 152, 75 (1971).

[19]These results were first derived from the phenomenon of superradiance by order of magnitude estimates by A. A. Starobinskiĭ (Zh. Eksp. Teor. Fiz. 64, 48 (1973) [Sov. Phys.—JETP 37, 28 (1973)]) and independently by Don Page (private communication).

[20]k is actually a function of ω, m, and some integer which we can take as equal to the number of zeros of the function $S(\theta)$. This integer and the value of the eigenvalue both will be represented by the symbol k. In cases of possible confusion we will write the eigenvalue explicitly as $k(\omega, m)$.

[21]These currents are derived from the complex Lagrangians

$$\mathfrak{L}_{\mathrm{cm}} = \int \sqrt{-g}\, A^{*}_{[\mu,\nu]} A^{\mu;\nu} d^3x\,,$$

$$\mathfrak{L}_{G} = \int \sqrt{-g}\, (2k^{*\mu\nu}{}_{;}{}^{\rho} k_{\mu\rho;\nu} - k^{*\mu\nu}{}_{;}{}^{\rho} k_{\mu\nu;\rho} + \tfrac{1}{2} k^{*\alpha}{}_{\alpha;\mu} k^{\beta}{}_{\beta;}{}^{\mu}) d^3x\,.$$

(See R. A. Isaacson [Phys. Rev. 166, 1263 (1968)] for the form of the Lagrangian.) The gauge invariance of the inner product holds only if the electromagnetic and gravitational fields (including the background space-time for gravitational fields) are source free.

[22]B. Carter, Phys. Rev. 174, 1559 (1968). See also R. Penrose, Riv. Nuovo Cimento 1, 252 (1969).

[23]W. Unruh, Phys. Rev. Lett. 31, 1265 (1973).

[24]V. Bargmann, Berl. Ber. 346 (1932).

[25]D. Brill and J. A. Wheeler, Rev. Mod. Phys. 29, 465 (1957).

[26]T. D. Lee and C. N. Yang, Phys. Rev. 104, 254 (1956); C. S. Wu *et al.*, *ibid.* 105, 1413 (1957).

[27]I thank Professor A. T. Taub for reminding me of this fact.

[28]Note that the (anti-) commutation relations of the creation and annihilation operators are forced by the canonical field (anti-) commutation relations.

[29]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, New York, 1973), and references therein.

[30]W. Press, S. Teukolsky, and D. Page (private communication).

[31]See, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Chap. 15, for the definition of the "in" vacuum.

[32]S. Fulling, Phys. Rev. D 7, 2850 (1973).

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[34]In particular for neutrinos, if we defined our vacuum so that all states, $\lambda = +, -$ with $\omega < 0$ are filled (i.e., $\omega > 0$ states correspond to particle states), all spontaneous creation is suppressed and $dE/dt = dL/dt = 0$ in this vacuum state.

[35]S. Hawking, Nature 248, 30 (1974); Cambridge University report, 1974 (unpublished).