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Current-algebra theorems for π-π scattering*

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The Adler theorem for π-π scattering is studied with the pions on the mass shell. A technique for evaluating the correction terms is described, and they are estimated using previous hard-pion results. This evaluation is then used to determine fairly stringent bounds on the S-wave scattering lengths. We find it likely that $0.56 \lesssim (2a_0 - 5a_2)m_\pi \lesssim 0.66$. We also investigate the on-mass-shell sum rules for the σ terms.

I. INTRODUCTION

One of the unsolved problems in the theory of the pion-pion interaction is the question of the compatibility of the Adler theorem[1,2] with the phenomenological description of the process. Thus, Adler found that a large $I=0$ S-wave interaction, which he attributed to a large scattering length, was needed to satisfy the sum rule. In contrast with this, Weinberg's soft-pion analysis[3] of $\pi\pi$ scattering found that small scattering lengths were possible.

Another indication that current algebra favors small $\pi\pi$ scattering lengths follows from the work of Meiere and Sugawara,[4] who related the Adler integral to the combination $2a_0 - 5a_2$ of $I=0$ and $I=2$ S-wave scattering lengths. The sum rule can be satisfied for quite small values of the scattering lengths.

An alternative approach[5] postulates the existence of a broad $I=0$ S-wave resonance (σ or ϵ) which, with the well-established ρ and f^0 mesons, would saturate the Adler integral, allowing the sum rule to be satisfied. There is now some preliminary indication[6] that such a resonance may exist.

Left unresolved in the above treatments is the size of the error resulting from relating on-mass-shell data by means of a sum rule which holds only for zero-mass pions. In order to test his relation for $\pi\pi$ scattering, Adler made[1] certain "kinematic" corrections and smoothness assumptions. Meiere and Sugawara estimate the effect of off-mass-shell extrapolations by means of a dispersive technique which takes into account low-lying, induced cuts in the pion mass.[7]

It would clearly be more satisfying if we could utilize the low-energy theorem with all amplitudes fully on the mass shell. This, of course, could possibly trade one difficulty for another. We would no longer need to know how to extrapolate off shell, but instead would have to evaluate certain additional terms which vanish in the soft-pion limit. In our opinion, however, this is an advantage, since, in principle, the on-mass-shell sum rule can be tested using experimental data and conventional dispersive techniques.

Such an approach has been successfully carried out recently by Brown, Pardee, and Peccei.[8] In a study of the on-mass-shell version of the low-energy theorems for πN scattering, they have confirmed that corrections to the πN Adler-Weisberger relation are of order m_π^2/m_N^2.

In this paper we investigate the on-mass-shell version of the current-algebra theorems for $\pi\pi$ scattering. A dispersive approach is introduced for the evaluation of the correction terms, which allows them to be evaluated in terms of on-mass-shell matrix elements of the isovector axial-vector current. To estimate the correction terms, we assume that the dominant contributions come from the ρ and an $I=0$ $J^P=0^+$ σ meson. Previous hard-pion analyses are used to obtain the relevant matrix elements of the axial-vector current between π and ρ,[9] and between π and σ.[10] The requirement that the current-algebra theorems manifest the correct pole structure gives rise to conditions which determine the unknown ρ and σ parameters. It is found that the correction to the Adler relation, which is obtained from *unsubtracted* dispersion relations, is less than ~25% of the other terms. The on-mass-shell Adler relation, when combined with a standard dispersion relation for the crossing-odd amplitude evaluated at threshold, gives rise to a sum rule for the combination of S-wave scattering lengths, $2a_0 - 5a_2$. This sum rule is the corrected version of one given by Weinberg.[3,4] From it we arrive at very restrictive bounds on the S-wave scattering lengths. These bounds are consistent with the low values found by Weinberg.[3] In particular, we find that the $I=0$ S-wave scattering length cannot be greater than $0.33\ m_\pi^{-1}$.

In addition to the Adler sum rule, which involves amplitudes having a t-channel isospin of one ($I_t=1$), we also study the corresponding $I_t=0$ and $I_t=2$ $\pi\pi$ sum rules which can be used to determine the $\pi\pi$ σ terms. A knowledge of the latter quantities would shed light on the way chiral SU(3)×SU(3) symmetry is broken. We relate the $I=0$ σ term to the scattering length combination a_0+5a_2, plus a correction term. The dispersion relations which determine this correction term may need subtractions. Nevertheless, we are encouraged[11] to present results based on unsubtracted dispersion relations and to compare them with other predictions.

The present investigation has some features in common with previous hard-pion treatments[12] of π-π scattering. However, in contrast with this earlier work, in which the construction of the complete low-energy π-π amplitude was the goal, our focus is much narrower. While there is a rough consistency between our results and the results of the other calculations, we believe that our determination of the correction terms is more reliable, since it appears to involve fewer assumptions.

In Sec. II we write down the hard-pion form of the current-algebra sum rules. A detailed discussion is given of our method of calculating the correction terms. The ρ and σ contributions to the correction terms are evaluated by means of hard-pion three-point functions in Secs. III and IV, respectively. In Sec. V we discuss the corrected Adler sum rule and use it to impose a bound on the S-wave scattering lengths. The $I_t=0$ and 2 sum rules are dealt with in Sec. VI. We present our conclusions in Sec. VII.

II. DEFINITION OF THE AMPLITUDES

Using the PCAC relation

$$\partial_\nu A_a^\nu(x)=F_\pi\mu^2\phi_a(x)\,, \tag{1}$$

where μ is the pion mass, and the standard reduction technique, the S-matrix element corresponding to the on-mass-shell process

$$\pi_a(q)+\pi_c(p_\alpha)\to\pi_b(p)+\pi_d(p_\beta) \tag{2}$$

can be written in the form[13,14]

$$\begin{aligned}S_{bd,ac}=&-(2\pi)^{-3}(2p^0 2q^0)^{-1/2}\\&\times\int d^4x\,d^4y\,e^{-ipy+iqx}\langle\pi_d(p_\beta)|\big(F_\pi^{-2}p_\mu q_\nu T\{\tilde A_b^\mu(y)\tilde A_a^\nu(x)\}+F_\pi^{-2}\delta(y^0-x^0)[A_b^0(y),\partial_\nu A_a^\nu(x)]\\&+iq_\nu F_\pi^{-2}\delta(y^0-x^0)[A_b^0(y),A_a^\nu(x)]\big)|\pi_c(p_\alpha)\rangle\,.\end{aligned} \tag{3}$$

Defining the commutators

$$\begin{aligned}&\delta(y^0-x^0)[A_b^0(y),A_a^\nu(x)]=i\epsilon_{bac}V_c^\nu(y)\delta^4(x-y)\,,\\&\delta(y^0-x^0)[A_b^0(y),\partial_\nu A_a^\nu(x)]=i\sigma_{ab}(y)\delta^4(x-y)\end{aligned} \tag{4}$$

and defining the invariant amplitude M by

$$S_{bd,ac}=\frac{-i(2\pi)^4\delta(q+p_\alpha-p-p_\beta)}{(2\pi)^6(16p^0q^0p_\alpha^0p_\beta^0)^{1/2}}M_{bd,ac}\,, \tag{5}$$

we can write Eq. (3) as

$$F_\pi^2M_{bd,ac}(\nu,\nu_B)=p_\lambda q_\rho R_{bd,ac}^{\lambda\rho}(\nu,\nu_B)+\sigma_{ab,cd}(t)-2\mu\nu\epsilon_{bae}\epsilon_{dce}F(t)\,, \tag{6}$$

where

$$\sigma_{ab,cd}(t)=(2\pi)^3(2p_\alpha^0 2p_\beta^0)^{1/2}\langle\pi_d(p_\beta)|\sigma_{ab}(0)|\pi_c(p_\alpha)\rangle\,, \tag{7}$$

$$\frac{i(2\pi)^4\delta(q+p_\alpha-p-p_\beta)}{(2\pi)^3(2p_\alpha^0 2p_\beta^0)^{1/2}}R_{bd,ac}^{\mu\nu}=\int d^4x\,d^4y\,e^{-ipy+iqx}\times\langle\pi_d(p_\beta)|T\{\tilde A_b^\mu(y)\tilde A_a^\nu(x)\}|\pi_c(p_\alpha)\rangle\,, \tag{8}$$

and $F(t)$ is the pion electromagnetic form factor. We have introduced the invariants

$$\begin{aligned}\nu&=\frac{-p\cdot(p_\alpha+p_\beta)}{2\mu}=\frac{s-u}{4\mu}\,,\\ \nu_B&=\frac{p\cdot q}{2\mu}=\frac{t-2\mu^2}{4\mu}\,,\end{aligned} \tag{9}$$

with $s=-(q+p_\alpha)^2$, $u=-(q-p_\beta)^2$, $t=-(p-q)^2$.

By projecting out the t-channel isospins $I_t=0$, 1, and 2 in Eq. (6) we obtain the three low-energy theorems

$$F_\pi^2M^{(0)}(0,\nu_B)=[p^\lambda q^\rho R_{\lambda\rho}^{(0)}(\nu,\nu_B)]|_{\nu=0}+\sigma^{(0)}(t)\,, \tag{10}$$

$$F_\pi^2\frac{\partial}{\partial\nu}M^{(1)}(\nu,-\mu/2)\Big|_{\nu=0}=-4\mu+\frac{\partial}{\partial\nu}[p^\lambda q^\rho R_{\lambda\rho}^{(1)}(\nu,\nu_B)]\Big|_{\nu=0;\,\nu_B=-\mu/2}\,, \tag{11}$$

$$F_\pi^2M^{(2)}(0,\nu_B)=[p^\lambda q^\rho R_{\lambda\rho}^{(2)}(\nu,\nu_B)]|_{\nu=0}+\sigma^{(2)}(t)\,, \tag{12}$$

where the superscript refers to the t-channel isospin.

Relation (11) is the Adler theorem for the on-mass-shell amplitude. If we assume an unsubtracted dispersion relation in ν for the odd amplitude $M^{(1)}$ for fixed ν_B, we can write Eq. (11) as

$$\frac{2F_\pi^2}{\pi}\int_0^\infty dq\frac{q^2}{\omega^3}[\sigma_{+-}(\omega)-\sigma(\omega)_{++}]$$
$$=2-\frac{1}{2\mu}\frac{\partial}{\partial\nu}[p^\lambda q^\rho R^{(1)}_{\lambda\rho}(\nu,\nu_B)]\Big|_{\nu=0;\ \nu_B=-\mu/2}, \quad (13)$$

where $\sigma_{+\mp}$ is the on-mass-shell total cross-section for $\pi^\pm\pi^\mp$ scattering and use has been made of the relation

$$4\mu q[\sigma_{+-}(\omega)-\sigma_{++}(\omega)]=\mathrm{Im}\,M^{(1)}(\nu,-\mu/2).$$

The variable $\omega=(s-2\mu^2)/2\mu$ is equal to ν for $t=0$, and q is defined by $\omega^2=q^2+\mu^2$, these being the lab energy and momentum of the pion, respectively. In Eq. (13) all particles are on their mass shell, so that there is no ambiguity as to comparison with experiment, provided that the additional term

$$R^{(1)}\equiv\frac{1}{2\mu}\frac{\partial}{\partial\nu}[p^\lambda q^\rho R^{(1)}_{\lambda\rho}(\nu,\nu_B)]\Big|_{\nu=0;\ \nu_B=-\mu/2} \quad (14)$$

can be evaluated.

It should be pointed out that the choice of ν and ν_B indicated in Eqs. (10)–(12) and to be used below is a matter of convenience since Eq. (6) is valid for all ν and ν_B. In the case of the Adler sum rule we have made the conventional assignments in order to be able to discuss previous work. We have more freedom in treating the other relations, however.

Equations (10) and (12) are on-mass-shell sum rules for the $I_t=0$ and $I_t=2$ contributions, respectively, of the σ commutator given in Eq. (4). The σ terms can, in principle, provide information on how chiral symmetry is broken. For example, when the symmetry-breaking term in the strong-interaction Hamiltonian belongs to the $(3,3^*)+(3^*,3)$ representation[15,16] of SU(3)×SU(3), $\sigma^{(2)}(t)\equiv0$. However, if there are terms belonging to, e.g., (8, 8) then, in general, both $\sigma^{(0)}$ and $\sigma^{(2)}$ will be nonvanishing. In particular, it is the quantities $\sigma^{(0)}(0)$ and $\sigma^{(2)}(0)$ that are predicted by the various symmetry-breaking models. To test the prediction, we should set $t=0$ ($\nu_B=-\mu/2$) in Eqs. (10) and (12). There may be some interest, however, in setting $\nu_B=0$ ($t=2\mu^2$) in these relations, since this is the soft-pion point, as well as the one utilized by Cheng and Dashen.[17]

Again, assuming that the data were good enough to reliably determine $M^{(0)}$ and $M^{(2)}$, we would still need to know $R^{(0)}(0,\nu_B)$ and $R^{(2)}(0,\nu_B)$, where

$$R^{(0)}(\nu,\nu_B)\equiv p^\mu q^\nu R^{(0)}_{\mu\nu}(\nu,\nu_B)$$

and (15)

$$R^{(2)}(\nu,\nu_B)\equiv p^\mu q^\nu R^{(2)}_{\mu\nu}(\nu,\nu_B),$$

in order to evaluate $\sigma^{(0)}$ and $\sigma^{(2)}$.

To lay the groundwork for the determination of the correction terms we first consider the decomposition of $R^{I_t}_{\mu\nu}(\nu,\nu_B)$ into invariant amplitudes. On the basis of general covariance arguments, the most general form for $R^{I_t}_{\mu\nu}$ is

$$R^{I_t}_{\mu\nu}(\nu,\nu_B)=A^{I_t}(\nu,\nu_B)P_\mu P_\nu+B_1^{I_t}(\nu,\nu_B)P_\mu Q_\nu+B_2^{I_t}(\nu,\nu_B)P_\mu\Delta_\nu+B_3^{I_t}(\nu,\nu_B)P_\nu Q_\mu+B_4^{I_t}(\nu,\nu_B)P_\nu\Delta_\mu+C_1^{I_t}(\nu,\nu_B)Q_\mu Q_\nu$$
$$+C_2^{I_t}(\nu,\nu_B)\Delta_\mu\Delta_\nu+C_3^{I_t}(\nu,\nu_B)Q_\mu\Delta_\nu+C_4^{I_t}(\nu,\nu_B)g_{\mu\nu}+C_5^{I_t}(\nu,\nu_B)Q_\nu\Delta_\mu, \quad (16)$$

where we have introduced the combinations

$$P=\tfrac{1}{2}(p_\alpha+p_\beta),\quad Q=\tfrac{1}{2}(p+q),\quad \Delta=q-p.$$

The relation

$$R^{\mu\nu}_{bd,ac}(\nu,\nu_B)=R^{\nu\mu}_{ac,bd}(\nu,\nu_B), \quad (17)$$

which follows from PT invariance, leads to the relations

$$B_1^{I_t}(\nu,\nu_B)=B_3^{I_t}(\nu,\nu_B),$$
$$B_2^{I_t}(\nu,\nu_B)=-B_4^{I_t}(\nu,\nu_B), \quad (18)$$
$$C_3^{I_t}(\nu,\nu_B)=-C_5^{I_t}(\nu,\nu_B),$$

so that we are left with seven independent, invariant amplitudes for each of $I_t=0,1,2$, and we can now write

$$R^{I_t}_{\mu\nu}(\nu,\nu_B)=A^{I_t}(\nu,\nu_B)P_\mu P_\nu+B_1^{I_t}(\nu,\nu_B)(P_\mu Q_\nu+Q_\mu P_\nu)$$
$$+B_2^{I_t}(\nu,\nu_B)(P_\mu\Delta_\nu-\Delta_\mu P_\nu)+C_1^{I_t}(\nu,\nu_B)Q_\mu Q_\nu$$
$$+C_2^{I_t}(\nu,\nu_B)\Delta_\mu\Delta_\nu+C_3^{I_t}(\nu,\nu_B)(Q_\mu\Delta_\nu-Q_\nu\Delta_\mu)$$
$$+C_4^{I_t}g_{\mu\nu}. \quad (19)$$

From the property

$$R^{\mu\nu}_{bd,ac}(\nu,\nu_B)=R^{\nu\mu}_{ad,bc}(-\nu,\nu_B) \quad (20)$$

we arrive at the crossing relations

$$A^{(+)}(\nu,\nu_B)=A^{(+)}(-\nu,\nu_B),$$
$$A^{(1)}(\nu,\nu_B)=-A^{(1)}(-\nu,\nu_B),$$
$$B_i^{(+)}(\nu,\nu_B)=-B_i^{(+)}(-\nu,\nu_B),\quad i=1,2$$
$$B_i^{(1)}(\nu,\nu_B)=B_i^{(1)}(-\nu,\nu_B), \quad (21)$$
$$C_j^{(+)}(\nu,\nu_B)=C_j^{(+)}(-\nu,\nu_B),\quad j=1,\dots,4$$
$$C_j^{(1)}(\nu,\nu_B)=-C_j^{(1)}(-\nu,\nu_B),$$

where (+) stands for either $I_t=0$ or 2. The relevant correction terms may now be written as

$$R^{(+)}(0,\nu_B)=\tfrac{1}{4}(\mu^2-2\mu\nu_B)^2C_1^{(+)}(0,\nu_B)$$
$$-(\mu^2+2\mu\nu_B)^2C_2^{(+)}(0,\nu_B)$$
$$+(\mu^4-4\mu^2\nu_B^2)C_3^{(+)}(0,\nu_B)+2\mu\nu_B C_4^{(+)}(0,\nu_B) \quad (22)$$

and

$$2\mu R^{(1)}=2\mu^3 B_1^{(1)}(0,-\mu/2)-\mu^2\frac{\partial}{\partial\nu}C_4^{(1)}(\nu,-\mu/2)\Big|_{\nu=0}+\mu^4\frac{\partial}{\partial\nu}C_1^{(1)}(\nu,-\mu/2)\Big|_{\nu=0}. \quad (23)$$

In order to determine the amplitudes $A^{I_t}(\nu,\nu_B)\ldots$, $C_4^{I_t}(\nu,\nu_B)$, we assume that they satisfy fixed-t dispersion relations in ν. As will be argued below, the $I_t=1$ amplitudes may be taken to satisfy unsubtracted dispersion relations. We assume, in addition, that the $I_t=0$ and 2 amplitudes also satisfy unsubtracted dispersion relations.

Thus we have the representation

$$A^{I_t}(\nu,\nu_B)=\frac{1}{\pi}\int_0^\infty d\nu'\,\mathrm{Im}\,A^{I_t}(\nu',\nu_B)\left(\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right),$$

$$B_i^{I_t}(\nu,\nu_B)=\frac{1}{\pi}\int_0^\infty d\nu'\,\mathrm{Im}B_i^{I_t}(\nu',\nu_B)\left(\frac{1}{\nu'-\nu}\mp\frac{1}{\nu'+\nu}\right), \quad (24)$$

$$C_j^{I_t}(\nu,\nu_B)=\frac{1}{\pi}\int_0^\infty d\nu'\,\mathrm{Im}C_j^{I_t}(\nu',\nu_B)\left(\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right),$$

where the upper (lower) sign refers to $I_t=0,2$ ($I_t=1$).

On the basis of Regge theory, we would expect[18] that, as $\nu\to\infty$,

$$A^{I_t}\sim\nu^{\alpha_{I_t}-2},$$

$$B^{I_t}\sim\nu^{\alpha_{I_t}-1},$$

$$C^{I_t}\sim\nu^{\alpha_{I_t}},$$

where $\alpha_{I_t}(t)$ represents the leading Regge singularity in the t channel having isospin I_t. For $I_t=1$, we have $\alpha_1(t)=\alpha_\rho(t)$ with $\alpha_\rho(0)\cong\frac{1}{2}$. Thus we would expect that the amplitudes contributing to $R^{(1)}(0,-\mu/2)$ [see Eqs. (23) and (24)] would satisfy unsubtracted dispersion relations (although the dispersion integrals may converge slowly). Also, since $\alpha_2(0)$ is probably less than zero, we might expect, from Eq. (22), that the amplitudes contributing to $R^{(2)}$ would satisfy unsubtracted dispersion relations for $t\cong 0$. However, conventional Regge theory suggests that the dispersion relations determining $R^{(0)}$ may need to be subtracted.

In order to make use of the dispersion relations of Eq. (24) we must find the relevant discontinuities of the invariant functions in the decomposition of Eq. (19). We first rewrite $R^{\mu\nu}$ in Eq. (8) as

$$R^{\mu\nu}_{bd,ac}(\nu,\nu_B)=-i(2\pi)^3(4p^0_\alpha p^0_\beta)^{1/2}\int d^4x\,e^{iq\cdot x}\theta(-x^0)\langle\pi_d(p_\beta)|[\tilde A^\mu_b(0),\tilde A^\nu_a(x)]|\pi_c(p_\alpha)\rangle\,. \quad (25)$$

Denoting the absorptive part of $R^{\mu\nu}$ by $r^{\mu\nu}$, we have

$$\begin{aligned} r^{\mu\nu}_{bd,ac}(\nu,\nu_B)&=-\tfrac{1}{2}[(2\pi)^3](4p^0_\alpha p^0_\beta)^{1/2}\int d^4x\,e^{iqx}\langle\pi_d(p_\beta)|[\tilde A^\mu_b(0),\tilde A^\nu_a(x)]|\pi_c(p_\alpha)\rangle\\ &=-\pi(2\pi)^6(4p^0_\alpha p^0_\beta)^{1/2}\Big[\sum_m 2k^0_m\langle\pi_d(p_\beta)|\tilde A^\mu_b(0)|m\rangle\langle m|\tilde A^\nu_a(0)|\pi_c(p_\alpha)\rangle\delta(s+k_m{}^2)\\ &\qquad-\sum_n 2k^0_n\langle\pi_d(p_\beta)|\tilde A^\nu_a(0)|n\rangle\langle n|\tilde A^\mu_b(0)|\pi_c(p_\alpha)\rangle\delta(u+k_n{}^2)\Big], \end{aligned} \quad (26)$$

where $k_m=p_\alpha+q$ and $k_n=p_\beta-q$.

The absorptive part of the invariant amplitudes may be identified from an expansion of $r^{I_t}_{\mu\nu}$ analogous to that for $R^{I_t}_{\mu\nu}$ in Eq. (19). We see that they are determined from on-mass-shell matrix elements of the form $\langle\pi|\tilde A^\mu_b(0)|m\rangle$, where $|m\rangle$ represents some multihadron state. In principle, these matrix elements can be determined experimentally. The form factors associated with the corresponding matrix elements $\langle\pi|A^\mu_b(0)|m\rangle$ could be obtained from the appropriate experimental data and then, with the help of dispersion relations (in p^2), the pion pole could be removed from the particular form factors which contain it. The remaining contributions to $\langle\pi|\tilde A|m\rangle$ should then be evaluated at $p^2=-\mu^2$.

In the following we will discuss the contributions to Eq. (26) of the ρ meson and a $J^P=0^+$, $I=0$ σ meson whose mass and width will be allowed to vary to some extent. An important constraint on these contributions is obtained from Eq. (6), which implies that the residues of the ρ and σ poles in the correction terms must equal the residues of the corresponding poles in the amplitude. Use will be made of this constraint in Secs. III and IV to help determine several of the parameters of the ρ and σ contributions.

The lack of experimental data prevents a consideration of higher intermediate states. Nevertheless, the S- and P-wave contributions up to ~1 GeV should be well accounted for by the σ and ρ states, respectively, and we feel that the correction term $R^{(1)}$, at least, will be reliably determined in this approximation.

III. CONTRIBUTION OF THE INTERMEDIATE ρ STATE

The contribution of the ρ state to $r^{\mu\nu}$ is given from Eq. (26) by

$$r_\rho^{\mu\nu} = -\pi(2\pi)^6(16p_\alpha^0 p_\beta^0 k^0 k^0)^{1/2}\Big[\sum_{\text{pol}}\langle \pi_d(p_\beta)|\tilde{A}_b^\mu(0)|\rho_e(k)\rangle\langle\rho_e(k)|\tilde{A}_a^\nu(0)|\pi_c(p_\alpha)\rangle\delta(s-m^2)$$
$$-\sum_{\text{pol}}\langle\pi_d(p_\beta)|\tilde{A}_a^\nu(0)|\rho_e(k')\rangle\langle\rho_e(k')|\tilde{A}_b^\mu(0)|\pi_c(p_\alpha)\rangle\delta(u-m^2)\Big], \quad (27)$$

where m is the ρ mass and the summation is over polarization states of the ρ.

We determine the matrix elements in Eq. (27) from the hard-pion analysis[9] of the three-point function

$$\int d^4x\, d^4y\, e^{-iqx+ipy}\langle T\{\partial_\mu A_a^\mu(x), A_b^\nu(y), V_c^\lambda(0)\}\rangle_0$$
$$=\frac{\epsilon_{abc}F_\pi\mu^2 g_\rho^{-1}g_A^{-1}}{q^2+\mu^2}\Delta_A^{\nu\sigma}(p)\Delta_\rho^{\lambda\eta}(k)\Gamma_{\sigma\eta}(q,p)$$
$$+\frac{\epsilon_{abc}F_\pi^2\mu^2 g_\rho^{-1}}{(q^2+\mu^2)(p^2+\mu^2)}p^\nu\Delta_\rho^{\lambda\eta}(k)\Gamma_\eta(q,p), \quad (28)$$

where $k=p-q$. The constants g_ρ and g_A are defined by

$$(2\pi)^{3/2}(2q^0)^{1/2}\langle 0|\,V_a^\mu(0)|\rho_b(q)\rangle=\delta_{ab}g_\rho\epsilon_\rho^\mu(q)$$

and

$$(2\pi)^{3/2}(2q^0)^{1/2}\langle 0|A_a^\mu(0)|A_{1b}(q)\rangle=\delta_{ab}g_A\epsilon_A^\mu(q),$$

respectively, where the A_1 "meson" has a mass $m_A = 1070$ MeV $\cong\sqrt{2}\,m$, and where ϵ_ρ^μ (ϵ_A^μ) is the polarization four-vector of the ρ (A_1). $\Delta_\rho^{\lambda\eta}(k)$ and $\Delta_A^{\nu\sigma}(p)$ are the covariant spin-1 parts of the renormalized vector and axial-vector propagators:

$$\Delta_\rho^{\lambda\eta}(k)\equiv\int d\mu^2\frac{\rho_V(\mu^2)}{\mu^2+k^2}\left(g^{\lambda\eta}+\frac{k^\lambda k^\eta}{\mu^2}\right) \quad (29)$$

and similarly for A_1. We will make use of the single-meson saturation hypothesis[9] to write these as

$$\Delta_\rho^{\lambda\eta}(k)\cong\frac{g_\rho^2}{k^2+m^2}\left(g^{\lambda\eta}+\frac{k^\lambda k^\eta}{m^2}\right),$$
$$\Delta_A^{\nu\sigma}(p)\cong\frac{g_A^2}{p^2+m_A^2}\left(g^{\nu\sigma}+\frac{p^\nu p^\sigma}{m_A^2}\right). \quad (30)$$

From Eq. (28) we derive the matrix element

$$(2\pi)^3(2q^0 2k^0)^{1/2}\langle\pi_a(q)|\tilde{A}_b^\nu(0)|\rho_e(k)\rangle$$
$$=\frac{\epsilon_{abc}}{g_A}\epsilon_\rho^\eta(k)\Delta_A^{\nu\sigma}(p)\Gamma_{\sigma\eta}(q,p). \quad (31)$$

Using Eq. (31) in Eq. (27) we get

$$r_\rho^{\mu\nu}=-\pi\sum_{\text{pol}}\frac{\epsilon_{dbe}\epsilon_{cae}}{g_A^2}\epsilon_\rho^\eta(k)\epsilon_\rho^\zeta(k)\Delta_A^{\mu\sigma}(p)$$
$$\times\Gamma_{\sigma\eta}(-p_\beta,p)\Delta_A^{\nu\tau}(q)\Gamma_{\tau\zeta}(-p_\alpha,q)\delta(s-m^2)$$
$$+\pi\sum_{\text{pol}}\frac{\epsilon_{dae}\epsilon_{cbe}}{g_A^2}\epsilon_\rho^\eta(k')\epsilon_\rho^\zeta(k')\Delta_A^{\nu\sigma}(q)$$
$$\times\Gamma_{\sigma\eta}(-p_\beta,-q)\Delta_A^{\mu\tau}(p)\Gamma_{\tau\zeta}(-p_\alpha,-p)\delta(u-m^2), \quad (32)$$

where the vertex function is given by[9]

$$\Gamma_{\nu\lambda}(q,p)=\frac{F_\pi m^2 m_A^2}{g_\rho g_A}\left[-g_{\nu\lambda}-\frac{1}{m_A^2}(g_{\nu\lambda}p^2-p_\nu p_\lambda)+\frac{g_A^2}{m^2F_\pi^2}\left(\frac{1}{m^2}-\frac{1}{m_A^2}\right)(g_{\nu\lambda}k^2-k_\nu k_\lambda)+\frac{\delta g_A^2}{F_\pi^2 m_A^4}(g_{\nu\lambda}q\cdot k-q_\lambda k_\nu)\right]$$
$$=\frac{m^2}{g_\rho\sqrt{\lambda}}\left[-m_A^2 g_{\nu\lambda}-(g_{\nu\lambda}p^2-p_\nu p_\lambda)+\lambda\left(\frac{1}{m^2}-\frac{1}{m_A^2}\right)(g_{\nu\lambda}k^2-k_\nu k_\lambda)+\frac{\delta\lambda}{m_A^2}(g_{\nu\lambda}q\cdot k-q_\lambda k_\nu)\right] \quad (33)$$

where $\lambda=g_A^2F_\pi^{-2}$, and δ is the parameter defined in Ref. 9.

In terms of this parameterization we get the result

$$r_\rho^{\mu\nu}=-\pi(\delta_{ab}\delta_{cd}-\delta_{ad}\delta_{bc})\delta(s-m^2)\frac{g_A^2}{(m_A^2-\mu^2)^2}\frac{m^4}{g_\rho^2\lambda}$$
$$\times[a^2g^{\mu\nu}+(a^2/m^2)k^\mu k^\nu+abq^\mu q^\nu+abp^\mu p^\nu-\tfrac{1}{2}abk^\mu q^\nu-\tfrac{1}{2}abp^\mu k^\nu+b^2(\tfrac{1}{4}m^2+2\mu\nu_B)p^\mu q^\nu]$$
$$+\pi(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd})\delta(u-m^2)\frac{g_A^2}{(m_A^2-\mu^2)^2}\frac{m^4}{g_\rho^2\lambda}[a^2g^{\mu\nu}+(a^2/m^2)k'^\mu k'^\nu+abq^\mu q^\nu+abp^\mu p^\nu+\tfrac{1}{2}abp^\mu k'^\nu$$
$$+\tfrac{1}{2}abk'^\mu q^\nu+b^2(\tfrac{1}{4}m^2+2\mu\nu_B)p^\mu q^\nu], \quad (34)$$

where

$$a=m_A^2+\lambda\frac{(m_A^2-m^2)}{m_A^2}-\frac{\delta\lambda m^2}{2m_A^2}-\mu^2,\qquad b=\frac{\lambda}{m_A^4}(m_A^2-m^2-\tfrac{1}{2}\delta m^2).$$

If we write Eq. (34) in terms of the combinations P, Q, Δ for direct comparison with Eq. (19), we get

$$
\begin{aligned}
r_\rho^{\mu\nu} = & -\pi(\delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc})\frac{\delta(s-m^2)g_A{}^2}{(m_A{}^2-\mu^2)^2 g_\rho{}^2\lambda}\, m^4 \\
& \times\Big\{\frac{a^2}{m^2}P^\mu P^\nu + \Big(\frac{a^2}{m^2}-\tfrac{1}{2}ab\Big)(P^\mu Q^\nu + Q^\mu P^\nu) - \tfrac{1}{4}ab(P^\mu\Delta^\nu - \Delta^\mu P^\nu) + \Big[\frac{a^2}{m^2}+ab+b^2(\tfrac{1}{4}m^2+2\mu\nu_B)\Big]Q^\mu Q^\nu \\
& + [-\tfrac{1}{4}ab + \tfrac{1}{2}b^2(\tfrac{1}{4}m^2+2\mu\nu_B)](Q^\mu\Delta^\nu - \Delta^\mu Q^\nu) + [\tfrac{1}{2}ab - \tfrac{1}{4}b^2(\tfrac{1}{4}m^2+2\mu\nu_B)]\Delta^\mu\Delta^\nu + a^2 g^{\mu\nu}\Big\} \\
& + \pi(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd})\frac{\delta(u-m^2)}{(m_A{}^2-\mu^2)^2}\frac{g_A{}^2 m^4}{g_\rho{}^2\lambda} \\
& \times\Big\{\frac{a^2}{m^2}P^\mu P^\nu - \Big(\frac{a^2}{m^2}-\tfrac{1}{2}ab\Big)(P^\mu Q^\nu + Q^\mu P^\nu) + \tfrac{1}{4}ab(P^\mu\Delta^\nu - \Delta^\mu P^\nu) + \Big[\frac{a^2}{m^2}+ab+b^2(\tfrac{1}{4}m^2+2\mu\nu_B)\Big]Q^\mu Q^\nu \\
& + [-\tfrac{1}{4}ab + \tfrac{1}{2}b^2(\tfrac{1}{4}m^2+2\mu\nu_B)](Q^\mu\Delta^\nu - \Delta^\mu Q^\nu) + [\tfrac{1}{2}ab - \tfrac{1}{4}b^2(\tfrac{1}{4}m^2+2\mu\nu_B)]\Delta^\mu\Delta^\nu + a^2 g^{\mu\nu}\Big\}\,. \qquad (35)
\end{aligned}
$$

From the unsubtracted dispersion relations for the invariant amplitudes given in Eq. (24)[19] we find

$$
\begin{aligned}
B_1^{(1)}(\nu,-\mu/2) &= -K\frac{2(m^2-2\mu^2)}{(m^2-s)(m^2-u)}\Big(\frac{a^2}{m^2}-\tfrac{1}{2}ab\Big)\,, \\
C_1^{(1)}(\nu,-\mu/2) &= -K\frac{4\mu\nu}{(m^2-s)(m^2-u)} \\
&\quad\times\Big[\frac{a^2}{m^2}+ab+b^2(\tfrac{1}{4}m^2-\mu^2)\Big]\,, \qquad (36) \\
C_4^{(1)}(\nu,-\mu/2) &= -K\frac{4\mu\nu}{(m^2-s)(m^2-u)}\,a^2\,,
\end{aligned}
$$

where

$$K = \frac{(g_A/g_\rho)^2 m^4}{(m_A{}^2-\mu^2)^2\lambda}\,.$$

This gives, from Eq. (23),

$$
\begin{aligned}
R_\rho^{(1)} = \mu^2 & \frac{(g_A/g_\rho)^2 2m^4}{(m_A{}^2-\mu^2)^2(m^2-2\mu^2)\lambda} \\
& \times\Big\{\tfrac{1}{2}ab + \frac{\mu^2}{m^2-2\mu^2}\Big[\frac{a^2}{m^2}-ab-b^2(\tfrac{1}{4}m^2-\mu^2)\Big]\Big\}. \qquad (37)
\end{aligned}
$$

Defining the parameter ξ by[20]

$$g_\rho{}^2 = \xi^2 2m^2 F_\pi{}^2\,, \qquad (38)$$

then

$$\Big(\frac{g_A}{g_\rho}\Big)^2 = \frac{2\xi^2-1}{\xi^2}$$

and (39)

$$\lambda = 2m^2(2\xi^2-1)\,,$$

where we have used the first Weinberg sum rule[21] to get Eq. (39), and have set $m_A{}^2 = 2m^2$. In terms of ξ, Eq. (37) becomes

$$
\begin{aligned}
R_\rho^{(1)} = \frac{\mu^2}{\xi^2}[&0.124(2\xi^2-1)^2(1-\delta/2)^2 \\
&+0.277(2\xi^2-1)(1-\delta/2)+0.064]\,. \qquad (40)
\end{aligned}
$$

The parameter ξ can be determined experimentally from the leptonic decay of the ρ (Ref. 22) by

$$
\begin{aligned}
\xi^2 = & \frac{3m_\rho\Gamma(\rho\to l^+l^-)}{4\pi 2F_\pi{}^2\alpha^2}\Big[1+2\Big(\frac{m_l}{m}\Big)^2\Big]^{-1} \\
& \times\Big[1-4\Big(\frac{m_l}{m}\Big)^2\Big]^{-1/2}\,. \qquad (41)
\end{aligned}
$$

Using the experimental values,[23] $m = 0.770$ GeV, $\Gamma(\rho\to 2\pi) \equiv \Gamma_{\rho\pi\pi} = 0.145$ GeV, and a $\rho\to e^+e^-/\rho\to 2\pi$ branching ratio of 5×10^{-5} gives the value $\xi^2 = 1.41$. If we take $\Gamma_{\rho\pi\pi} = 0.155$ GeV and use the $\rho\to\mu^+\mu^-/\rho\to 2\pi$ branching ratio of 6.5×10^{-5} we find $\xi^2 = 2$.

On the other hand, the constraint [see Eq. (6)] that the ρ poles in the correction term and amplitude have the same residue leads to a value of $\xi^2 = 1.89$ for $\Gamma_{\rho\pi\pi} = 0.145$ GeV. Consistency with the work of Schnitzer and Weinberg leads to $\delta = -\frac{1}{4}$ in this case. The residues of the poles can also be made equal for $\xi^2 = 1.41$ (with $\delta = -\frac{1}{2}$), if we are willing to give up the relation[24] between g_A and g_ρ expressed in Eq. (39). The values of $R_\rho^{(1)}$ corresponding to the cases $\xi^2 = 1.89$ and 1.41 are given in Table I.

IV. CONTRIBUTION OF THE INTERMEDIATE σ STATE

To calculate the contribution of a possible $I = J = 0$ enhancement in the π-π interaction, we assume the existence of a σ meson. Analogously to the previous calculation we consider the three-point function[10,25]

TABLE I. Variation of $R_\rho^{(1)}$ and $R_\sigma^{(1)}$ with ξ^2 and with $\Gamma_{\sigma\pi\pi}$.

		$R_\sigma^{(1)}$		
ξ^2	$R_\rho^{(1)}$	$\Gamma_{\sigma\pi\pi} = 250$ MeV	$\Gamma_{\sigma\pi\pi} = 400$ MeV	$\Gamma_{\sigma\pi\pi} = 600$ MeV
1.41	0.025	−0.147	−0.235	−0.352
1.89	0.021	−0.174	−0.278	−0.432

$$\int d^4x\, d^4y\, e^{-iqx+ipy}\langle T\{\partial_\mu A_a^\mu(x), A_b^\nu(y), \sigma(0)\}\rangle_0 = -\frac{iF_\pi\mu^2}{q^2+\mu^2}\left[F^\nu(q,p)+\frac{F_\pi}{p^2+\mu^2}p^\nu F(q,p)\right], \quad (42)$$

where $\sigma(x)$ is defined by [see Eq. (4)] $\sigma_{ab}(x) = \delta_{ab}\sigma(x)$. From Eq. (42) we find

$$(2\pi)^3(2q^0 2k^0)^{1/2}\langle \pi_a(q)|\tilde{A}_b^\nu(0)|\sigma(k)\rangle = i\delta_{ab}\frac{g_\sigma}{F_\pi\mu^2}\left[p^\nu+\frac{Ag_A{}^2\mu^2}{m_A{}^2(p^2+m_A{}^2)}\left(p^\nu\frac{p\cdot q}{m_A{}^2}+q^\nu\right)\right], \quad (43)$$

where again $k=p-q$, and g_σ is defined analogously to g_A and g_ρ. The constant A is introduced through the smoothness assumption of the vertex function, and is determined[10] from the Ward identities to be

$$A=-\frac{m_A{}^2}{(F_\pi\mu)^2(2\xi^2-1)^2}, \quad (44)$$

where ξ is the parameter of Sec. III.

From Eqs. (26) and (43) we get for the contribution of the σ

$$\begin{aligned}r_\sigma^{\mu\nu} = &-\pi\delta_{ac}\delta_{bd}\delta(s-m_\sigma{}^2)\left(\frac{g_\sigma}{F_\pi\mu^2}\right)^2[\gamma^2P^\mu P^\nu-\gamma d(P^\mu Q^\nu+Q^\mu P^\nu)-\tfrac{1}{2}(\gamma d+\gamma^2)(P^\mu\Delta^\nu-\Delta^\mu P^\nu)\\ &+d^2Q^\mu Q^\nu+\tfrac{1}{2}(d^2+\gamma d)(Q^\mu\Delta^\nu-\Delta^\mu Q^\nu)-\tfrac{1}{4}(d+\gamma)^2\Delta^\mu\Delta^\nu]\\ &+\pi\delta_{ad}\delta_{bc}\delta(u-m_\sigma{}^2)\left(\frac{g_\sigma}{F_\pi\mu^2}\right)^2[\gamma^2P^\mu P^\nu+\gamma d(P^\mu Q^\nu+Q^\mu P^\nu)+\tfrac{1}{2}(\gamma d+\gamma^2)(P^\mu\Delta^\nu-\Delta^\mu P^\nu)\\ &+d^2Q^\mu Q^\nu+\tfrac{1}{2}(d^2+\gamma d)(Q^\mu\Delta^\nu-\Delta^\mu Q^\nu)-\tfrac{1}{4}(d+\gamma)^2\Delta^\mu\Delta^\nu],\end{aligned} \quad (45)$$

where

$$\gamma=-\frac{2m^2}{(m_A{}^2-\mu^2)(2\xi^2-1)}$$

and

$$d=1-\frac{m^2(m_\sigma{}^2-2\mu^2)}{m_A{}^2(m_A{}^2-\mu^2)(2\xi^2-1)}.$$

This gives

$$B_1^{(1)}(\nu,-\mu/2)=\frac{2(m_\sigma{}^2-2\mu^2)\gamma d}{(m_\sigma{}^2-s)(m_\sigma{}^2-u)}\left(\frac{g_\sigma}{F_\pi\mu^2}\right)^2,$$

$$C_1^{(1)}(\nu,-\mu/2)=\frac{-4\mu\nu}{(m_\sigma{}^2-s)(m_\sigma{}^2-u)}d^2\left(\frac{g_\sigma}{F_\pi\mu^2}\right)^2,$$

and

$$C_4^{(1)}=0, \quad (46)$$

so that

$$R_\sigma^{(1)}=\frac{2\mu^2}{m_\sigma{}^2-2\mu^2}\left(\frac{g_\sigma}{F_\pi\mu^2}\right)^2\left[\gamma d-\frac{\mu^2d^2}{(m_\sigma{}^2-2\mu^2)^2}\right]. \quad (47)$$

g_σ can be determined from Eq. (43) by means of a generalized Goldberger-Treiman relation. We find

$$\frac{g_\sigma}{F_\pi\mu^2}=\frac{F_\pi m_\sigma g_{\sigma\pi\pi}}{\frac{1}{2}\gamma(m_\sigma{}^2-2\mu^2)(1-\mu^2/m_A{}^2)-\mu^2} \cong \frac{2F_\pi m_\sigma g_{\sigma\pi\pi}}{\gamma(m_\sigma{}^2-\mu^2)}, \quad (48)$$

where $g_{\sigma\pi\pi}$ is related to the $\sigma\to 2\pi$ width, $\Gamma_{\sigma\pi\pi}$, by

$$\frac{g_{\sigma\pi\pi}{}^2}{4\pi}=\frac{4}{3}\frac{\Gamma_{\sigma\pi\pi}}{|\vec{p}|}, \quad (49)$$

with $|\vec{p}|$ the c.m. momentum of the pions.

We will take the σ mass to be 700 MeV for the purpose of our calculation. If we require that the residues of the σ poles in the amplitude and the correction term be equal, this gives $\xi^2=1.12$. However, relaxation of the Goldberger-Treiman relation, Eq. (48), by ~15% allows equality of the residues for $\xi^2=1.89$. We regard this as the most likely consistent solution for the ρ and σ contributions. It is not possible, however, to rule out the "compromise" solution with $\xi^2=1.41$, which requires a smaller breaking of the Goldberger-Treiman relation (~10%) but gives up the relation (39).

In Table I, we give the values of $R_\sigma^{(1)}$ corresponding to $\xi^2=1.41$ and 1.89 for the choices $\Gamma_{\sigma\pi\pi}=250$, 400, and 600 MeV. The possibility of m_σ being much less than 700 MeV appears to be excluded in the present model because this would lead to unacceptably large violations of the Goldberger-Treiman relation. Larger values of m_σ lead to smaller values of $R_\sigma^{(1)}$.

V. BOUNDS ON THE S-WAVE SCATTERING LENGTHS

The above evaluation of $R^{(1)}$ may be used to provide a bound on the $\pi\pi$ S-wave scattering lengths. To do this we write an unsubtracted dispersion relation for the odd amplitude which can be expressed in the form[26]

$$(2a_0-5a_2)\mu=\frac{6\mu^2}{8\pi^2}\int_0^\infty\frac{dq}{\omega}[\sigma_{+-}(\omega)-\sigma_{++}(\omega)] \quad (50)$$

in terms of the variables defined in Sec. II. Here

a_0 and a_2 are the isospin-0 and -2 S-wave scattering lengths, respectively. Using Eq. (50) we can write Eq. (13) as[4]

$$(2a_0-5a_2)\mu=\frac{3\mu^2}{8\pi F_\pi{}^2}(2-R^{(1)})+\frac{3\mu^4}{4\pi^2}\int_0^\infty\frac{dq}{\omega^3}[\sigma_{+-}(\omega)-\sigma_{++}(\omega)]\,. \tag{51}$$

If we estimate the rapidly convergent integral in Eq. (51) in the narrow-resonance approximation including the ρ, σ, and f^0 mesons, we find a value of 0.005. Taking a lower bound for $-R^{(1)}$ of 0.0 and the upper bound of 0.41, which corresponds to $\xi^2=1.89$ and $\Gamma_{\sigma\pi\pi}=600$ MeV, gives the bounds on the scattering lengths

$$0.56\lesssim(2a_0-5a_2)\mu\lesssim0.66\,. \tag{52}$$

Assuming that a_2 is small and negative gives the upper bound

$$\mu a_0\lesssim0.33\,,$$

while if we also use the Weinberg value[3] for the ratio of the scattering lengths

$$\frac{a_0}{a_2}=-\frac{7}{2}\,,$$

we get the stricter bounds

$$0.16\lesssim\mu a_0\lesssim0.19$$

and

$$-0.055\lesssim\mu a_2\lesssim-0.047\,.$$

An alternative way of deriving a bound on the scattering lengths is to evaluate Eq. (6) at the point $\nu=\mu$ and $\nu_B=-\mu/2$. The amplitude $M^{(1)}(\mu,-\mu/2)$ is proportional to $2a_0-5a_2$ and we obtain

$$(2a_0-5a_2)\mu=\frac{3\mu^2}{8\pi F_\pi{}^2}(2-R_\mu^{(1)}), \tag{53}$$

where

$$R_\mu^{(1)}\equiv\frac{1}{2\mu^2}[p^\lambda q^\rho R_{\lambda\rho}^{(1)}(\nu,\nu_B)]\Big|_{\nu=\mu;\;\nu_B=-\mu/2}\,.$$

Note that this is the corrected, on-mass-shell version of Weinberg's scattering length sum rule.[3] By this method we avoid the necessity of evaluating the integral in Eq. (51). The value of $R_\mu^{(1)}$ for $\xi^2=1.89$ and $\Gamma_{\sigma\pi\pi}=600$ MeV is -0.400, which gives the bounds on the scattering lengths

$$0.56\lesssim(2a_0-5a_2)\mu\lesssim0.66\,.$$

Consistency between Eq. (53) and Eq. (51) then requires that the integral in Eq. (51) have the value +0.004, which is close to the narrow-resonance value of 0.005 quoted above.

If we evaluate the integral in Eq. (50) in the narrow-resonance approximation with the parameters

$$m=0.770\,,\quad\Gamma_{\rho\pi\pi}=0.145\,,$$
$$m_f=1.260\,,\quad\Gamma_{f\pi\pi}=0.150\,,$$
$$m_\sigma=0.700\,,\quad\Gamma_{\sigma\pi\pi}=0.400\,,$$

in GeV units, we find the value

$$(2a_0-5a_2)\mu=0.58\,,$$

which is consistent in this approximation (i.e., neglect of continuum and $I=2$ contribution) with the sum rule given in Eq. (51).

We can now attempt to check the consistency of the soft-pion Adler theorem with our hard-pion version. The off-mass-shell form of the sum rule can be written as[1]

$$1=\frac{F_\pi{}^2}{\pi}\int_0^\infty\frac{d\nu}{\nu+\frac{3}{4}}[\sigma_0^{+-}(\nu)-\sigma_0^{++}(\nu)]\,, \tag{54}$$

where $\nu=\frac{1}{4}s-1$ in units of μ^2. We approximate the cross sections by the ρ, f, and σ contributions in the narrow-resonance limit, with the Adler correction[1] for mass-shell dependence given by

$$\sigma_0^{I,l}=\left[\frac{(\nu+\frac{3}{4})^2}{\nu(\nu+1)}\right]^l\pi^2\frac{2l+1}{\nu}m\Gamma\delta(\nu-\nu_R) \tag{55}$$

for the total cross section associated with partial wave l. Using the above parameters for the resonances we obtain for the right-hand side of Eq. (54) the value 1.09. If the on-mass-shell cross sections are used [by setting the quantity in brackets in Eq. (55) equal to one] a smaller value for the integral is obtained.

The on-mass-shell version of the theorem reads

$$1=\frac{F_\pi{}^2}{\pi}\int_0^\infty d\nu\,\frac{4[\nu(\nu+1)]^{1/2}}{(2\nu+1)^2}\times[\sigma_{+-}(\nu)-\sigma_{++}(\nu)]+\tfrac{1}{2}R^{(1)}\,. \tag{56}$$

The narrow-resonance approximation gives the value of 1.08 for the integral itself. The correction terms, being negative, would seem to allow for a small positive asymptotic contribution to the sum rule.[27]

Thus, for the particular resonance parameters employed, the correction to the soft-pion sum rule [the difference between the integral in Eq. (54) evaluated using Eq. (55) and using on-mass-shell cross sections] is in the wrong direction. The correction, $\frac{1}{2}R^{(1)}$, in Eq. (56) is in the right direction. It is possible, of course, that Eq. (55) does not accurately describe the off-shell extrapolation of the cross section.

Before concluding the present section, a comment is in order concerning Adler's use[1] of an effective range approximation to the $I=0$ S-wave part of the amplitude, in which the large value, $\mu a_0 \cong 1.3$, is required for the saturation (with the ρ and f^0 contributions) of Eq. (54). As Adler pointed out, the choice of an S-wave amplitude which saturates the sum rule is not unique; thus, his large value of a_0 cannot be taken as a prediction. While our analysis cannot, of course, absolutely rule out large scattering lengths and the large, low-energy $\pi\pi$ interaction that they imply, the σ "meson" dominance approach adopted here appears[6] capable of providing an accurate representation of the $I=0$ S-wave $\pi\pi$ amplitude up to ~ 1 GeV. Moreover, the result[6] $\mu a_0 = 0.34 \pm 0.18$, which is consistent with our bounds, completely excludes Adler's value. We might add that, were $\mu a_0 \approx 1.0$, $R^{(1)}$ would be underestimated by a factor of ~ 15, a possibility we find extremely unlikely. Finally, it is worth emphasizing the self-consistency of the present calculation represented by the agreement between the dispersive and current-algebra results for $2a_0 - 5a_2$.

VI. THE σ TERM

The sigma term $\sigma_{ab,cd}(t)$ given by Eq. (7) has the isospin decomposition

$$\sigma_{ab,cd}(t) = \tfrac{1}{3}\,\delta_{ab}\,\delta_{cd}\,\sigma^{(0)}(t) + \tfrac{1}{2}(\delta_{ad}\delta_{bc} + \delta_{bd}\,\delta_{ac} - \tfrac{2}{3}\,\delta_{ab}\,\delta_{cd})\sigma^{(2)}(t) \tag{57}$$

including both $I=0$ and $I=2$ components. In terms of the above form factors one obtains the two low-energy theorems given by Eqs. (10) and (12). Since our calculation in Sec. IV of the σ-meson contribution to the correction terms was based on the assumption that $\sigma^{(2)}(t) \equiv 0$, we will restrict our considerations to $\sigma^{(0)}(t)$.

At the current-algebra point $\nu = \nu_B = 0$, Eq. (10) becomes, with the definition in Eq. (15),

$$\sigma^{(0)}(2\mu^2) = F_\pi^{\,2} M^{(0)}(0,0) - R^{(0)}(0,0)\,, \tag{58}$$

where the expression for $R^{(0)} \equiv R^{(0)}(0,0)$ in terms of the invariant amplitudes is given by Eq. (22). Using the results of Secs. III and IV we find

$$R_\rho^{(0)} = -\frac{\mu^4 (g_A/g_\rho)^2 m^4}{(2m^2-\mu^2)^2(m^2-\mu^2)\lambda}\left(\frac{a^2}{m^2} + 2ab + \tfrac{1}{2}b^2 m^2\right) \tag{59}$$

and

$$R_\sigma^{(0)} = -\frac{\mu^4}{m_\sigma^{\,2}-\mu^2}\left(\frac{g_\sigma}{F_\pi \mu^2}\right)^2 (2d^2 + \gamma^2 + 2\gamma d)\,. \tag{60}$$

Taking the values $\xi^2 = 1.89$ and $\Gamma_{\sigma\pi\pi} = 600$ MeV gives

$$R_\rho^{(0)} = -0.114\mu^2\,, \quad R_\sigma^{(0)} = -0.706\mu^2$$

and therefore

$$R^{(0)} = -0.820\mu^2\,.$$

In order to determine $\sigma^{(0)}(2\mu^2)$ we need the value of the amplitude $M^{(0)}$ at $s=u=\mu^2$, $t=2\mu^2$. This will clearly depend on the choice of a model for the $\pi\pi$ amplitude, and we will therefore just quote the results of two such models[28,29] which seem to be in agreement. The model of Moffat and Weisman[28] gives the value

$$F_\pi^{\,2} M^{(0)}(0,0) = -1.58\mu^2\,,$$

while that of Prasad and Brehm[29] gives

$$F_\pi^{\,2} M^{(0)}(0,0) = -1.83\mu^2\,.$$

These models would suggest a value[30] for $\sigma^{(0)}(2\mu^2)$ between $-0.40\mu^2$ and $-1.00\,\mu^2$.

To avoid the use of models for the scattering amplitude, we can evaluate Eq. (10) at the point $\nu=\mu$, $\nu_B = -\mu/2$ to give

$$\sigma^{(0)}(0) = F_\pi^{\,2} M^{(0)}(\mu, -\mu/2) - R^{(0)}(\mu, -\mu/2)\,. \tag{61}$$

The threshold value of the amplitude is given by

$$M^{(0)}(\mu, -\mu/2) = -32\pi\mu \times \tfrac{1}{3}\,(a_0 + 5a_2)$$

in terms of the S-wave scattering lengths, and Eq. (61) becomes

$$\sigma^{(0)}(0) = -32\pi\mu\tfrac{1}{3}\,F_\pi^{\,2}(a_0 + 5a_2) - R^{(0)}(\mu, -\mu/2)\,. \tag{62}$$

Evaluating $R^{(0)}$ for $\xi^2 = 1.89$ and $\Gamma_{\sigma\pi\pi} = 600$ MeV gives

$$R_\rho^{(0)}(\mu, -\mu/2) = 7.64\mu^2\,,$$

$$R_\sigma^{(0)}(\mu, -\mu/2) = -1.30\mu^2\,,$$

so that

$$R^{(0)}(\mu, -\mu/2) = 6.34\mu^2\,.$$

This value is much larger than that found at the point $\nu = \nu_B = 0$, and as such may be subject to larger errors, the greatest uncertainty in its evaluation being the possible need for a subtraction in the dispersion integral for the invariant amplitude $C_4^{(0)}(\nu, \nu_B)$.

Since we do not know the scattering lengths, we cannot evaluate Eq. (62) directly, but we can check the consistency of the results. If we take the value for $\sigma^{(0)}(0)$ which holds in the model of Gell-Mann, Oakes, and Renner,[15]

$$\sigma^{(0)}(0) = -3\mu^2\,,$$

we find that

$$\tfrac{1}{3}(a_0+5a_2)\mu=-0.067\,.$$

Combining this with the constraint given by Eq. (52) gives the bounds

$$0.12\lesssim a_0\mu\lesssim 0.15,\quad -0.07\lesssim a_2\mu\lesssim -0.06$$

corresponding to the ratio $a_0/a_2\cong -2.0$. These results would be compatible with the experimental situation.

VII. CONCLUSIONS

We have attempted to focus on several rather narrow aspects of the current-algebra approach to $\pi\pi$ scattering. Instead of trying to construct the complete, low-energy $\pi\pi$ amplitude using current algebra and unitarity constraints, an interesting and useful program which has been pursued by others,[12] we have confined ourselves to the study of the three ($I_t=0, 1, 2$) current-algebra sum rules. These relations, which include the Adler sum rule and two σ-term sum rules, have been formulated with all pions on the mass shell. Written in this way, they contain additional correction terms which are not present in the soft-pion limit.

We have presented a general technique for calculating these correction terms based on their decomposition into a complete set of invariant amplitudes which are assumed to satisfy fixed-t dispersion relations. In principle, the correction terms can be evaluated using on-mass-shell experimental data. It seems preferable to us to estimate the correction terms in the on-mass-shell relations rather than to guess at the off-mass-shell behavior of physical amplitudes in the soft-pion approach.

In order to test the on-mass-shell sum rules, we have estimated the correction terms by approximating the relevant dispersion integrals by their ρ- and σ-meson contributions. The discontinuities are calculated using hard-meson models for the contributing three-point functions. The correct pole structure of the current-algebra theorems is not guaranteed when these models are used to calculate the correction terms; in particular the residues of a particular pole in the correction term and amplitude must be equal. It is encouraging therefore that the ρ and σ residues approximately satisfy this requirement for experimentally reasonable ranges of the determining parameters. We have used this residue condition to furnish constraints on otherwise free parameters. For example, it is found that a σ mass of much less than 700 MeV would be incompatible with the present model.

For the Adler sum rule at least, we feel that we have obtained a reliable upper bound on the correction term (by allowing for a large σ contribution). This conclusion is based on the fact that Regge theory suggests that the dispersion relations associated with the Adler sum-rule corrections are unsubtracted. Thus the ρ and σ contributions may indeed saturate these dispersion integrals. We have also assumed unsubtracted dispersion relations for the corrections to the $I_t=0$ σ-term sum rule, although this is not supported by Regge theory.

By recasting the Adler sum rule into a corrected form of the Weinberg sum rule[3] for the combination, $2a_0-5a_2$, of S-wave scattering lengths, we have established what we feel are the reliable bounds:

$$0.56\lesssim(2a_0-5a_2)m_\pi\lesssim 0.66\,. \tag{63}$$

Then, assuming $a_2\leq 0$, we can predict with some confidence that

$$m_\pi a_0\leq 0.33\,. \tag{64}$$

The interesting $I_t=0$ sum rule can be written in such a way that the σ term is related to the combination a_0+5a_2 plus the relevant correction term. Since this correction term is estimated on the basis of unsubtracted dispersion relations, it may not be well determined. However, if we accept this estimate for it, we find that the $I_t=0$ sum rule is consistent with the Gell-Mann–Oakes–Renner value[15] of the σ term for values of a_0 and a_2 which satisfy the constraints in Eqs. (63) and (64).

Finally, we believe that the analysis presented here gives some new insight into the magnitude of off-mass-shell extrapolation effects in the $\pi\pi$ amplitude. The correction terms represent the error made in using physical, on-mass-shell amplitudes in soft-pion theorems. While these effects are not embarrassingly large, they may nevertheless be appreciable,[31] $\lesssim 25\%$.

An estimate of the size of these effects is also available from the more general hard-pion treatments of $\pi\pi$ scattering,[12] although usually little more than an order of magnitude can be inferred. We feel that our results are more definitive; fewer assumptions are made in the present work and the strongest of these are equivalent to ones found in the previous hard-pion analyses.

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[14]Here a, b, c, and d are isospin indices. The momentum associated with each pion is indicated in parentheses in Eq. (2). We work in the metric +, +, +, −. F_π, the pion decay constant, is taken to be 92 MeV. The operator $\tilde{A}_a^\mu(x)$ is defined by the requirement that its matrix elements be obtained from the corresponding matrix elements of the axial-vector current $A_a^\mu(x)$ by subtracting off the pion pole contributions to the latter (see Ref. 13).

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[25]In adopting the analysis of Ref. 10 we give up some of the generality of the present method, since the σ commutator is now assumed to be an isosinglet operator. For that reason we will not estimate the corrections to the $I_t = 2$ sum rule.

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[30]However, if we combine $R^{(0)}(0, \nu_B)$ with the $M^{(0)}(0, \nu_B)$ of Prasad and Brehm (Ref. 29), we obtain a form factor $\sigma^{(0)}(t)$ which varies much more rapidly than the one determined in Ref. 29.

[31]We are presently studying, with D. Boal, the on-mass-shell current-algebra sum rules in πK scattering. This affords the opportunity to compare the magnitude of the correction term when the kaons are contracted [in the expression analogous to Eq. (6)] with the size of the corresponding term when the pions are contracted.