Comments on alternate forms for the breaking of chiral SU(3)⊗SU(3)*

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Possible alternatives to the usual $(3^*, 3) \oplus (3, 3^*)$ form for the symmetry breaking of chiral SU(3) ⊗ SU(3) are discussed. The width for the decay $X \to \eta\pi\pi$ is calculated in the pure $(6^*, 6) \oplus (6, 6^*)$ breaking model, and the result is compared with $(3^*, 3) \oplus (3, 3^*)$ and (8, 8) models. The symmetry-breaking parameters are shown to be restricted to certain allowed regions for both $(6^*, 6) \oplus (6, 6^*)$ and (8, 8) representations on the basis of positivity of spectral functions. These restrictions are shown to be particularly severe if the additional assumption $f_K \geq f_\pi$ is imposed. Based on $X \to \eta\pi\pi$ and present $\pi\pi$ scattering data we conclude that a symmetry breaking which is reducible into $(3^*, 3) \oplus (3, 3^*)$ and (8, 8) is favored and we estimate the relative amounts of the two types of breaking.

I. INTRODUCTION

Success of SU(3)⊗SU(3) current algebra has prompted a study of the nature of the breaking of chiral symmetry. The most popular model for symmetry breaking has been the $(3^*, 3) \oplus (3, 3^*)$ model, which was investigated by Gell-Mann, Oakes, and Renner[1], and by Glashow and Weinberg.[2] In this model the symmetry-breaking Hamiltonian, H_{SB}, transforms like the quark mass terms in a quark model of current algebra. The symmetry is realized in the Goldstone manner by the vacuum being approximately SU(3)-invariant, to ensure an SU(3) spectrum of states, and the Hamiltonian being approximately SU(3)⊗SU(3)-invariant to ensure smallness of the pion mass in relation to other pseudoscalar mesons. This latter property is specific to the $(3^*, 3) \oplus (3, 3^*)$-breaking model. Thus, if we write the total Hamiltonian density as

$$H = H_0 + H_{SB}, \tag{1.1}$$

where H_0 is SU(3)×SU(3)-invariant, and H_{SB} is the symmetry-breaking term, then in the $(3^*, 3)$ model we have

$$H_{SB} = \epsilon_0 U_0 + \epsilon_8 U_8, \tag{1.2}$$

where U_0 and U_8 are the unitary singlet and the $I=0$, $Y=0$ member of an octet. Smallness of the pion mass requires $a (\equiv \epsilon_8/\epsilon_0) \approx -\sqrt{2}$ yielding approximate SU(2)⊗SU(2) invariance. H_{SB} can be decomposed into an SU(2)⊗SU(2)-invariant part, H_1, and a part that breaks SU(2)⊗SU(2) invariance, H_2:

$$\begin{aligned} H_{SB} &= H_1 + H_2, \\ H_1 &= \tfrac{1}{3}\epsilon_0(1-\sqrt{2}\,a)(U_0 - \sqrt{2}\,U_8), \\ H_2 &= \tfrac{1}{3}\epsilon_0(a+\sqrt{2})(\sqrt{2}U_0 + U_8). \end{aligned} \tag{1.3}$$

The pattern of symmetry breaking is thus necessarily

$$H_0 \gg H_1 \gg H_2. \tag{1.4}$$

Tests of such a model of symmetry breaking involve the verification of Eq. (1.4). This is experimentally feasible since the σ term involved in current-algebra calculation of processes involving two pions is proportional to H_2;

$$H_2 = i[F_5^\alpha, \partial^\mu A_\mu^\alpha],$$

$$\alpha = 1,\ 2,\ \text{or}\ 3\ \text{(no summation)}. \tag{1.5}$$

Expectation values of H_2 have been investigated in three processes in recent years: (a) the nucleon σ term extracted from $\pi + N$ scattering data, (b) the expectation value $\langle X|H_2|\eta\rangle$ in the decay $X \to \eta\pi\pi$, and (c) the σ term in $\pi+\pi\to\pi+\pi$ scattering. In all the three processes a serious discrepancy exists between predictions of the $(3^*, 3)$ model and experiment. In process (a) the experimental values fall in the range[3-7]

$$\sigma_{NN}^{\text{expt}} = 70 \pm 30 \text{ MeV}.$$

Theoretically, if we assume H_2 is ~10% of H_1, and H_1 is of the order of SU(3) breaking, then

$$\sigma_{NN}^{\text{theory}} = 10\text{–}15 \text{ MeV}.$$

In process (b) the experimental width which is proportional to the square of the σ term[8] is

$$\Gamma_{X\to\eta\pi\pi}^{\text{expt}} \lesssim 1 \text{ MeV},$$

while the theoretical range[9] is

$$\Gamma_{X\to\eta\pi\pi}^{\text{theory}} \approx 0.2\text{–}20 \text{ keV}.$$

It has been emphasized in recent years that the π-π σ term is not purely $I=0$, as required by the $(3^*, 3)$ model,[10] but that a sizable $I=2$ admixture exists.

It thus becomes natural to ask if the symmetry-breaking Hamiltonian could transform according to some other irreducible representation. Among the simplest viable alternatives[11] are the (8, 8)- and the $(6^*, 6)\oplus(6, 6^*)$-breaking models.

The nucleon σ term has been calculated with both pure (8, 8) (Ref. 12) and pure $(6^*, 6)\oplus(6, 6^*)$ symmetry breaking,[13,14] and a larger σ_{NN} can be obtained in either model. Predictions based on a pure $(6^*, 6)$ model, however, do not seem to agree with σ terms obtained from KN and $\pi\Sigma$ experiments.[15] The $\pi\pi$ scattering process becomes worse in either pure (8, 8) or in pure $(6, 6^*)\oplus(6^*, 6)$ because of the sign of the contribution.[12,13]

For the process $X\to\eta\pi\pi$ a symmetry breaking which transforms as (8, 8) yields a very reasonable value.[16] We calculate this process for $(6^*, 6)\oplus(6, 6^*)$ symmetry breaking and find a value larger than that found in the $(3^*, 3)$ model but still much smaller than the experimental number.

The more difficult problem of Hamiltonian densities with mixed representations seems intractable in general because of several new parameters that enter the theory. Even in the meson sector, and assuming octet dominance, four new parameters enter for every added representation (the vacuum expectation values and strengths of the singlet and octet parts). If we assume, however, that the mixing satisfies an additivity property

$$H_{SB} = \alpha H^{(3^*,3)} + (1-\alpha)H', \tag{1.6}$$

where $H^{(3^*,3)}$ is the (normalized) contribution from $(3^*, 3)\oplus(3, 3^*)$ breaking and H' is the contribution from $(6^*, 6)\oplus(6, 6^*)$ or (8, 8), then it makes sense to consider the two contributions separately. The expectation values of the σ terms are the weighted sum of the expectation values of the σ terms calculated in models with H_{SB} belonging to the different irreducible representations of the symmetry.

In Sec. II we define our notation and discuss the $(6^*, 6)$-breaking model. The results of Sec. II are then used in Sec. III to evaluate the $X\to\eta\pi\pi$ decay rate. This rate is compared with the results for the rate obtained from other breaking models.

Nontrivial restrictions on the parameters of the symmetry breaking have been obtained for the $(3^*, 3)$ model from the requirement of positivity of the spectral functions.[17] In Sec. IV we calculate the similar restrictions for the cases of pure $(6^*, 6)\oplus(6, 6^*)$ or pure (8, 8) symmetry breaking. Surprisingly, if we also add the assumption of one-meson dominance and make the reasonable requirement that $f_K \geq f_\pi$, the parameters are severely restricted in all models.

Our conclusions are contained in Sec. V.

II. THE $(6,6^*)\oplus(6^*,6)$-BREAKING MODEL

The $(6, 6^*)\oplus(6^*, 6)$ model has been investigated by Auvil[13] as a possible alternative to $(3^*, 3)\oplus(3, 3^*)$ model. The commutation rules of the scalar and pseudoscalar densities with the charges are given by

$$[F_\alpha, U_0] = 0, \tag{2.1a}$$

$$[F_\alpha, U_\beta] = if_{\alpha\beta\gamma}U_\gamma, \tag{2.1b}$$

$$[F_\alpha, U_\theta] = -iF_{\theta\alpha\Gamma}U_\Gamma, \tag{2.1c}$$

$$[F_\alpha, V_0] = 0, \tag{2.1d}$$

$$[F_\alpha, V_\beta] = if_{\alpha\beta\gamma}V_\gamma, \tag{2.1e}$$

$$[F_\alpha, V_\theta] = -iF_{\theta\alpha\Gamma}V_\Gamma, \tag{2.1f}$$

$$[F^5_\alpha, U_0] = -i\,(\tfrac{5}{3})^{1/2}V_\alpha, \tag{2.1g}$$

$$[F^5_\alpha, U_\beta] = -i\,(\tfrac{5}{3})^{1/2}\delta_{\alpha\beta}V_0 - \frac{7i}{5}d_{\alpha\beta\gamma}V_\gamma - \frac{i}{\sqrt{10}}D_{\alpha\beta\theta}V_\theta, \tag{2.1h}$$

$$[F^5_\alpha, U_\theta] = -\frac{i}{\sqrt{10}}D_{\alpha\beta\theta}V_\beta - iE_{\alpha\theta\Gamma}V_\Gamma, \tag{2.1i}$$

$$[F^5_\alpha, V_0] = i\,(\tfrac{5}{3})^{1/2}U_\alpha, \tag{2.1j}$$

$$[F^5_\alpha, V_\beta] = i\,(\tfrac{5}{3})^{1/2}\delta_{\alpha\beta}U_0 + \frac{i7}{5}d_{\alpha\beta\gamma}U_\gamma + \frac{i}{\sqrt{10}}D_{\alpha\beta\theta}U_\theta, \tag{2.1k}$$

$$[F^5_\alpha, V_\theta] = \frac{i}{\sqrt{10}}D_{\alpha\beta\theta}U_\beta + iE_{\alpha\theta\Gamma}U_\Gamma, \tag{2.1l}$$

where F_α is the vector charge; F^5_α is the axial charge; U_0, U_α, U_θ are the scalar densities which transform as $\underline{1}, \underline{8}, \underline{27}$ representations of SU(3), respectively; and V_0, V_α, V_θ are the corresponding pseudoscalar densities. Note that $\alpha, \beta, \gamma = 1, \ldots, 8$ while $\theta, \Gamma = 1, \ldots, 27$. The structure constants are defined in terms of the algebra of 6×6 matrices that decomposed into singlet Δ, octet $\{S^\alpha\}$, and 27-plet matrices $\{T^\theta\}$. The orthonormality relations chosen for these matrices are

$$\begin{aligned}&\mathrm{Tr}(\Delta\Delta) = 6, \quad \mathrm{Tr}(\Delta S^\alpha) = \mathrm{Tr}(\delta T^\theta) = 0,\\ &\mathrm{Tr}(S^\alpha S^\beta) = 10\delta^{\alpha\beta}, \quad \mathrm{Tr}(S^\alpha T^\theta) = 0,\\ &\mathrm{Tr}(T^\theta T^\Gamma) = \delta_{\theta\Gamma}.\end{aligned} \tag{2.2}$$

The algebra obeyed by the matrices is

$$[S_\alpha, S_\beta] = 2if_{\alpha\beta\gamma}S_\gamma, \tag{2.3a}$$

$$\{S_\alpha, S_\beta\} = \tfrac{10}{3}\delta_{\alpha\beta} + \tfrac{14}{5}d_{\alpha\beta\gamma}S_\gamma + 2D_{\alpha\beta\theta}T_\theta, \tag{2.3b}$$

$$\{S_\alpha, T_\theta\} = 2E_{\alpha\theta\Gamma}T_\Gamma + \tfrac{1}{5}D_{\alpha\beta\theta}S_\beta, \tag{2.3c}$$

$$[S_\alpha, T_\theta] = -iF_{\theta\alpha\Gamma}T_\Gamma. \tag{2.3d}$$

The symbols $f_{\alpha\beta\gamma}$ and $d_{\alpha\beta\gamma}$ are the usual SU(3) structure constants, while $D_{\alpha\beta\theta}$, $F_{\theta\alpha\Gamma}$, and $E_{\alpha\theta\Gamma}$ can be evaluated from the explicit representation for the matrices given in the Appendix.

For the symmetry-breaking Hamiltonian we take

$$H = C_{6,0}U_0 + C_{6,8}U_8 + C_{6,27}U_{27}, \qquad (2.4)$$

where U^8 and U^{27} are the $I=0$, $Y=0$ members of the octet and 27-plet representations, respectively.

Using the commutation relations above it is easy to write the σ commutators:

$$[F^\alpha,[F^\beta,H]] = C_{68}f^{\alpha 8\gamma}f^{\gamma\beta\delta}U_\delta + C_{6,27}F^{27\alpha\theta}F^{\theta\beta\Gamma}U_\Gamma, \qquad (2.5)$$

$$\begin{aligned}[F_5^\alpha,[F_5^\beta,H]] = {} & C_{60}\left(\tfrac{5}{3}\delta^{\alpha\beta}U_0 + \frac{7}{\sqrt{15}}d^{\alpha\beta\gamma}U_\gamma + \frac{1}{\sqrt{6}}D^{\alpha\beta\Gamma}U_\Gamma\right) \\ & + C_{68}\left[\tfrac{5}{3}\delta^{\alpha 8}U^\beta + \tfrac{7}{5}d^{8\alpha\gamma}\left((\tfrac{5}{3})^{1/2}\delta^{\gamma\beta}U_0 + \tfrac{7}{5}d^{\gamma\lambda\beta}U_\lambda + \frac{1}{\sqrt{10}}D^{\gamma\beta\Gamma}U_\Gamma\right) + \frac{1}{\sqrt{10}}D^{8\alpha\theta}\left(\frac{1}{\sqrt{10}}D^{\theta\beta\lambda}U_\lambda + E^{\theta\beta\Gamma}U_\Gamma\right)\right] \\ & + C_{6,27}\left[\frac{1}{\sqrt{10}}D^{27\alpha\gamma}\left((\tfrac{5}{3})^{1/2}\delta^{\gamma\beta}U_0 + \tfrac{7}{5}d^{\beta\gamma\lambda}U_\lambda + \frac{1}{\sqrt{10}}D^{\gamma\beta\Gamma}U_\Gamma\right) + E^{27\alpha\theta}\left(\frac{1}{\sqrt{10}}D^{\theta\beta\lambda}U_\lambda + E^{\theta\beta\Gamma}U_\Gamma\right)\right].\end{aligned} \qquad (2.6)$$

The contributions of the σ terms to $\pi\pi$ scattering and πN scattering have been considered before.[13,14] We shall work out the consequences for $X\to\eta\pi\pi$ decay in the next section.

III. THE DECAY $X\to\eta\pi\pi$

The width for X decay[18] is given in terms of the slope parameter g as

$$\Gamma(X\to\eta\pi\pi) = 3(1.00+0.24g+0.27g^2)|M|^2 \qquad (3.1)$$

in units of keV. The matrix element is

$$M = \frac{2}{f_\pi^2}\,\frac{\sigma_{X\eta}}{1+2g}, \qquad (3.2)$$

where $\sigma_{X\eta}$ is the matrix element of the σ commutator (2.6):

$$\sigma_{X\eta} = \langle\eta|[F_5^3,\ [F_5^3,\ H]]|X\rangle. \qquad (3.3)$$

For the purposes of calculating this decay we will assume that the symmetry breaking should be dominated by the octet term and set $C_{6,27}=0$ in H.

The scalar density U_8 where 8 is the member of the 27-plet transforms as $I=2$ and thus does not contribute to $\sigma_{X\eta}$. The structure constants which we need are given in the Appendix.

We evaluate (3.3) by parameterizing the matrix elements of the scalar densities as

$$\begin{aligned}\langle M_k|U_i|M_j\rangle = {} & \alpha'\,\delta_{k0}\delta_{i0}\delta_{j0} + \alpha\delta_{i0}\delta_{jk} + \beta d_{ijk} \\ & + \frac{1}{\sqrt{6}}\beta'(\delta_{k0}\delta_{ij} + \delta_{ik}\delta_{j0}) \\ & + \gamma' D_{ijk},\end{aligned} \qquad (3.4)$$

where j, k equal $0, 1, \ldots, 8$ and i equals 0 (singlet), $1,\ldots,8$ (octet), or $1,\ldots,27$ (27-plet). The physical η and X states are defined as

$$\begin{aligned}|\eta\rangle &= p|\eta_8\rangle + q|\eta_0\rangle, \\ |X\rangle &= q|\eta_8\rangle - p|\eta_0\rangle,\end{aligned} \qquad (3.5)$$

with $p^2+q^2=1$.

In terms of our decomposition (2.10) the masses of the pseudoscalar mesons are

$$m_\pi^2 = C_{6,0}[\alpha+(\tfrac{5}{3})^{1/2}\beta] + C_{6,8}\frac{7}{5\sqrt{3}}\beta, \qquad (3.6a)$$

$$m_K^2 = C_{6,0}[\alpha+(\tfrac{5}{3})^{1/2}\beta] - C_{6,8}\frac{7}{10\sqrt{3}}\beta, \qquad (3.6b)$$

$$m_{\eta_8}^2 = C_{6,0}[\alpha+(\tfrac{5}{3})^{1/2}\beta] - C_{6,8}\frac{7}{5\sqrt{3}}\beta, \qquad (3.6c)$$

$$m_{\eta_8-\eta_0}^2 = C_{6,8}\left[\left(\frac{5}{3}\right)^{1/2}\beta + \frac{1}{\sqrt{6}}\beta'\right], \qquad (3.6d)$$

$$m_{\eta_0}^2 = \mu_0^2 + C_{6,0}\left[\alpha'+\alpha+\left(\frac{5}{3}\right)^{1/2}\beta + \frac{2}{\sqrt{6}}\beta'\right]. \qquad (3.6e)$$

The term μ_0^2 is added to (3.6c) so that the mass of η_0 will not be zero in the limit of exact SU(3) $\otimes$ SU(3) symmetry. We cannot determine the value of μ_0^2 but it is expected to be around 1 GeV. The masses m_π^2, m_K^2, and m_8^2 satisfy the Gell-Mann–Okubo mass formula, as they must since $C_{6,27}$ has been set equal to zero.

These equations (3.6), together with the equations from diagonalizing the mass matrix

$$m_X^2 + m_\eta^2 = m_{\eta_8}^2 + m_{\eta_0}^2, \qquad (3.7a)$$

$$(m_X^2 - m_8^2)(m_8^2 - m_\eta^2) = m_{\eta_8-\eta_0}^4, \qquad (3.7b)$$

give us six equations on the eight unknowns C_{60}, C_{68}, α, α', β, β', γ', and p.

We may get one more equation by considering the divergence of the axial-vector current

$$\partial_\mu A^\mu_\alpha = i[F^\alpha_5, H]. \tag{3.8}$$

The matrix element of this between the vacuum and the one-pion state gives

$$\begin{aligned}\langle 0|\partial_\mu A^\mu_3|\pi\rangle &= m_\pi{}^2 f_\pi \\ &= \left[C_{60}\left(\frac{5}{3}\right)^{1/2} + C_{68}\frac{7}{5\sqrt{3}}\right]\langle 0|V_3|\pi\rangle .\end{aligned} \tag{3.9}$$

This implies that the pion mass is zero if $C_{68} = -\frac{5}{7}\sqrt{5}\,C_{60}$, which, together with (3.6a), implies $\alpha = 0$.

Finally an eighth equation may be found by using soft-pion techniques and comparing

$$\lim_{p\to 0}\langle M_3(p)|\,U_{27}|M_3(p')\rangle \tag{3.10a}$$

with

$$\lim_{p\to 0}\langle M_3(p)|\,U_8|M_3(p')\rangle . \tag{3.10b}$$

If we assume that $\langle 0|V_\Gamma|M_3\rangle$, $\Gamma = 3$ can be neglected compared with $\langle 0|V_\gamma|M_3\rangle$, $\gamma = 3$, where M_3 is the member of the octet (pion), then (3.10a) and (3.10b) give

$$\gamma' = \frac{1}{\sqrt{10}}\beta. \tag{3.11}$$

Solving the eight equations gives

$$\sigma_{X\eta} = m_\pi{}^2(0.47 + 1.80\mu_0{}^2). \tag{3.12}$$

If $\mu_0{}^2$ is 1 GeV^2 we obtain

$$\Gamma(X\to\eta\pi\pi) = (64\text{–}254)\ \text{keV} \tag{3.13}$$

for a range of the slope parameter $g = 0$ to $g = -0.2$.

This can be compared with the result derived by assuming $(3,3^*)\oplus(3^*,3)$ symmetry breaking[9]:

$$\Gamma(X\to\eta\pi\pi) \simeq (0.2 - 0.8)\ \text{keV}, \tag{3.14}$$

where again $\mu_0{}^2$ is taken to be 1 GeV^2 and we have taken the case of no dilaton.[18]

The symmetry breaking (8, 8) gives a much more reasonable value for the width,[16]

$$\Gamma(X\to\eta\pi\pi) \simeq (1\text{–}5)\ \text{MeV}. \tag{3.15}$$

A symmetry breaking which transforms as (1, 8) or (8, 1) gives zero contribution to $X\to\eta\pi\pi$. Thus, if we are to add an alternate symmetry breaking to $(3,3^*)\oplus(3^*,3)$ which will give a reasonable value for the width of X decay, it would seem that it is required to transform as (8, 8).

IV. RESTRICTIONS DUE TO POSITIVITY OF SPECTRAL FUNCTIONS

If we write a Lehmann-Källén spectral representation for the vacuum expectation value of the commutation of two vector or two axial-vector currents then the integral of the scalar spectral function is the vacuum expectation value of the σ commutators, (2.5) or (2.6). Let us call this vacuum matrix element of (2.5) $K_{\alpha\beta}$, and the similar matrix element of (2.6) $I_{\alpha\beta}$. Positivity of the spectral function then requires $K_{\alpha\beta}$ and $I_{\alpha\beta}$ to be positive. This requirement puts nontrivial restrictions on the values of the symmetry-breaking parameters.

The restrictions for the $(3^*,3)\oplus(3,3^*)$ symmetry-breaking model are well known. There the Hamiltonian is

$$H = \epsilon_0 U_0 + \epsilon_8 U_8, \tag{4.1}$$

and if we define

$$\begin{aligned} &a = \frac{\epsilon_8}{\epsilon_0},\quad b = \frac{\langle U_8\rangle_0}{\langle U_0\rangle_0},\\ &\gamma = -\tfrac{2}{3}\epsilon_0\langle U_0\rangle_0,\end{aligned} \tag{4.2}$$

we have

$$I_{33} = \gamma(1 + a + b + ab), \tag{4.3a}$$

$$I_{44} = \gamma(1 - \tfrac{1}{2}a - \tfrac{1}{2}b + \tfrac{1}{4}ab), \tag{4.3b}$$

$$I_{88} = \gamma(1 - a - b + 3ab), \tag{4.3c}$$

$$K_{44} = \tfrac{9}{4}\gamma ab. \tag{4.3d}$$

Requiring these to be positive restricts a and b to lie in the domains shown in Fig. 1.

If the symmetry breaking is purely $(6,6^*)\oplus(6^*,6)$ then the restrictions are similar. In this case the Hamiltonian is given by (2.4), and if we assume $C_{6,27}$ and $\langle U_{27}\rangle_0$ are zero and define

$$\begin{aligned} &a = \frac{7}{5\sqrt{5}}\,\frac{C_{68}}{C_{60}},\quad b = \frac{7}{5\sqrt{5}}\,\frac{\langle U_8\rangle_0}{\langle U_0\rangle_0},\\ &\gamma = -\tfrac{5}{3}C_{60}\langle U_0\rangle_0,\end{aligned} \tag{4.4}$$

then the matrix elements of the σ commutators give

$$I_{33} = \gamma(1 + a + b + \tfrac{85}{49}ab), \tag{4.5a}$$

$$I_{44} = \gamma(1 - \tfrac{1}{2}a - \tfrac{1}{2}b + \tfrac{265}{196}ab), \tag{4.5b}$$

$$I_{88} = \gamma(1 - a - b + \tfrac{225}{49}ab), \tag{4.5c}$$

$$K_{44} = \tfrac{225}{196}\gamma ab. \tag{4.5d}$$

The structure functions $D^{\alpha\beta\theta}$ and $E^{\alpha\beta\theta}$ needed to obtain these are listed in the Appendix.

The restrictions on the parameters a and b for this case are shown in Fig. 2.

We can also work out the restrictions for the (8, 8) symmetry-breaking model. Here there are 64 operators $S^{\alpha\beta}$, $\alpha, \beta = 1, \ldots, 8$. In terms of these the Hamiltonian can be written as

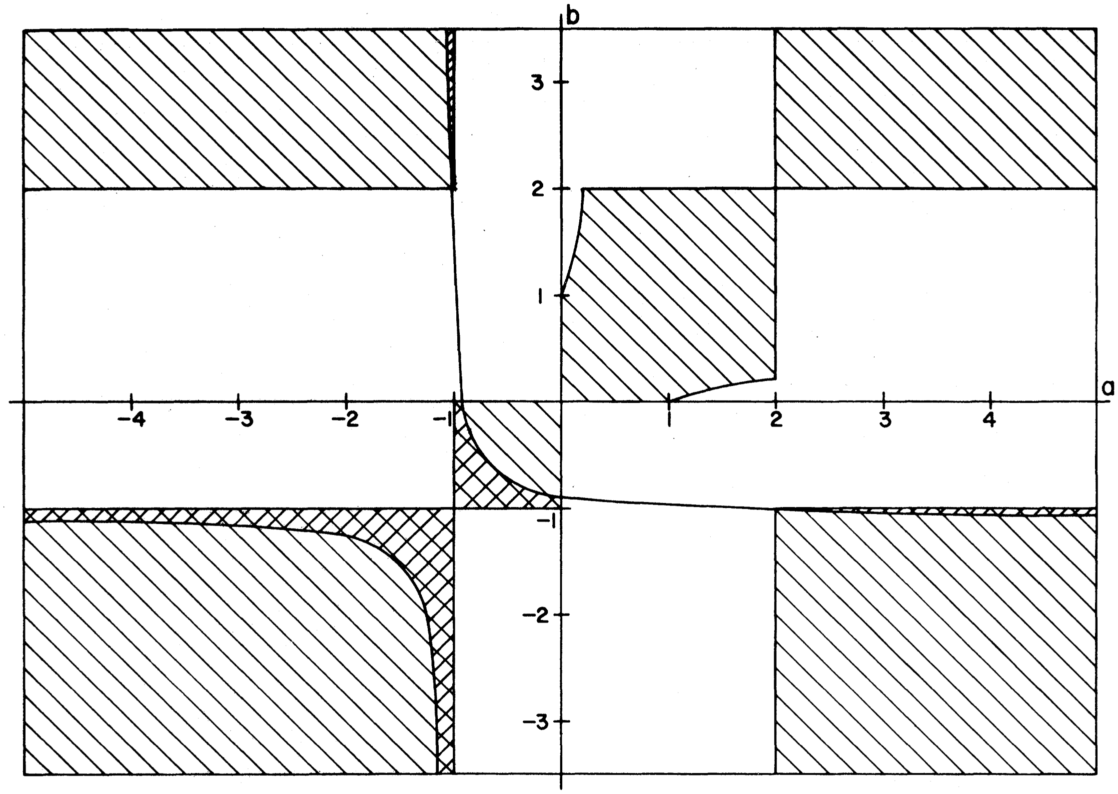

FIG. 1. The allowed values of the symmetry-breaking parameters a and b for the $(3^*,3)\oplus(3,3^*)$ model. The cross-hatched regions are the only allowed values if we require $f_K \geq f_\pi$.

$$H(x)=\frac{A}{2\sqrt{8}}\,S^{\alpha\alpha}(x)+\frac{B\sqrt{3}}{2\sqrt{5}}\,d_{8\alpha\beta}S^{\alpha\beta}(x). \tag{4.6}$$

The $S^{\alpha\beta}$ can be expanded as

$$S^{\alpha\beta}=\tfrac{1}{8}\,S^{\alpha\beta}\overline{S}+d_{8\alpha\beta}S^{8}+S_{27,\alpha\beta}S^{27}, \tag{4.7}$$

with $\langle S^{27}\rangle_0$ taken to be zero. In this case we define

$$a=\frac{1}{2}\frac{B}{A},\quad b=\frac{4}{\sqrt{3}}\,\frac{\langle S^8\rangle_0}{\langle \overline{S}\rangle_0},\qquad \gamma=-\frac{3}{4\sqrt{5}}\,A\langle\overline{S}\rangle_0, \tag{4.8}$$

and the independent $I_{\alpha\beta}$ and $K_{\alpha\beta}$ are

$$I_{33}=\gamma(1+a+b+3ab), \tag{4.9a}$$

$$I_{44}=\gamma(1-\tfrac{1}{2}a-\tfrac{1}{2}b+\tfrac{5}{4}ab), \tag{4.9b}$$

$$I_{88}=\gamma(1-a-b+ab), \tag{4.9c}$$

$$K_{44}=\tfrac{5}{4}\gamma ab. \tag{4.9d}$$

The restrictions on a and b are similar to those shown in Fig. 2 for the $(6,6^*)\oplus(6^*,6)$ model

If we introduce a set of intermediate states in the σ commutator and assume that the one-meson state, where the mesons belong to the octet, saturates the relation then we may identify

$$I_{33}=\tfrac{1}{2}f_\pi{}^2m_\pi{}^2, \tag{4.10a}$$

$$I_{44}=\tfrac{1}{2}f_K{}^2m_K{}^2, \tag{4.10b}$$

$$I_{88}=\tfrac{1}{2}f_8{}^2m_8{}^2, \tag{4.10c}$$

$$K_{44}=\tfrac{1}{2}f_\kappa{}^2m_\kappa{}^2, \tag{4.10d}$$

where, for example, f_π is defined by the matrix element of the axial-vector current

$$\langle 0|A_\mu^{(3)}(0)|\pi(k)\rangle=i\frac{1}{\sqrt{2}}f_\pi k_\mu\,. \tag{4.11}$$

Now experimentally $f_K{}^2/f_\pi{}^2$ is greater than one. The interesting thing is that, in the $(6,6^*)\oplus(6^*,6)$ model where we use (4.10) in (4.5), requiring

$$\frac{f_K{}^2}{f_\pi{}^2}\geq 1 \tag{4.12}$$

and using experimental values for m_K and m_π eliminates almost all of the allowed values of a and b of Fig. 2. The only values of a and b consistent with (4.12) are the narrow regions shown in Fig. 3.

The same thing happens in the (8, 8) model and also in the $(3^*,3)$ model; requiring (4.12) restricts a and b of (4.8) to lie in narrow bands similar to

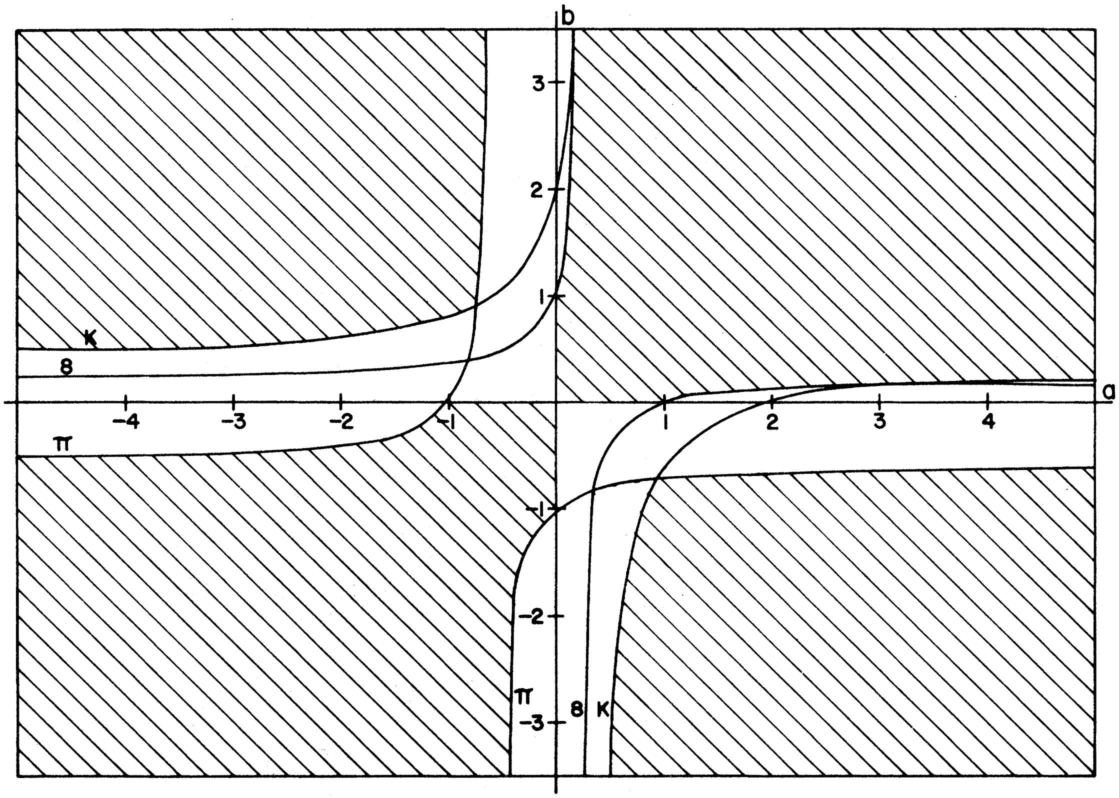

FIG. 2. The allowed values of the symmetry-breaking parameters a and b for the $(6^*, 6) \oplus (6, 6^*)$ model. The (8, 8) model gives a similar graph.

those of Fig. 3. For the $(3^*, 3)$ model the allowed values of a and b are shown by the cross-hatched areas in Fig. 1.

These restrictions may be stated in a different way by not using the experimental values for the masses. In any calculation involving only $(6, 6^*) \oplus (6^*, 6)$ or (8, 8) symmetry breaking, any assumption of SU(3) symmetry of a matrix element like, e.g.,

$$\langle 0 | V_\alpha | M_\beta \rangle \sim \delta_{\alpha\beta} \tag{4.13}$$

or

$$\langle M_a | U_b | M_a \rangle = \alpha \delta_{b0} + \beta d_{aab} \tag{4.14}$$

is sufficient, when taken together with (4.10) and positivity, to require

$$\frac{f_K^2}{f_\pi^2} \leq 1. \tag{4.15}$$

For example, (4.14) leads to the GMOR formula[1]

$$\frac{m_K^2}{m_\pi^2} = \frac{1 - \frac{1}{2}a}{1 + a} . \tag{4.16}$$

This, together with (4.5) and (4.10), is enough to require (4.15). This is not true in the $(3^*, 3) \oplus (3, 3^*)$ model, however, since for that model (4.16) gives

$$\frac{f_K^2}{f_\pi^2} = \frac{1 - \frac{1}{2}b}{1 + b}, \tag{4.17}$$

which does not necessarily require (4.15). This difference in the models can easily be seen from Fig. 1 and Fig. 3. Using the experimental values for m_K and m_π in (4.16) gives $a \approx -0.89$. This value of a requires $b = 0$ in Fig. 3, but all b between 0 and -1 are allowed in Fig. 1.

In most of the discussions in the recent literature the authors have been careful to set $f_K = f_\pi$ when using assumptions like (4.13) or (4.14), but it is possible to find calculations where this was not done.[19] We do not know, however, of any cases where incorrect conclusions were reached because of this mistake. It does not seem to have been realized, however, that $f_K = f_\pi$ is in fact required by positivity.

Finally we could plot f_8^2/f_π^2 on the same graph

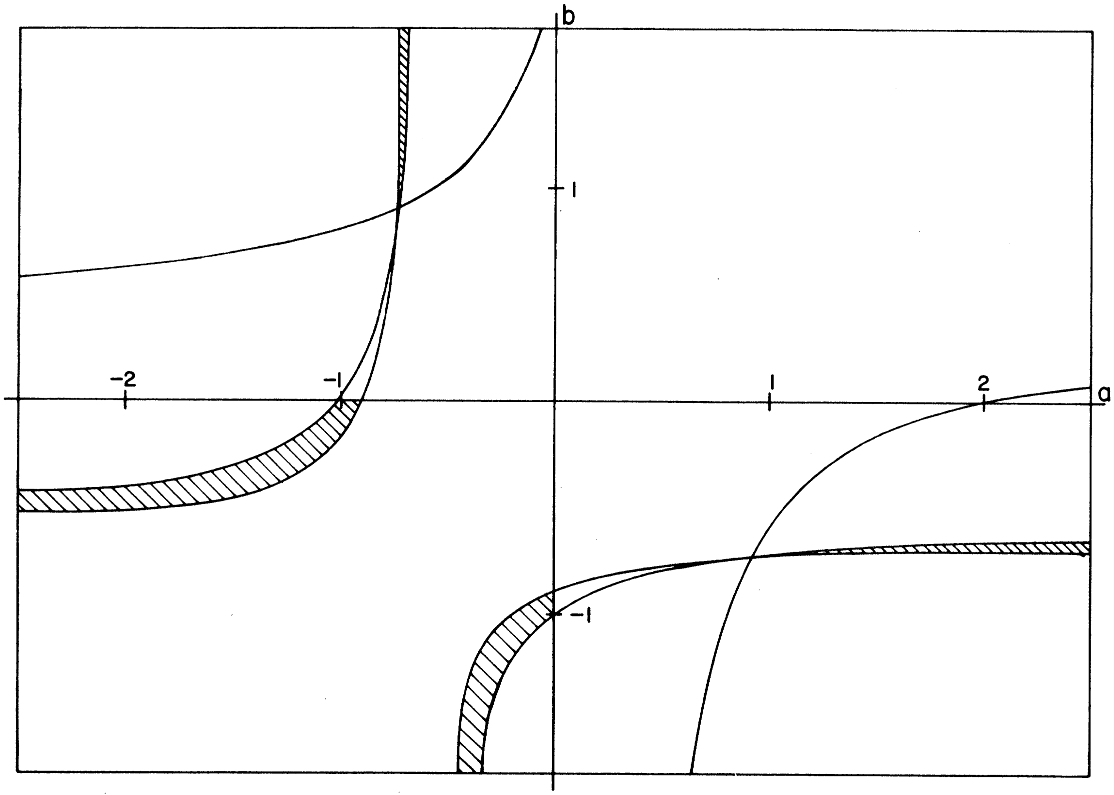

FIG. 3. The allowed values of a and b if we require $f_K \geq f_\pi$. The (8, 8) model has similar allowed regions.

where we have required $f_K \geq f_\pi$ to determine the allowed values of f_8. It turns out that the only allowed values of f_8 are $f_8 \geq f_K$.

V. CONCLUSIONS

On the basis of present experimental data it is unlikely that any single irreducible representation of SU(3)×SU(3) can describe the symmetry breaking. If we assume that the symmetry breaking contains $(3^*, 3)\oplus(3, 3^*)$ plus one additional representation, then (8, 8) would be the preferred choice because the rate for $X \to \eta\pi\pi$ is large. Assuming the additivity discussed in Sec. I, we write the symmetry-breaking Hamiltonian as

$$H_{SB} = \alpha H^{(3^*,3)} + (1-\alpha) H^{(8,8)} . \qquad (5.1)$$

The expression for the X rate is now

$$\Gamma_{X\to\eta\pi\pi} \cong (1-\alpha)^2 (1.2 \text{ to } 5) \text{ MeV}. \qquad (5.2)$$

To obtain the correct sign for the $\pi\pi$ scattering lengths α should be greater than one. A value of $\alpha \approx \frac{4}{3}$ would then be consistent with data:

$$\Gamma_{X\to\eta\pi\pi} \approx 130\text{–}500 \text{ keV}. \qquad (5.3)$$

The $I = 0$ and $I = 2$ $\pi\pi$ scattering lengths are related to A, which is defined by

$$A = \frac{1}{F^4}\langle 0|[F^5_{(3)},[F^5_{(3)},[F^5_{(3)},[F^5_{(3)},H]]]]|0\rangle . \qquad (5.4)$$

The same value of α ($\approx \frac{4}{3}$) gives

$$A \approx \tfrac{4}{3}A^{(3^*,3)} - \tfrac{1}{3}A^{(8,8)} \qquad (5.5)$$

$$= 5\frac{m_\pi^2}{F^2} . \qquad (5.6)$$

Such a value is in agreement with experiment. The (8, 8) contribution to the nucleon σ term can be sufficiently large now to fit the data. Thus a consistent picture of symmetry breaking emerges, in which the dominant symmetry breaking is contained in $(3^*, 3)$ with a small (8, 8) admixture.

APPENDIX

The $(6^*, 6)\oplus(6, 6^*)$ symmetry breaking is conveniently discussed in terms of the 6×6 matrices defined in Sec. II. If by δ_{ab} we mean that the element in the a row and b column is 1 then one possible choice for these matrices is

$$S_1=\sqrt{2}(\delta_{12}+\delta_{23})+\delta_{45}+\text{H.c.},$$
$$S_2=\sqrt{2}i\,(\delta_{12}+\delta_{23})+i\delta_{45}+\text{H.c.},$$
$$S_3=2(\delta_{11}-\delta_{33})+\delta_{44}-\delta_{55},$$
$$S_4=\sqrt{2}(\delta_{14}+\delta_{46})+\delta_{25}+\text{H.c.},$$
$$S_5=\sqrt{2}i\,(\delta_{14}+\delta_{46})+i\delta_{25}+\text{H.c.},$$
$$S_6=\sqrt{2}(\delta_{35}+\delta_{56})+\delta_{24}+\text{H.c.},$$
$$S_7=\sqrt{2}\,i\,(\delta_{35}+\delta_{56})+i\delta_{24}+\text{H.c.},$$
$$S_8=\frac{2}{\sqrt{3}}(\delta_{11}+\delta_{22}+\delta_{33})-\frac{1}{\sqrt{3}}(\delta_{44}+\delta_{55})-\frac{4}{\sqrt{3}}\delta_{66},$$
$$\sqrt{40}\,T_1=\sqrt{2}\,i\,(\delta_{12}+\delta_{23})-4i\delta_{45}+\text{H.c.},$$
$$\sqrt{40}\,T_2=\sqrt{2}(\delta_{12}+\delta_{23})-4\delta_{45}+\text{H.c.},$$
$$\sqrt{10}\,T_3=\delta_{11}-\delta_{33}-2\delta_{44}+2\delta_{55},$$
$$\sqrt{40}\,T_4=\sqrt{2}(\delta_{14}+\delta_{46})-4\delta_{25}+\text{H.c.},$$
$$\sqrt{40}\,T_5=i\sqrt{2}(\delta_{14}+\delta_{46})-4i\delta_{25}+\text{H.c.},$$
$$\sqrt{40}\,T_6=\sqrt{2}(\delta_{35}+\delta_{56})-4\delta_{24}+\text{H.c.},$$
$$\sqrt{40}\,T_7=\sqrt{2}i\,(\delta_{35}+\delta_{56})-4i\delta_{24}+\text{H.c.},$$
$$\sqrt{6}\,T_8=\delta_{11}+\delta_{33}-2\delta_{22},$$
$$\sqrt{2}\,T_9=i\delta_{13}+\text{H.c.},$$
$$\sqrt{2}\,T_{10}=\delta_{13}+\text{H.c.},$$
$$2T_{11}=i(\delta_{12}-\delta_{23})+\text{H.c.},$$
$$2T_{12}=\delta_{12}-\delta_{23}+\text{H.c.},$$
$$\sqrt{2}\,T_{13}=i\delta_{36}+\text{H.c.},$$
$$2T_{14}=\delta_{14}-\delta_{46}+\text{H.c.},$$
$$2T_{15}=i(\delta_{14}-\delta_{46})+\text{H.c.},$$
$$2T_{16}=\delta_{35}-\delta_{56}+\text{H.c.},$$
$$2T_{17}=i(\delta_{35}-\delta_{56})+\text{H.c.},$$
$$\sqrt{2}\,T_{18}=\delta_{15}+\text{H.c.},$$
$$\sqrt{2}\,T_{19}=i\delta_{15}+\text{H.c.},$$
$$\sqrt{2}\,T_{20}=\delta_{16}+\text{H.c.},$$
$$\sqrt{2}\,T_{21}=i\delta_{16}+\text{H.c.},$$
$$\sqrt{2}\,T_{22}=\delta_{26}+\text{H.c.},$$
$$\sqrt{2}\,T_{23}=i\delta_{26}+\text{H.c.},$$
$$\sqrt{2}\,T_{24}=\delta_{34}+\text{H.c.},$$
$$\sqrt{2}\,T_{25}=i\delta_{34}+\text{H.c.},$$
$$\sqrt{2}\,T_{26}=\delta_{36}+\text{H.c.},$$
$$\sqrt{30}\,T_{27}=\delta_{11}+\delta_{22}+\delta_{33}-3(\delta_{44}+\delta_{55}-\delta_{66}).$$

Using these matrices it is easy to calculate $D^{\alpha\beta\theta}$, $E^{\alpha\theta\Gamma}$, and $F^{\theta\alpha\Gamma}$ from (2.2). The ones we need in Sec. III are

$$D^{8,3,3}=2(\tfrac{6}{5})^{1/2},$$
$$D^{3,3,27}=\frac{2}{\sqrt{30}},$$
$$D^{3,3,8}=4(\tfrac{2}{3})^{1/2},$$
$$E^{3,3,27}=\frac{8}{5\sqrt{3}},$$

where the third index refers to the 27-plet.

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