Predictions of three-body decays of mesons from pole-dominated dispersion relations*

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The meson decays $\eta\to\pi\pi\gamma$, $\eta'\to\pi\pi\gamma$, $\eta'\to\eta\pi\pi$, $\omega\to 3\pi$, $\phi\to 3\pi$, and $K^*\to K\pi\pi$ are studied, using dispersion relations for the related two-body "scattering" amplitudes. Pole dominance of the low-energy part of the dispersion integral with Regge-pole dominance of the high-energy part leads to relations between these three-body decays and various meson two-body decays. When these are known, the approach leads to predictions for the three-body decays in agreement with experiment. In other cases, consistency is shown with the results of vector-meson-dominance couplings, SU(3)-symmetric couplings with η-η' mixing, and upper limits for unobserved decays.

I. INTRODUCTION

For two-body scattering reactions, the principle of crossing relates the three channels (s,t,u) via analytic continuation of the scattering amplitudes. If, in addition, the masses are such that one particle can decay into the other three, the same amplitudes, continued into an appropriate kinematic region, will also describe the decay. We apply this to three-body decays of pseudoscalar and vector mesons. The analytic continuation is accomplished with a fixed-t dispersion relation for the scattering amplitude. The low-energy region is assumed to be dominated by the s- and u-channel resonances, and the high-energy part by the t-channel Regge-pole exchanges. Finite-energy sum rules (FESR) are used to relate the Regge residues to the resonance couplings, so that the three-body decay is predicted in terms of various two-body decays. Note that this method provides an unambiguous way to combine pole terms in various channels without double-counting. Comparisons with various pole models will be made for individual reactions. In Sec. II the formalism for the dispersion relation and continuation will be developed. Section III deals with the individual reactions. General conclusions and comparisons are made in Sec. IV. An appendix on three-body decay kinematics is added for completeness.

II. FORMALISM

We use the method of finite dispersion relations (FDR) as developed by Aviv and Nussinov.[1]

Consider a three-body decay with momenta and masses

$$M\to m+\mu+\mu\,,$$

$$p\to K+q_1+q_2\,,$$

where we always will have at least two equal-mass particles in the final state. The usual Mandelstam variables are defined as

$$s=(K+q_2)^2=(p-q_1)^2\,,$$

$$u=(K+q_1)^2=(p-q_2)^2\,,$$

$$t=(q_1+q_2)^2=(p-K)^2\,.$$

Following Ref. 1, we write a fixed-t dispersion relation for the appropriate invariant amplitude $A^{\pm}(\nu,t)$, where $\nu=\frac{1}{2}(s-u)$ and $\pm$ means amplitudes even or odd in ν. The contour is a circle of radius N in the ν plane, with indentations to avoid crossing the cuts on real ν axis for the s- and u-channel thresholds:

$$A^{\pm}(\nu,t)=\frac{1}{2\pi i}\oint_c\frac{A^{\pm}(\nu',t)}{\nu'-\nu}\,d\nu'\,. \tag{1}$$

It is convenient to separate the circular integral from the integral around the cuts, which is proportional to the discontinuity, or imaginary part of the amplitude:

$$A^{\pm}(\nu,t)\equiv A_1^{\pm}+A_2^{\pm}\,, \tag{2}$$

$$A_1^{\pm}=\frac{2}{\pi}\int_0^N\frac{\binom{\nu'}{\nu}\,\mathrm{Im}A^{\pm}(\nu',t)}{\nu'^2-\nu^2}\,d\nu'\,, \tag{3}$$

$$A_2^{\pm}=\frac{1}{2\pi i}\int_{c_N}\frac{\binom{\nu'}{\nu}A^{\pm}(\nu',t)}{\nu'^2-\nu^2}\,d\nu'\,. \tag{4}$$

One makes the usual assumption that a value of N can be found such that resonance-pole saturation in the s and u channels dominates the imaginary part of the amplitude for $|\nu'|<N$, and that a t-channel Regge-pole expansion dominates for $|\nu'|\geq N$.

The resonance contributions have the form

$$A_{\mathrm{res}}^{\pm}(\nu,t)=f(\nu,t)\left(\frac{1}{\nu-\nu_B}\pm\frac{1}{-\nu-\nu_B}\right)\,, \tag{5}$$

which yields

$$A_1^\pm(\nu,t)=2\begin{pmatrix}\nu_B\\ \nu\end{pmatrix}\frac{f(\nu_B,t)}{\nu^2-\nu_B^2}\,, \tag{6}$$

where $\nu_B=M_R^2+\frac{1}{2}t-\frac{1}{2}\Sigma$ is the position of the resonance; M_R is the resonance mass and Σ is the usual sum of the squares of the four external masses. $f(\nu,t)$ contains resonance couplings and spin factors appropriate to the external particles.

We parameterize the Regge contribution

$$A^\pm_{\text{Regge}}(\nu,t)=\frac{\pi\beta^\pm}{\sin\pi\alpha}\left[\nu^\alpha\pm(-\nu)^\alpha\right] \tag{7}$$

or

$$A^\pm_{\text{Regge}}(\nu,t)=\frac{\pi\beta^\pm}{\sin\pi\alpha}\left[\nu^{\alpha-1}\pm(-\nu)^{\alpha-1}\right], \tag{8}$$

where the two possible forms are for different external spins. Upon integration around the circular contour, they yield

$$A_2^\pm(\nu,t)=2\beta^\pm N^{\alpha-1}\begin{pmatrix}N\\ -\nu\end{pmatrix}\sum_{n=0}^{\infty}\left(\frac{\nu^2}{N^2}\right)^n\frac{1}{\alpha-\binom{0}{1}-2n} \tag{9}$$

or

$$A_2^\pm(\nu,t)=2\beta^\pm N^{\alpha-2}\begin{pmatrix}-N\\ \nu\end{pmatrix}\sum_{n=0}^{\infty}\left(\frac{\nu^2}{N^2}\right)^n\frac{1}{\alpha-\binom{1}{2}-2n}\,, \tag{10}$$

where we have expanded the denominator of the integral in a power series which converges for $|\nu|<N$. Since the amplitude is to be evaluated in the three-body decay region and N is typically of the order of a few GeV^2, this requirement will be satisfied for all cases to be considered.

The functions $f(\nu,t)$ and β can be related by requiring the amplitude to satisfy a finite-energy sum rule (FESR). One uses

$$\int_0^N\begin{pmatrix}\nu\\ 1\end{pmatrix}d\nu\,\text{Im}[A^\pm(\nu,t)-A^\pm_{\text{Regge}}(\nu,t)]=0\,, \tag{11}$$

which yields

$$\nu_B f^\pm(\nu_B,t)=\beta^\pm N^{\alpha+1}\begin{pmatrix}N\\ -\nu_B\end{pmatrix}\frac{1}{\alpha+\binom{2}{1}} \tag{12}$$

or

$$\nu_B f^\pm(\nu_B,t)=\beta^\pm N^{\alpha}\begin{pmatrix}-N\\ \nu_B\end{pmatrix}\frac{1}{\alpha+\binom{1}{0}}\,. \tag{13}$$

Remember that in general a sum over resonances and/or Regge poles is implied. The same N value as in the FDR is used here.

III. MESON RESONANCE DECAYS

A. $\eta\to\pi\pi\gamma$

This reaction has been considered using the FDR and FESR procedures by Lassila and Young[2] with positive results, with emphasis on determining the single- or double-pole nature of the s- and u-channel A_2-resonance pole. Here we shall review this reaction, concentrating on the FDR and FESR procedures.

The s- and u-channel quantum numbers allow only the A_2 pole of the well-established resonances. The t channel similarly allows only the ρ Regge pole. The invariant amplitude $A(\nu,t)$ (see Appendix for definition) is an even function of ν and has one unit of spin flip, so we use (6), (8), and (10) with the upper sign. The quantities of interest are

$$f(\nu,t)=\frac{4g_{A_2\eta\pi}g_{A_2\pi\gamma}}{M_R^6}\left[\nu+\tfrac{3}{2}t-\Sigma-\frac{\mu^2(M^2-\mu^2)}{M_R^2}\right], \tag{14}$$

where μ = pion mass, $M=\eta$ mass, and $M_R=A_2$ mass. The couplings g are related to the two-body decays of the A_2:

$$\Gamma(A_2\to\eta\pi)=\frac{16}{15}\frac{g_{A_2\eta\pi}^2}{4\pi}\frac{q_0^5}{M_R^4}\,, \tag{15}$$

$$\Gamma(A_2\to\gamma\pi)=\frac{8}{5}\frac{g_{A_2\pi\gamma}^2}{4\pi}\frac{q_0^5}{M_R^4}\,, \tag{16}$$

where q_0 is the magnitude of the momentum of one of the decay products in the A_2 rest frame. From the known $A_2\to\eta\pi$ rate,[3] one calculates $g_{A_2\eta\pi}^2/4\pi=1.06\pm0.20$. Since the $A_2\to\gamma\pi$ rate has not been accurately measured, one must resort to the vector-meson-dominance model (VDM) to relate it to the well-known $A_2\to\rho\pi$ decay. If we use the γ-ρ coupling e/f_ρ with $f_\rho^2/4\pi=2.56\pm0.22$,[4] the result is $g_{A_2\pi\gamma}^2/4\pi=0.0146\pm0.0033$. It only remains to determine the cutoff value N. According to usual procedure, and as suggested by semi-local duality, the cutoff point is taken halfway between the last resonance pole included in the s or u channel and the next Regge recurrence. The effect of varying the N value by an amount corresponding to the resonance width on the trajectory changes the predicted β values by about 15%. We take the A_2 trajectory

$$\alpha_A(t)=0.5+1.5t/M_R^2 \tag{17}$$

so that

$$N=\tfrac{5}{3}M_R^2-\tfrac{1}{2}M^2-\mu^2+\tfrac{1}{2}t\,. \tag{18}$$

An alternate parameterization

$$\alpha_A(t)=2.0-M_R^2+t \tag{19}$$

and the corresponding

$$N=1.0+M_R^2-\tfrac{1}{2}M^2-\mu^2+\tfrac{1}{2}t \tag{20}$$

turn out to have no significant effect on the results. We use similar parameterization for the t-channel

ρ Regge trajectory:

$$\alpha_\rho(t)=0.5+t/2m_\rho^2 \tag{21}$$

or

$$\alpha_\rho(t)=1.0-m_\rho^2+t\ , \tag{22}$$

which also give essentially the same results. The FESR (11) is evaluated for $t=0$ and β^+ from (13) is taken as constant over the decay region $4\mu^2\leq t\leq M^2$. The FESR could be used to determine $\beta(t)$ for all t in this case where only one trajectory dominates, but the variation in t obtained is insignificant when compared with the over-all uncertainty in the magnitude of β due to cutting off the FESR after only one resonance pole. We estimate the uncertainty in β to be of the order of 15–20% based on the standard investigations of FESR in πN amplitudes.[5] The final calculated value is $\Gamma(\eta\to\pi\pi\gamma)=0.141$ keV, with an estimated uncertainty due to all previously mentioned sources of about 25%. This compares well with the experimental average value of 0.131 $\pm$0.029 keV.[3]

We have also investigated the effect of including the width of the ρ by modifying the denominator of (10) to read $\alpha-1+i\Gamma_\rho/2m_\rho$. This changes the calculated value of $\Gamma(\eta\to\pi\pi\gamma)$ by about 3% as Γ_ρ is varied from 0 to 145 MeV. The reason is that the ρ pole is outside the decay region, but this effect will turn out to be very important in the η' decay.

The $\pi\pi$ spectrum is shown in Fig. 1, along with pure phase space and the experimental points. It is seen that the FDR amplitude with the ρ pole enhances the large-t (invariant $\pi\pi$ mass) part of the spectrum, in agreement with the data.[6]

One can further assume that the constant β assumption is valid out to $t=m_\rho^2$ and relate it to decay couplings in the t channel. The relation is

$$\beta(t)\xrightarrow[t\to m_\rho^2]{}\alpha' g_{\eta\rho\gamma}g_{\rho\pi\pi}\ , \tag{23}$$

where α' is the ρ trajectory slope, and

$$\Gamma(\rho\to\pi\pi)=\frac{2}{3}\,\frac{g_{\rho\pi\pi}^2}{4\pi}\,\frac{q_0^3}{m_\rho^2}\ , \tag{24}$$

$$\Gamma(\rho\to\eta\gamma)=\frac{1}{3}\,\frac{g_{\eta\rho\gamma}^2}{4\pi}\,q_0^3 \tag{25}$$

define the couplings. From the latest ρ width of 146 $\pm$ 10 MeV (see Ref. 3) we fix $g_{\rho\pi\pi}^2/4\pi=2.8$. From the FESR at $t=0$, $\beta=3.53$ GeV^{-3}, which predicts $g_{\eta\rho\gamma}^2/4\pi=0.042$ GeV^{-2} and $\Gamma(\rho\to\eta\gamma)=89$ keV. This small value is consistent with the absence of this mode in the experimental surveys.[3] One can, however, compare this with the fitted values using SU(3) and η-η' mixing, which give $g_{\eta\rho\gamma}^2/4\pi=0.038$ and 0.043 GeV^{-2} for linear and

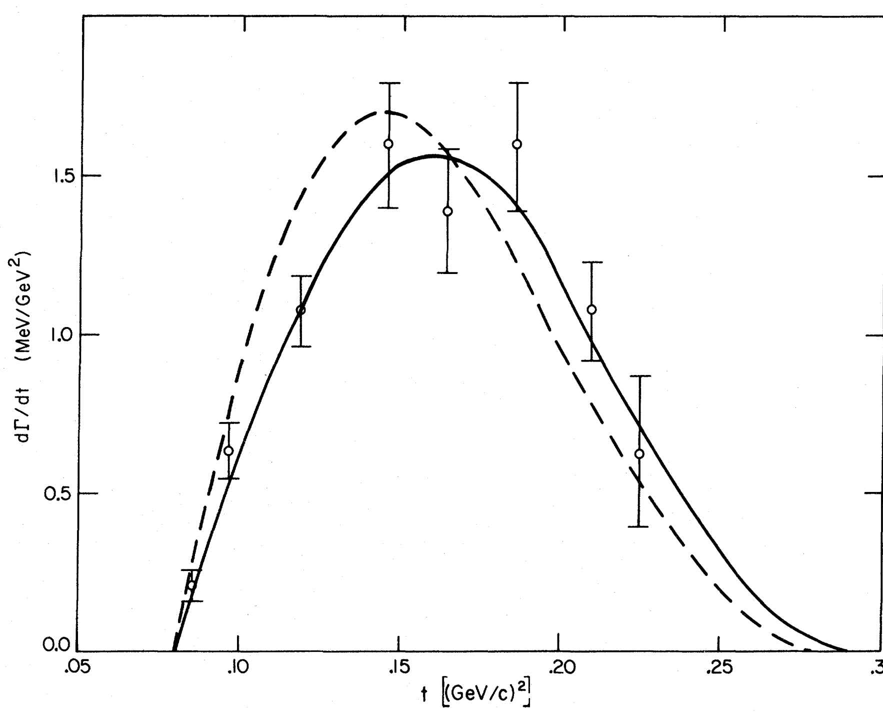

FIG. 1. $\pi\pi$-invariant mass spectrum for $\eta\to\pi\pi\gamma$ (arbitrary units). The dashed curve is phase space, the solid line the FDR prediction, and the data points are from Cnops *et al.*, Ref. 6.

quadratic mixing, respectively.[7] This gives one some added confidence in the procedure of obtaining β via FESR.

B. $\eta' \to \pi\pi\gamma$

Superficially, this reaction is the same as the previous one and one need merely replace masses and couplings appropriate to η' rather than η to get the results. However, there are two important differences. One is that all of the couplings are not known in this case. Only an upper bound is known for the total η' width (1.9 MeV) (see Ref. 3) and only branching ratios are known accurately. Also, the $A_2 \to \eta'\pi$ decay is only known as an upper limit, thus placing only an upper limit on $g_{A_2\eta'\pi}$. The second and more important difference is that due to the higher mass of the η', the t-channel ρ resonance pole is in the physical decay region, i.e., the decay $\eta' \to \rho^0\gamma$ followed by $\rho^0 \to \pi^+\pi^-$ contributes to the over-all three-body decay. In fact, this mode is certainly the major part of the decay, with a branching ratio estimated to be $(94 \pm 20)\%$.[3] Thus one should not necessarily expect that the ρ residue function determined at $t=0$ by the FESR should remain constant throughout the ρ resonance region. In fact, one can calculate the ratio of the residues at $t = m_\rho^2$ by

$$\beta(t) \xrightarrow[t \to m_\rho^2]{} \alpha' g_{\eta'\rho\gamma} g_{\rho\pi\pi} \tag{26}$$

and

$$\Gamma(\eta' \to \rho\gamma) = \frac{g_{\eta'\rho\gamma}^{\;2}}{4\pi} q_0^{\;3} \tag{27}$$

to that determined by the FESR at $t=0$ via (13). Remember that the $\beta(m_\rho^2)$ is the appropriate value for the $\eta' \to \rho\gamma$ decay if it stays constant over the ρ width. The values for this ratio are 2.66, 4.65, and 3.06 for upper limits,[3] quadratic, and linear mixing,[8] respectively. It is obvious from these numbers that β cannot remain constant over the entire t range. One cannot calculate the decay rate in this case, but merely show consistency with the FDR approach. Since the ratios just discussed are approximately a measure of the relative size of the ρ-pole to the A_2-pole contributions to the amplitude, it is clear that the A_2 poles will give a contribution of the order of 10%, which is certainly consistent with the $(94 \pm 20)\%$ branching ratio. The actual procedure was to fix the residue at $t=0$ by FESR and let it vary (either linearly or exponentially, both give equivalent results) to a value at $t = m_\rho^2$ which produces the correct decay width for the case under consideration (upper limit, quadratic, or linear). The t spectrum is

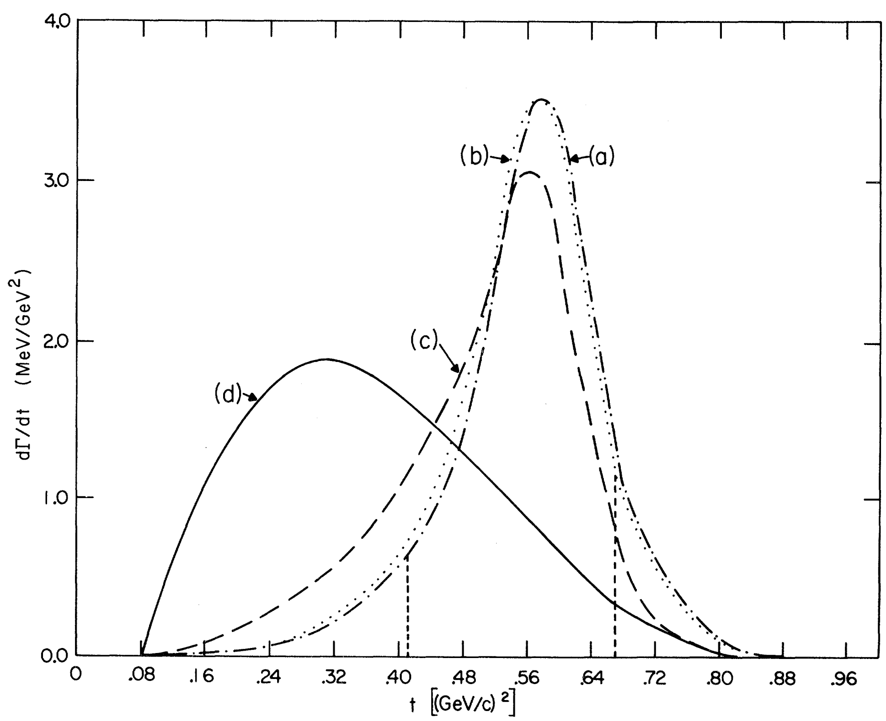

FIG. 2. $\pi\pi$-invariant mass spectrum for $\eta' \to \pi\pi\gamma$ (arbitrary units). Curves (a) and (b) are the FDR predictions for $\Gamma(\eta') = 100$ keV or 160 keV. Curve (c) is described in the text. Curve (d) is pure phase space.

then plotted and checked to ensure the $(94 \pm 20)\%$ population of the ρ-meson region. In Fig. 2 such spectra are shown, along with pure phase space (no ρ pole). Note that the two curves (a) and (b) (corresponding to decay widths of 100 and 160 keV but normalized to the same area) give essentially the same shape, so that no prediction of the decay width is possible. Both of them have approximately 85% of the area in the ρ band (shown with dotted lines). Note, however, in (c) which corresponds to a linear mixing total width combined with the upper limit for the $A_2 \to \eta'\pi$ coupling, the curve is shifted substantially toward low t values, so that only about 78% of the area is in the ρ band. In this sense there is a direct correlation between the total η' width and the size of the $A_2 \to \eta'\pi$ decay, both in the three-body-decay analysis here and in the SU(3)-mixing schemes from two-body decays alone.[7,8]

C. $\omega \to 3\pi$

This reaction is symmetric in all three channels, with the $\pi\pi \to \pi\omega$ having the quantum numbers of the ρ meson. The quantities of interest are

$$f(\nu, t) = 2g_{\omega\rho\pi}\, g_{\rho\pi\pi} \tag{28}$$

and a crossing-even spin-flip ρ Regge amplitude, according to the top sign of (8) and (10) with a residue evaluated at $t=0$ from FESR (13) as

$$\beta = -\frac{\alpha+1}{N^{\alpha+1}} \times 2\nu_B\, g_{\omega\rho\pi}\, g_{\rho\pi\pi}\,, \tag{29}$$

where the effective $\omega\rho\pi$ coupling is defined by

$$\epsilon^{\alpha}(\omega)\, \epsilon^{\beta}(\rho)\, \epsilon_{\alpha\beta\lambda\mu}(K_\omega)^{\lambda}(K_\rho)^{\mu} g_{\omega\rho\pi}\,.$$

The coupling constant $g_{\omega\rho\pi}$ cannot be directly determined from a decay, but must be inferred from the $\omega \to \pi\gamma$ decay via vector meson (ρ) dominance. This leads to

$$\Gamma(\omega \to \pi\gamma) = \frac{1}{3}\, \frac{q_0^{\,3}}{4\pi} \left(\frac{e g_{\omega\rho\pi}}{f_\rho}\right)^2 = 0.9 \text{ MeV}\,, \tag{30}$$

which yields $g_{\omega\rho\pi} = 14.8$ GeV^{-1} for f_ρ from leptonic decays or $g_{\omega\rho\pi} = 15.4$ GeV^{-1} for $f_\rho = g_{\rho\pi\pi}$ universality. Both of these are in reasonable agreement with the SU(3) value of 15.8 GeV^{-1} from two-body meson decays.[7] We again use a linear ρ trajectory and take N halfway between the ρ and its first recurrence, the $g(1680)$. When these numbers are inserted into the amplitudes and integrated over phase space, the resulting decay width is predicted to be $\Gamma(\omega \to 3\pi) = 5.84$ MeV, compared with the experimental 8.8 ± 0.4 MeV.[3]

This value is quite low, and we must look at the amplitudes more carefully. One obvious remark is that this amplitude is almost the same as the vector-dominance case, which predicts $\Gamma(\omega \to \pi\gamma)/\Gamma(\omega \to 3\pi) \approx 0.11$ for $f_\rho^{\,2}/4\pi = 2.6$, in agreement with experiment. However, that approach includes some obvious double-counting, since it adds the ρ-pole amplitudes in all three channels simultaneously. The FDR approach avoids this difficulty, but seems to give too small an amplitude. Since the $\omega \to 3\pi$ amplitude is completely crossing-symmetric, the FDR approach also makes an approximation in singling out one channel (t) to use as the Regge channel, while treating the resonance channels (s and u) symmetrically. This is obvious when one looks at (10), which exhibits the complete pole structure of every resonance on the ρ trajectory, whereas we have cut off the resonance contribution to the s and u channels with only the ρ. Theoretically, the N value chosen should take this into account and make the amplitudes at least numerically crossing-symmetric. In fact, one can compare the β value at $t=0$ (β is approximately independent of t from the FESR evaluation) to the value at $t = m_\rho^{\,2}$ directly since both are proportional to $g_{\omega\rho\pi}$. One gets

$$\frac{\beta(m_\rho^{\,2})}{\beta(0)} = \frac{3\sqrt{\pi}\;\, \nu_B(t=0)}{(N_0)^{3/2}\alpha'} = 0.92\,,$$

which indicates approximate numerical crossing symmetry. It is obvious, however, that a larger N value in the FESR better satisfies the approximation of using the Regge amplitude for the circular integral. To this end, we extend the cutoff point above the next resonance on the ρ trajectory, the spin-3 $g(1680)$ meson. The appropriate quantities are

$$f(\nu_B, t) = g_{g\pi\pi}\, g_{g\omega\pi} \left[4(2t + M_g^{\,2} - M_\omega^{\,2} - 3\mu^2)^2 + \tfrac{4}{5}(M_g^{\,2} - 4\mu^2)\left(2M_\omega^{\,2} + 2\mu^2 - M_g^{\,2} + \frac{(M_\omega^{\,2} - \mu^2)^2}{M_g^{\,2}}\right)\right], \tag{31}$$

$$\Gamma(g \to \omega\pi) = \frac{256}{105}\, \frac{g_{g\omega\pi}^{\;2}}{4\pi}\, q_0^{\,7}\,, \tag{32}$$

$$\Gamma(g \to \pi\pi) = \frac{64}{35}\, \frac{g_{g\pi\pi}^{\;2}}{4\pi}\, \frac{q_0^{\,7}}{M_g^{\,2}}\,. \tag{33}$$

From the latest values[3] $\Gamma_g(\text{total}) = 160 \pm 30$ MeV, $\Gamma(g \to 2\pi)/\Gamma_g \approx 40\%$, $\Gamma(g \to \omega\pi)/\Gamma_g < \Gamma(g \to 4\pi)/\Gamma_g \approx 50\%$ one gets $g_{g\pi\pi} = 2.2$ GeV^{-2} and $g_{g\omega\pi} \lesssim 3.0$ GeV^{-3}.

When these values are inserted into (6) it is seen that the g gives a negligible contribution (compared with the ρ) to the s- and u-channel amplitudes, merely because the poles are far away from the decay region. However, inclusion of the g meson does change the t-channel amplitudes via the $\beta(0)$ determined by the FESR and the increased N value. We evaluate β at $t=0$ and get $\beta(0)=-69$ GeV^{-3} (where we have taken $g_{g\omega\pi}$ at its maximum value and all couplings to have the same sign). We compare this with the values at the t-channel poles, $\beta(m_\rho{}^2)=-85$ GeV^{-3} and $\beta(M_g{}^2)=-51$ GeV^{-3}. The approximate agreement of these numbers indicates that the approximation of using $\beta=$constant in the decay region is not a bad one, and in addition that approximate crossing symmetry is maintained. When one integrates these new amplitudes over the decay region, the resulting $\Gamma(\omega\to 3\pi)$ values are 6.65 and 7.37 MeV for assumed branching ratios of $(g\to\omega\pi)/(g\to 4\pi)$ of 50% and 100%, respectively. These values are substantially higher than the results with no g meson (5.84 MeV) but still somewhat lower than the 8.8-MeV experimental value. However, the typical uncertainty in the resonance-pole saturated FESR again gives an estimated 25% uncertainty in the calculated decay widths, so that the values with the g meson included are consistent with the experimental value. As a check on crossing symmetry, the amplitudes were calculated twice more, with the s and u channels as the Regge-pole channel. The three amplitudes were then averaged to impose crossing symmetry and the decay width calculated again. The result was $\Gamma(\omega\to 3\pi)=7.36$ MeV, compared with the single-amplitude value of 7.37 MeV.

D. $\phi\to 3\pi$

This reaction bears the same resemblance to $\omega\to 3\pi$ as the $\eta'\to\pi\pi\gamma$ does to the $\eta\to\pi\pi\gamma$ decay. The main differences again are that the ρ meson is in the physical decay regions and accounts for a substantial (but unknown precisely) fraction of the total decay, and also the coupling constant $g_{\phi\rho\pi}$ is not experimentally determined. We need only alter the $\omega\to 3\pi$ case to include the ρ region by replacing $t-m_\rho{}^2$ with $t-m_\rho{}^2+im_\rho\Gamma$ in the s and u poles, and $\alpha-1$ with $\alpha-1+i\Gamma/2m_\rho$ in the t-channel contribution to the amplitude. Since the amplitude is completely crossing-symmetric, only one product of coupling constants $g_{\phi\rho\pi}\,g_{\rho\pi\pi}$ enters, and the FESR predicts β at $t=0$ to be

$$\beta(0)=\frac{-3\sqrt{\pi}\,g_{\phi\rho\pi}\,g_{\rho\pi\pi}\,\nu_B}{N^{3/2}}\left(1-\frac{2}{\pi}\,\frac{m_\rho\Gamma_\rho}{N}\right), \tag{34}$$

where the last factor is the finite-width correction (about 6%) and ν_B and N are to be evaluated at $t=0$ in GeV units. Alternatively, β at $t=m_\rho{}^2$ is directly related to the couplings

$$\beta(m_\rho{}^2)=-\frac{g_{\phi\rho\pi}\,g_{\rho\pi\pi}}{2m_\rho{}^2}, \tag{35}$$

so that we can see that $\beta(0)/\beta(m_\rho{}^2)=0.16$ and the residue function must change with t quite dramatically over the decay phase-space region. Note that this variation with t is not predicted by FESR (β here is again approximately independent of t as for $\omega\to 3\pi$) but required by crossing symmetry imposed at $t=m_\rho{}^2$. This variation was put in both linearly and exponentially, and the difference between the two forms was negligible. The result for the decay width is

$$\Gamma(\phi\to 3\pi)=0.714\,(g_{\phi\rho\pi})^2\ \text{MeV}, \tag{36}$$

where $g_{\phi\rho\pi}$ is in units of GeV^{-1}. Alternatively, one can express the two-body decay $\phi\to\rho\pi$ in terms of the same coupling:

$$\Gamma(\phi\to\rho\pi)=\frac{1}{3}\,q_0{}^3\,\frac{g_{\phi\rho\pi}{}^2}{4\pi}\times 3\ \text{(for all charge modes)}$$
$$=0.504\,(g_{\phi\rho\pi})^2\ \text{MeV}. \tag{37}$$

Thus the ratio is independent of the unknown $g_{\phi\rho\pi}$ and tests the FDR approach, $\Gamma(\phi\to\rho\pi)/\Gamma(\phi\to 3\pi)=0.706$. Note that it can be considered a success that this number is less than unity, since there is no guarantee that an interference of the t-channel ρ pole with the s- and u-channel terms could not predict a decrease of the 3π decay rate below that for $\phi\to\rho\pi$ alone. To get the absolute decay rate, one must again appeal to the VDM for the coupling $g_{\phi\rho\pi}$. This is related via the VDM to $\phi\to\pi\gamma$, but only an upper limit (0.35%) is known for this branching fraction.[3] Hence we adjust $g_{\phi\rho\pi}$ to fit the $\phi\to 3\pi$ rate and compare with the upper-limit values. Experimentally, $\Gamma(\phi\to 3\pi)=0.64\pm 0.16$ MeV, which yields $g_{\phi\rho\pi}=0.947$ GeV^{-1}. From the VDM we get

$$\Gamma(\phi\to\pi\gamma)=\frac{1}{3}\,q_0{}^3\,\frac{e^2}{f_\rho{}^2}\,\frac{g_{\phi\rho\pi}{}^2}{4\pi}, \tag{38}$$

which gives $\Gamma(\phi\to\pi\gamma)=8.3$ keV. We compare this with the upper-limit value, $\Gamma(\phi\to\pi\gamma)<(.35\%)(4.2\ \text{MeV})=14.7$ keV to show consistency of this number with the FDR approach to the $\phi\to 3\pi$ decay.

E. $K^*\to K\pi\pi$

This is the SU(3) partner of ω, $\phi\to 3\pi$ and the same amplitude structure applies. Experimentally this decay is not observed, and an upper limit of 0.2% of the total width is estimated, i.e., $\Gamma(K^*\to K\pi\pi)<0.1$ MeV. Part of this suppression is due to phase space. From (A15) one can calculate the ratio of phase space for this decay to that for $\omega\to 3\pi$. This result is approximately $\frac{1}{50}$, which would predict

$$\Gamma(K^*\to K\pi\pi)=\tfrac{1}{50}(10\ \text{MeV})=0.2\ \text{MeV}$$

if the amplitudes were equal. Hence a suppression of at least $\sqrt{2}$ is needed in this amplitude relative to that for ω decay, and must be provided by the pole coupling factors in the FDR approach.

We consider the K^{*+} decay in three possible charge modes. Mode I, $K^+\pi^+\pi^-$, has K^* and K_N poles in the s channel, no poles in the u channel, and vacuum[9] and ρ Regge trajectories in the t channel. Mode II, $K^+\pi^0\pi^0$, has K^* and K_N poles in both s and u channels with only the vacuum[9] Regge trajectories in the t channel. Mode III, $K^0\pi^+\pi^0$, has only the ρ Regge trajectory in the t channel, with the K^* and K_N poles in the s and u channels. From isospin invariance, one can write

$$A_{\rm I} = \frac{1}{\sqrt{2}} A_{\rm III} + A_{\rm II} \tag{39}$$

so that only two of the three charge modes are independent.

The s- and u-channel pole factors are

for K^*,

$$f(\nu_B, t) = 2 g_{K^*K^*\pi}\, g_{K^*K\pi}, \tag{40}$$

for K_N,

$$f(\nu_B, t) = 4 g_{K_N K^*\pi} g_{K_N K\pi} \times (2t - m_K^2 - 2\mu^2 + M_N^2 - M^{*2}) \times \left(1 - \frac{\mu^2 - m_K^2}{M_N^2}\right), \tag{41}$$

where M_N and M^* are masses of K_N and K^*, and the couplings are related to two-body decays by

$$\Gamma(K_N \to K\pi) = \frac{16}{15} \frac{g_{K_N K\pi}^2}{4\pi} \frac{q_0^5}{M_N^2} = 55 \pm 3 \text{ MeV}, \tag{42}$$

$$\Gamma(K_N \to K^*\pi) = \frac{8}{5} \frac{g_{K_N K^*\pi}^2}{4\pi} q_0^5 = 30 \pm 3 \text{ MeV}, \tag{43}$$

$$\Gamma(K^* \to K\pi) = \frac{2}{3} \frac{g_{K^*K\pi}^2}{4\pi} \frac{q_0^3}{M^{*2}} = 50 \pm 1 \text{ MeV}. \tag{44}$$

For $g_{K^*K^*\pi}$, we use the SU(3) value from $g_{\omega\rho\pi}$ in the previous work, yielding $g_{K^*K^*\pi}^2/4\pi = 5.5 \text{ GeV}^{-2}$.

The FESR relation for the ρ trajectory alone leads to

$$\beta_\rho(0) = -\pi[f_V(\nu_V, t)\,\nu_V + f_T(\nu_T, t)\,\nu_T] \frac{\alpha_\rho + 1}{N^{\alpha_\rho+1}}, \tag{45}$$

where V, T refer to vector or tensor pole factors and resonance positions. One can compare this with the factors at the t-channel pole point ($t = m_\rho^2$) directly when $\alpha_\rho \to 1$, which implies

$$\beta_\rho(m_\rho^2) = -\pi\alpha' f_V(\nu_V, t). \tag{46}$$

Using the usual cutoff value at halfway between the K_N and the next Regge recurrence ($N \approx 2$ GeV^2), we get $\beta(m_\rho^2)/\beta(0) \approx 2.0$. Hence the residue function probably does not vary much over the physical decay region $4\mu^2 \leqslant t \leqslant (M^* - m_K)^2$, we assume it is constant $\beta = \beta(0)$ in this region. This leads to a decay width of

$$\Gamma_{\rm III} = \begin{cases} 0.023 \text{ MeV} \\ 0.010 \text{ MeV} \end{cases}$$

for the $K^0\pi^+\pi^0$ mode, where the two values are for the products of K^* or K_N couplings with the same or opposite signs, respectively.

For the charge mode II ($K^+\pi^0\pi^0$), one must use the FESR for the vacuum trajectories. In the K^* production cross section $KN \to K^*N$, there is no evidence for an energy-independent component. This indicates that the Pomeron-K-K^* coupling is very small, and we can neglect the Pomeron contribution to the t channel in this decay. The lower-lying vacuum poles give an FESR result

$$\beta(0) = \pi[f_V(\nu_V, t) + f_T(\nu_T, t)] \frac{\alpha_0}{N^{\alpha_0}}, \tag{47}$$

where the ν factor is missing [compare with (45)] because this amplitude is crossing-odd. This pole, along with the corresponding crossing-odd combination of s- and u-channel poles gives a decay rate

$$\Gamma_{\rm II} = \begin{cases} 81 \text{ eV} \\ 35 \text{ eV} \end{cases}$$

for the $K^+\pi^0\pi^0$ mode. Since this is so small (due to the crossing-odd amplitude) we need not investigate its sensitivity to the parameters of the vacuum trajectories. The SU(2) relation (39) then implies $\Gamma_{\rm I} \approx \frac{1}{2}\Gamma_{\rm III}$, so that we predict

$$\Gamma(K^* \to K\pi\pi) = \begin{cases} 0.034 \text{ MeV} \\ 0.015 \text{ MeV}. \end{cases} \tag{48}$$

Both possible values are well below the experimental upper limit, so that a confirmation or contradiction with the FDR approach must await a lowering of the upper bound by at least a factor of 3.

F. $\eta' \to \eta\pi\pi$

This strong decay shares the same s- and u-channel structure with the $\eta, \eta' \to \pi\pi\gamma$ decays, i.e., only the A_2 poles contribute. The pole factor is

$$f(\nu_B, t) = \frac{g_{A_2\eta\pi}\, g_{A_2\eta'\pi}}{M_R^2}\left[\left(M_R^2 - \Sigma + 2t + \frac{(M^2-\mu^2)(m^2-\mu^2)}{M_R^2}\right)^2 - \frac{1}{3}\left(M_R^2 - 2M^2 - 2\mu^2 + \frac{(M^2-\mu^2)^2}{M_R^2}\right)\left(M_R^2 - 2m^2 - 2\mu^2 + \frac{(m^2-\mu^2)^2}{M_R^2}\right)\right], \tag{49}$$

where M_R, M, m, and μ are the A_2, η', η, and π masses, respectively, and the couplings are defined by (15). Since only an upper limit is known for the decay $A_2 \to \eta'\pi$, we consider three cases: (1) The upper limit value[3] of the A_2 branching ratio of 1.1% which implies $g_{A_2\eta'\pi} = 4.7$ [must be consistent with $\Gamma(\eta') < 1.9$ MeV]. (2) The SU(3)-fitted value[7,8] of the branching ratio assuming quadratic η, η' mixing which leads to $g_{A_2\eta'\pi} = 3.2$ and $\Gamma(\eta') = 2.7 \pm 1.1$ MeV. (3) The SU(3)-fitted value[7,8] for linear mixing which leads to $g_{A_2\eta'\pi} = 1.6$ and $\Gamma(\eta') = 0.43 \pm 0.19$ MeV.

The t channel has vacuum quantum numbers. The residues of the vacuum[9] trajectories (except for the Pomeron according to the Harari-Freund conjecture) are constrained by FESR. For this crossing-even amplitude, the t-channel contributions are quite significant and their magnitude depends sensitively on the trajectory and residue parameterization. We consider four different possibilities:

(a) One trajectory only with $\alpha = 0.5 + t$, which leads to

$$\beta(t) = \frac{\alpha+2}{N^{\alpha+2}}\, \nu_B f(\nu_B, t).$$

(b) One trajectory $\alpha_1 = 0.5 + t$ and another $\alpha_2 = -0.5 + t$, which leads to $\beta_1 = 3.12\, g_1 g_2$ and $\beta_2 = -3.82\, g_1 g_2$, where $g_1 g_2 \equiv g_{A_2\eta\pi} g_{A_2\eta'\pi}$.

(c) One trajectory $\alpha_1 = 0.5 + t$, another $\alpha_2 = -0.5 + t$ with a factor of α_2 in β_2, which leads to $\beta_1 = 0.677\, g_1 g_2$, $\beta_2 = 1.54\, g_1 g_2$.

(d) One trajectory $\alpha_1 = 0.5 + t$, another $\alpha_2 = 0.0 + t$ with a factor of α_2 in β_2, which leads to $\beta_1 = 0.062\, g_1 g_2$, $\beta_2 = 1.72\, g_1 g_2$.

The α_1 trajectory is meant to include the usual P' (or f, f') trajectory, while the lower-lying α_2 includes a possible scalar meson (σ) trajectory. For a single trajectory [case (a)] the t dependence of β from the FESR is substantial over the allowed decay range $4\mu^2 \le t \le (M-m)^2$ and is kept in the calculations. For the two trajectory cases [(b), (c), and (d)], the β values are determined by a simultaneous solution of the FESR equation at both end values of t, thus ensuring approximate compliance of the amplitude with the t dependence implied by the FESR. Unfortunately, none of these cases give a consistent calculated decay width. All of the case (a) values are too small, while all of the remaining cases give widths much too large. The pattern that emerges is that a t-channel trajectory with $\alpha = 0.5$ predicts amplitudes much too small, while the inclusion of a lower-lying trajectory ($\alpha_2 < 0$) reverses the situation but predicts amplitudes sometimes orders of magnitude too large. However, one must still examine the effect of including the Pomeron.[11]

The Pomeron contribution cannot be assumed negligible here, since there are no high-energy data for $\eta\pi \to \eta'\pi$. To estimate the contribution, we use a U(3)-coupling model, with the octet part of the Pomeron adjusted to account for the difference between πN and KN total cross sections, and the singlet part determined via factorization using these plus the NN total cross section. The relevant expressions are (as $s \to \infty$)

$$\sigma_{\pi\pi} = \pi\left(\beta_0 + \frac{1}{\sqrt{3}}\beta_8\right)^2, \tag{50a}$$

$$\sigma_{KK} = \pi\left(\beta_0 - \frac{1}{2\sqrt{3}}\beta_8\right)^2, \tag{50b}$$

$$\sigma_{\eta_8\eta_8} = \pi\left(\beta_0 - \frac{1}{\sqrt{3}}\beta_8\right)^2, \tag{50c}$$

$$\sigma_{\eta_0\eta_0} = \pi\beta_0^2, \tag{50d}$$

$$\sigma_{\eta_0\eta_8} = \pi \times \tfrac{2}{3}\beta_8^2, \tag{50e}$$

$$\sigma_{\pi N} = \pi\left(\beta_0 + \frac{1}{\sqrt{3}}\beta_0\right)\beta_{NN} = 22.8 \text{ mb}, \tag{50f}$$

$$\sigma_{KN} = \pi\left(\beta_0 - \frac{1}{2\sqrt{3}}\beta_8\right)\beta_{NN} = 18.9 \text{ mb}, \tag{50g}$$

$$\sigma_{NN} = \pi(\beta_{NN})^2 = 39 \text{ mb}. \tag{50h}$$

It is well known that these values are not in fact asymptotic, but the cross sections rise at larger energies. It is still possible, however, that the Pomeron contribution (pole plus cuts) approximately factors. We take the cross-section values in an energy range where all are well known and presumably at their minimum values. We conjecture that this procedure will indicate approximately (factor of 2) the magnitudes of the Pomeron-η-η' and Pomeron-π-π couplings. The result is

$$\begin{aligned}\beta_{\text{Pom}} &= (\tfrac{2}{3})^{1/2}\beta_8\left(\beta_0 + \frac{1}{\sqrt{3}}\beta_8\right) \times \left(\cos^2\theta - \sin^2\theta - \frac{\cos\theta\sin\theta}{\sqrt{2}}\right)\\ &= 1.61 \text{ GeV}^{-2} \text{ for linear mixing}\\ &= 1.81 \text{ GeV}^{-2} \text{ for quadratic mixing}.\end{aligned} \tag{51}$$

The Pomeron contribution gives an amplitude $A_p = 2\beta N^{\alpha}/\alpha = 2\beta N$ for $\alpha = 1$. We take $N = 2.1$ GeV^2 (halfway between the A_2 and the next recurrence) and find the Pomeron contribution alone to the total η' width to be 0.195 MeV for linear or 0.247 MeV for quadratic mixing. We next combine the Pomeron amplitude with the A_2 pole and other vacuum trajectory amplitudes. There is a sign ambiguity since the relative sign of the Pomeron to A_2-pole amplitude is not determined by an FESR, so that we have calculated $2\times3\times4 = 24$ possible cases. Out of these, there are three which may be consistent solutions. Two are for the upper limit values and case (a) for the trajectories. Depending on the relative Pomeron sign, one gets 0.34 or 0.09 MeV for the total η' width, both of which are consistent with the upper bound $\Gamma(\eta') < 1.9$ MeV. However, the small $\Gamma(\eta')$ combined with a large $A_2\eta'\pi$ coupling may be inconsistent with ρ dominance of the $\eta' \to \pi\pi\gamma$ decay, as mentioned in part B. The third possible solution is for case (a) vacuum trajectory, linear mixing, and constructive Pomeron interference. It predicts $\Gamma(\eta') = 0.25$ MeV, which is consistent with the linear mixing value from SU(3)-fitted two-body decays. There are no consistent solutions for quadratic mixing. All remaining 21 cases give inconsistent results, either above the bound for $\Gamma(\eta')$, or unequal to the SU(3)-fitted values for the linear or quadratic mixing cases. It is interesting to note that although the A_2-pole contributions are negligible in the amplitude, the essential scale of the decay is still determined by the $A_2\eta'\pi$ coupling through the FESR and its interference with the Pomeron amplitude.

One might think that the unknown coupling $A_2\eta'\pi$ might be eliminated by considering the ratio $\Gamma(\eta' \to \eta\pi\pi)/\Gamma(\eta' \to \pi\pi\gamma)$. However, one must remember that, unlike this case, the scale of the $\Gamma(\eta' \to \pi\pi\gamma)$ is not determined by the $A_2\eta'\pi$ coupling. This only determines β near $t = 0$ and is in disagreement with the β at $t = m_\rho^2$ inferred from the existence of a two-step decay through a real ρ meson [see discussion after Eq. (27)].

IV. CONCLUSIONS

The main result of this study is that the FDR approach with pole dominance is consistent with a wide range of three-body meson decays. The specific results vary with each reaction. For $\eta \to \pi\pi\gamma$, an absolute total width prediction is made which is in agreement with experiment.[10] For $\eta' \to \pi\pi\gamma$ and $\phi \to 3\pi$ absolute branching fractions are predicted for the ρ-dominated part. For $\omega \to 3\pi$ an absolute decay rate consistency requires a substantial branching ratio for the $\omega\pi$ decay of the g meson. For $K^* \to K\pi\pi$ an absolute decay rate is predicted which is lower than the present upper bound, but may be accessible in the future. For $\eta' \to \eta\pi\pi$ several solutions are possible, but one with linear η-η' mixing is preferred. For each possibility, however, a definite form for the t-channel Regge-pole terms is selected. The form of the FDR amplitude may be considered as vector-meson pole dominance plus corrections ($\eta, \eta' \to \pi\pi\gamma$), or a method of consistently adding poles in all three channels of a crossing-symmetric amplitude without double-counting (ω, $\phi \to 3\pi$), or as a constraint on possible mixing schemes ($\eta' \to \eta\pi\pi$). It is interesting to note the consistency of the SU(3)-fitted values for $A_2 \to \eta'\pi$ and $\Gamma(\eta')$ from two-body decays with the three-body decays $\eta' \to \pi\pi\gamma$ and $\eta' \to \eta\pi\pi$ via the FDR amplitude constrained by FESRS.

We take these successes as an indication of the basic reliability of the FDR+ FESR approach, and propose to extend it to other reactions. Other less well known meson decays as well as the baryon decays $N^* \to N\pi\pi$ will be considered in a subsequent paper.

APPENDIX: THREE-BODY DECAYS

We consider the general decay,

mass: $M \to m_1 m_2 m_3$,

momentum: $p \to q_1 q_2 q_3$

and specialize to $m_1 = m_2 = \mu$, $m_3 = m$. The decay rate is given by

$$\Gamma = \frac{1}{2M}\int \frac{d^3q_1}{2E_1}\frac{d^3q_2}{2E_2}\frac{d^3q_3}{2E_3}\frac{(2\pi)^4}{(2\pi)^9} \times \delta^{(4)}(p - q_1 - q_2 - q_3)\sum_{\text{spins}}|M|^2$$

$$= \frac{1}{2M}\frac{\pi^2}{(2\pi)^5}\int dE_1\, dE_2\, F(E_1, E_2), \tag{A1}$$

where $F(E_1, E_2) \equiv \sum_{\text{spins}}|M|^2$. We define the usual Mandelstam variables and evaluate in the rest frame of M, where $p = (M, 0, 0, 0)$:

$$s \equiv (p - q_3)^2 = (q_1 + q_2)^2 = M^2 + m^2 - 2ME_3, \tag{A2}$$

$$t = (p - q_1)^2 = (q_2 + q_3) = M^2 + \mu^2 - 2ME_1, \tag{A3}$$

$$u = (p - q_2)^2 = (q_2 + q_3)^2 = M^2 + \mu^2 - 2ME_2, \tag{A4}$$

limits

$$4\mu^2 \le s \le (M - m)^2$$

$$(m + \mu)^2 \le t, u \le (M - \mu)^2.$$

In terms of the s, t variables,

$$\Gamma = \frac{1}{2M}\frac{\pi^2}{(2\pi)^5}\frac{1}{4M^2}\int\int ds\, dt\; F(s, t). \tag{A5}$$

We first consider all 0^- particles ($\eta\to 3\pi, \eta'\to\eta\pi\pi$). Since there are no spins involved, assume $F(s,t)$ = constant.

The phase-space boundary is given by zeros of the Kibble determinant

$$\Delta \equiv \begin{vmatrix} q_1^2 & q_1\cdot q_2 & q_1\cdot q_3 \\ q_1\cdot q_2 & q_2^2 & q_2\cdot q_3 \\ q_1\cdot q_3 & q_2\cdot q_3 & q_3^2 \end{vmatrix}, \tag{A6}$$

$$\Delta = \tfrac{1}{4}[st(M^2+m^2+2\mu^2) - s^2t - st^2 + s(M^2-\mu^2)(\mu^2-m^2) - \mu^2(M^2-m^2)^2],$$

$\Delta = 0$ when $t = t_\pm(s)$:

$$t_\pm = \frac{M^2+m^2+2\mu^2-s}{2} \pm \frac{1}{2}\left[\frac{(s-4\mu^2)[s-(M+m)^2][s-(M-m)^2]}{s}\right]^{1/2}. \tag{A7}$$

We need

$$\int\int ds\,dt = \int_{4\mu^2}^{(M-m)^2} ds(t_+ - t_-).$$

To evaluate this, we let

$$y \equiv \frac{s-4\mu^2}{(M-m)^2-4\mu^2} \equiv \frac{s-4\mu^2}{a},$$

with

$$a = (M-m-2\mu)(M-m+2\mu) \equiv Q(M-m+2\mu).$$

Then

$$\int\int ds\,dt = a^2\left[\frac{(M+m)^2-4\mu^2}{4\mu^2}\right]^{1/2}\int_0^1 dy\left[\frac{y(1-y)\{1-ay/[(M+m)^2-4\mu^2]\}}{1+(a/4\mu^2)y}\right]^{1/2}$$

$$= Q^2(M-m+2\mu)^2\left[\frac{(M+m)^2}{4\mu^2}-1\right]^{1/2} B(\tfrac{3}{2},\tfrac{3}{2})\,F_1(\tfrac{3}{2},-\tfrac{1}{2},\tfrac{1}{2};3;x,y), \tag{A8}$$

where

$$x \equiv \frac{Q(M-m+2\mu)}{(M+m)^2-4\mu^2},\quad y\equiv -\frac{Q(M-m+2\mu)}{4\mu^2},$$

$$B(x,y) \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

and F_1 is a generalized hypergeometric function,[12] defined by a series expansion

$$F_1(\alpha,\beta,\beta';\gamma;x,y) \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!\,n!}x^m y^n, \tag{A9}$$

which converges for $|x|<1$, $|y|<1$.

For most mass configurations, $|y|>1$ and $|x|<\tfrac{1}{2}$, so we must use the joining relation

$$F_1(\alpha,\beta,\beta';\gamma;x,y) = (1-x)^{-\beta}(1-y)^{-\beta'}F_1\left(\gamma-\alpha,\beta,\beta';\gamma;\frac{x}{x-1},\frac{y}{y-1}\right). \tag{A10}$$

This gives

$$\Gamma = \frac{|A|^2Q^2}{(32)^2\pi^2M}\frac{(M-m+2\mu)^2}{M^2}\frac{(Mm)^{1/2}}{M-m} \times F_1(\tfrac{3}{2},-\tfrac{1}{2},\tfrac{1}{2};3;u,v), \tag{A11}$$

where

$$u = -\frac{Q(M-m+2\mu)}{4Mm},\quad v = \frac{Q(M-m+2\mu)}{(M-m)^2}.$$

We evaluate this for $\eta\to 3\pi$ in the nonrelativistic (N.R.) limit, $m=\mu$, $M=3\mu$, $u=v=0$, $F_1=1$, to get the usual expression

$$\Gamma^{\text{N.R.}}_{\eta\to 3\pi} = \frac{|A|^2Q^2}{(32)^2\pi^2M}\frac{(4\mu)^2}{(3\mu)^2}\frac{(3\mu^2)^{1/2}}{2\mu} = \frac{Q^2|A|^2}{\sqrt{3}\,\pi^2M\times 384}. \tag{A12}$$

For $\eta'\to\eta\pi\pi$, we evaluate F_1 by the power series

$$F_1 = 1 - \tfrac{1}{4}u + \tfrac{1}{4}v - \tfrac{5}{64}uv - \tfrac{5}{128}u^2 + \tfrac{15}{128}v^2 + \cdots.$$

This leads to

$$\Gamma_{\eta'\to\eta\pi^+\pi^-} = 1.97\times 10^{-3}|A|^2\ \text{MeV}$$

or

$$\Gamma_{\eta'\to\eta\pi\pi} \approx 3|A|^2\ \text{keV}$$

for all charge modes.

For one vector and three pseudoscalar particles ($\omega\to 3\pi$, η, $\eta'\to\pi\pi\gamma$), the most general form for the amplitude is

$$\sum_{\text{spins}}|M|^2 = \sum_\lambda|\epsilon_\alpha^{(\lambda)}(v)\,\epsilon^{\alpha\beta\gamma\delta}(q_1)_\beta(q_2)_\gamma(q_3)_\delta A|^2, \tag{A13}$$

which leads to

$$\sum_{\text{spins}}|M|^2 = |A|^2\Delta,$$

where Δ is the Kibble determinant previously calculated. We need

$$\int\int ds\,dt\;\Delta(s,t)=\int_{4\mu^2}^{(M-m)^2} ds\int_{t_-}^{t_+} dt\,\frac{-s}{4}\,(t-t_+)(t-t_-)$$

$$=\frac{1}{24}\int_{4\mu^2}^{(M-m)^2} s\,ds\left[\frac{(s-4\mu^2)(s-(M+m)^2)(s-(M-m)^2)}{s}\right]^{3/2}. \qquad \text{(A14)}$$

We again use $y=(s-4\mu^2)/a$, and express the integral as

$$I=\frac{Q^4(M-m+2\mu)^4}{48\mu}\,[(M+m)^2-4\mu^2]^{3/2}\,B(\tfrac{5}{2},\tfrac{5}{2})\,F_1(\tfrac{5}{2},-\tfrac{3}{2},\tfrac{1}{2};5;x,y). \qquad \text{(A15)}$$

For most equal-mass configurations, we use the joining relation

$$F_1(\tfrac{5}{2},-\tfrac{3}{2},\tfrac{1}{2};5;x,y)=\left[\frac{4\mu M}{(M-\mu)(M+3\mu)}\right]^{3/2}\left[\frac{(M-\mu)^2}{4\mu^2}\right]^{-1/2}F_1\left(\tfrac{5}{2},-\tfrac{3}{2},\tfrac{1}{2};5;-\frac{Q(M+\mu)}{4\mu M},\frac{Q(M+\mu)}{(M-\mu)^2}\right). \qquad \text{(A16)}$$

For vector-meson decay, the result is

$$\Gamma_{V\to 3\pi}=\frac{2}{3}\,\frac{Q^4(M+\mu)^4\mu\sqrt{\mu M}\,|A|^2}{\pi^2(16)^4M^2(M-\mu)}\,F_1. \qquad \text{(A17)}$$

For $\omega\to 3\pi$, we find numerically $F_1=2.08$, so that if if we express

$$\Gamma_{\omega\to 3\pi}\equiv\frac{|A|^2M^7}{(2\pi)^3}\,Y_\omega \qquad \text{(A18)}$$

we find the phase-space factor

$$Y_\omega=4.57\times 10^{-6}.$$

For the decay $\phi\to 3\pi$, the masses are such that we must use the joining relations

$$F_1(\tfrac{5}{2},-\tfrac{3}{2},\tfrac{1}{2};5;x,y)=(1-x)^4(1-y)^{-1/2}\times F_1\left(\tfrac{5}{2},6,\tfrac{1}{2};5;x,\frac{x-y}{1-y}\right), \qquad \text{(A19)}$$

with $x=0.552$ and $(x-y)/(1-y)=0.955$. We evaluate F_1 by a power series and find a value 19.3. This yields $Y_\phi=0.78\times 10^{-5}$, where Y is defined in the same way as in (A18). It is instructive to compare this value with the extreme relativistic case, where $M\gg\mu$. The phase-space integral is then

$$I=-\int_0^{M^2}\frac{t\,dt}{24}\,(t-M^2)^3=\frac{M^{10}}{480},$$

so that

$$\Gamma_{\text{rel}}=\frac{1}{3}\,\frac{1}{64\pi^3M}\,\frac{1}{4M^2}\,\frac{M^{10}}{480}\,|A|^2,$$

or

$$Y_{\text{rel}}=2.17\times 10^{-5},$$

almost a factor of 3 greater than for ϕ. For the decays $\eta,\eta'\to\pi\pi\gamma$, the zero-mass photon makes phase-space integrals simplify somewhat. We have

$$\int\int ds\,dt\;\Delta(s,t)=\frac{1}{24}\int_{4\mu^2}^{M^2} s\,ds\left[\frac{(s-4\mu^2)(s-M^2)^2}{s}\right]^{3/2}$$

$$=\frac{(M^2-4\mu^2)^{11/2}}{24M}\,B(\tfrac{5}{2},4)\,{}_2F_1\left(4,\tfrac{1}{2};\tfrac{13}{2};1-\frac{4\mu^2}{M^2}\right), \qquad \text{(A20)}$$

where ${}_2F_1$ is the usual hypergeometric function of one variable. This leads to

$$\Gamma_{P\to\pi\pi\gamma}=\frac{|A|^2M^7(1-4\mu^2/M^2)^{11/2}}{(2\pi)^3\cdot 24\cdot 33\cdot 35}\times{}_2F_1\left(4,\tfrac{1}{2};\tfrac{13}{2};1-\frac{4\mu^2}{M^2}\right). \qquad \text{(A21)}$$

A numerical evaluation of ${}_2F_1$ yields 1.35 for the η and 1.51 for the η', leading to the usual dimensionless phase-space factor of

$$Y_\eta=0.98\times 10^{-5}$$

and

$$Y_{\eta'}=3.4\times 10^{-5}.$$

It is often useful to have a pole-dominated amplitude integrated over phase space rather than a constant matrix element. For $\omega\to 3\pi$, the ρ pole is outside the physical region, so we multiply the original phase-space density by a factor $[(4\mu^2-m_\rho^2)/(s-m_\rho^2)]^2$ and expand around $s=4\mu^2$

(the normalization point). The result is a series of hypergeometric functions, and we can express the ratio of this to the original phase space as an enhancement factor

$$f=\sum_{n=0}^{\infty}\frac{(n+1)[(M-3\mu)(M+\mu)]^n}{(m_\rho{}^2-4\mu^2)^n}\frac{B(\frac{5}{2}+n,\frac{5}{2})}{B(\frac{5}{2},\frac{5}{2})}\times\frac{F_1(\frac{5}{2},-\frac{3}{2},\frac{1}{2};5+n;u,v)}{F_1(\frac{5}{2},-\frac{3}{2},\frac{1}{2};5;u,v)}\,, \qquad \text{(A22)}$$

where

$$u=-\frac{(M-3\mu)(M+\mu)}{4M\mu}\,,\quad v=\frac{(M-3\mu)(M+\mu)}{(M-\mu)^2}\,.$$

A numerical evaluation gives 2.3 for this pole enhancement factor for $\omega\to3\pi$. For $\phi\to3\pi$, the ρ pole is in the physical region, so we must include a width. We multiply by a factor

$$\frac{m_\rho{}^4+m_\rho{}^2\Gamma^2}{(t-m_\rho{}^2)^2+m_\rho{}^2\Gamma^2}$$

and do the extreme relativistic case and get

$$I=\frac{M^{10}}{24}\left(\frac{m_\rho{}^4}{M^4}+\frac{m_\rho{}^2\Gamma^2}{M^4}\right)J\left(\frac{m_\rho{}^2}{M^2},\frac{m_\rho{}^2\Gamma^2}{M^4}\right),$$

where

$$J(a,b)\equiv\int_0^1\frac{dz(1-z)z^3}{z^2+2z(a-1)+(a-1)^2+b} \qquad \text{(A23)}$$

can be evaluated in terms of elementary functions. For $\phi\to3\pi$ and $\Gamma_\rho=125$ MeV, $J=1.27$, which leads to an enhancement factor of 8.2 over the constant matrix element case.

For $\eta\to\pi\pi\gamma$ the ρ pole is outside the physical region so we use the same factor as for $\omega\to3\pi$. The phase-space integral becomes

$$\int\int ds\,dt\,\Delta(s,t)=\frac{(M^2-4\mu^2)^{11/2}}{24M}\left(\frac{m_\rho{}^2-4\mu^2}{m_\rho{}^2-M^2}\right)^2B(4,\tfrac{5}{2})\,F_1\left(4,2,\tfrac{1}{2};\tfrac{13}{2};\frac{M^2-4\mu^2}{M^2-m_\rho{}^2},1-\frac{4\mu^2}{M^2}\right). \qquad \text{(A24)}$$

The F_1 function is evaluated for $M=m_\eta$ and yields 0.62, leading to an enhancement factor of 1.4 over the constant matrix element result. For $\eta'\to\pi\pi\gamma$, the ρ pole is in the physical region, and we use the same factor as for $\phi\to3\pi$. This leads to a phase-space integral which must be approximated:

$$I\approx\frac{(M^2-4\mu^2)^{11/2}}{48\mu}\left[\frac{4\mu^2(m_\rho{}^2-4\mu^2)}{m_\rho{}^2(M^2-4\mu^2)}\right]^{1/2}\frac{m_\rho{}^4+m_\rho{}^2\Gamma^2}{(M^2-4\mu^2)^2}J\left(\frac{m_\rho{}^2-4\mu^2}{M^2-4\mu^2},\frac{m_\rho{}^2\Gamma^2}{(M^2-4\mu^2)^2}\right), \qquad \text{(A25)}$$

where J is again defined by (A23). The approximation involved using an average value of a slowly-varying part of the phase-space factor, but keeping the exact pole factor. For $M=m_{\eta'}$ and $\Gamma=125$ MeV, $J=0.645$, which yields an enhancement factor of 10.0.

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[9]By vacuum trajectories we mean the Pomeron, the lower-lying P' with $\alpha(0)\approx0.5$, and a possible scalar meson trajectory P'' or σ which has $-0.5\le\alpha(0)\le0.0$.

[10]A recent experiment [A. Browman *et al.*, Phys. Rev. Lett. 32, 1067 (1974)] has obtained a width for $\eta\to2\gamma$ lower than previous work by about a factor of 3, which would imply both $\Gamma(\eta)$ and hence $\Gamma(\eta\to\pi\pi\gamma)$ lower by a factor of 3. If this is correct, then either (a) the FDR +FESR calculation cannot correctly explain $\eta\to\pi\pi\gamma$, or (b) the rate for $A_2\to\eta\pi$ is also a factor of 3 lower than presently accepted, or (c) the VDM estimate of $A_2\to\pi\gamma$ was a factor of 3 too large.

[11]R. H. Graham and Toaning Ng [Phys. Rev. D 8, 2957 (1973)] have investigated this process without including the Pomeron and found $\Gamma(\eta')=4.25$ MeV.

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