# Applications of chiral perturbation theory: Mass formulas and the decay $\eta \rightarrow 3\pi$[†]

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Several new results on the breaking of chiral SU(3) × SU(3) are presented within the theoretical framework of chiral perturbation theory. (a) The leading-order corrections to the Gell-Mann–Okubo formula for the baryon octet are shown to be of order $\epsilon^{3/2}$, where $\epsilon$ is a chiral symmetry breaking parameter. An explicit exact expression is given for the leading-order corrections, which provides a new development in understanding why this formula works so well. Similarly the corrections to the Gell-Mann–Okubo formula for the ground-state pseudoscalar octet are shown to be of order $\epsilon^2 \ln\epsilon$ (including $\eta$-$\eta'$ mixing). On the basis of these exact results it is argued that SU(3) × SU(3) symmetry is as good as SU(3) symmetry ~30% except when one considers electromagnetic interactions. (b) We examine the $\eta \rightarrow 3\pi$ decay on the assumption that it is regulated by a nonelectromagnetic isospin-violating term of the type $\epsilon_3 u_3$ with $u_3$ a member of $\overline{3}3 \oplus 3\overline{3}$. The strength $\epsilon_3$ of this term is related to the experimental rate including all leading-order chiral-symmetry corrections. This estimate of $\epsilon_3$ leads to $\Delta I = 1$ hadron level shifts about a factor of 2 or 3 too large, although our estimate of $\epsilon_3$ depends sensitively on the experimental details. (c) Octet enhancement, an exact formalism to describe $\eta$-$\eta'$ mixing, and other topics are discussed.

## I. INTRODUCTION

The ideas of current algebra, partial conservation of axial-vector current, and the approximate SU(3) classification of hadronic states were elegantly unified by Gell-Mann, Oakes, and Renner[1] and by Glashow and Weinberg.[2] They suggested that the strong-interaction Hamiltonian could be written as

$$H = H_0 + \epsilon_0^0 u_0^0 + \epsilon_8^0 u_8^0, \qquad (1.1)$$

where $H_0$ is invariant under the group SU(3)×SU(3), $\epsilon_0^0 u_0^0$ breaks SU(3)×SU(3) but preserves SU(3), and $\epsilon_8^0 u_8^0$ breaks both. The unrenormalized operators $u_0^0$ and $u_8^0$ are assumed to transform as members of the $(\overline{3}, 3) + (3, \overline{3})$ representation (which contains scalar operators $u_i^0$, $i = 0, \ldots, 8$ and pseudoscalar operators $\phi_i^0$, $i = 0, \ldots, 8$) in analogy with the bare-mass terms in a quark model.

It was further suggested that, when the *explicit* symmetry breaking is turned off, the symmetry is spontaneously broken so that the vacuum (and hence the spectrum of physical states) is only SU(3)-invariant. This is implemented by the assumption $\langle u_0^0 \rangle_0 \neq 0$ and $\langle u_8^0 \rangle_0 = 0$ for $\epsilon_0^0$ and $\epsilon_8^0 = 0$. Associated with this spontaneous symmetry breaking would be an octet of massless Nambu-Goldstone bosons, identified as the $\pi$, $K$, and $\eta$ mesons.

Gell-Mann, Oakes, and Renner then suggested that the explicit symmetry breaking could be turned on in two steps. In the first step the parameter $c \equiv \epsilon_8^0/\epsilon_0^0$ is held fixed at $c = -\sqrt{2}$. (This corresponds to giving only the strange quark a bare mass.) The term $\epsilon_0^0(u_0^0 - \sqrt{2}\, u_8^0)$ breaks the symmetry down to SU(2)×SU(2). The $K$ and $\eta$ mesons are therefore given small masses, and the degeneracy of the SU(3) multiplets is broken. In the second step, $c$ is allowed to deviate slightly from $-\sqrt{2}$. This breaks the SU(2)×SU(2) symmetry, giving the pions a very small mass.[3] These low-mass mesons dominate the axial-vector current divergences at low momentum transfer (PCAC).

The unrenormalized explicit symmetry-breaking parameters $\epsilon_0^0$ and $\epsilon_8^0$ are related to the pseudoscalar masses by

$$\begin{aligned} \epsilon_0^0 &= -\frac{1}{\sqrt{6}}\left( \frac{2 f_K \mu_K^2}{Z_K^{1/2}} + \frac{f_\pi \mu_\pi^2}{Z_\pi^{1/2}} \right), \\ \epsilon_8^0 &= \frac{2}{\sqrt{3}}\left( \frac{f_K \mu_K^2}{Z_K^{1/2}} - \frac{f_\pi \mu_\pi^2}{Z_\pi^{1/2}} \right), \end{aligned} \qquad (1.2)$$

where $f_K$ and $f_\pi$ are the kaon and pion decay constants ($f_\pi \sim 93$ MeV) and $Z_K^{1/2}$ and $Z_\pi^{1/2}$ are their wave-function renormalization constants. If one approximates $Z_K \approx Z_\pi$ and $f_K \approx f_\pi$ [which corresponds to neglecting $\langle u_8^0 \rangle_0$, the SU(3) breaking in the vacuum], then one finds $c \approx -1.25$, close to the SU(2)×SU(2) value.

For later convenience we define renormalized symmetry-breaking parameters $\epsilon_0$ and $\epsilon_8$ by

$$\begin{aligned} \frac{\epsilon_0}{f_\pi} &= -\frac{1}{\sqrt{6}}(2\mu_K^2 + \mu_\pi^2) = -0.21\ \text{GeV}^2, \\ \frac{\epsilon_8}{f_\pi} &= \frac{2}{\sqrt{3}}(\mu_K^2 - \mu_\pi^2) = 0.26\ \text{GeV}^2. \end{aligned} \qquad (1.3)$$

The formalism of the $(\overline{3}, 3) + (3, \overline{3})$ model is more thoroughly discussed in Ref. 4, which we shall refer to as I.

We take the viewpoint that SU(2)×SU(2) is accurate[4] to around 7%, making it by far the best symmetry of the strong Hamiltonian after isospin. This can be seen roughly by the smallness of $\mu_\pi^2$ compared to other hadronic masses. Other tests of SU(2)×SU(2), as well as of the $(\bar{3},3)+(3,\bar{3})$ model of SU(3)×SU(3) breaking, are reviewed in Ref. 5.

One of the aims of this paper is to consider how good a symmetry SU(3)×SU(3) really is. It is not enough to know the values of $\epsilon_0$ and $\epsilon_8$; one must also know how to implement the chiral symmetry and *how to make quantitative estimates of the dependence of physical quantities on the symmetry-breaking terms in (1.1).*

There have been two major approaches to this problem. The first approach, pioneered by Glashow and Weinberg,[2] is based upon the idea of smoothness. One writes down Ward identities for the Green's functions of the theory, saturates the two-point functions with meson poles or resonances, and then assumes that the remaining quantities are "smooth" in their momentum dependence. The difficulty with this approach is that *there is no way to determine the reliability of the approximations.* One cannot argue, for example, that the dominance of the two-point function $\langle T(u_i^0 u_j^0)\rangle_0$ by a scalar resonance would become exact in the chiral limit, or that the errors are of any definite order in $\epsilon_0$ and $\epsilon_8$.

The difficulty in controlling the extrapolations explicit in this technique may be seen by the differing results in the literature due to different methods of implementing "smoothness."

The other major approach, stressed by Dashen[6] and by Dashen and Weinstein,[7] is to do a perturbation expansion in the parameters $\epsilon_0^0$ and $\epsilon_8^0$. Dashen and Weinstein showed that pion-pole dominance (PCAC) would become *exact* in the chiral symmetry limit. They argued that low-energy theorems can be interpreted as exact statements concerning the threshold behavior of massless (on-shell) Goldstone bosons in an SU(2)×SU(2)- or SU(3)×SU(3)-symmetric world. Higher-order terms in $\epsilon_0^0$ and $\epsilon_8^0$ would be corrections to the low-energy theorems. Hence PCAC is a consequence of symmetry.

It was later pointed out by Li and Pagels[8] that a perturbation expansion in $\epsilon_0^0$ and $\epsilon_8^0$ must contain nonanalytic terms. This is due to the fact that Goldstone bosons become massless in the chiral limit, producing a long-range component in the strong interactions. Because of the Adler zero the associated infrared singularities are usually finite, but they lead to nonanalytic terms such as $\epsilon\ln\epsilon$ or $\epsilon^{1/2}$ in the expansion. *These nonanalytic terms are often the leading corrections to the chiral limit*, and because they are due to a finite number of diagrams, *they can usually be calculated exactly without assuming any knowledge of the structure of the symmetric part of the Hamiltonian, $H_0$.*

*One can therefore make quantitative statements about the dependence of physical quantities on $\epsilon_0$ and $\epsilon_8$.* For example, one can prove[9] the model-independent result

$$\frac{f_K}{f_\pi}-1=\frac{3(\mu_K^2-\mu_\pi^2)}{64\pi^2 f_\pi^2}\ln\frac{\Lambda^2}{4\mu^2}+O(\epsilon_8),\qquad \frac{f_\eta}{f_\pi}-1=\tfrac{4}{3}\left(\frac{f_K}{f_\pi}-1\right), \tag{1.4}$$

where $\mu^2\approx 0.17\ \text{GeV}^2$ is the average pseudoscalar mass, and $\Lambda^2$ is some cutoff (e.g., $4m_N^2$). Changing $\Lambda$ merely changes the $O(\epsilon_8)$ part of (1.4). For $\ln(\Lambda^2/4\mu^2)\approx 2$, (1.4) predicts $f_K/f_\pi\approx 1.22$, as compared with the experimental value $1.26\pm0.02$. Another result[4] [which depends on the $(\bar{3},3)+(3,\bar{3})$ model] is

$$\frac{Z_K^{1/2}}{Z_\pi^{1/2}}-1=\frac{(\mu_K^2-\mu_\pi^2)}{192\pi^2 f_\pi^2}\ln\frac{\Lambda^2}{4\mu^2}+O(\epsilon_8)\approx 0.025,\qquad \frac{Z_\eta^{1/2}}{Z_\pi^{1/2}}-1=\tfrac{4}{3}\left(\frac{Z_K^{1/2}}{Z_\pi^{1/2}}-1\right). \tag{1.5}$$

One may utilize (1.4) to eliminate the logarithm, yielding

$$\frac{Z_K^{1/2}}{Z_\pi^{1/2}}-1=\tfrac{1}{9}\left(\frac{f_K}{f_\pi}-1\right)+O(\epsilon_8). \tag{1.6}$$

In various other applications the singularity is $O(1/\epsilon)$, $O(\epsilon^{1/2})$, or $O(\epsilon^{3/2})$. The arbitrary cutoff $\Lambda$ that appears for the logarithmic singularities is absent in these cases, so the scaling ambiguity is not present.

In I we developed a set of techniques to enable one to determine the leading nonanalytic terms in $S$-matrix elements and other quantities, including such technical difficulties as renormalization and mass-shell constraints. A fairly complete list of applications of these ideas may be found there.[10]

These nonanalytic terms are frequently the *formally* leading terms. Whether they are numerically dominant over the higher-order analytic corrections (which cannot be calculated without a detailed knowledge of $H_0$) is an open question. We make the optimistic assumption that they do in fact dominate in most cases.[11] However, *our attitude is to use these calculations more as an estimate of how good the zero-order* (chiral-symmetric) *terms are rather than as a numerically accurate estimate of the perturbations.* These remarks apply particularly to those cases for which the leading correction has a logarithm. For those cases for which the leading term is $O(\epsilon^{1/2})$ or $O(\epsilon^{3/2})$ we have more confidence.

The nonanalytic terms are usually due to the divergence of a dispersion integral at the lower limit of some two-meson cut. In those applications in which the leading terms are *analytic*, one cannot prove any exact theorems, but one might hope that the dispersion integrals are still approximately dominated by the thresholds of the two-meson channels. Li and Pagels applied this idea to the meson and baryon mass differences[12] and found that *octet enhancement for matrix elements that vanish if SU(3) is exact emerged naturally*. They reproduced all of the tadpole-model results of Coleman and Glashow,[13] plus an additional result, and gave a possible solution to the puzzle raised by Dashen[6] of how to understand octet enhancement in the chiral-symmetry context.

In Sec. II of this paper we will present some new results on nonanalytic terms including the leading corrections to the meson Gell-Mann–Okubo (GMO) formula. For this we discuss $\eta$-$\eta'$ mixing and the existence of a mixing angle. Also considered are matrix elements of the $u_3^0$ operators between single-meson states, and the $\langle 3\pi|u_3|\eta\rangle$ matrix element. When these results are combined with old results a rather definite picture emerges: *SU(3) × SU(3) is reliable to within 30% except when computing matrix elements of the effective electromagnetic Hamiltonian. Its accuracy is comparable to the accuracy of SU(3)*.[1]

We then generalize the Goldstone boson pair mechanism of Ref. 12. We find that from $M_N - M_\Sigma$ and the meson masses, all of the other baryon medium strong and $I=1$ mass differences can be roughly predicted, supporting the idea of threshold dominance.

It is well known that in the current-algebra approach to symmetry breaking the corrections to the GMO formula for baryons can be represented as a continuum integral. However, one cannot achieve quantitative control on the integral. From the framework of chiral perturbation theory, however, the corrections can be shown to be of order $\epsilon^{3/2}$, and hence one may calculate the leading-order term exactly. We obtain the exact result

$$\tfrac{3}{4}M_\Lambda + \tfrac{1}{4}M_\Sigma - \tfrac{1}{2}(M_N + M_\Xi) = \frac{g_A{}^2(\mu_K{}^2 - \mu_\pi{}^2)(\mu_\eta - \mu_\pi)}{192\pi f_\pi{}^2} \times (3 - 6\alpha + 2\alpha^2), \qquad (1.7)$$

where $(f/d)_A = (1-\alpha)/\alpha$ is the $(f/d)$ ratio for axial-vector baryon $\underline{8}$ coupling. The right-hand side is very small for the experimental value of $\alpha \approx \frac{2}{3}$ but could be large for other values, suggestive that the observed octet enhancement is a consequence of the dynamics of the symmetric world or a higher symmetry which determines $\alpha$.

In Sec. III we consider the $\eta \to 3\pi$ decay. It has long been known that the decay width is much too large to be electromagnetic in origin. This is still true in spite of the new Cornell experiment[14] which has reduced the width by a factor of 3. The theoretical problem can be summarized by Sutherland's theorem,[15] which states that the electromagnetic amplitude vanishes in the SU(2)×SU(2) limit.

It has often been suggested[16,17] that the decay might be explained by adding a small isospin-violating term $\epsilon_3^0 u_3^0$ to the strong Hamiltonian (1.1). The relevant matrix element of $u_3^0$ does not vanish in the SU(2)×SU(2) limit. In Sec. III we make a careful study of whether such a scheme can, in fact, work [in the $(\overline{3},3)+(3,\overline{3})$ model]. The matrix element $\langle 3\pi|u_3^0|\eta\rangle$ is calculated in an SU(3)×SU(3) expansion up to $O(\epsilon \ln\epsilon)$ and is found to be reliable to ~30%. The value of $\epsilon_3$ needed to explain the decay is determined. This value implies an $\epsilon_3 u_3$ contribution to the kaon and baryon $I=1$ mass shifts that is a factor 2 or 3 too large to be easily believable. However, the result is very sensitive to the experimental $\eta$-decay parameters (the combined experimental and theoretical uncertainty in our determination of $\epsilon_3$ is $\gtrsim 50\%$), so the $u_3$ explanation (perhaps with a different representation for $u_3$) cannot be definitely ruled out.

In our conclusion we suggest some other possible applications of chiral perturbation theory, and in a technical appendix we describe an exact formalism to treat $\eta$-$\eta'$ mixing.

## II. MASS FORMULAS AND THE ACCURACY OF SU(3)×SU(3)

### A. Nonanalytic terms

In this section we will present a few new applications of chiral perturbation theory and mention some old ones. Our main goal is to get an idea of the accuracy of SU(3)×SU(3). Methods of derivation may be found in I and in Appendixes A and B. We list results in decreasing order of singularity.

*(a)* $[O(1/\epsilon_0)]$. We have shown[18] that the leading renormalization of the (zero momentum transfer) form factors of the $K^+$ vector currents is of order $\epsilon_8{}^2/\epsilon_0$ and is exactly computable. The result for meson form factors (including a generalization due to Wada[18]) is

$$f_+(0) = 1 - \frac{\mu_K{}^2}{64\pi^2 f_\pi{}^2}\left(\tfrac{5}{2} - 6\ln\tfrac{4}{3}\right) + O(\epsilon^{3/2}) \sim 0.97, \qquad (2.1)$$

where we have set the pion mass equal to zero for simplicity. For baryons, the renormalization is less than 13%.

*(b)* $[O(1/\epsilon_0{}^{1/2})]$. The leading corrections to the baryon Gell-Mann–Okubo formula are discussed later in this section. They are of order $\epsilon_8{}^2/\epsilon_0{}^{1/2}$ and represent a 3-MeV correction to the GMO for-

mula.

*(c)* $[O(\epsilon \ln\epsilon)]$. The meson decay constants and renormalization constants are renormalized[4] from their SU(3)×SU(3) values by 10–35% (see Appendix A). The ratios $f_K/f_\pi$ and $Z_K^{1/2}/Z_\pi^{1/2}$ [Eqs. (1.4) and (1.5)] are 1.22 and 1.025.

The parameter $c=\epsilon_8^0/\epsilon_0^0$ [see Eq. (1.2)] is often estimated to be −1.25. Cornwell[19] has computed the leading corrections, yielding $c=-1.29$, a 3% effect. This result can be easily rederived from (1.4) and (1.5):

$$c=\frac{\epsilon_8^0}{\epsilon_0^0}=-\sqrt{2}\left(\frac{1-a\mu_\pi^2/\mu_K^2}{1+a\mu_\pi^2/2\mu_K^2}\right),$$
$$a\equiv\frac{f_\pi Z_K^{1/2}}{f_K Z_\pi^{1/2}}=1-\tfrac{8}{9}\left(\frac{f_K}{f_\pi}-1\right)+O(\epsilon_8)\simeq 0.805\,. \tag{2.2}$$

Now, consider the Gell-Mann–Okubo (GMO) formula[20] for the pseudoscalar octet. In addition to the defining equations (1.2) there is a third relation between $\epsilon_0^0$, $\epsilon_8^0$ and physical quantities (Appendix A):

$$(\tfrac{2}{3})^{1/2}\epsilon_0^0-\frac{1}{\sqrt{3}}\epsilon_8^0=-\frac{\mu_\eta^2 f_\eta}{Z_{8\eta}^{1/2}}-(\tfrac{2}{3})^{1/2}\epsilon_8^0\,\frac{Z_{0\eta}^{1/2}}{Z_{8\eta}^{1/2}}\,, \tag{2.3}$$

where $Z_{i\eta}^{1/2}\equiv\langle 0|\phi_i^0|\eta\rangle$, $i=0, 8$. [The quantity $Z_\eta^{1/2}$ in (1.5) is the same as $Z_{8\eta}^{1/2}$.] Combining (1.2) and (2.3) one can prove the following relation, which is exact in the Gell-Mann, Oakes, Renner model:

$$\frac{3}{4}\left[\frac{f_\eta\mu_\eta^2}{Z_{8\eta}^{1/2}}+\left(\frac{2}{3}\right)^{1/2}\frac{\epsilon_8^0 Z_{0\eta}^{1/2}}{Z_{8\eta}^{1/2}}\right]+\frac{f_\pi\mu_\pi^2}{4Z_\pi^{1/2}}-\frac{f_K\mu_K^2}{Z_K^{1/2}}=0\,. \tag{2.4}$$

The term involving $\epsilon_8^0 Z_{0\eta}^{1/2}$ represents the $\eta$-$\eta'$ mixing effect. In Appendix A, $\epsilon_8^0 Z_{0\eta}^{1/2}$ is calculated and shown to be of $O(\epsilon_8^2\ln\epsilon_0)$. If one sets $f_K=f_\pi=f_\eta$, etc., in (2.4) one obtains the GMO formula:

$$\tfrac{3}{4}\mu_\eta^2+\tfrac{1}{4}\mu_\pi^2-\mu_K^2=0\,. \tag{2.5}$$

This relation works quite well. The left-hand side is −0.017 GeV², as compared with the average pseudoscalar mass, $\mu^2=0.17$ GeV².

The leading corrections to (2.5) can be determined using (1.4), (1.5), and the results on $\epsilon_8^0 Z_{0\eta}^{1/2}$:

$$\tfrac{3}{4}\mu_\eta^2+\tfrac{1}{4}\mu_\pi^2-\mu_K^2-\frac{\epsilon_8\ln(\Lambda^2/4\mu^2)}{96\pi^2 f_\pi^3}\left(\frac{5\epsilon_8}{2f_\pi}-\sqrt{3}\,\mu_\eta^2+\frac{\mu_\pi^2}{\sqrt{3}}+\frac{2\mu_K^2}{\sqrt{3}}\right)=0+O(\epsilon_8^2)\,. \tag{2.6}$$

The first term in the brackets is due to $\eta$-$\eta'$ mixing; the other terms are from the ratios of the $f$'s and $Z$'s (the vacuum symmetry breaking). Putting in numbers, we find

$$(-0.017-0.038+0.013)\ \text{GeV}^2=-0.042\ \text{GeV}^2$$
$$=O(\epsilon_8^2),$$

where −0.017 is the GMO combination, −0.038 is from the mixing, and 0.013 is from the vacuum symmetry breaking.

*The corrections have made the formula worse, although (2.5) is still small compared to* $\mu^2=0.17$ GeV². What could be the source of the difficulty? It could be an indication that the $(\bar{3},3)+(3,\bar{3})$ model (or the whole chiral-symmetry scheme) is not correct. Our suspicion, however, is that *in this application* the $\epsilon\ln\epsilon$ terms are not correctly estimating $Z_{0\eta}^{1/2}$.

It is amusing to note that (2.6) can be rewritten

$$\tfrac{3}{4}\mu_\eta^2+\tfrac{1}{4}\mu_\pi^2-\mu_K^2=\frac{(\mu_K^2-\mu_\pi^2)^2}{48\pi^2 f_\pi^2}\ln\left(\frac{\Lambda^2}{4\mu^2}\right)+O(\epsilon_8^2)$$
$$=\tfrac{4}{9}\left(\frac{f_K}{f_\pi}-1\right)(\mu_K^2-\mu_\pi^2)+O(\epsilon_8^2)\,. \tag{2.7}$$

Equation (2.7) shows that the *formally* leading-order contribution to the ratio of 27- to 8-dimensional components of the pseudoscalar masses is

$$\tfrac{4}{9}\left(\frac{f_K}{f_\pi}-1\right),$$

which numerically is around 0.1. This contribution to the ratio is reasonably small, but it differs in sign from the true ratio.

We would like to comment on the standard mixing angle formula

$$\tfrac{3}{4}(\mu_\eta^2\cos^2\theta+\mu_{\eta'}^2\sin^2\theta)+\tfrac{1}{4}\mu_\pi^2-\mu_K^2=O(\epsilon_8^2)\,. \tag{2.8}$$

Can such a formula (which has no predictive power) be theoretically justified? One can certainly *define* $\theta$ as the angle which satisfies (2.8). However, it is shown in Appendix A that $\theta$ can only be interpreted as the angle which rotates the $\eta'$ and $\eta$ states into "singlet" and "octet" states if one *turns off all strong interactions*. This interpretation cannot be justified by dominating any dispersion relation by a pole or by keeping the leading symmetry-breaking terms. In other words, a mixing angle is appropriate for the bare-mass terms in a Lagrangian, but not for physical hadron states.

Even if one does ignore the strong interactions and introduce a mixing angle in terms of a rotation of the $\eta'$ and $\eta$, one can only obtain (2.8) by ignoring vacuum symmetry breaking. Finally, the GMO formula is supposed to be correct only to order $\epsilon_8$, so there is no justification for keeping the term $\sin^2\theta\mu_{\eta'}{}^2$ in (2.8) which is of order $\epsilon_8{}^2$.

In Sec. III we will need the matrix element of $u_3^0$ between single-meson states. Up to $O(\epsilon\ln\epsilon)$, the perturbation of the matrix elements around SU(3) $\times$SU(3) is

$$\begin{aligned}\langle K^+|u_3^0|K^+\rangle &= -\langle K^0|u_3^0|K^0\rangle \\ &= -\frac{Z_s{}^{1/2}(0)}{2f_K(\epsilon)}\left[1-\frac{(\frac{2}{3})^{1/2}\epsilon_0\ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi{}^3}-\frac{7}{12}\frac{1}{\sqrt{3}}\frac{\epsilon_8\ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi{}^3}\right] \\ &= -\frac{Z_s{}^{1/2}(0)}{2f_K(\epsilon)}(1.06) \\ &= -\frac{Z_s{}^{1/2}(0)}{2f_\pi(\epsilon)}(0.87), \end{aligned} \tag{2.9}$$

where $f_K(\epsilon)$ and $f_\pi(\epsilon)$ are the physical values of the decay constants, and $Z_s{}^{1/2}(0)$ is the value of $Z_\pi{}^{1/2}$, $Z_K{}^{1/2}$, and $Z_{8\eta}{}^{1/2}$ in the SU(3)$\times$SU(3) limit. Similarly,

$$\begin{aligned}\langle\pi^0|u_3^0|\eta\rangle &= -\frac{Z_s{}^{1/2}(0)}{\sqrt{3}\,f_\eta(\epsilon)}\left[1-(\tfrac{2}{3})^{1/2}\frac{\epsilon_0\ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi{}^3}-\frac{11}{6\sqrt{3}}\frac{\epsilon_8\ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi{}^3}\right] \\ &= -\frac{Z_s{}^{1/2}(0)}{\sqrt{3}\,f_\eta(\epsilon)}(0.94) \\ &= -\frac{Z_s{}^{1/2}(0)}{\sqrt{3}\,f_\pi(\epsilon)}(0.73). \end{aligned} \tag{2.10}$$

We see that these matrix elements are changed by less than 30% by the leading symmetry-breaking terms. Formulas (2.9) and (2.10) are exact to leading order. They incorporate *all* effects, including internal dynamics, mass-shell constraints, renormalization, and vacuum symmetry breaking.

In Sec. III and Appendix B we consider a still more complicated matrix element: $\langle 3\pi|u_3^0|\eta\rangle$. The leading corrections to its SU(3)$\times$SU(3) symmetric value are ~34%.

Finally, we mention the one known example in which the perturbation expansion completely breaks down. Dashen[6] has shown that in the SU(3) $\times$SU(3) limit the electromagnetic contributions to $\mu_{K^+}{}^2-\mu_{K^0}{}^2$ and $\mu_{\pi^+}{}^2-\mu_{\pi^0}{}^2$ are equal. That is, the ratio of the octet part of the effective electromagnetic Hamiltonian $(\mu_{K^+}{}^2-\mu_{K^0}{}^2)$ to the 27-dimensional part $(\mu_{\pi^+}{}^2-\mu_{\pi^0}{}^2)$ becomes a Clebsch-Gordan coefficient (unity) in the SU(3)$\times$SU(3) limit. Hence, the tadpole mechanism[13] (tadpole is used in its *original* meaning: a dynamical enhancement of the octet part of $H_{\rm em}$) cannot occur in the chiral symmetry limit.

Since, in fact, octet enhancement does occur $(\mu_{K^+}{}^2-\mu_{K^0}{}^2=-0.004\ \text{GeV}^2$ while $\mu_{\pi^+}{}^2-\mu_{\pi^0}{}^2=+0.0013\ \text{GeV}^2)$, Dashen concluded that the perturbation expansion must break down in this application.

This was shown explicitly in Ref. 21; the leading $\epsilon\ln\epsilon$ corrections to the electromagnetic mass shifts were computed and shown to be *larger than the leading term*. (The large corrections were due to the internal meson loops, not the off-shell extrapolations of the external particles.) Hence the SU(3)$\times$SU(3) perturbation expansion breaks down in this application,[22] and the Dashen theorem does not correctly estimate the electromagnetic mass shifts.

*The meson mass shifts therefore provide no evidence for or against the existence of an explicit* $\epsilon_3 u_3$ *term in the Hamiltonian.* The origin of octet enhancement is left unexplained.

The conclusion to be drawn from all of this is that *regardless of the order of the leading corrections, SU(3)×SU(3) is reliable in almost all cases to around 30% or better, making its accuracy comparable to that of SU(3).* The only exception seems to be the matrix elements of the time-ordered product of two currents.

### B. Octet enhancement

We have seen that octet enhancement cannot be explained as a chiral-limit theorem. Li and Pagels have shown[9,12] that octet enhancement for quantities which vanish in the SU(3) symmetry limit emerges naturally from the assumption of the Goldstone boson pair mechanism. This assumption is that the dispersion integrals for $\langle a|\partial\cdot V|b\rangle$ ($V$ is a vector current) are dominated by the thresholds of the two-meson cuts. For mesons, the GMO relation (2.5) and the (octet) Coleman-Glashow formula

$$\sqrt{3}\,\Delta_{\pi^0\eta} = \mu_{K^+}{}^2 - \mu_{K^0}{}^2 \qquad (2.11)$$

*follow without any assumption concerning the origin of the symmetry breaking.* Moreover, the 27-dimensional $\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2$ is predicted to vanish.

When this idea is extended to the baryons, the GMO formula, the Coleman-Glashow formula, and the tadpole results[13] all follow. Baryon mass differences are given by formulas like

$$M_a - M_b \approx A(\mu_c{}^2 - \mu_d{}^2)\int_{(\mu_c+\mu_d)^2}^{4\Lambda^2} dt\,\sqrt{t}/t$$

$$\simeq 4A(\mu_c{}^2 - \mu_d{}^2)\Lambda\,, \qquad (2.12)$$

where $\Lambda$ is a cutoff, $A$ is a constant, and $\mu_c$ and $\mu_d$ are meson masses. Here we would like to extend these results by retaining the $\epsilon^{3/2}$ terms associated with the lower limit of (2.12). We obtain a one-parameter fit to all baryon mass differences. The results are

$$M_N - M_\Sigma = \frac{g_A{}^2}{192\pi f_\pi{}^2}(\mu_K{}^2 - \mu_\pi{}^2)[(2\Lambda - \mu_K - \mu_\pi)(15 - 48\alpha + 28\alpha^2) + (2\Lambda - \mu_K - \mu_\eta)(3 - 8\alpha + 4\alpha^2)]\,,$$

$$M_p - M_n = \frac{g_A{}^2}{16\pi f_\pi{}^2}(\mu_{K^+}{}^2 - \mu_{K^0}{}^2)[(2\Lambda - 2\mu_K)\tfrac{1}{8}(3 - 4\alpha^2) + (2\Lambda - \mu_\pi - \mu_\eta)\tfrac{1}{3}(3 - 4\alpha)]\,, \qquad (2.13)$$

and similar formulas for $m_\Sigma - m_\Xi$, $m_\Lambda - m_\Xi$, $m_{\Sigma^+} - m_{\Sigma^-}$, and $m_{\Xi^-} - m_{\Xi^0}$ which can easily be inferred from Eq. (4.1) of Ref. 9. The cutoff $\Lambda$ is chosen to be 705 MeV, to fit $m_N - m_\Sigma$, $g_A$ is 1.24, and $(f/d)_A = (1-\alpha)/\alpha$ is the $f/d$ ratio for axial-vector-baryon couplings. Experimentally, $\alpha \approx 0.66 \pm 0.02$. Using $\alpha = \frac{2}{3}$ and the experimental meson masses, the predictions for the baryon mass shifts are given in Table I.

These numbers are in rough agreement with experiment, *supporting the view that the two-meson thresholds are dominating the dispersion integrals.*[23] Several comments are in order: (a) These predictions are *not* exact theorems of the type discussed earlier. (b) The results are *independent* of the origin of SU(3) and SU(2) breaking. (c) The numbers are rather sensitive to the value of $\alpha$ used and to the validity of the meson Coleman-Glashow formula (2.11). (d) The $I=1$ predictions probably *do not properly include the electromagnetic Born terms* (for $M_p - M_n$ the Born term is 0.8 MeV).

Although the contributions of order $\epsilon_8$ to (2.13) are only approximate, the pieces of order $\epsilon_8{}^2/\epsilon_0{}^{1/2}$ are in fact exact [and independent of the $(\overline{3},3)+(3,\overline{3})$ model]. Forming the GMO combination we find the leading corrections (which can also be found from the methods of I):

$$\tfrac{3}{4}M_\Lambda + \tfrac{1}{4}M_\Sigma - \tfrac{1}{2}(M_N + M_\Xi) = \frac{g_A{}^2(\mu_K{}^2 - \mu_\pi{}^2)(\mu_\eta - \mu_\pi)}{192\pi f_\pi{}^2} \times (3 - 6\alpha + 2\alpha^2) + O(\epsilon_8{}^2)\,. \qquad (2.14)$$

The left-hand side is experimentally $7.5 \pm 5.0$ MeV, while the right-hand side is 28 MeV$\times(3 - 6\alpha + 2\alpha^2) = -3.1$ MeV for $\alpha = \frac{2}{3}$. The correction is tiny and of the wrong sign. It is interesting that $3 - 6\alpha + 2\alpha^2$ vanishes for $\alpha = 0.63$. It has always been puzzling that the GMO formula works so well. The leading correction (2.14) is a large number times a function of $\alpha$ which nearly vanishes due to an apparent accident[24] of the closeness of $\alpha$ to 0.63. *It appears that the success of the octet formula is being governed by the value of $\alpha$, which can only be understood on the basis of the dynamics of the SU(3)-symmetric interactions or a higher symmetry, such as SU(6).*

The $\epsilon^{3/2}$ terms in (2.13) are almost purely octet and they work fairly well (Table I). It is only the tiny 27-dimensional piece in (2.14) that is of the wrong sign[25] if $\alpha > 0.63$.

The Coleman-Glashow formula

$$M_p - M_n + M_{\Sigma^-} - M_{\Sigma^+} + M_{\Xi^0} - M_{\Xi^-} = 0\,, \qquad (2.15)$$

which is satisfied to within experimental errors, does not have any $\epsilon_8/\epsilon_0{}^{1/2}$ corrections even though the individual mass differences in (2.13) do. The corrections are of order $\alpha\epsilon_8 \ln\epsilon_0$ (here, $\alpha$ is the fine structure constant).

In summary, SU(3)×SU(3) seems to work well

TABLE I. The baryon mass differences predicted from the threshold dominance relations (2.13).

| Quantity | Predicted value (MeV) | Experimental value (MeV) |
|---|---|---|
| $M_N - M_\Sigma$ | −252 (input) | −252 |
| $M_\Sigma - M_\Xi$ | −188 | −123 |
| $M_\Lambda - M_\Xi$ | −235 | −199 |
| $M_p - M_n$ | −2.3 | −1.3 |
| $M_{\Sigma^+} - M_{\Sigma^-}$ | −5.6 | −7.9 |
| $M_{\Xi^-} - M_{\Xi^0}$ | +3.3 | +6.6 |

in most cases. Octet enhancement is predicted from the Goldstone boson pair mechanism, but its *extreme* success in the baryon GMO formula seems to be an accident of the value of $\alpha$. The leading corrections to the GMO formulas for both the mesons and the baryons are predicted to be small, but of the wrong sign.[25]

## III. THE $\eta\to 3\pi$ DECAY

The large experimental widths for the principal decay modes of the $\eta$ have long been a puzzle. The $\eta\to 2\gamma$ width was a factor of 5 greater than expected[26] from SU(3) and the $\pi^0$ decay rate. Similarly, the $\eta\to 3\pi$ rates seemed far too large to be compatible with electromagnetism.

A recent Cornell experiment[14] has reduced the $2\gamma$ rate by a welcome factor of 3.[27] Since the $2\gamma$ decay is used to determine the total width, the $\eta\to 3\pi$ rates are also reduced by a factor of 3. Nevertheless, the $3\pi$ widths are still two orders of magnitude larger than expected from electromagnetism.[15,17] The basic problem is summarized by Sutherland's theorem[15]: The electromagnetic amplitude vanishes in the $SU(2)\times SU(2)$ limit.

Various authors[16,17] have suggested that the decay could be explained by adding a small isospin-violating term $\epsilon_3^0 u_3^0$ to the strong Hamiltonian (1.1). The decay amplitude due to this term need not vanish in the $SU(2)\times SU(2)$ limit. The $u_3^0$ term can be thought of as a difference between the bare masses of the nonstrange quarks or as a Coleman-Glashow tadpole.[13] It could also be the effect of new interactions, such as the neutral gauge bosons used in some gauge theories to cancel the divergences associated with electromagnetism.

In order to predict the $\eta$ decay rate one must know $\epsilon_3^0$ and must compute the matrix element $\langle 3\pi|u_3^0|\eta\rangle$.

A typical approach[17] has been to estimate $\epsilon_3^0$ from the observed mass difference $\mu_{K^+}^2-\mu_{K^0}^2$, assuming that the purely electromagnetic part of the splitting is given by Dashen's theorem[6] (see Sec. II). As we have shown[21] that Dashen's theorem does not correctly estimate the electromagnetic mass difference, this approach must be abandoned.

The calculation of $\langle 3\pi|u_3^0|\eta\rangle$ involves going to the $SU(3)\times SU(3)$ limit and possesses some subtleties.[28] In Sec. III B we will compute this matrix element [in the $(\bar{3},3)+(3,\bar{3})$ model] in an $SU(3)\times SU(3)$ expansion correct up to $O(\epsilon\ln\epsilon)$. *These leading corrections are reasonably small (34%), so the chiral calculation is fairly reliable.* We then determine $\epsilon_3$ from the observed decay parameters. Unfortunately, the number obtained depends very sensitively on the experimental numbers. In Sec. III C we use this value to determine the $\epsilon_3 u_3$ contribution to the proton-neutron and other mass differences. They turn out to be uncomfortably large (for example, $m_p-m_n|_{u_3}\sim -7.9$ MeV), although the combined experimental and theoretical uncertainty in this number is at least 50%. The implications of this result are then discussed.

### A. Experimental results and electromagnetic contribution

The branching ratios for the $3\pi^0$ and the $\pi^+\pi^-\pi^0$ decays are $\Gamma_{000}/\Gamma_{tot}=30.0\pm 1.1$ and $\Gamma_{+-0}/\Gamma_{tot}=23.9\pm 0.6$, where[14] $\Gamma_{tot}=0.85\pm 0.12$ keV.

Experiment is consistent with a linear form for the $\pi^+\pi^-\pi^0$ matrix element:

$$T_{+-0}=A+BE_0\ , \tag{3.1}$$

where $E_0$ is the energy of the $\pi^0$. Experimentally,[29] the slope parameter $B/A$ is around $-2/\mu_\eta$.

It is usually assumed that the final state has isospin one. This is partly a theoretical prejudice: If the decay is electromagnetic, then only the $I=1$ part can contribute because of $G$ parity. Of course, an $\epsilon_3 u_3$ term is also $I=1$.

If the final state is $I=1$, then from Bose statistics

$$\begin{aligned} T_{000}&=T_{+-0}+T_{-0+}+T_{0+-}\\ &=3A+B\mu_\eta\ . \end{aligned} \tag{3.2}$$

The widths for the $3\pi^0$ and $\pi^+\pi^-\pi^0$ decays are then given by[30]

$$\begin{aligned} \Gamma_{000}&=827|3A+B\mu_\eta|^2\ \text{eV}\ ,\\ \Gamma_{+-0}&=489|3A+B\mu_\eta|^2[1+0.02y(1+y)]\ \text{eV}\ , \end{aligned} \tag{3.3}$$

where $y=(\mu_\eta-3\mu_\pi)/(\mu_\eta+3A/B)$.

Using the experimental value $B/A\approx -2/\mu_\eta$ one can then predict that $\Gamma_{+-0}/\Gamma_{000}=0.59$. (The prediction would be $\frac{2}{3}$ if one neglected the $\pi^+$-$\pi^0$ mass difference and set $B=0$. It is not very sensitive to $B/A$.) Experimentally, $\Gamma_{+-0}/\Gamma_{000}=0.80\pm 0.05$. This is in crude agreement with the $I=1$ assumption, but an $I\geq 3$ final state clearly cannot be ruled out.[31]

Consider the possibility that the decay is electromagnetic. Then the amplitude is

$$T_{ijk}=-\langle ijk|H_{em}|\eta\rangle\ , \tag{3.4}$$

and the quantities $A$ and $B$ of (3.1) and (3.2) are functions of the chiral symmetry breaking parameters $\epsilon_0$ and $\epsilon_8$. In the $SU(2)\times SU(2)$ limit both $T_{+-0}$ and $T_{000}$ must vanish when the four-momentum of the (on-shell) $\pi^0$ goes to zero, because[6,7]

$$T_{ij0}\xrightarrow[p_{\pi^0}\to 0]{}\frac{i}{f_\pi}\langle ij|[{}^5F_3,H_{em}]|\eta\rangle=0\ . \tag{3.5}$$

But for $p_{\pi^0}=0$, $T_{+-0}=A$ and $T_{000}=3A+B\mu_\eta$. Hence, in the $SU(2)\times SU(2)$ limit

$$A = B = 0\ , \qquad (3.6)$$

so the electromagnetic decay is suppressed by a factor of $\mu_\pi^2$ (or $\mu_\pi^2 \ln \mu_\pi^2$). This result is independent of any detailed model and depends only on the assumption that neutral axial charges commute with $H_{\rm em}$ (up to anomalous terms which are higher order in $\alpha \simeq \frac{1}{137}$) and that SU(2) × SU(2) is a good hadron symmetry. This is Sutherland's theorem[15] stated as a chiral-limit theorem. (The original statement was that $T_{+-0}$ and $T_{000}$ must vanish at the off-mass-shell point $p_{\pi^0} = 0$.)

Various authors[15,17] have tried to estimate these $\mu_\pi^2$ terms by assuming linear extrapolation formulas in the masses of the three pions, which they vary independently (via PCAC). In this approach the slope $B/A \approx -2/\mu_\eta$ is predicted correctly, but the magnitude of $A$ always comes out far too small. Dittner, Dondi, and Eliezer,[17] for example, find $\Gamma_{+-0} = 0.6$ eV in the $(\bar{3}, 3) + (3, \bar{3})$ model and similar results in other models [one must make a model-dependent SU(3) × SU(3) calculation to compute $A$].

Therefore, *the vanishing of A in the SU(2) × SU(2) limit seems to rule out the possibility that the decay is electromagnetic* (unless, perhaps, there is some unexpected very strong final-state enhancement of the amplitude).

### B. Addition of an $\epsilon_3 u_3$ term

Due to the failure of electromagnetism to account for the decay, it has been suggested[16,17] that one should add an $I=1$ term to the strong-symmetry-breaking Hamiltonian. Then (1.1) becomes

$$H = H_0 + \epsilon_0^0 u_0^0 + \epsilon_3^0 u_3^0 + \epsilon_8^0 u_8^0\ , \qquad (3.7)$$

where, one hopes, $|\epsilon_3^0| \ll |\epsilon_0^0|, |\epsilon_8^0|$. *In contrast to the electromagnetic case the $u_3$ contribution to the $\eta \to 3\pi$ decay does not vanish in the SU(2) × SU(2) limit,* so it appears to be a good candidate to explain the decay. In this section we will calculate the parameters $A$ and $B$ in an expansion about the SU(3) × SU(3) limit. Throughout we will assume that $u_0^0$, $u_3^0$, and $u_8^0$ belong to the $(\bar{3}, 3) + (3, \bar{3})$ representation.

We must compute the matrix element $T_{ijk}(p_i, p_j, p_k; \epsilon_0^0; \epsilon_8^0) = -\epsilon_3^0 \langle ijk | u_3^0 | \eta \rangle$, assuming the form (3.1) for $T_{+-0}$. Now, in the SU(2) × SU(2) limit, $T_{+-0}$ vanishes when $p_{\pi^+} = 0$ because $[{}^5F_{\pi^+}, u_3^0] = 0$. But $E_0 = \mu_\eta/2$ when $p_{\pi^+} = 0$, so from (3.1) *the slope B/A is correctly predicted to be $-2/\mu_\eta$ in the SU(2) × SU(2) limit.* We have argued in the Introduction that SU(2) × SU(2) results are valid to ~7%, *so we will work in the* SU(2) × SU(2) *limit from now on.* From (3.1) and (3.2) we have

$$T_{+-0} = A(\epsilon_0^0)\left(1 - \frac{2E_0}{\mu_\eta}\right)\ ,$$
$$T_{000} = A(\epsilon_0^0)\ . \qquad (3.8)$$

To determine $A$ we must go to the SU(3) × SU(3) limit (always maintaining the ratio $\epsilon_8^0/\epsilon_0^0 = -\sqrt{2}$). The ratio $E_0/\mu_\eta$ is ambiguous in the chiral limit, so rather than work directly[32] with $T_{+-0}$ we will utilize the exact result that for $p_{\pi^0} \to 0$,

$$\begin{aligned} T_{+-0}(p_{\pi^0} = 0, \epsilon_0^0) &= A(\epsilon_0^0) \\ &= \frac{i\epsilon_3^0}{f_\pi} \langle +-|[{}^5F_3 u_3^0]|\eta\rangle \\ &= \frac{\epsilon_3^0}{f_\pi}\left\langle +-\left|\left[(\tfrac{2}{3})^{1/2}\phi_0^0 + \frac{1}{\sqrt{3}}\phi_8^0\right]\right|\eta\right\rangle\ , \end{aligned} \qquad (3.9)$$

where the $\phi_0^0$ and $\phi_8^0$ operators carry zero momentum. Notice that if the underlying symmetry group were U(3) × U(3), as is naively expected in quark models but not present in fact,[33] then $\phi_0^0$ and $\phi_8^0$ could be written as linear combinations of $\partial \cdot A_8$ and $\partial \cdot A_0$ so that $A$ (and $B$) would vanish in the SU(2) × SU(2) limit.

We will evaluate (3.9) exactly in the SU(3) × SU(3) limit. In Appendix B we sketch the calculation of the leading $\epsilon \ln \epsilon$ corrections.

$A(0)$ is determined from the following *unambiguous*[34] prescription:

(a) Define the off-mass-shell Green's function

$$G(p_\eta^2, \epsilon_0^0) \equiv \frac{i\epsilon_3^0}{f_\pi} \int d^4x\, e^{ip_\eta\cdot x}(\Box + \mu_\eta^2)\left\langle +-\left| T\left(\phi_\eta(x)\left[(\tfrac{2}{3})^{1/2}\phi_0^0(0) + \frac{1}{\sqrt{3}}\phi_8^0(0)\right]\right)\right|0\right\rangle. \qquad (3.10)$$

A suitable interpolating field $\phi_\eta$ (including the effects of $\eta$-$\eta'$ mixing) is defined in Appendix A.

(b) Approximate $A(\epsilon_0^0) = G(\mu_\eta^2, \epsilon_0^0)$ by $G(0, \epsilon_0^0)$. The error induced is of $O(\epsilon_0 \ln \epsilon_0)$ and is calculated in Appendix B.

(c) For $\epsilon_0^0 \neq 0$, use the exact Dashen-Weinstein[6,7] method for SU(2)×SU(2) to evaluate $G(0, \epsilon_0^0)$:

$$G(0, \epsilon_0^0) = -\frac{i\epsilon_3^0}{f_\pi^3}\mu_\eta^2 \int d^4x \left\langle 0\left|\left[{}^5F_{\pi^+}, \left[{}^5F_{\pi^-}, T\left((\tfrac{2}{3})^{1/2}\phi_\eta(x)\phi_0^0(0) + \frac{1}{\sqrt{3}}\phi_\eta(x)\phi_8^0(0)\right)\right]\right]\right|0\right\rangle\ , \qquad (3.11)$$

where $[{}^5F, [{}^5F, T(\ )]]$ is a shorthand notation meaning to keep the various equal-time commutators.

(d) Drop all terms which vanish as $\epsilon_0^0 \to 0$. The result of all this is

$$G(0, \epsilon_0^0) = -\frac{4i\epsilon_3^0}{3\sqrt{3}f_\pi^{\,3}} \frac{\mu_\eta^{\,2}}{Z_{8\eta}^{\,1/2}} \int d^4x \langle T(\phi_8^0(x)\phi_8^0(0))\rangle_0 + O(\epsilon \ln\epsilon) . \qquad (3.12)$$

But the propagator in (3.12) is just $-iZ_{8\eta}/\mu_\eta^{\,2}$, where $Z_{8\eta}$ is defined after (2.3), so we have

$$A(0) = G(0, 0) = -\frac{4\epsilon_3^0 Z_{8\eta}^{\,1/2}}{3\sqrt{3}f_\pi^{\,3}} \equiv -\frac{4\epsilon_3}{3\sqrt{3}f_\pi^{\,3}} . \qquad (3.13)$$

For $\epsilon_0 = 0$, $Z_{8\eta}^{\,1/2}$ is equal to $Z_s^{\,1/2}(0)$ defined in Appendix A, and we have defined the renormalized $\epsilon_3$ as $\epsilon_3^0 Z_s^{\,1/2}(0)$.

In Appendix B we calculate the $\epsilon_0 \ln\epsilon_0$ corrections to $A(0)$ exactly. The result is

$$A(\epsilon_0) = -\frac{4\epsilon_3}{3\sqrt{3}f_\pi^{\,3}(\epsilon)} \times \left[1 + (\tfrac{2}{3})^{1/2} \frac{45}{16} \frac{\epsilon_0 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^{\,3}} + O(\epsilon_0)\right] , \qquad (3.14)$$

where $\epsilon_3$ is still defined as $\epsilon_3^0 Z_s^{\,1/2}(0)$, and $f_\pi(\epsilon)$ is the *physical* value of the pion decay constant. For $\ln(\Lambda^2/4\mu^2) = 2$, the correction term is $-34\%$, *which indicates that SU(3) × SU(3) is reasonably dependable in this application.*

From (3.3) and the experimental width $\Gamma_{+-0}$, one obtains[29] $|A| = 0.703$ for $B/A = -2.08/\mu_\eta$ and $|A| = 0.572$ for $B/A = -1.87/\mu_\eta$. These numbers would be reduced by 15% if $\Gamma_{000}$ were used instead of $\Gamma_{+-0}$. These two values yield

$$\left|\frac{\epsilon_3}{f_\pi}\right| = \begin{cases} 1.19 \times 10^{-2}\ \text{GeV}^2 , \\ 0.97 \times 10^{-2}\ \text{GeV}^2 , \end{cases} \qquad (3.15)$$

respectively. This is to be compared with the SU(3) × SU(3)-breaking parameters $\epsilon_8/f_\pi = 0.26$ GeV$^2$ and $\epsilon_0/f_\pi = -0.21$ GeV$^2$, and the SU(2) × SU(2)-breaking parameter[35] $\epsilon/f_\pi = -\mu_\pi^{\,2} = -1.8 \times 10^{-2}$ GeV$^2$.

### C. Implications of the $\epsilon_3 u_3$ terms

Assuming that the $\eta \to 3\pi$ decay is due to an $\epsilon_3 u_3$ term, we have given what we believe is a reliable determination of $\epsilon_3$.

We now consider the other implications of an $\epsilon_3 u_3$ term. Using the matrix element (2.9), and assuming $\epsilon_3 > 0$, we can find the $\epsilon_3 u_3$ contribution to $\mu_{K^+}^{\,2} - \mu_{K^0}^{\,2}$. The result, listed in Table II for the two values of $\epsilon_3$ in (3.15), is about twice the experimental mass difference.

Since $u_3^0$ and $u_8^0$ are members of the same octet, the $\epsilon_3^0 u_3^0$ baryon mass differences can be predicted from the medium strong differences:

$$\begin{aligned} (M_p - M_n)|_{u_3} &= E[(f/d)_B + 1] , \\ (M_{\Sigma^+} - M_{\Sigma^-})|_{u_3} &= E[2(f/d)_B] , \\ (M_{\Xi^-} - M_{\Xi^0})|_{u_3} &= E[-(f/d)_B + 1] , \end{aligned} \qquad (3.16)$$

where $(f/d)_B \sim -3.29$ from the medium strong differences, and $E \equiv \sqrt{3}\epsilon_3 (M_\Sigma - M_\Lambda)/2\epsilon_8^0 Z_s^{\,1/2}(0)$. The small deviation of $\epsilon_8^0 Z_s^{\,1/2}(0)$ from $\epsilon_8$ can be determined from Appendix A. The numbers predicted from (3.16) are given in Table II.

We see that *the $u_3$ contributions predicted for the mass differences are probably too large*[36] *to be believable, although there is no reliable way to calculate the electromagnetic contributions.*

What are the errors in the numbers in Table II? The numbers are uncertain by 20% because of $B/A$. They would be reduced 15% had we used $\Gamma_{000}$ rather than $\Gamma_{+-0}$ to determine $\epsilon_3$ (old values for the widths would increase the numbers by $\sqrt{3}$). There is also a 15% error due to the *quoted* uncertainties in $\Gamma_{\text{tot}}$, the branching ratios, and the baryon masses. Finally, if the $\epsilon \ln\epsilon$ corrections in (3.14) had been ignored, the numbers in Table II would be reduced by a factor $\frac{2}{3}$. Hence, *the uncertainty in the numbers is at least* 50%.

What can be concluded from all this? The decay seems clearly to be nonelectromagnetic. Also, an $\epsilon_3 u_3$ term *in the $(\bar{3}, 3) + (3, \bar{3})$ model* probably cannot account for the decay[37] using the present experimental parameters. The problem cannot be blamed on the SU(3) × SU(3) extrapolation; the corrections are ~30%.

The numbers predicted from the $\epsilon_3 u_3$ term are in error only by a factor of 2 or 3, not an order of magnitude. Also, they depend sensitively on the dynamical details. Perhaps $u_3^0$ is not a member of the $(\bar{3}, 3) + (3, \bar{3})$ multiplet, or contains a piece which is not. If the $u_3$ term is the effect of some neutral gauge bosons, for example, it may not have simple SU(3) × SU(3) transformation properties. This could account for the failure of the prediction on $\Gamma_{+-0}/\Gamma_{000}$ [after (3.3)] based on the

TABLE II. The predicted $\epsilon_3 u_3$ contributions to the $\Delta I = 1$ mass differences: (a) The quantity. (b) Prediction for $B/A = -2.08/\mu_\eta$. (c) Prediction for $B/A = -1.87/\mu_\eta$. (d) Experimental mass difference.

| (a) | (b) | (c) | (d) |
|---|---|---|---|
| $\mu_{K^+}^{\,2} - \mu_{K^0}^{\,2}$ | −0.0104 GeV$^2$ | −0.0084 GeV$^2$ | −0.004 GeV$^2$ |
| $M_p - M_n$ | −7.9 MeV | −6.4 MeV | −1.3 MeV |
| $M_{\Sigma^+} - M_{\Sigma^-}$ | −22.7 MeV | −18.5 MeV | −7.9 MeV |
| $M_{\Xi^-} - M_{\Xi^0}$ | +14.8 MeV | +12.0 MeV | +6.6 MeV |

$I=1$ assumption. Of course, detailed predictions would be difficult to make in such a case.

One could also abandon the $(\overline{3},3)+(3,\overline{3})$ model for all the terms in (3.7). But the (8, 8) model seems no better,[17] and the $(\overline{6},6)+(6,\overline{6})$ model has serious problems of its own.[38]

Perhaps the decay is basically due to the $u_3$ term (or even electromagnetism), but is enhanced by some strong final-state interaction (such as a heavy pion)[39] to which the chiral symmetry arguments are blind.

It would be worthwhile to study (both experimentally and theoretically) the other implications of a $u_3$ term. Possibilities include violations of SU(2) symmetry in $\rho$ decays and in meson-baryon coupling constants.[40] Baĉe[41] has considered the effects on nonleptonic $K$ decays, and Osborn and Wallace[30] have considered the $\pi N$ scattering length.

## IV. CONCLUSION

We have argued that SU(2)× SU(2) is the best symmetry of the strong Hamiltonian after isospin. In Sec. II we observed that SU(3)× SU(3) and SU(3) are comparable and valid to ~30% in most cases. Some applications of chiral perturbation theory were given. We also argued that octet enhancement for matrix elements which vanish in the SU(3) limit could be understood from the Goldstone boson pair mechanism.

In Sec. III we showed that an $\epsilon_3 u_3$ term is of the right order of magnitude to explain the $\eta\to 3\pi$ decay, but the details depend sensitively on the experimental parameters and on the representation content of $u_3$.

Other possible applications of these ideas (in addition to those at the end of Sec. III) might include the decays $\eta'\to\eta\pi\pi$ and $K_s\to 2\pi$, the effects of symmetry breaking on $g_A$ and the meson-baryon couplings, and the extension of these ideas to other symmetry-breaking models.

## APPENDIX A: THE $\eta$-$\eta'$ SYSTEM

In the third section of I we presented a formalism to describe the $\eta$-$\eta'$ system which assumed the existence of an $\eta$-$\eta'$ mixing angle. Such a mixing angle exists only if one drops a certain continuum integral. As no such assumption was made anywhere else in I, our treatment was inconsistent.[42] In this appendix we will correct this inconsistency by giving an exact formalism for the $\eta$-$\eta'$ system.[43]

In the $(\overline{3},3)+(3,\overline{3})$ model there are two unrenormalized fields $\phi_8^0$ and $\phi_0^0$ and two states $|\eta\rangle$ and $|\eta'\rangle$ with $I=Y=0$. We expect that for $\epsilon_8=0$ there will be no mixing effects and that $\phi_8^0$ ($\phi_0^0$) will correspond to the $\eta$ ($\eta'$) state. Furthermore, as $\epsilon_0\to 0$ we expect $\mu_\eta{}^2\to 0$ but $\mu_{\eta'}{}^2\neq 0$: The $\eta$ becomes a Goldstone boson but not the $\eta'$.

Let us define the renormalization constant $Z_{ia}{}^{1/2}\equiv\langle 0|\phi_i^0|a\rangle$ for $i=0$ or 8 and $a=\eta$ or $\eta'$. They are real by $T$ invariance. In the $\epsilon_8=0$ limit, $Z_{0\eta}{}^{1/2}$ and $Z_{8\eta'}{}^{1/2}$ vanish, while $Z_{8\eta}{}^{1/2}=Z_\pi{}^{1/2}=Z_K{}^{1/2}=Z_s{}^{1/2}(\epsilon_0^0)$; $Z_{0\eta'}{}^{1/2}$ approaches some different value. Assuming canonical quantization rules for $\phi_0^0$ and $\phi_8^0$ one can prove formally that

$$\begin{aligned} 0&=Z_{0\eta}{}^{1/2}Z_{8\eta}{}^{1/2}+Z_{0\eta'}{}^{1/2}Z_{8\eta'}{}^{1/2}+\int_{s_{08}}^\infty ds\,\rho_{08}^0(s)\,,\\ 1&=Z_{i\eta}+Z_{i\eta'}+\int_{s_{ii}}^\infty ds\,\rho_{ii}^0(s)\,, \end{aligned}\qquad\text{(A1)}$$

for $i=0$ or 8, where

$$\rho_{ij}^0(s)=\frac{(2\pi)^3}{2}\sum_n\delta^4(q-p_n)\{\langle 0|\phi_i^0|n\rangle\langle n|\phi_j^0|0\rangle+(i\leftrightarrow j)\}\,,\qquad\text{(A2)}$$

with $q^2=s$. Of course, in a real field theory these integrals will in general diverge.

The $\eta$-$\eta'$ mixing effects are incorporated in the $Z_{ia}{}^{1/2}$. An $\eta$-$\eta'$ mixing angle $\theta$ could be *defined* in various ways. For example, it could be defined as the angle which satisfies Eq. (2.8). A theoretically useful definition, however, would be to define "singlet" and "octet" states $|\overline{0}\rangle$ and $|8\rangle$ as a rotation by $\theta$ of the physical states $|\eta'\rangle$ and $|\eta\rangle$ such that

$$\langle 0|\phi_8^0|\overline{0}\rangle=\langle 0|\phi_0^0|8\rangle=0\,.\qquad\text{(A3)}$$

But (A3) implies

$$\tan\theta=\frac{Z_{8\eta'}{}^{1/2}}{Z_{8\eta}{}^{1/2}}=-\frac{Z_{0\eta}{}^{1/2}}{Z_{0\eta'}{}^{1/2}}\,.\qquad\text{(A4)}$$

This last equality is true if and only if the $\rho_{08}^0$ integral in (A1) can be neglected relative to the renormalization constants. However, we shall see that $Z_{0\eta}{}^{1/2}$, $\rho_{08}^0$, and probably $Z_{8\eta'}{}^{1/2}$ all vanish like $\epsilon_8\ln\epsilon_0$. Hence, *one cannot justify dropping the continuum integral in (A1) on the basis of a small* $\epsilon_8$. *A mixing angle, as defined in (A3), can only be justified if the interactions in the symmetric Hamiltonian* $H_0$ *are neglected.*

Now define $f_\eta$ and $f_{\eta'}$ by

$$\langle 0|\partial\cdot A_8|a\rangle=\mu_a{}^2 f_a\,,\qquad\text{(A5)}$$

where $a=\eta$ or $\eta'$. For $\epsilon_8 \to 0$, $f_{\eta'}$ vanishes and $f_\eta = f_\pi = f_K \equiv f_s(\epsilon_0)$. Then using the result[1]

$$\partial \cdot A_8 = -\left[(\tfrac{2}{3})^{1/2}\epsilon_0^0 - \frac{1}{\sqrt{3}}\epsilon_8^0\right]\phi_8^0 - (\tfrac{2}{3})^{1/2}\epsilon_8^0\phi_0^0 \,, \quad \text{(A6)}$$

one has

$$\frac{\mu_a{}^2 f_a}{Z_{8a}{}^{1/2}} = -\left[(\tfrac{2}{3})^{1/2}\epsilon_0^0 - \frac{1}{\sqrt{3}}\epsilon_8^0\right] - (\tfrac{2}{3})^{1/2}\epsilon_8^0 \frac{Z_{0a}{}^{1/2}}{Z_{8a}{}^{1/2}} \quad \text{(A7)}$$

for $a=\eta$ or $\eta'$. The Ward identities (3.11) of I become

$$(\tfrac{2}{3})^{1/2}\langle u_0^0\rangle_0 - \frac{1}{\sqrt{3}}\langle u_8^0\rangle_0 = Z_{8\eta}{}^{1/2}f_\eta + Z_{8\eta'}{}^{1/2}f_{\eta'} - \left[(\tfrac{2}{3})^{1/2}\epsilon_0^0 - \frac{1}{\sqrt{3}}\epsilon_8^0\right]\int_{s_{88}}^\infty ds\,\rho_{88}^0(s)/s - (\tfrac{2}{3})^{1/2}\epsilon_8^0\int_{s_{08}}^\infty ds\,\rho_{08}^0(s)/s \,,$$

$$(\tfrac{2}{3})^{1/2}\langle u_8^0\rangle_0 = Z_{0\eta}{}^{1/2}f_\eta + Z_{0\eta'}{}^{1/2}f_{\eta'} - \left[(\tfrac{2}{3})^{1/2}\epsilon_0^0 - \frac{1}{\sqrt{3}}\epsilon_8^0\right]\int_{s_{08}}^\infty ds\,\rho_{08}^0(s)/s - (\tfrac{2}{3})^{1/2}\epsilon_8^0\int_{s_{00}}^\infty ds\,\rho_{00}^0(s)/s \,. \quad \text{(A8)}$$

Now, from (A7) one has

$$\mu_{\eta'}{}^2 f_{\eta'} = -(\tfrac{2}{3})^{1/2}\epsilon_8^0 Z_{0\eta'}{}^{1/2} + O(\epsilon^2\ln\epsilon)\,, \quad \text{(A9)}$$

which implies that $f_{\eta'}$ vanishes like $\epsilon_8$, not like $\epsilon_8\ln\epsilon_0$. Then, using the fact that the continuum integrals in (A8) have no threshold singularities,[44] we have

$$(\tfrac{2}{3})^{1/2}\langle u_0^0\rangle_0 - \frac{1}{\sqrt{3}}\langle u_8^0\rangle_0 = Z_{8\eta}{}^{1/2}f_\eta + O(\epsilon)\,, \quad \text{(A10a)}$$

$$(\tfrac{2}{3})^{1/2}\langle u_8^0\rangle_0 = Z_{0\eta}{}^{1/2}f_\eta + O(\epsilon_8)\,. \quad \text{(A10b)}$$

Equation (A10a) will be used to determine $Z_{8\eta}{}^{1/2}$ and $f_\eta$; from (A10b) and (3.18) of I we can determine the mixing quantity $Z_{0\eta}{}^{1/2}$ which was needed for the corrections to the Gell-Mann-Okubo formula in Sec. II:

$$\frac{Z_{0\eta}{}^{1/2}}{[Z_s(0)]^{1/2}} = -\left(\frac{2}{3}\right)^{1/2}\frac{5\epsilon_8\ln(\Lambda^2/4\mu^2)}{96\pi^2 f_\pi{}^3} + O(\epsilon_8)\,. \quad \text{(A11)}$$

We would now like to define a pair of suitable interpolating fields to define off-shell Green's function for the $\eta$ and $\eta'$. They are

$$\phi_\eta \equiv (Z_{0\eta'}{}^{1/2}\phi_8^0 - Z_{8\eta'}{}^{1/2}\phi_0^0)/D\,,$$
$$\phi_{\eta'} \equiv (-Z_{0\eta}{}^{1/2}\phi_8^0 + Z_{8\eta}{}^{1/2}\phi_0^0)/D\,, \quad \text{(A12)}$$
$$D \equiv Z_{0\eta'}{}^{1/2}Z_{8\eta}{}^{1/2} - Z_{8\eta'}{}^{1/2}Z_{0\eta}{}^{1/2}\,.$$

In the $\epsilon_8 = 0$ limit, $\phi_\eta = \phi_8^0/Z_{8\eta}{}^{1/2}$, $\phi_{\eta'} = \phi_0^0/Z_{0\eta'}{}^{1/2}$.

The fields are constructed so that

$$\langle 0|\phi_a|b\rangle = \delta_{ab}\,, \quad \text{(A13)}$$

for $(a,b)\epsilon(\eta,\eta')$, and

$$\partial\cdot A_8 = f_\eta\mu_\eta{}^2\phi_\eta + f_{\eta'}\mu_{\eta'}{}^2\phi_{\eta'}\,. \quad \text{(A14)}$$

One can use PCAC techniques for $\phi_\eta$ (or $\phi_8^0$) by writing them in terms of $\partial\cdot A_8$ and $\phi_{\eta'}$ (or $\phi_0^0$) from (A14) and (A6), and then verifying that the $\phi_{\eta'}$ ($\phi_0^0$) pieces vanish to the required order.

Equation (A12) can be inverted to yield

$$\phi_8^0 = Z_{8\eta}{}^{1/2}\phi_\eta + Z_{8\eta'}{}^{1/2}\phi_{\eta'}\,,$$
$$\phi_0^0 = Z_{0\eta}{}^{1/2}\phi_\eta + Z_{0\eta'}{}^{1/2}\phi_{\eta'}\,. \quad \text{(A15)}$$

Since $Z_{0\eta}{}^{1/2}$ and probably also $Z_{8\eta'}{}^{1/2}$ vanish like $\epsilon_8\ln\epsilon_0$, we see that $\rho_{08}^0$ [defined in (A2)] must also vanish like $\epsilon_8\ln\epsilon_0$, as was asserted after (A4).

In analogy to (3.22) of I one can easily derive

$$\langle a|u_l^0(0)|a\rangle = \frac{d\mu_a{}^2}{d\epsilon_l^0} \quad \text{(A16)}$$

for $l=0$ or 8, $a=\eta$ or $\eta'$, with $u_l^0$ carrying zero momentum. Also,

$$\langle\eta'(k^2)|u_l^0|\eta(k^2)\rangle = 0\,, \quad \text{(A17)}$$

where $u_l^0$ carries no momentum and $k^2 = \mu_\eta{}^2$ *or* $\mu_{\eta'}{}^2$.

Finally, as in (3.23) of I one can show

$$\frac{d}{d\epsilon_l^0}(Z_{ia}{}^{1/2}Z_{ja}{}^{1/2}) = -\frac{d}{dk^2}\left[(k^2-\mu_a{}^2)^2\int d^4x\,d^4y\,e^{ik\cdot(x-y)}\langle 0|T(\phi_i^0(x)\phi_j^0(y)u_l^0(0))|0\rangle\right]\Bigg|_{k^2=\mu_a{}^2}\,. \quad \text{(A18)}$$

From (A18) one can derive various results, such as

$$D_Z^l = \frac{1}{2}\frac{d}{dk^2}\langle\eta(k^2)|u_l^0(0)|\eta(k^2)\rangle|_{k^2=\mu_\eta{}^2}\,, \quad \text{(A19)}$$

where

$$D_Z^l = \frac{1}{D}\left(Z_{0\eta'}{}^{1/2}\frac{dZ_{8\eta}{}^{1/2}}{d\epsilon_l^0} - Z_{8\eta'}{}^{1/2}\frac{dZ_{0\eta}{}^{1/2}}{d\epsilon_l^0}\right)$$
$$= \frac{1}{Z_{8\eta}{}^{1/2}}\frac{dZ_{8\eta}{}^{1/2}}{d\epsilon_l^0} + O(\epsilon\ln^2\epsilon) \quad \text{(A20)}$$

[$D$ is defined in (A12)].

From Eqs. (A10a), (A 19), and (A20) one can use the techniques described in I to show

$$\frac{f_i}{f_s(0)} - 1 = -\frac{\ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^{\,3}}(\sqrt{6}\,\epsilon_0 + \tfrac{3}{2}d_i\epsilon_8) + O(\epsilon)\,, \tag{A21}$$

$$\frac{Z_i^{\,1/2}}{Z_i^{\,1/2}(0)} - 1 = -\frac{\ln(\Lambda^2/4\mu^2)}{192\pi^2 f_\pi^{\,3}}[14(\tfrac{2}{3})^{1/2}\epsilon_0 + d_i\epsilon_8] + O(\epsilon)\,,$$

for $i=1,\ldots,8$. By $f_8(Z_8^{\,1/2})$ we mean $f_\eta(Z_{8\eta}^{\,1/2})$; $d_i$ is the Gell-Mann $d$ symbol $d_{ii8}$.

If $T(K^2;\epsilon_0,\epsilon_8)$ is a Green's function involving an $\eta$ field carrying momentum $K$, then [as in (3.30) and (3.31) of I] the derivative of the on-mass-shell matrix element $T(\mu_\eta^{\,2};\epsilon_0,\epsilon_8)$ is

$$\begin{aligned}\frac{dT(\mu_\eta^{\,2};\epsilon_0,\epsilon_8)}{d\epsilon_l^0} &= -iT^{s}_{u_l^0}(\mu_\eta^{\,2};\epsilon_0,\epsilon_8)\\ &= -iT^{I}_{u_l^0}(\mu_\eta^{\,2};\epsilon_0,\epsilon_8) + D_Z^l\,T(\mu_\eta^{\,2};\epsilon_0,\epsilon_8)\\ &\quad + \frac{d\mu_\eta^{\,2}}{d\epsilon_l^0}\,T_\eta(\mu_\eta^{\,2};\epsilon_0,\epsilon_8)\,,\end{aligned} \tag{A22}$$

where $D_Z^l$ is defined in (A20). $T^{s}_{u_l^0}$ and $T^{I}_{u_l^0}$ are the matrix element $T$ with a $u_l^0$ operator inserted everywhere except on the external legs. The technical difference between the two is defined in I. Both still contain $\eta'$ poles in the $(\eta, u_l^0)$ channel. The quantity $T_\eta$ is defined as

$$T_\eta(k^2;\epsilon_0,\epsilon_8) \equiv \frac{T(k^2;\epsilon_0,\epsilon_8) - T(\mu_\eta^{\,2};\epsilon_0,\epsilon_8)}{k^2-\mu_\eta^{\,2}}\,. \tag{A23}$$

At $K^2=\mu_\eta^{\,2}$, $T_\eta$ is just $dT/dK^2$. Of course, one must also add the counterterms associated with other external legs to (A22).

As in (3.32) of I, one can approximate (A22) by a formula in which the quantities on the right are evaluated at $K^2=0$ [the $\eta'$ poles in the $(\eta, u_l^0)$ channel present no difficulty]. The result is

$$\begin{aligned}\frac{dT(\mu_\eta^{\,2};\epsilon_0,\epsilon_8)}{d\epsilon_l^0} &= -iT_{u_l^0}(0;\epsilon_0,\epsilon_8)\\ &\quad+\left(\frac{1}{\mu_\eta^{\,2}}\frac{d\mu_\eta^{\,2}}{d\epsilon_l^0} - D_Z^l\right)T(0;\epsilon_0,\epsilon_8)\\ &\quad+\frac{d\mu_\eta^{\,2}}{d\epsilon_l^0}\,T_\eta(0;\epsilon_0,\epsilon_8) + O(\text{constant})\,,\end{aligned} \tag{A24}$$

where $T_{u_l^0}$ includes $u_l^0$ insertions everywhere, even on the external leg. One may usually[45] set $\epsilon_8=0$ on the right, inducing an error of $O(\epsilon_8\ln\epsilon_0)$.

The error in (A24) would be $O(\epsilon\ln\epsilon)$ if $T_\eta$ were evaluated at $\mu_\eta^{\,2}$. If there are "dependent variable singularities" (see Sec. II C of I) the error can be $O(1)$ when $T_\eta$ is evaluated at $K^2=0$. This presents no difficulty in practice since we are usually only interested in the $\ln\epsilon_0$ part of the derivative. The same comments apply to Eqs. (2.71) and (3.32) of I.

## APPENDIX B: CORRECTIONS TO THE SU(3)×SU(3) CALCULATION OF THE $\eta\to3\pi$ AMPLITUDE

In this appendix we will sketch the calculation of the leading ($\epsilon\ln\epsilon$) corrections to the result (3.13) for $A$. We will stay on the line $\epsilon_8^0=-\sqrt{2}\epsilon_0^0$ of exact SU(2)× SU(2) symmetry.

We will work with the expression (3.10) for $A(\epsilon_0^0)=G(\mu_\eta^{\,2},\epsilon_0^0)$. Since $f_\pi$ is an explicit factor in (3.10) there is no need to expand it around $\epsilon_0=0$. From (3.10) and (A24) we find

$$\begin{aligned}\frac{1}{f_\pi}\frac{d}{d\epsilon_0^0}[f_\pi G(\mu_\eta^{\,2},\epsilon_0^0)] &= -iG_{u_0^0-\sqrt{2}u_8^0}(0,\epsilon_0^0) - \frac{2(\frac{2}{3})^{1/2}Z_{8\eta}^{\,1/2}}{\mu_\eta^{\,2}f_\eta}G(0,\epsilon_0^0)\\ &\quad+G(0,\epsilon_0^0)\left[-\frac{2}{Z_\pi^{\,1/2}}\frac{dZ_\pi^{\,1/2}}{d\epsilon_0^0} - \frac{1}{f_\eta}\frac{df_\eta}{d\epsilon_0^0} + \frac{2}{\sqrt{3}\,\mu_\eta^{\,2}}\frac{d}{d\epsilon_0^0}\left(\frac{\epsilon_0^0 Z_{0\eta}^{\,1/2}}{f_\eta}\right)\right]\\ &\quad-\frac{2(\frac{2}{3})^{1/2}Z_{8\eta}^{\,1/2}}{f_\eta}\frac{d}{dp^2}G(p^2,\epsilon_0^0)\bigg|_{p^2=\mu_\eta^2} + O(\epsilon\ln\epsilon)\,.\end{aligned} \tag{B1}$$

The interpretation of (B1) is that $-iG_{u_0^0-\sqrt{2}u_8^0}$ represents the $\epsilon_0$ dependence of the internal dynamics. The $G/\mu_\eta^{\,2}$ comes from differentiating the Klein-Gordon operator; it cancels the $u_0^0-\sqrt{2}u_8^0$ insertions on the external $\eta$ leg in the first term. The $G(0,\epsilon_0^0)$ [ ] term represents renormalization effects, and the last term is due to the variation of the mass of the external $\eta$ with $\epsilon$.

In (B1) we have chosen to evaluate $dG/dp^2$ at $\mu_\eta^{\,2}$ rather than zero. We have rewritten the

$[G(0, \epsilon_0^0) d\mu_\eta^2/d\epsilon_0^0]/\mu_\eta^2$ term of (A24) using the expression (A7) for $\mu_\eta^2$. The $dZ_\pi^{1/2}/d\epsilon_0^0$ term in (B1) is the counterterm associated with the external pions. [This is clear from (2.46) of I since $d\mu_\pi^2/d\epsilon_0^0 = 0$.]

Using (A11) and (A21) one easily finds that $\epsilon_0^0$ times the bracketed counterterms in (B1) is

$$\epsilon_0^0 G(0, \epsilon_0^0)[\ \ ] = \frac{11}{2}\left(\frac{2}{3}\right)^{1/2} \frac{\epsilon_0 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^{\ 3}} A(0) + O(\epsilon_0)\ . \tag{(B2)}$$

To calculate the first term in (B1) we use the exact SU(2)×SU(2) expression

$$\begin{aligned} -i\, G_{u_0^0 - \sqrt{2} u_8^0}(0, \epsilon_0^0) &= \frac{\epsilon_3^0 \mu_\eta^{\ 2}}{f_\pi} \int d^4x\, d^4z \left\langle + - \left| T\left( [u_0^0(z) - \sqrt{2}\, u_8^0(z)] \phi_\eta(x) \left[ (\tfrac{2}{3})^{1/2} \phi_0^0(0) + \frac{1}{\sqrt{3}} \phi_8^0(0) \right] \right) \right| 0 \right\rangle \\ &= -\frac{\epsilon_3^0 \mu_\eta^{\ 2}}{f_\pi^{\ 3}} \int d^4x\, d^4z \left\langle 0 \left| \left[ {}^5F_{\pi^+}, \left[ {}^5F_{\pi^-}, T\left\{ (u_0^0 - \sqrt{2}\, u_8^0) \phi_\eta \left[ \sqrt{(\tfrac{2}{3})} \phi_0^0 + \frac{1}{\sqrt{3}} \phi_8^0 \right] \right\} \right] \right] \right| 0 \right\rangle , \end{aligned} \tag{B3}$$

where, just as in (3.11), we mean take all the equal-time commutators. For the second term in (B1) we use (3.11). We now compute all the equal-time commutators for these two terms [using the expression[46] (A12) for $\phi_\eta$], utilize Eqs.[46] (A6) and (A14), and then drop all terms which vanish or go to constants as $\epsilon_0 \to 0$. The result is that the first two terms in (B1) are

$$\begin{aligned} \frac{i\epsilon_3^0}{f_\pi^{\ 4}} \int d^4z\, [ & -\tfrac{4}{3}\sqrt{2} \langle T(u_{\pi^+}^0(z) u_{\pi^-}^0(0)) \rangle_0 \\ & + \tfrac{4}{9}\sqrt{2} \langle T(u_0^0(z)[u_0^0(0) - \sqrt{2}\, u_8^0(0)]) | 0 \rangle_0 \\ & + \tfrac{14}{9} \langle T(u_8^0(z)[u_0^0(0) - \sqrt{2}\, u_8^0(0)]) | 0 \rangle_0 ] \\ & + O(\text{constant})\ . \end{aligned} \tag{B4}$$

Each of these terms diverges like $\epsilon_0$ due to the two-meson cut (none have a two-pion cut). The divergent part is calculated as in (3.15) of I to give (when multiplied by $\epsilon_0^0$) the same expression as in (B2).

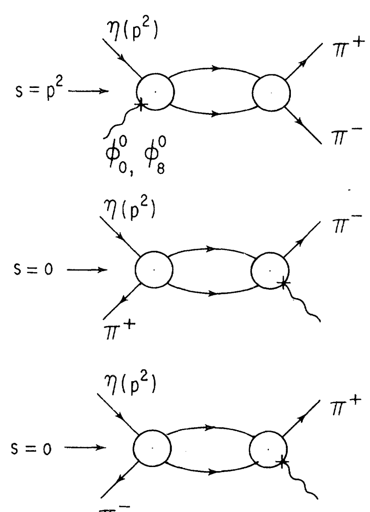


FIG. 1. The three diagrams contributing to the singular part of $dG/dp^2$ in Eq. (B1).

Finally, the $dG/dp^2$ term in (B1) has singularities due to the three diagrams in Fig. 1. In the first diagram the invariant mass of the cut is of the dependent-variable type (see I). The contribution of these diagrams (when multiplied by $\epsilon_0^0$) is

$$\begin{aligned} & -\frac{2(\frac{2}{3})^{1/2} Z_{8\eta}^{\ 1/2} \epsilon_0^0}{f_\eta} \frac{d}{dp^2} G(p^2, \epsilon_0^0) \Big|_{p^2 = \mu_\eta^2} \\ & = -\frac{83}{16}\left(\frac{2}{3}\right)^{1/2} \frac{\epsilon_0 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^{\ 3}} A(0) + O(\epsilon)\ . \end{aligned} \tag{B5}$$

(The first diagram contributes $\frac{75}{16}$, and the last two $\frac{1}{2}$.)

Putting the various pieces together, we have

$$A(\epsilon_0) = -\frac{4\epsilon_3}{3\sqrt{3} f_\pi(\epsilon) f_\pi^{\ 2}(0)} \left[ 1 + \frac{93}{16}\left(\frac{2}{3}\right)^{1/2} \frac{\epsilon_0 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^{\ 3}} + O(\epsilon_0) \right]. \tag{B6}$$

Part of the $\epsilon \ln\epsilon$ correction can be absorbed into the renormalization of $f_\pi(0)$ by the symmetry-breaking interaction [see (A21)] to give[47]

$$A(\epsilon_0) = -\frac{4\epsilon_3}{3\sqrt{3} f_\pi^{\ 3}(\epsilon)} \left[ 1 + \frac{45}{16}\left(\frac{2}{3}\right)^{1/2} \frac{\epsilon_0 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^{\ 3}} + O(\epsilon_0) \right], \tag{B7}$$

where $f_\pi(\epsilon)$ is the physical (renormalized) value of $f_\pi$. For $\ln(\Lambda^2/4\mu^2) = 2$, the $\epsilon \ln\epsilon$ term is $\approx -0.34$.

We conclude by commenting that the relatively small (34%) corrections in (B7) are due to the cancellation of much larger factors. We do not consider this to be a lucky accident. The different terms in (B1) merely single out for separate consideration different aspects of the same Feynman diagrams (e.g., the $\epsilon$ dependence of the masses of the internal and external lines, of the vertices, etc.). Their near cancellation is not accidental; it simply means that the relevant graphs are small.

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[32]Using PCAC to evaluate $T_{+-0}$ at the off-shell point at which all four mesons have zero momentum, one obtains an approximate result for $A$ which is $1+\mu_\pi{}^2 f_\pi Z_{8\eta}{}^{1/2}/(4\mu_\eta{}^2 f_\eta Z_\pi{}^{1/2})$ times the result in (3.13).

[33]P. Langacker and H. Pagels, Phys. Rev. D 9, 3413 (1974).

[34]The chiral-limit techniques of Dashen and Weinstein (Refs. 6 and 7) would yield $\frac{3}{4}$ of (3.13). The problem is that the $\varphi_8^0$ part of (3.9) is proportional to the $\eta\eta\to\pi^+\pi^-$ amplitude in the chiral limit, except one of the Klein-Gordon operators is missing. Hence, it is of the ambiguous form $\infty\times 0$.

[35]The Hamiltonian (1.1) can be rewritten as $H_1+\epsilon^0\sigma^0$, where $H_1$ is $SU(2)\times SU(2)$-symmetric, $\sigma^0\equiv(\frac{2}{3})^{1/2}u_0^0+(\frac{1}{3})^{1/2}u_8^0$, and $\epsilon^0=\epsilon/Z_\pi{}^{1/2}=(\frac{2}{3})^{1/2}\epsilon_0^0+(\frac{1}{3})^{1/2}\epsilon_8^0$.

[36]As the authors of Ref. 30 have remarked, the $I=1$ mass differences should rule out the scheme of R. J. Oakes [Phys. Lett. 29B, 683 (1969)], which predicts an even larger value for $\epsilon_3$ than (3.15).

[37]We do not think the results are very sensitive to the linear matrix element assumption (3.1).

[38]P. R. Auvil, Phys. Rev. D 6, 3209 (1972).

[39]The usefulness of a heavy pion in other contexts has been suggested by C. Michael, Phys. Rev. 166, 1826 (1967); H. Pagels and A. Zepeda, Phys. Rev. D 5, 3262 (1972); and S. D. Drell, *ibid.* 7, 2190 (1973).

[40]We are indebted to G. Feinberg for suggesting these possibilities.

[41]M. Bace, Phys. Rev. D 4, 2838 (1971).

[42]The calculations in Ref. 21 were correct to the stated order.

[43]Some of the results in this appendix not requiring chiral perturbation theory have previously been given by P. R. Auvil and N. G. Deshpande [Phys. Rev. 183, 1463 (1969), and Phys. Lett. 49B, 73 (1974)].

[44]H. Pagels and A. Zepeda, Phys. Rev. D 5, 3262 (1972).

[45]In the examples checked there has never been any problem in setting $\epsilon_8=0$ at this point when dealing with $\epsilon\ln\epsilon$ singularities.

[46]The $\eta$-$\eta'$ mixing effects are incorporated into these equations.

[47]If in Ref. 21 we had written the zero-order contribution to $\mu_{K^+}{}^2-\mu_{K^0}{}^2$ in terms of the physical $f_K{}^{-2}(\epsilon)$ instead of $f_s{}^{-2}(0)$, the remaining corrections would have been *larger*: 200% rather than 130%.