PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Eigenvalue conditions and asymptotic freedom for Higgs-scalar gauge theories

Ngee-Pong Chang*
Physics Department, City College of the City University of New York, New York, New York 10031
and College of General Education,† University of Tokyo (Komaba), Tokyo, Japan
and Research Institute for Fundamental Physics,† Kyoto University, Kyoto, Japan
(Received 23 May 1974)

Eigenvalue conditions are obtained from a study of the renormalization-group equations for a non-Abelian gauge theory with Higgs scalars. With these conditions, it is found that the theory is asymptotically free. For the purely leptonic SO(3) model of Georgi and Glashow, the eigenvalue conditions fix completely the parameters of the theory.

It has become a common belief very recently that spontaneously broken gauge theories, with Higgs scalars, are not asymptotically free.[1,2] In the work of Gross and Wilczek,[1] an initial investigation of this problem was made, without, however, considering in detail the effect of Yukawa couplings which generate fermion masses. In a subsequent paper by Cheng, Eichten, and Li,[3] the full problem was discussed and the possibility of asymptotic freedom was quickly dismissed. In this note we wish to point out a simple eigenvalue condition that was not considered in their investigation, and, with it, we show that asymptotic freedom is restored for the Georgi-Glashow–type gauge theories of weak and electromagnetic interactions.[4] Theories of the Weinberg-Salam type,[5] involving mixing with an Abelian gauge group, are not asymptotically free.

For simplicity, we have considered the Georgi-Glashow model with only leptons present. Let h_1, h_2 be the Yukawa coupling constants such that the lowest-order masses for e^-, E^+, and X_0 are $m_0 - h_1 v$, $m_0 + h_1 v$, and $+h_2 v/\sin\theta$, respectively (θ is the ν_e, X mixing angle in the model, while v is the vacuum expectation of the neutral component of the Higgs scalar field). For the muon system, corresponding coupling constants are denoted by H_1, H_2. Let λ be the quartic self-interaction coupling constant for the Higgs scalar field $[\mathcal{L}_\phi \propto -(\lambda/4!)\,\phi_0^{\,4} + \cdots]$. Then the lowest-order coupled equations for the effective coupling constants read in the usual notation[1,6,7]

$$16\pi^2 \frac{dg^2}{dt} = -[\tfrac{44}{3} - \tfrac{16}{3}(2) - \tfrac{2}{3}(1)]g^4 \equiv -\beta_0 g^4\,, \tag{1a}$$

$$16\pi^2 \frac{dh_1^{\,2}}{dt} = 16h_1^{\,4} + 4h_1^{\,2}(2H_1^{\,2} + H_2^{\,2}) + h_1^{\,2}h_2^{\,2} - 24g^2h_1^{\,2}\,, \tag{1b}$$

$$16\pi^2 \frac{dh_2^{\,2}}{dt} = 12h_2^{\,4} + 4h_2^{\,2}(2H_1^{\,2} + H_2^{\,2}) + 2h_1^{\,2}h_2^{\,2} - 12g^2h_2^{\,2}\,, \tag{1c}$$

$$16\pi^2 \frac{d\theta}{dt} = 0\,, \tag{1d}$$

$$16\pi^2 \frac{d\Theta}{dt} = 0\,, \tag{1e}$$

$$16\pi^2 \frac{d\lambda}{dt} = \tfrac{11}{3}\lambda^2 + (16h_1^{\,2} + 8h_2^{\,2} + 16H_1^{\,2} + 8H_2^{\,2} - 24g^2)\lambda + 72g^4 - 96(h_1^{\,4} + H_1^{\,4}) - 48(h_2^{\,4} + H_2^{\,4})\,. \tag{1f}$$

The equation for $H_1^{\,2}$ and $H_2^{\,2}$ can, by μ-e symmetry, be obtained from Eqs. (1b) and (1c).

For clarity, let us suppose first that the h_2, H_1, H_2 couplings are absent in the theory and examine the prototype equation

$$16\pi^2 \frac{dh^2}{dt} = Ah^4 - Bg^2h^2\,, \tag{2}$$

with A, B strictly positive constants. Call $\bar{h}^2 \equiv h^2/g^2$ and $u(t) \equiv \int_0^t d\tau g^2(\tau)$; then Eq. (2) reduces to

$$16\pi^2 \frac{d\bar{h}^2}{du} = \bar{h}^2[A\bar{h}^2 - (B - \beta_0)]\,. \tag{3}$$

As Cheng, Eichten, and Li[3] have pointed out, the critical point $\bar{h}^2 = 0$ is ultraviolet-stable, for $B > \beta_0$. This solution has $\bar{h}^2$ vanishing as $t^{-\nu}$, $\nu = (B - \beta_0)/\beta_0$. Since this implies that h^2 vanishes faster than g^2, in the asymptotic domain, it is the same situation as that considered by Gross and Wilczek,[1] and a largely pessimistic conclusion results.

However, there *exists* a solution to Eq. (3) which is identically satisfied, viz., when $\bar{h}^2$ is a constant, $\kappa \equiv (B - \beta_0)/A$. It is a trivial solution to Eq. (3); however, by virtue of the fact that the proportionality between h^2 and g^2 holds for all t, it is an *eigenvalue* condition that must be imposed on the renormalized theory. That is to say, if the renormalized Yukawa coupling constant $[h^2(0)]$ is chosen initially to be an arbitrary value smaller than $\kappa g^2(0)$, then in the deep Euclidean region $h^2(t)$ vanishes faster than $g^2(t)$. However, if the initial value assigned to $h^2(0)$ is exactly equal to $\kappa g^2(0)$, then in the deep Euclidean region $h^2(t)$ is always proportional to $g^2(t)$. This behavior is absolutely essential to make the quartic coupling $\lambda(t)$ asymptotically free.

We turn now to the coupled set of equations in (1b) and (1c). To look for an eigenvalue solution that is asymptotically free, we make the substitutions $h_1^2=\kappa_1 g^2$, $h_2^2=\kappa_2 g^2$, $H_1^2=K_1 g^2$, $H_2^2=K_2 g^2$, and look for the allowed nonvanishing values for the constants for κ_1, κ_2, K_1, K_2.[8] The solution is

$$\kappa_1 = K_1 = (324-11\beta_0)/334\,, \tag{4}$$

$$\kappa_2 = K_2 = (24-7\beta_0)/167\,. \tag{5}$$

Since, in a purely leptonic world, $\beta_0=\frac{10}{3}$, the solution here is physical (for κ_2 to be >0, β_0 has to be $<\frac{24}{7}$).

To check that the quartic Higgs scalar self-coupling is asymptotically free, substitute $\lambda=\Lambda g^2$ in Eq. (1f) and find that a positive root for Λ exists in the condition[9]

$$\tfrac{11}{3}\Lambda^2+(32\kappa_1+16\kappa_2-24+\beta_0)\Lambda-(192\kappa_1^2+96\kappa_2-72)=0\,. \tag{6}$$

This is guaranteed so long as $192\kappa_1^2+96\kappa_2^2>72$, which is amply satisfied in a pure leptonic world.

The discussion thus far has been confined to the set of lowest-order coupled equations for the various coupling constants. Higher-order corrections to Eqs. (1) result in a correction to the eigenvalue condition of the form

$$h^2=\kappa^{(0)}g^2+\kappa^{(1)}g^4+\kappa^{(2)}g^6+\cdots\,, \tag{7}$$

the coefficients of which can obviously be determined in an iterative way. At $t=0$, the initial value of h^2 is thus also modified; however, for theories of weak and electromagnetic interactions such corrections to $h^2(0)$ are of higher order in $g^2(0)/4\pi(\cong\frac{1}{137})$.

At this point we make a remark concerning the analyticity of the solutions we found. Solutions given by Eqs. (4)–(7) are regular at $g^2=0$ in the complex g^2 plane, and are fully consistent with the perturbative expansion of the effective coupling constants with respect to g^2. To be complete, we should mention that there exist other solutions to Eqs. (1) which are asymptotically free. These are solutions for which h_1^2/g^2 approaches constants, while h_2^2/g^2 vanishes with a power of g^2 when $g^2\to 0$. That the solution is asymptotically free follows from the fact that the nonvanishing h_1^2/g^2 alone was sufficient to make λ asymptotically free.[8] This solution, however, is not analytic in g^2 at the point $g^2=0$. This mild criticism aside, the real relative shortcoming of this solution is that the theory is still "arbitrary." $h_2^2(0)$ is not determined by the theory. Therefore we have relegated a discussion of this solution to the Appendix.

Finally, we list the physical mass constraints which result from the analytic solution of the renormalization group equations for a purely leptonic Georgi-Glashow model.[4] In this model, μ-e symmetry is strictly observed, so we use as input an "average" mass for $m_e=m_\mu=0.053$ GeV. This, together with the identification $g(0)=e$ electromagnetic coupling constant, leads immediately to the following results:

$$m_X=m_Y=3.34\ \text{GeV}/c^2\,,$$

$$m_W=3.58\ \text{GeV}/c^2\,,$$

$$\theta\cong 0.0669\left(1-\frac{0.001}{0.97}\right),$$

$$m_{E^+}=m_{M^+}=6.64\ \text{GeV}/c^2\,,$$

$$\lambda(0)\simeq 3.52g^2(0)\,.$$

It is surprising that all the particles turn out to be unexpectedly light, with the charged heavy leptons, however, constrained to be nearly double the vector-boson mass. Of course since we cannot in this model include hadronic quarks the mass predictions here cannot be taken too seriously.

In conclusion, it is clear that there exist unified weak and electromagnetic gauge theories that are asymptotically free. It is gratifying that analytic solutions exist which are regular at $g^2=0$ (therefore rigorously allowing for a perturbative expansion in g^2 for all effective coupling constants), and that these solutions fix uniquely the heretofore arbitrary mass and mixing angle parameters that are present in the Georgi-Glashow model. While asymptotic freedom, per se, is not essential to the "success" of weak and electromagnetic gauge theories (they are already convergent), it is hoped that with the help of the eigenvalue conditions obtained from the renormalization group equations they can finally be "calculable" and compared against experiment.

It is a pleasure to acknowledge the warm hospitality of Professor Kawarabayashi and his colleagues at the College of General Education, University of Tokyo (Komaba) where most of this work was done, and to thank K. Shizuya and H. Wada for discussions. We would also like to thank Professor Maki for the hospitality at the Research Institute for Fundamental Physics, Kyoto University.

APPENDIX

Solutions exist to the lowest-order coupled equations (1) that are not regular at $g^2=0$ in the complex g^2 plane. We indicate in this appendix the properties of this solution.

We first rewrite the equations for h_1, h_2 as ($\tau \equiv 1/g^2$)

$$\beta_0 \frac{dh_1^2}{d\tau} = 24h_1^4 + \left(5h_2^2 - \frac{24}{\tau}\right)h_1^2, \tag{A1a}$$

$$\beta_0 \frac{dh_2^2}{d\tau} = 16h_2^4 + \left(10h_1^2 - \frac{12}{\tau}\right)h_2^2, \tag{A1b}$$

and note that they are of the Bernoulli type,

$$\frac{dy}{d\tau} = a(\tau)y + b(\tau)y^2, \tag{A2}$$

whose solution is given by

$$\begin{aligned} y^{-1} &= Y_1 + Y_2, \\ Y_1 &= Ce^{\phi}, \quad \phi = -\int_{\tau_0}^{\tau} a(x)dx, \\ Y_2 &= -e^{\phi}\int_{\tau_0}^{\tau} e^{-\phi(x)} b(x)dx. \end{aligned} \tag{A3}$$

We look for a solution for which h_1^2/g^2 approaches a constant as $g^2 \to 0$, while h_2^2/g^2 vanishes in the same limit [i.e., solution (I), Ref. 8]. From Eq. (A1a) it is easy to find that

$$h_1^2\tau \xrightarrow[\tau\to\infty]{} (1 - \tfrac{1}{24}\beta_0) + o(1). \tag{A4}$$

The solution for h_2^2 can, in general, be written as

$$[h_2^2]^{-1} = e^{-\phi_2}\left(\frac{\tau}{\tau_0}\right)^{\alpha}\left[C_2 - \frac{16}{\beta_0}\int_{\tau_0}^{\tau} e^{\phi_2(x)}\left(\frac{\tau_0}{x}\right)^{\alpha} dx\right], \tag{A5}$$

where we have parametrized h_1^2 as $(1 - \frac{1}{24}\beta_0)g^2 + z$, and

$$\phi_2 \equiv \frac{10}{\beta_0}\int_{\tau_0}^{\tau} z(x)dx, \quad \alpha = \tfrac{5}{12} + 2/\beta_0. \tag{A6}$$

$z(\tau)$ vanishes faster than $1/\tau$ as $\tau \to \infty$. In order that the solution (A5) be physical, the constant C_2 must obey the constraint ($\alpha = \frac{61}{60}$ in this model)

$$C_2 > \frac{16}{\beta_0}\int_{\tau_0}^{\infty} e^{\phi_2(x)}(g_0^2 x)^{-\alpha} dx \tag{A7}$$

in an iterative solution to the coupled set of equations we have in the zeroth-order solution $\phi_2 = 0$, and therefore

$$C_2^{(0)} > \frac{16}{\beta_0}\frac{1}{\alpha - 1} g_0^{-2}. \tag{A8}$$

This in turn implies that the renormalized coupling constant $h_2^2(0)$ must, to lowest order, satisfy the constraint

$$h_2^2(0) < \beta_0(\alpha - 1)g_0^2/16. \tag{A9}$$

Going back to the equation for h_1^2, the solution for h_1^2 including the first iteration now reads

$$[h_1^2]^{-1} = (\tau)^{24/\beta_0} e^{-\phi_1}\int_{\tau}^{\infty} dx\, e^{\phi_1}(x)^{-24/\beta_0}\frac{24}{\beta_0}, \tag{A10}$$

$$\phi_1 = \frac{5}{\beta_0}\int_{\tau_0}^{\tau} h_2^2(x)dx. \tag{A11}$$

If we call the asymptotic behavior of $h_2^2 \to \kappa(g^2)^{\alpha}$ as $\tau \to \infty$, then the correction to the asymptotic behavior of h_1^2 reads ($\kappa_1 \equiv 1 - \beta_0/24$)

$$[h_1^2]^{-1} \underset{g^2\to 0}{\sim} \kappa_1^{-1}g^{-2} + \frac{5\kappa}{\kappa_1}(g^2)^{\alpha-2}/(24 + (\alpha - 2)\beta_0). \tag{A12}$$

As far as positivity of the solutions is concerned, so long as g^2 is real, the solutions (A5) and (A10) remain positive definite, and insofar as they vanish as $g^2 \to 0$ are asymptotically free. It is easy to check that the solution for λ, because $h_1^2/g^2 \to (1 - \frac{1}{24}\beta_0)$, is also well behaved as $g^2 \to 0$, remaining positive. In the earlier analysis of Cheng, Eichten, and Li both h_1^2 and h_2^2 vanished faster than g^2 and in the asymptotic domain did not provide the damping needed against the positive $72g^4$ term in Eq. (1f).

From Eq. (A10) the eigenvalue condition for $h_1^2(0)$ is in principle obtainable. In the presence of h_2^2, however, it is not obtainable in a simple closed form.

So far we have discussed only solution (I) mentioned in Ref. 8. The situation for solutions (II) and (III) and their $\mu \leftrightarrow e$ images is entirely analogous.

Finally, a question can be raised on whether the higher-order corrections to Eqs. (1) will destroy the asymptotic freedom of the solutions (A5) and (A10). Intuitively speaking, since the lowest-order solutions already vanish in the asymptotic domain, the higher-order terms, at least in an iterative solution, are clearly much smaller, in the same asymptotic domain. This expectation is borne out in Levinson's theorem, but we refer the interested reader to the mathematical literature for reference.[10]

*On Sabbatical leave from the City College of New York, Spring, 1974.

†Visiting Professor under the sponsorship of the Japan Society for the Promotion of Science.

[1]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); Phys. Rev. D 8, 3633 (1973).

[2]H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).

[3]T. P. Cheng, E. Eichten, and L.-F. Li, Phys. Rev. D 9, 2259 (1974).

[4]H. Georgi and S. Glashow, Phys. Rev. Lett. 28, 1494 (1972).

[5]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8)*, edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968).

[6]S. Coleman, in *Properties of the Fundamental Interactions*, proceedings of the 1971 International Summer School "Ettore Majorana," edited by A. Zichichi (Editrice Compositori, Bologna, 1973), p. 359.

[7]The equations for Yukawa coupling and quartic boson self-couplings were first studied by A. Zee in the absence of gauge couplings: A. Zee, Phys. Rev. D 7, 3630 (1973).

[8]There exist solutions with zero eigenvalues:

(I) $\kappa_2 = K_2 = 0$, $\kappa_1 = K_1 = 1 - \beta_0/24$

(II) $\kappa_2 = 0$, $\kappa_1 = \frac{111}{118} - 23\beta_0/944$, $K_1 = (240 - 11\beta_0)/236$, $K_2 = (24 - 7\beta_0)/118$

(III) $\kappa_1 = 0$, $\kappa_2 = (4032 - 413\beta_0)/1491$, $K_1 = (2268 - 77\beta_0)/1491$, $K_2 = (180 - 14\beta_0)/213$ plus $\mu \longleftrightarrow e$ mirror image of solutions (II and III).

Zero *eigenvalues* are not acceptable for the Georgi-Glashow model [e.g. (III) requires e^- to be degenerate with E^+]. However, they can be used as a zeroth approximation in an iterative solution of the type discussed in the appendix. (II) and (III) are obviously $m_\mu \neq m_e$ solutions.

[9]In Eq. (2.8) of Ref. 3, the coefficient for H_{ijkl} should be -48 instead of -12.

[10]See, e.g., S. Lefschetz, *Differential Equations: Geometric Theory* (Interscience, New York, 1957), Chap. V.

PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Spin-statistics connection for dyonium

A. O. Barut*
International Centre for Theoretical Physics, Trieste, Italy

(Received 3 August 1973)

The spin-$\frac{1}{2}$ bound state of two spinless dyons (dyonium) has additional *independent* (gauge) degrees of freedom associated with an arbitrary unit vector $\hat{n}$ in the Hamiltonian. Hence it does not constitute a counterexample of the normal spin-statistics connection.

There has been a recent assertion in this journal[1] that the spin-$\frac{1}{2}$ bound state of two spinless dyons (particles with both electric and magnetic charges), called *dyonium*, constitutes a counterexample to the normal spin-statistics connection, i.e., it is a boson, or the spin-zero constituent dyons must be fermions. This assertion is based on the assumption that the configuration space of two dyons in dyonium is $R_3 \times R_3$, hence the wave function of the dyonium is a square integrable function in the space $L^2(R_3) \times L^2(R_3)$. This would be the case for an ordinary atom of two bosons. But due to the singular structure of the vector potential along a string for Dirac monopoles, I wish to show, both mathematically and physically, that dyonium has additional degrees of freedom. Consequently, the exchange of particle coordinates *alone* is not the complete exchange operation of the system as a whole.

Consider first the exactly soluble case where one of the dyons is very heavy. The nonrelativistic motion of a dyon in the field of another fixed dyon is governed by the Hamiltonian

$$H = \frac{1}{2m}[\vec{p} - \mu\vec{D}(\vec{n}, \vec{r})]^2 - \frac{\alpha}{r}, \qquad (1)$$

where $\mu = e_1 g_2 - e_2 g_1$ and $\alpha = e_1 e_2 + g_1 g_2$ in terms of the electric and magnetic charges of the dyons, and the Dirac vector potential

$$\vec{D}(\vec{n}, \vec{r}) = \frac{\vec{r} \times \vec{n}\, \hat{r} \cdot \hat{n}}{r[1 - (\hat{r} \cdot \hat{n})^2]} \qquad (2)$$

is singular along the half-line $\hat{r} = \hat{n}$. The singular vector potential is a necessity and may be taken as the definition of the Dirac monopole.[2] The singularity line can be interpreted as a string of magnetic dipoles[3]; the end points behave like positive and negative magnetic charges, and the string as the return line of the magnetic flux:

$$\nabla \times \vec{D} = \frac{\hat{r}}{r^2} - 4\pi\delta(\vec{r} - r\hat{n}),$$