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T matrix for electromagnetic scattering from an arbitrary number of scatterers with continuously varying electromagnetic properties

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We consider monochromatic electromagnetic scattering from a fixed configuration of an arbitrary number of separate scatterers which are immersed in a medium with constant electric and magnetic properties. Within the scatterers the electric and magnetic properties are assumed to vary smoothly. By considering the integral equations for the scattering we show that the total T matrix for the configuration of the scatterers can be expressed in terms of the T matrices for the individual scatterers in the same way as was previously found for a similar configuration of scatterers, each having a discontinuity in the electric and magnetic properties at the surface and constant electric and magnetic properties in its interior.

I. INTRODUCTION

In the present article we consider scattering of a monochromatic electromagnetic field from a configuration of an arbitrary number of separate scatterers which are assumed to be immersed in a homogeneous and isotropic medium. Within the scatterers the electric and magnetic properties are assumed to have a sufficiently smooth variation (see Ref. 1 and below). The electromagnetic scattering from a single scatterer of this kind is governed by by an integral equation (see, e.g., Refs. 1, 2) which thus determines also, e.g., the T matrix. Our main objective in this article is to show that if we assume the T matrix for each of the individual scatterers of the kind described above to be known then the total T matrix for a given configuration of an arbitrary number of such scatterers can be obtained in the same way as was found previously for the case of a configuration of homogeneous isotropic scatterers.[3] In other words, the multiple-scattering aspect of the problem can be clearly separated from that of determining the T matrices of the individual scatterers, as might be expected in the stationary description of the scattering.

Our results follow from a consideration of the structure of the integral equations determining the scattering. In Sec. II we make a few remarks on the interrelationship between various forms of these equations and in Sec. III we derive the main results. In Sec. IV we make a few concluding remarks on the relevance of the results of Sec. III for other types of mathematically similar but physically different multiple-scattering problems.

II. PRELIMINARY REMARKS ON THE INTEGRAL EQUATIONS FOR THE ELECTROMAGNETIC FIELD

For later reference and in order to establish our notations we recall some aspects of the integral equations for the electromagnetic field. Throughout this article we shall consider a stationary field and we shall omit a time factor $\exp(-i\omega t)$ from all equations. Thus we may write

$$\nabla \times \vec{H} + i\omega\epsilon_0 n^2 \vec{E} = 0, \qquad (2.1)$$

$$\nabla \times \vec{E} - i\omega\mu_0\mu\vec{H} = 0, \qquad (2.2)$$

where $n^2 \equiv \epsilon + i\omega^{-1}\epsilon_0^{-1}\sigma$ (in an obvious notation), for the electromagnetic fields in a medium characterized by ϵ, μ, and σ. For the wave vectors we have $k^2 = k_0^2\mu n^2$, $k_0 = \omega\epsilon_0^{1/2}\mu_0^{1/2}$. Equations (2.1) and (2.2) imply

$$\nabla\cdot(\mu\vec{H}) = 0, \quad \nabla\cdot(n^2\vec{E}) = 0 \qquad (2.3)$$

and for $\vec{E}$ and $\vec{H}$ we have the separate differential equations (the functions μ and n^2 are assumed to be continuously differentiable)

$$\nabla\times(\nabla\times\vec{E}) - k^2\vec{E} = \mu^{-1}(\nabla\mu)\times(\nabla\times\vec{E}), \qquad (2.4)$$

$$\nabla\times(\nabla\times\vec{H}) - k^2\vec{H} = n^{-2}(\nabla n^2)\times(\nabla\times\vec{H}). \qquad (2.5)$$

The corresponding integral equations are obtained, e.g., from the vector analog of Green's theorem (see, e.g., Ref. 4). We shall consider a scattering situation, and thus we choose the outgoing free-space Green's function

$$G(k_a; \vec{r}, \vec{r}\,') = (4\pi|\vec{r} - \vec{r}\,'|)^{-1}\exp(ik_a|\vec{r} - \vec{r}\,'|),$$

satisfying $(\nabla^2 + k_a^2)G = -\delta(\vec{r} - \vec{r}\,')$ with constant k_a.

Using the auxiliary vector

$$\vec{M}_{\vec{E}}(\vec{r},\vec{r}\,') \equiv G\vec{a}\times[\nabla'\times\vec{E}(\vec{r}\,')] - \vec{E}(\vec{r}\,')\times[\nabla'\times(G\vec{a})] + [\nabla'\cdot\vec{E}(\vec{r}\,')]G\vec{a} - \vec{E}(\vec{r}\,')\nabla'\cdot(G\vec{a})$$

($\vec{a}$ is a constant vector) which satisfies

$$\nabla'\cdot\vec{M}_{\vec{E}}(\vec{r},\vec{r}\,') = \vec{a}\cdot((k_a{}^2-k^2)G\vec{E}(\vec{r}\,') + \delta(\vec{r}-\vec{r}\,')\vec{E}(\vec{r}\,') - \mu^{-1}(\nabla'\mu)\times[\nabla'\times\vec{E}(\vec{r}\,')]G + \{\nabla'[\nabla'\cdot\vec{E}(\vec{r}\,')]\}G)$$

and

$$\hat{n}\cdot\vec{M}_{\vec{E}}(\vec{r},\vec{r}\,') = \vec{a}\cdot(-\{\hat{n}\times[\nabla'\times\vec{E}(\vec{r}\,')]\}G - [\hat{n}\times\vec{E}(\vec{r}\,')]\times\nabla'G + \hat{n}\cdot[\nabla'\cdot\vec{E}(\vec{r}\,')]G - \hat{n}\cdot\vec{E}(\vec{r}\,')\nabla'G)$$

we get by means of Gauss's theorem (S_0 encloses V_0 and $\hat{n}$ is the inward-pointing unit normal vector)

$$\left.\begin{matrix}\vec{E}(\vec{r})\\ 0\end{matrix}\right\} = \int_{V_0}((k^2-k_a{}^2)G\vec{E}(\vec{r}\,') + \mu^{-1}\nabla'\mu\times[\nabla'\times\vec{E}(\vec{r}\,')]G - \{\nabla'[\nabla'\cdot\vec{E}(\vec{r}\,')]\}G)d\vec{r}\,' + \int_{S_0}\{\hat{n}\times[\nabla'\times\vec{E}(\vec{r}\,')]G + [\hat{n}\times\vec{E}(\vec{r}\,')]\times\nabla'G - \hat{n}[\nabla'\cdot\vec{E}(\vec{r}\,')]G + \hat{n}\cdot\vec{E}(\vec{r}\,')\nabla'G\}dS' \quad \text{for}\left\{\begin{matrix}\vec{r}\text{ inside }V_0\\ \vec{r}\text{ outside }V_0.\end{matrix}\right. \tag{2.6}$$

We now apply (2.6) to the following scattering situation. We assume that the whole space is filled with a source-free medium characterized by $k_a{}^2 = k_0{}^2\mu_a\epsilon_a$ with constant μ_a and ϵ_a, except for the following two facts. Inside a finite volume V, bounded by the surface S, ϵ, μ, and σ vary but in such a way that they are continuously differentiable everywhere. Furthermore, in another finite region far away from V, a field is generated which in the vicinity of V will be designated the incoming field $\vec{E}^i$. In (2.6) we now take $\vec{r}$ inside V_0 and let S_0 recede to infinity. In the process it encloses the region where $\vec{E}^i$ is generated. The contribution from the surface integral at infinity vanishes and thus we may write

$$\vec{E}(\vec{r}) = \vec{E}^i(\vec{r}) + \int_V((k^2-k_a{}^2)G\vec{E}(\vec{r}\,') + \mu^{-1}\nabla'\mu\times[\nabla'\times\vec{E}(\vec{r}\,')]G - \{\nabla'[\nabla'\cdot\vec{E}(\vec{r}\,')]\}G)d\vec{r}\,'. \tag{2.7}$$

This integral equation for $\vec{E}(\vec{r})$ could of course also be obtained directly from (2.4). We note that in (2.7) $\vec{r}$ may be taken both inside and outside V. Another common representation of $\vec{E}$ is obtained from a different choice of V_0 in (2.6): Let V_0 consist only of a part of space where $\epsilon=\epsilon_a$, $\mu=\mu_a$, $\sigma=0$ and let V_0 enclose V so that S_0 consists of an inner and an outer part. Let the outer part recede to infinity, which again results in a contribution $\vec{E}^i$, and let the inner part tend to S. With this choice of V_0 the volume integral in (2.6) vanishes and from the integral over S_0 there remains $\vec{E}^i$ and an integral over S. Using $\nabla'G=-\nabla G$ we may write

$$\left.\begin{matrix}\vec{E}(\vec{r})\\ 0\end{matrix}\right\} = \vec{E}^i(\vec{r}) + \int_S\hat{n}\times[\nabla'\times\vec{E}(\vec{r}\,')]GdS' + \nabla\times\int_S[\hat{n}\times\vec{E}(\vec{r}\,')]GdS' - \nabla\int_S\hat{n}\cdot\vec{E}(\vec{r}\,')GdS' \quad\text{for}\left\{\begin{matrix}\vec{r}\text{ outside }S\\ \vec{r}\text{ inside }S,\end{matrix}\right. \tag{2.8}$$

which contains both the normal and tangential components of $\vec{E}$. The normal components appear only in the gradient term and may therefore be eliminated by means of the curl operator. In this way we get

$$\left.\begin{matrix}\vec{E}(\vec{r})\\ 0\end{matrix}\right\} = \vec{E}^i(\vec{r}) + \nabla\times\int_S[\hat{n}\times\vec{E}(\vec{r}\,')]GdS' + k_a{}^{-2}\nabla\times\left(\nabla\times\int_S\hat{n}\times[\nabla'\times\vec{E}(\vec{r}\,')]GdS'\right) \quad\text{for}\left\{\begin{matrix}\vec{r}\text{ outside }S\\ \vec{r}\text{ inside }S.\end{matrix}\right. \tag{2.9}$$

Thus, with $\vec{E}=\vec{E}^i+\vec{E}^{sc}$, where $\vec{E}^{sc}$ is the scattered field, we have a representation for $\vec{E}^{sc}$ either as a volume integral as in (2.7) or as a surface integral as in (2.8) and (2.9). Alternatively, the representation in terms of a surface integral follows from (2.6) for the choice $V_0=V$ and a consiaeration of $\vec{r}$ outside V.

It is instructive to consider another aspect of the relation between (2.7) and (2.8). First we note that (2.8) does not depend on any regularity assumptions for μ and n^2 inside V; the only assumption is that, when approaching S from the outside, the field is sufficiently regular for the vector theorems to be applicable. Thus, if S is a surface where μ and n^2 have discontinuities and if S itself is sufficiently regular, we have the representation (2.8) but not (2.7). However, in this case (2.8) can also be obtained from (2.7) by means of a limiting procedure whereby a continuously differentiable distribution of μ and n^2 approaches a distribution where these quantities have discontinuities at the surface. We note that the field in (2.8) is the field obtained by approaching S from the outside and to obtain (2.8) from (2.7) also in the case of a discontinuity the boundary conditions have to be introduced.

The incoming field and the scattered field, outside the circumscribed sphere of the scatterer, can be expanded as

$$\vec{E}^i=\sum_n a_n\,\mathrm{Re}\vec{\psi}_n \tag{2.10}$$

and

$$\vec{E}^{sc}=\sum_n f_n\vec{\psi}_n, \tag{2.11}$$

where the fields $\vec{\psi}_n$ and $\mathrm{Re}\vec{\psi}_n$ are chosen as in Ref. 5 (apart from a factor $k^{1/2}$ due to a different normalization of the Green's function). The scattering is described by the T matrix, defined by the relation

$$f_n=\sum_{n'} T_{nn'}a_{n'}. \tag{2.12}$$

By expanding the Green's function in (2.9) and comparing coefficients for $\vec{\psi}_n$ and $\mathrm{Re}\vec{\psi}_n$, respectively, one obtains the equations[5]

$$f_n=i\int_S \{(\hat{n}\times\vec{E})\cdot[\nabla'\times\mathrm{Re}\vec{\psi}_n(k_a\vec{r}')] +[\hat{n}\times(\nabla'\times\vec{E})]\cdot\mathrm{Re}\vec{\psi}_n(k_a\vec{r}')\}dS' \tag{2.13}$$

and

$$a_n=-\int_S \{(\hat{n}\times\vec{E})\cdot[\nabla'\times\vec{\psi}_n(k_a\vec{r}')] +[\hat{n}\times(\nabla'\times\vec{E})]\cdot\vec{\psi}_n(k_a\vec{r}')\}dS' \tag{2.14}$$

which determine the T matrix, by means of elimination of the surface fields. Thus if we have a set of coefficients $\{f_n\}$ which are related to the surface field as in (2.13) and a set of coefficients $\{a_n\}$ related to the surface field as in (2.14), we write the relation between $\{f_n\}$ and $\{a_n\}$ as in (2.12), where $T_{nn'}$ is determined by the geometric and electromagnetic properties of the scatterer. In the case of a homogeneous scatterer, an explicit expression for the T matrix can be obtained from (2.13) and (2.14) under some additional geometric conditions on the surface S, as was shown in Ref. 5. However, in the present case of variable μ and ϵ inside S, the T matrix remains implicit in (2.13) and (2.14). This implicit relation can also be expressed in the volume-integral form of Eq. (2.7).

III. THE T MATRIX FOR AN ARBITRARY NUMBER OF SCATTERERS WITH CONTINUOUSLY VARYING ELECTROMAGNETIC PROPERTIES

Before going into the details of the derivations we recall a number of features which were essential in the treatment of the multiple-scattering problem for a configuration of homogeneous (and isotropic) scatterers given in Ref. 3:

(i) The scatterers are immersed in a homogeneous (and isotropic) medium.

(ii) We consider the wave propagation in a stationary state, i.e., outside the scatterers the wave propagation is governed by Helmholtz's equation (and expansions of the free-space Green's function for this equation is available in sufficiently explicit and easily handled form).

(iii) The geometry of the configuration of scatterers was assumed to be such that all the desired translations of the origin of the spherical wave expansions could be performed in a suitable fashion.

(iv) The scatterers were assumed to be passive and they could thus be characterized by a transition matrix, depending only on their geometric and electromagnetic properties.

In Ref. 3 an additional regularity assumption concerning the scatterer surfaces was used, namely that the radius $r(\theta,\phi)$ from an inner origin to a point on the surface of the scatterer was to be a continuous one-valued function of the spherical angles θ and ϕ. Because of this assumption and the homogeneity of the scatterers, explicit expansions for the surface fields can be introduced, as was shown in Ref. 5. It was then shown[3,6] that these assumptions led to a solution to the scattering problem for a configuration of an arbitrary number of homogeneous or multilayered scatterers.

As will be seen below, it is possible to separate the multiple scattering aspects of the problem also in the more general situation with scatterers having continuously varying electromagnetic properties, to be considered here. In order to display this structure it is not necessary to make any assumptions concerning the surface fields (besides those necessary for the validity of the vector theorems). We shall thus assume that the T matrix for each single scatterer in the configuration has been determined in some way or another (this will in general have to be done by means of an integral equation, as discussed in the previous section; we shall make no contribution to that part of the problem here). Several essential features of the multiscatterer problem are present in the case of two scatterers, and therefore we begin by considering this case in detail. The implications for the general case will then, in view of the results of Ref. 3, be obvious.

Thus we consider the configuration of two scatterers depicted in Fig. 1, where the notations are also defined. The multiple-scattering aspect is determined by the geometry of the boundaries of the "outside" region of constant electromagnetic properties and we use the representation (2.9) for the field, which makes no explicit reference to the internal properties of the scatterers. The surface S in (2.9) is now replaced by the sum of the separate surfaces S_1 and S_2, and in the left-hand side we get $\vec{E}(\vec{r})$ for $\vec{r}$ outside S_1 and S_2 and zero for $\vec{r}$ inside S_1 or S_2. Thus we have three equations corresponding to these three choices of $\vec{r}$. In particular, by choosing, respectively, $\vec{r}$ outside a sphere with center in O and containing S_1 and S_2, and $\vec{r}$ inside the inscribed spheres of $S_{1,2}$ with centers in $O_{1,2}$, and comparing coefficients in the relevant spherical wave expansions, we get three equations for these coefficients. We use ($\mathscr{I}$ is the unit dyad)

$$\mathscr{I}G(k_a|\vec{r}-\vec{r}'|)=i\sum_n \vec{\psi}_n(\vec{r}_>)\,\mathrm{Re}\vec{\psi}_n(\vec{r}_<)+\mathscr{I}_1$$

$$=i\sum_n \mathrm{Re}\vec{\psi}_n(\vec{r}_<)\vec{\psi}_n(\vec{r}_>)+\mathscr{I}_1, \qquad (3.1)$$

where the dyad $\mathscr{I}_1$ is formed out of irrotational vectors. n stands for the set $\tau\sigma m n$ of indices,[5] where $\tau=1, 2$ and one has $k_a{}^{-1}\nabla\times\vec{\psi}_{1\sigma mn}=\vec{\psi}_{2\sigma mn}$, $k_a{}^{-1}\nabla\times\vec{\psi}_{2\sigma mn}=\vec{\psi}_{1\sigma mn}$. If, e.g., $\vec{r}_>=\vec{r}$, $\vec{r}_<=\vec{r}'$ it then follows that

$$\nabla\times[\mathscr{I}G(k_a|\vec{r}-\vec{r}'|)]=i\sum_n \mathrm{Re}\vec{\psi}_n(\vec{r}')\nabla\times\vec{\psi}_n(\vec{r}')$$

$$=i\sum_n \nabla'\times\mathrm{Re}\vec{\psi}_n(\vec{r}')\vec{\psi}_n(\vec{r}).$$

With $\vec{r}$ in S_1 we have $\vec{r}=\vec{r}_1+\vec{a}_1$ and $\vec{E}^i$ can then be written

$$\vec{E}^i(\vec{r})=\sum_{n'} a_{n'}\,\mathrm{Re}\vec{\psi}_{n'}(\vec{r}_1+\vec{a}_1)$$

$$=\sum_{n'n} a_{n'}R_{n'n}(\vec{a}_1)\,\mathrm{Re}\vec{\psi}_{n'}(\vec{r}_1), \qquad (3.2)$$

where the translation matrices $R_{n'n}$ are those defined in Ref. 3. We want to introduce restrictions so that $\vec{r}_>$ and $\vec{r}_<$ in (3.1) can be characterized in a simple way for all $\vec{r}'$ on S_1 and S_2. Still taking $\vec{r}$ inside S_1 we write $\vec{r}-\vec{r}'\equiv\vec{r}-\vec{r}_1'=\vec{r}_1-\vec{r}_1''$ in the integral over S_1, and we have $\vec{r}_1=\vec{r}_<$, $\vec{r}_1''=\vec{r}_>$ if we furthermore restrict $\vec{r}$ to lie within the inscribed sphere of S_1, with center at O_1. In the integral over S_2 we write $\vec{r}-\vec{r}'\equiv\vec{r}-\vec{r}_2'=\vec{r}_1-(\vec{r}_2''+\vec{a}_2-\vec{a}_1)$ where $r_1<|\vec{r}_2''+\vec{a}_2-\vec{a}_1|$ since $\vec{r}$ is inside the inscribed sphere of S_1. An expansion according to (3.1) of G in the integral over S_2 thus involves $\vec{\psi}_n(\vec{r}_2''+\vec{a}_2-\vec{a}_1)\,\mathrm{Re}\vec{\psi}_n(\vec{r}_1)$. In order to be able to extract an explicit algebraic expression for the total T matrix, involving the individual T matrices, we now introduce a restriction on the form and relative position of the scatterers, namely that $r_2''<|\vec{a}_1-\vec{a}_2|$ (and correspondingly $r_1''<|\vec{a}_1-\vec{a}_2|$ when we consider $\vec{r}$ inside S_2; see Ref. 3). Under this condition we have

$$\vec{\psi}_n(\vec{r}_2''+\vec{a}_2-\vec{a}_1)=\sum_{n'}\sigma_{nn'}(\vec{a}_2-\vec{a}_1)\,\mathrm{Re}\vec{\psi}_{n'}(\vec{r}_2''), \quad (3.3)$$

where the translation matrices $\sigma_{nn'}$ are those defined in Ref. 3. Thus, by taking (3.1)–(3.3) into account we get from (2.9), for the case when $\vec{r}$ is taken inside the inscribed sphere of S_1,

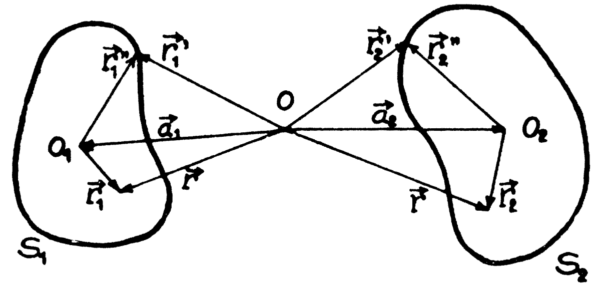

FIG. 1. Geometry and notations for the case of two scattering regions.

$$0=\sum_{nn'} a_{n'} R_{n'n}(\vec{a}_1)\,\mathrm{Re}\vec{\psi}_n(\vec{r}_1)$$
$$+i\sum_{n}\left(\int_{S_1}\{(\hat{n}_1\times\vec{E})\cdot\nabla''\times\vec{\psi}_n(\vec{r}_1'')+[\hat{n}_1\times(\nabla''\times\vec{E})]\cdot\vec{\psi}_n(\vec{r}_1'')\}dS''\right)\mathrm{Re}\vec{\psi}_n(\vec{r}_1)$$
$$+i\sum_{nn'}\sigma_{nn'}(\vec{a}_2-\vec{a}_1)\left(\int_{S_2}\{(\hat{n}_2\times\vec{E})\cdot\nabla''\times\mathrm{Re}\vec{\psi}_{n'}(\vec{r}_2'')+[\hat{n}_2\times(\nabla''\times\vec{E})]\cdot\mathrm{Re}\vec{\psi}_{n'}(\vec{r}_2'')\}dS''\right)\mathrm{Re}\vec{\psi}_n(\vec{r}_1), \tag{3.4}$$

where $\vec{r}_1$ may vary over a complete sphere, i.e., the total coefficient of each $\mathrm{Re}\vec{\psi}_n(\vec{r}_1)$ is zero. We introduce the notations

$$a_n^j\equiv -i\int_{S_j}\{(\hat{n}_j\times\vec{E})\cdot[\nabla''\times\vec{\psi}_n(\vec{r}_j'')] +[\hat{n}_j\times(\nabla''\times\vec{E})]\cdot\vec{\psi}_n(\vec{r}_j'')\}dS'',\quad j=1,2 \tag{3.5}$$

$$f_n^j\equiv i\int_{S_j}\{(\hat{n}_j\times\vec{E})\cdot[\nabla''\times\mathrm{Re}\vec{\psi}_n(\vec{r}_j'')] +[\hat{n}_j\times(\nabla''\times\vec{E})]\cdot\mathrm{Re}\vec{\psi}_n(\vec{r}_j'')\}dS'',\quad j=1,2. \tag{3.6}$$

From (3.4) we thus get

$$\sum_{n'}(R^t(\vec{a}_1))_{nn'}a_{n'}=a_n^1-\sum_{n'}\sigma_{nn'}(\vec{a}_2-\vec{a}_1)f_{n'}^2$$

(R^t is the transpose of R) which we write in a vector and matrix notation as

$$R^t(\vec{a}_1)\vec{a}=\vec{a}^1-\sigma(\vec{a}_2-\vec{a}_1)\vec{f}^2. \tag{3.7}$$

It is immediately clear that by choosing $\vec{r}$ inside the inscribed sphere of S_2 we get, under the assumption that $r_1''<|a_1-a_2|$, the equation

$$R^t(\vec{a}_2)\vec{a}=-\sigma(\vec{a}_1-\vec{a}_2)\vec{f}^1+\vec{a}^2. \tag{3.8}$$

To obtain the third equation, consider $\vec{r}$ outside a sphere with center in O and containing S_1 and S_2. Then $\vec{r}=\vec{r}_>$ and $\vec{r}_{1,2}'=\vec{r}_<$ and the relevant expansion of $\mathcal{G}G$ on S_j is

$$\mathcal{G}G=i\sum_n \mathrm{Re}\vec{\psi}_n(\vec{r}_j')\vec{\psi}_n(\vec{r})+\mathcal{G}_1,$$

where $r_j'=\vec{a}_j+\vec{r}_j''$. By an analogous calculation in which $\vec{E}^{sc}$ in (2.11) is equated with the expression obtained from (2.9), we get

$$\vec{f}=R(\vec{a}_1)\vec{f}^1+R(\vec{a}_2)\vec{f}^2. \tag{3.9}$$

The crucial observation is now that according to the definitions (3.5) and (3.6) and the discussion in Sec. II, we have the relations

$$\vec{f}^j=T(j)\vec{a}^j,\quad j=1,2 \tag{3.10}$$

where $T(j)$ denotes the T matrix corresponding to the single scatterer bounded by S_j [cf. (2.13) and (2.14)]. Introducing (3.10) into (3.7)–(3.9) we get

$$R^t(\vec{a}_1)\vec{a}=\vec{a}^1-\sigma(\vec{a}_2-\vec{a}_1)T(2)\vec{a}^2, \tag{3.11}$$

$$R^t(\vec{a}_2)\vec{a}=-\sigma(\vec{a}_1-\vec{a}_2)\vec{a}^1+\vec{a}^2, \tag{3.12}$$

$$\vec{f}=R(\vec{a}_1)T(1)\vec{a}^1+R(\vec{a}_2)T(2)\vec{a}^2. \tag{3.13}$$

Equations (3.11)–(3.13) constitute three relations among the four quantities $\vec{a}$, $\vec{f}$, $\vec{a}^1$, and $\vec{a}^2$, i.e., they can be used to extract a relation between any pair of these quantities, and we shall be interested in the relation between $\vec{a}$ and $\vec{f}$, a relation which, by definition, determines the total T matrix for the configuration of the two scatterers. We now note that Eqs. (3.11)–(3.13), regarded as equations which determine a relation between $\vec{a}$ and $\vec{f}$, have the same structure as Eqs. (3.11)–(3.13) in Ref. 3 [introduce $T(j)=-\mathrm{Re}Q^{jt}(Q^{jt})^{-1}$ into these equations in Ref. 3 and make the correspondence $\vec{a}^j\to Q^{jt}\vec{\alpha}^j$; in Ref. 3 explicit assumptions concerning the surface fields were introduced and in that case the equations can also be used to obtain an explicit relation between the surface fields and the incoming or scattered field].

Consider now the case of an arbitrary number N of scatterers with (independently) continuously varying electromagnetic properties, bounded by the closed nonoverlapping surfaces S_j ($j=1,2,\ldots,N$) and characterized by the individual T matrices $T(j)$, referring to centers defined by the position vectors $\vec{a}_j$ inside S_j, where $\vec{a}_j$ and S_j are such that $r_{i,j}''<|\vec{a}_i-\vec{a}_j|$ for all combinations i,j, $i\neq j$ [$\vec{r}_j''$ ($j=1,2,\ldots,N$) generalize $\vec{r}_{1,2}''$ in Fig. 1]. It is then easy to see that the generalization of (3.11)–(3.13) is the following set of $N+1$ equations:

$$R^t(\vec{a}_i)\vec{a}=\vec{a}^i-\sum_{\substack{j=1\\ j\neq i}}^{N}\sigma(\vec{a}_j-\vec{a}_i)T(j)\vec{a}^j, \tag{3.14}$$

$$\vec{f}=\sum_{j=1}^{N}R(\vec{a}_j)T(j)\vec{a}^j, \tag{3.15}$$

where $\vec{a}^j$ ($j=1,2,\ldots,N$) is defined in analogy with (3.5). Again, the structure of (3.14) and (3.15), regarded as equations determining a relation between $\vec{a}$ and $\vec{f}$, have the same structure as the corresponding equations (4.1) and (4.2) of Ref. 3. In other words, the multiple-scattering aspect of the present problem is the same as that obtained in Ref. 3 for the case of several homogeneous scatterers, and in Ref. 6 for several multilayered scatterers, in the sense that the total T matrix is the same algebraic function of the individual T matrices. This result is an expected one in view of the fact that we consider the scattering in a stationary situation and in this case each of the individual scatterers is fully characterized by its T matrix. Therefore we have here limited ourselves to pointing out the steps which are crucial for the conclusions and we refer to Ref. 3 for more details on the algebraic procedure for obtaining the total T matrix from (3.14) and (3.15). Here we only note that the solution is obtained by applying an explicit algorithm in N steps. The algorithm used is that devised by Ore[7] for obtaining the inverse of a matrix with noncommutative elements. For instance, in the case of two scatterers one finds for the total T matrix $T(1, 2)$ (see Ref. 3)

$$T(1,2)=R(\vec{a}_1)\{T(1)[1-\sigma(\vec{a}_2-\vec{a}_1)T(2)\sigma(\vec{a}_1-\vec{a}_2)T(1)]^{-1}[1+\sigma(\vec{a}_2-\vec{a}_1)T(2)R(\vec{a}_1-\vec{a}_2)]\}R(-\vec{a}_1)$$
$$+R(\vec{a}_2)\{T(2)[1-\sigma(\vec{a}_1-\vec{a}_2)T(1)\sigma(\vec{a}_2-\vec{a}_1)T(2)]^{-1}[1+\sigma(\vec{a}_1-\vec{a}_2)T(1)R(\vec{a}_2-\vec{a}_1)]\}R(-\vec{a}_2).$$

[Note that in formula (4.17) in Ref. 3 for the total T matrix $T(1, 2, 3)$ for three scatterers there is a misprint inasmuch as the next-to-last factor, $R(k, i)$, should read $R(i, k)$.]

IV. CONCLUDING REMARKS

In the previous section we have shown that the results of Ref. 3 have a wider range of applicability in electromagnetic scattering, in the sense explained before. However, from the enumeration of the properties (i)–(iv) in Sec. III it is clear that the results apply to many other types of multiple-scattering problems. Consider for instance the scattering of a scalar-wave field (for several homogeneous and multilayered scatterers this was treated in Refs. 8 and 9, respectively). It is then immediately clear that the results of this paper apply also to, e.g., acoustic scattering from several scatterers with continuously varying densities, etc. Another case of great interest to which the above also applies is quantum-mechanical potential scattering in the case of several nonoverlapping potentials with finite ranges. This case will be considered separately elsewhere. Among recent work which stresses the similarity between the quantummechanical and electromagnetic multiple-scattering problem we mention in particular Ref. 10.

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