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T-matrix formulation of electromagnetic scattering from multilayered scatterers*

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Recently a T-matrix formulation of classical electromagnetic scattering has been given by Waterman for the case of one homogeneous scatterer, and this formulation has subsequently been extended to the case of an arbitrary number of homogeneous scatterers by the present authors. In the present article we show that the T-matrix formulation is also well suited for the treatment of electromagnetic scattering from scatterers consisting of an arbitrary number of consecutively enclosing layers with constant electric and magnetic properties. We also show how the earlier results on the T-matrix formulation can be combined with these new results to apply to more general types of multilayered scatterers. Some numerical applications are presented.

I. INTRODUCTION

In Refs. 1 and 2 Waterman has given a T-matrix description of acoustic and electromagnetic scattering from a single homogeneous scatterer. The boundary conditions on the scatterer can be of a fairly general nature and the surface of the scatterer has to satisfy certain fairly weak geometrical conditions. Monochromatic waves are considered, and the T matrix refers to expansions in spherical-wave solutions to Helmholtz's equation. This T-matrix formulation has subsequently been extended to the case of an arbitrary number of scatterers,[3,4] the extension being valid under fairly weak conditions on the configuration of the scatterers.

In the present article we consider monochromatic electromagnetic scattering and we show that the T-matrix formulation is equally well suited for the description of scattering from a target consisting of several layers, each of which has constant electric and magnetic properties and which consecutively enclose each other. By invoking the results of Ref. 3 we then also obtain the T matrix for an arbitrary number of such scatterers, where the number of layers and the electric and magnetic properties of these layers can be chosen independently for each of the scatterers. The problem of calculating the total T matrix for one scatterer with several layers turns out to be algebraically much simpler than that of calculating the total T matrix for several homogeneous scatterers.[3] In the case of one multilayered scatterer a recursion formula for the T matrix itself is obtained. By using the results of Ref. 3 we also obtain the T matrix for one scatterer which contains several enclosures, which may themselves be multilayered in the above sense. Furthermore, the methods and results of Ref. 3 are shown to apply to the case of a scatterer consisting of several nonenclosing parts, each with arbitrary but constant electromagnetic properties. All the above-mentioned results have their counterpart in a matrix formulation of acoustic scattering. This case is treated in Ref. 5.

The plan of the present article is as follows. In Sec. II we consider first the case of a scatterer consisting of two layers and we obtain the total T matrix in terms of the T matrix for the inner surface and Q matrices[2] associated with the outer surface. A formal expansion of the total T matrix gives a series of terms which can be interpreted as multiple-scattering contributions to the scattered field. In Sec. III we extend the formalism to the case of a scatterer with an arbitrary number of consecutively enclosing layers and we discuss the iteration procedure for obtaining the total T matrix. In Sec. IV we discuss some examples of other types of scatterers to which the results of Ref. 3 and Secs. II and III of the present article can be applied. A discussion of the results and some numerical applications are given in Sec. V.

II. THE T MATRIX FOR A TWO-LAYERED SCATTERER

Throughout this paper the T matrices to be considered will refer to spherical waves, i.e., the scattered field outside the scatterer will be represented as

$$\vec{E}^{sc} = \sum_n f_n \vec{\psi}_n , \tag{2.1}$$

where $\vec{\psi}_n$ stands for an outgoing spherical-wave solution to

$$(\nabla^2 + k^2)\vec{\psi} = 0 \tag{2.2}$$

[a time factor $\exp(-i\omega t)$ will be suppressed throughout]. Explicitly, we choose a complete set of $\vec{\psi}_n$'s as follows:

$$\vec{\psi}_n \equiv \vec{\psi}_{\tau\sigma mn}(k\vec{r}) \equiv (\gamma_{mn})^{1/2}(k^{-1}\nabla\times)^{\tau}[k\vec{r}Y_{\sigma mn}(\hat{r})h_n^{(1)}(kr)], \quad (2.3)$$

where $\tau=1, 2$, $\sigma=e, o$ ("even" or "odd"), $n=1, 2, \ldots, m=0, 1, \ldots, n$, and

$$\gamma_{mn}=\frac{\epsilon_m}{4\pi}\frac{(2n+1)(n-m)!}{n(n+1)(n+m)!}, \quad \epsilon_0=1, \ \epsilon_m=2, \ m\neq 0. \quad (2.4)$$

$h^{(1)}(kr)$ is a spherical Hankel function and

$$Y_{emn}(\hat{r})=P_n^m(\cos\theta)\cos m\varphi,$$

$$Y_{omn}(\hat{r})=P_n^m(\cos\theta)\sin m\varphi,$$

where P_n^m is an associated Legendre function.[2,3,6] The incoming field is assumed to be regular everywhere inside the scatterer, i.e., it can be represented as

$$\vec{E}^{\text{inc}}=\sum_n a_n \text{Re}\vec{\psi}_n, \quad (2.5)$$

where $\text{Re}\vec{\psi}_n$ stands for the regular part of $\vec{\psi}_n$, i.e., the expression (2.3) with $h_n^{(1)}(kr)$ replaced by $j_n(kr)$, a spherical Bessel function. The solution of the scattering problem is given if the transition matrix T, with elements $T_{nn'}$, satisfying

$$f_n=\sum_{n'} T_{nn'}a_{n'} \quad (2.6)$$

is determined. In Ref. 2 it is shown that the T matrix for a single scatterer is of the form

$$T=-(\text{Re}Q)Q^{-1}, \quad (2.7)$$

where ReQ and Q are matrices which are functions of the surface S of the scatterer and of the nature of the boundary conditions. For example, if the scatterer is characterized by a relative dielectric constant ϵ and a relative permeability μ, Q is given by

$$Q_{nn'}\equiv k\int_S dS'\hat{n}\cdot\{[\nabla'\times\vec{\psi}_n(k\vec{r}')]\times\text{Re}\vec{\psi}_{n'}(k'\vec{r}') + \mu^{-1}\vec{\psi}_n(k\vec{r}')\times[\nabla'\times\text{Re}\vec{\psi}_{n'}(k'\vec{r}')]\}, \quad (2.8)$$

where $k'=k(\mu\epsilon)^{1/2}$ (note that in the present article we use conventions for the Q matrices which are slightly different from those in Refs. 2 and 3). ReQ is the matrix obtained by taking the functions $\text{Re}\vec{\psi}_n(k\vec{r})$ instead of $\vec{\psi}_n(k\vec{r})$ in (2.8). In order to obtain (2.7) one uses the Poincaré-Huygens principle,[7] which for a medium characterized by μ and ϵ reads (we put $\epsilon_0=\mu_0=1$ for vacuum)

$$\left.\begin{matrix}\vec{E}(\vec{r})\\ 0\end{matrix}\right\}=\nabla\times\int_S dS'k'[\hat{n}\times\vec{E}_+(\vec{r}')]G(k'|\vec{r}-\vec{r}'|)+\nabla\times\left(\nabla\times\int_S dS' i\mu^{1/2}\epsilon^{-1/2}[\hat{n}\times\vec{H}_+(\vec{r}')]G(k'|\vec{r}-\vec{r}'|)\right) \text{ for } \left\{\begin{matrix}\vec{r}\text{ inside }V,\\ \vec{r}\text{ outside }V,\end{matrix}\right. \quad (2.9)$$

where S now is the whole surface which encloses the volume V and where the unit normal vector $\hat{n}$ points into V. We note that in order to obtain (2.9) it is sufficient to require $G(k|\vec{r}-\vec{r}'|)$ to satisfy

$$(\nabla^2+k^2)G(k|\vec{r}-\vec{r}'|)=-k^{-1}\delta(\vec{r}-\vec{r}')$$

and $\nabla G=-\nabla' G$, i.e., the boundary conditions on G, and thus on $\vec{E}$, have to be introduced separately. $\vec{E}_+$ and $\vec{H}_+$ denote the boundary values of the total fields $\vec{E}$ and $\vec{H}$ on S. (2.7) and (2.8) are obtained by means of the Poincaré-Huygens principle, including a source term, applied to the region outside the scatterer bounded by the surface S (in this outside region the wave vector is k and $\mu=\epsilon=1$) and combined with the boundary conditions. From the resulting equations the surface fields can be eliminated.[2] Through the boundary conditions $\vec{E}_+$ and $\vec{H}_+$ are connected to the fields inside the scatterer and one then assumes expansions for these fields in terms of regular solutions to the interior wave equation.[2] This can be done under certain regularity assumptions on the surface S. We refer to Refs. 1 and 2 and further references cited therein for a discussion of these questions, and in the present article we shall always assume that the corresponding expansions of the surface fields are valid. We shall furthermore concentrate on the cases of infinite or zero conductivity (i.e., no losses). Many features of the formalism presented below are expected to be valid also in the case of finite nonzero conductivity. However, the description of translation of spherical waves characterized by a complex wave vector requires more general results concerning local representations of the three-dimensional Euclidean group E(3) than were explicitly given in 3, and therefore we shall presently concentrate on the lossless case [concerning these more general properties of the local representations of E(3), see, e.g., Ref. 8].

We shall consider a two-layered scatterer defined by the closed surfaces S_1 and S_2, where S_1 encloses S_2 according to Fig. 1. The scatterer is

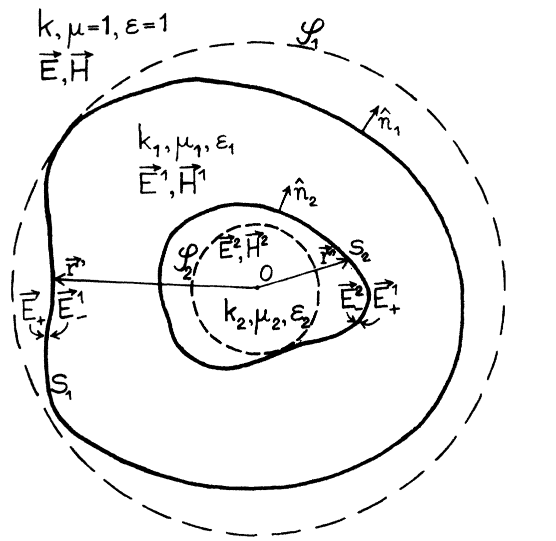

FIG. 1. Geometry and notations for a two-layered scatterer.

immersed in a medium with $\epsilon=\mu=1$ and the two layers are characterized by the relative constants ϵ_1, μ_1 and ϵ_2, μ_2, respectively, and the corresponding wave vectors are k_1 and k_2 (the small variations in the formulas which are obtained for the case of infinite conductivity will not be written down in the following; cf. Refs. 2 and 3). The boundary values of the fields on the outside and the inside of a surface are denoted by a subscript + and −, respectively. First, we apply Poincaré-Huygens principle to the region outside S_1 and proceed as in Ref. 2. However, in the boundary conditions

$$\hat{n}_1\times\vec{E}_+=\hat{n}_1\times\vec{E}^1_-, \tag{2.10}$$

$$\hat{n}_1\times\vec{H}_+=\hat{n}_1\times\vec{H}^1_-, \tag{2.11}$$

($\hat{n}_i$ is the unit normal vector on S_i, $i=1,2$), we now have to assume a more general expansion for the interior surface fields, containing both regular and irregular solutions. For convenience we shall write this expansion in terms of a regular part and an "outgoing" part, i.e., we assume that

$$\hat{n}_1\times\vec{E}^1_-=\sum_n \hat{n}_1\times[\alpha^1_n \operatorname{Re}\vec{\psi}_n(k_1\vec{r}\,')+\beta^1_n\vec{\psi}_n(k_1\vec{r}\,')], \tag{2.12}$$

and thus

$$\hat{n}_1\times\vec{H}^1_-=(ik\mu_1)^{-1}\sum_n \hat{n}_1\times[\alpha^1_n\nabla'\times\operatorname{Re}\vec{\psi}_n(k_1\vec{r}\,') +\beta^1_n\nabla'\times\vec{\psi}_n(k_1\vec{r}\,')]\,. \tag{2.13}$$

In this way we obtain

$$f_n=-i\sum_{n'}[Q^1_{nn'}(\mathrm{Re},\mathrm{Re})\alpha^1_{n'}+Q^1_{nn'}(\mathrm{Re},\mathrm{Out})\beta^1_{n'}] \tag{2.14}$$

and

$$a_n=i\sum_n[Q^1_{nn'}(\mathrm{Out},\mathrm{Re})\alpha^1_{n'}+Q^1_{nn'}(\mathrm{Out},\mathrm{Out})\beta^1_{n'}] \tag{2.15}$$

for the coefficients f_n and a_n in the expansions (2.1) and (2.5) of the scattered and incoming field. We have used the notation

$$Q^1_{nn'}(\mathrm{Re},\mathrm{Re})\equiv k\int_{S_1}dS'\hat{n}_1\cdot\{[\nabla'\times\operatorname{Re}\vec{\psi}_n(k\vec{r}\,')]\times\operatorname{Re}\vec{\psi}_{n'}(k_1\vec{r}\,')+\mu_1{}^{-1}\operatorname{Re}\vec{\psi}_n(k\vec{r}\,')\times[\nabla'\times\operatorname{Re}\vec{\psi}_{n'}(k_1\vec{r}\,')]\}\,. \tag{2.16}$$

The other combinations of "Re" and "Out" in the coefficients $Q^1_{nn'}$ in (2.14) and (2.15) refer to similar expressions, where Re or Out in the first place in the argument of $Q^1_{nn'}$ corresponds to taking the functions $\operatorname{Re}\vec{\psi}_n(k\vec{r}\,')$ or $\vec{\psi}_n(k\vec{r}\,')$, respectively, and similarly Re or Out in the second place corresponds to taking the functions $\operatorname{Re}\vec{\psi}_{n'}(k_1\vec{r}\,')$ or $\vec{\psi}_{n'}(k_1\vec{r}\,')$ [these replacements shall be made in both places in (2.16) where a function of $k\vec{r}\,'$ or $k_1\vec{r}\,'$ occurs].

In the next step we apply the Poincaré-Huygens principle to the volume between S_1 and S_2. It is easy to see that (2.9) is valid for a general Green's function

$$G(k|\vec{r}-\vec{r}\,'|)=\lambda_1 g(k|\vec{r}-\vec{r}\,'|)+\lambda_2\bar{g}(k|\vec{r}-\vec{r}\,'|),\quad \lambda_1+\lambda_2=1 \tag{2.17}$$

where

$$g=(4\pi k|\vec{r}-\vec{r}\,'|)^{-1}\exp(ik|\vec{r}-\vec{r}\,'|) \tag{2.18}$$

is an outgoing Green's function and thus $\bar{g}$ (where the bar denotes complex conjugation) is an ingoing Green's function. In the present situation the choice of the Green's function is dictated by the following requirements. It should be general enough to describe a field between S_1 and S_2 which is general enough, i.e., consisting of both regular and irregular parts. On the other hand, it should be such that in the limit of $\epsilon_1\to 1$, $\mu_1\to 1$ the solution of the scattering problem approaches the form given by the T matrix for one scatterer bounded by the surface S_2. As is easily seen (cf. the formulas below) the last requirement gives

$\lambda_2=0$ and the first requirement is then still fulfilled.

Thus, by applying the Poincaré-Huygens principle with the Green's function $g(k_1|\vec{r}-\vec{r}'|)$ and assuming that $\vec{E}^1$ has no sources between S_1 and S_2 we obtain

$$\left.\begin{matrix}\vec{E}^1(\vec{r})\\ 0\end{matrix}\right\}=\nabla\times\int_{S_1}k_1dS'[(-\hat{n}_1)\times\vec{E}^1_-]g(k_1|\vec{r}-\vec{r}'|)+\nabla\times\left(\nabla\times\int_{S_1}i\mu_1{}^{1/2}\epsilon_1{}^{-1/2}[(-\hat{n}_1)\times\vec{H}^1_-]g(k_1|\vec{r}-\vec{r}'|)\right)$$

$$+\nabla\times\int_{S_2}k_1dS'(\hat{n}_2\times\vec{E}^1_+)g(k_1|\vec{r}-\vec{r}'|)+\nabla\times\left(\nabla\times\int_{S_2}i\mu_1{}^{1/2}\epsilon_1{}^{-1/2}(\hat{n}_2\times\vec{H}^1_+)g(k_1|\vec{r}-\vec{r}'|)\right)$$

$$\text{for}\begin{cases}\vec{r}\text{ between }S_1\text{ and }S_2,\\ \vec{r}\text{ outside }S_1\text{ or inside }S_2.\end{cases}\qquad(2.19)$$

The expansions for $\hat{n}_1\times\vec{E}^1_+$ and $\hat{n}_1\times\vec{H}^1_+$ are given in (2.12) and (2.13). The boundary conditions on S_2 are

$$\begin{aligned}\hat{n}_2\times\vec{E}^1_+&=\hat{n}_2\times\vec{E}^2_-,\\ \hat{n}_2\times\vec{H}^1_+&=\hat{n}_2\times\vec{H}^2_-,\end{aligned}\qquad(2.20)$$

where we now assume that $\hat{n}_2\times\vec{E}^2_-$ has an expansion of the form (cf. Ref. 2)

$$\hat{n}_2\times\vec{E}^2_-=\sum_n\alpha_n^2\,\hat{n}_2\times\text{Re}\vec{\psi}_n(k_2\vec{r}'),\qquad(2.21)$$

and thus

$$\hat{n}_2\times\vec{H}^2_-=(ik\mu_2)^{-1}\sum_n\alpha_n^2\hat{n}_2\times[\nabla'\times\text{Re}\vec{\psi}_n(k_2\vec{r}')]\qquad(2.22)$$

(note that we have already used the boundary conditions on S_1). By considering the cases of $\vec{r}$ outside the circumscribing sphere $\mathcal{S}_1$ of S_1 (with center in 0) and of $\vec{r}$ inside the inscribed sphere $\mathcal{S}_2$ of S_2 (also with center in 0), we obtain two equations for α_n^1, β_n^1, and α_n^2 as follows. Let $\mathfrak{g}$ denote the unit dyad. We then have

$$\begin{aligned}\mathfrak{g}g(k_1|\vec{r}-\vec{r}'|)&=i\sum_n\vec{\psi}_n(k_1\vec{r}_>)\,\text{Re}\vec{\psi}_n(k_1\vec{r}_<)+\mathfrak{g}_1\\&=i\sum_n\text{Re}\vec{\psi}_n(k_1\vec{r}_<)\vec{\psi}_n(k_1\vec{r}_>)+\mathfrak{g}_1,\end{aligned}\qquad(2.23)$$

where the dyad $\mathfrak{g}_1$ is formed out of irrotational vectors. It will therefore disappear from all subsequent relations. With $\vec{r}$ outside $\mathcal{S}_1$ we have $\vec{r}_>=\vec{r}$, $\vec{r}_<=\vec{r}'$ on both S_1 and S_2 and with $\vec{r}$ inside $\mathcal{S}_2$ we have $\vec{r}_<=\vec{r}$, $\vec{r}_>=\vec{r}'$ on both S_1 and S_2. Introducing (2.12), (2.13), (2.21), (2.22), and (2.23) into (2.19) and using $\nabla\times\vec{\psi}_{\tau\sigma mn}(k_1\vec{r})=k_1\vec{\psi}_{\tau'\sigma mn}(k_1\vec{r})$, where $\tau\neq\tau'$, we obtain, from a consideration of $\vec{r}$ inside $\mathcal{S}_2$, by comparing the coefficients of $\text{Re}\vec{\psi}_n(k_1\vec{r})$, the equation

$$\begin{aligned}0=\sum_{n'}\Bigg(&\int_{S_1}dS'\hat{n}_1\cdot\{[\nabla'\times\vec{\psi}_n(k_1\vec{r}')]\times\text{Re}\vec{\psi}_{n'}(k_1\vec{r}')+\vec{\psi}_n(k_1\vec{r}')\times[\nabla'\times\text{Re}\vec{\psi}_{n'}(k_1\vec{r}')]\}\alpha^1_{n'}\\&+\int_{S_1}dS'\hat{n}_1\cdot\{[\nabla'\times\vec{\psi}_n(k_1\vec{r}')]\times\vec{\psi}_{n'}(k_1\vec{r}')+\vec{\psi}_n(k_1\vec{r}')\times[\nabla'\times\vec{\psi}_{n'}(k_1\vec{r}')]\}\beta^1_{n'}\\&-\int_{S_2}dS'\hat{n}_2\cdot\{[\nabla'\times\vec{\psi}_n(k_1\vec{r}')]\times\text{Re}\vec{\psi}_{n'}(k_2\vec{r}')+\mu_1\mu_2{}^{-1}\vec{\psi}_n(k_1\vec{r}')\times[\nabla'\times\text{Re}\vec{\psi}_{n'}(k_2\vec{r}')]\}\alpha^2_{n'}\Bigg).\end{aligned}\qquad(2.24)$$

Similarly we obtain from a consideration of $\vec{r}$ outside $\mathcal{S}_1$

$$\begin{aligned}0=\sum_{n'}\Bigg(&\int_{S_1}dS'\hat{n}_1\cdot\{[\nabla'\times\text{Re}\vec{\psi}_n(k_1\vec{r}')]\times\text{Re}\vec{\psi}_{n'}(k_1\vec{r}')+\text{Re}\vec{\psi}_n(k_1\vec{r}')\times[\nabla'\times\text{Re}\vec{\psi}_{n'}(k_1\vec{r}')]\}\alpha^1_{n'}\\&+\int_{S_1}dS'\hat{n}_1\cdot\{[\nabla'\times\text{Re}\vec{\psi}_n(k_1\vec{r}')]\times\vec{\psi}_{n'}(k_1\vec{r}')+\text{Re}\vec{\psi}_n(k_1\vec{r}')\times[\nabla'\times\vec{\psi}_{n'}(k_1\vec{r}')]\}\beta^1_{n'}\\&-\int_{S_2}dS'\hat{n}_2\cdot\{[\nabla'\times\text{Re}\vec{\psi}_n(k_1\vec{r}')]\times\text{Re}\vec{\psi}_{n'}(k_2\vec{r}')+\mu_1\mu_2{}^{-1}\,\text{Re}\vec{\psi}_n(k_1\vec{r}')\times[\nabla'\times\text{Re}\vec{\psi}_{n'}(k_2\vec{r}')]\}\alpha^2_{n'}\Bigg).\end{aligned}\qquad(2.25)$$

A direct calculation, using Gauss's theorem and the Wronskian relation for j_n and $h_n^{(1)}$, yields

$$\int_S dS'\hat{n}\cdot\{[\nabla'\times \mathrm{Re}\vec{\psi}_n(k\vec{r}')]\times \mathrm{Re}\vec{\psi}_{n'}(k\vec{r}')+\mathrm{Re}\vec{\psi}_n(k\vec{r}')\times[\nabla'\times \mathrm{Re}\vec{\psi}_{n'}(k\vec{r}')]\}$$
$$=\int_S dS'\hat{n}\cdot\{[\nabla'\times\vec{\psi}_n(k\vec{r}')]\times\vec{\psi}_{n'}(k\vec{r}')+\vec{\psi}_n(k\vec{r}')\times[\nabla'\times\vec{\psi}_{n'}(k\vec{r}')]\}$$
$$=0, \tag{2.26}$$

$$k\int_S dS'\hat{n}\cdot\{[\nabla'\times \mathrm{Re}\vec{\psi}_n(k\vec{r}')]\times\vec{\psi}_{n'}(k\vec{r}')+\mathrm{Re}\vec{\psi}_n(k\vec{r}')\times[\nabla'\times\vec{\psi}_{n'}(k\vec{r}')]\}=i\delta_{nn'}\,. \tag{2.27}$$

With the notation

$$Q^2_{nn'}(\mathrm{Re},\mathrm{Re})\equiv k_1\int_{S_2} dS'\hat{n}_2\cdot\{[\nabla'\times\mathrm{Re}\vec{\psi}_n(k_1\vec{r}')]\times\mathrm{Re}\vec{\psi}_{n'}(k_2\vec{r}')+\mu_1\mu_2^{-1}\,\mathrm{Re}\vec{\psi}_n(k_1\vec{r}')\times[\nabla'\times\mathrm{Re}\vec{\psi}_{n'}(k_2\vec{r}')]\} \tag{2.28}$$

[note that this is a direct analog of (2.16)], and similarly for other combinations of Re and Out according to the prescriptions given in connection with (2.16), we get

$$-i\alpha_n^1=\sum_{n'}Q^2_{nn'}(\mathrm{Out},\mathrm{Re})\alpha^2_{n'}\,, \tag{2.29}$$

$$i\beta_n^1=\sum_{n'}Q^2_{nn'}(\mathrm{Re},\mathrm{Re})\alpha^2_{n'}\,. \tag{2.30}$$

In a vector and matrix notation ($\{a_n\}\equiv\vec{a}$, etc.) we have thus obtained the following system of equations:

$$\vec{a}=i[Q^1(\mathrm{Out},\mathrm{Re})\vec{\alpha}^1+Q^1(\mathrm{Out},\mathrm{Out})\vec{\beta}^1]\,, \tag{2.31}$$

$$\vec{f}=-i[Q^1(\mathrm{Re},\mathrm{Re})\vec{\alpha}^1+Q^1(\mathrm{Re},\mathrm{Out})\vec{\beta}^1]\,, \tag{2.32}$$

$$\vec{\alpha}^1=iQ^2(\mathrm{Out},\mathrm{Re})\vec{\alpha}^2\,, \tag{2.33}$$

$$\vec{\beta}^1=-iQ^2(\mathrm{Re},\mathrm{Re})\vec{\alpha}^2\,. \tag{2.34}$$

Here we may remark that if we had started with a more general Green's function (2.17), this would have affected Eq. (2.33) but not (2.34), and this fact can be used to show that one has to choose $\lambda_2=0$. From the four equations (2.31)–(2.34) we can extract relations between any two of the five vectors $\vec{a}$, $\vec{f}$, $\vec{\alpha}^1$, $\vec{\beta}^1$, and $\vec{\alpha}^2$ and we shall consider the relation between $\vec{a}$ and $\vec{f}$, i.e., the total T matrix for the two-layered scatterer which will be denoted $T(1,2)$. A useful expression for $T(1,2)$ is

$$T(1,2)=\{T(1)-Q^1(\mathrm{Re},\mathrm{Out})T(2)[Q^1(\mathrm{Out},\mathrm{Re})]^{-1}\}\times\{1+Q^1(\mathrm{Out},\mathrm{Out})T(2)[Q^1(\mathrm{Out},\mathrm{Re})]^{-1}\}^{-1}\,. \tag{2.35}$$

Here

$$T(i)=-Q^i(\mathrm{Re},\mathrm{Re})[Q^i(\mathrm{Out},\mathrm{Re})]^{-1} \tag{2.36}$$

is the T matrix of a homogeneous scatterer bounded by the surface S_i and with media constants ϵ_i, μ_i. As in the case of two separate homogeneous scatterers,[3] the various terms obtained by a formal expansion of the inverse in (2.35) can be interpreted as various multiple-scattering contributions to the total T matrix. In such an expansion there occur, besides the individual $T(i)$ matrices, Q^1 matrices which, in accordance with the form of $T(1)$, can be associated with a passage of a wave out through S_1 [and thus a $(Q^1)^{-1}$ factor is associated with a passage in through S_1, and a factor (with the appropriate arguments of the Q^1's) $(Q^1)^{-1}Q^1$ represents a reflection at the inside of S_1]. The first few terms in the expansion of (2.35) can be depicted as in Fig. 2 and one easily sees that in general one has exactly the terms expected from a multiple-scattering picture.

III. THE T MATRIX FOR A MULTILAYERED SCATTERER

The extension of the results of Sec. II to the case of a scatterer which consists of an arbitrary number of consecutively enclosing surfaces S_i, $i=1,\ldots,N$, is straightforward. The notations of Fig. 1 are generalized in an obvious way so that we now have the wave vector k_i, the relative media constants ϵ_i, μ_i, and the fields $\vec{E}^i$, $\vec{H}^i$, etc. in the space between S_i and S_{i+1}. As before, it follows that we must consider the Green's function $g(k_i|\vec{r}-\vec{r}'|)$. For the expansions of the surface fields we now assume

$$\hat{n}_i\times\vec{E}^i_-=\sum_n\hat{n}_i\times[\alpha_n^i\,\mathrm{Re}\vec{\psi}_n(k_i\vec{r})+\beta_n^i\vec{\psi}_n(k_i\vec{r})]\,, \tag{3.1}$$

and thus

$$\hat{n}_i\times\vec{H}^i_-=(ik\mu_i)^{-1}\sum_n\hat{n}_i\times[\alpha_n^i\nabla\times\mathrm{Re}\vec{\psi}_n(k_i\vec{r})+\beta_n^i\nabla\times\vec{\psi}_n(k_i\vec{r})] \tag{3.2}$$

for $i=1,2,\ldots,N-1$, while for $i=N$ we assume

$$\hat{n}_N\times\vec{E}^N_-=\sum_n\alpha_n^N\hat{n}_N\times\mathrm{Re}\vec{\psi}_n(k_N\vec{r})\,, \tag{3.3}$$

and thus

$$\hat{n}_N\times\vec{H}^N_-=(ik\mu_N)^{-1}\sum_n\alpha_n^N\hat{n}_N\times[\nabla\times\mathrm{Re}\vec{\psi}_n(k_N\vec{r})]\,. \tag{3.4}$$

By considering Poincaré-Huygens principle for the volume between S_i and S_{i+1} we obtain, in complete analogy with the derivation of (2.31)–(2.34), the following set of equations

$$\vec{a} = i[Q^1(\mathrm{Out},\mathrm{Re})\vec{\alpha}^{\,1} + Q^1(\mathrm{Out},\mathrm{Out})\vec{\beta}^{\,1}], \quad (3.5)$$

$$\vec{f} = -i[Q^1(\mathrm{Re},\mathrm{Re})\vec{\alpha}^{\,1} + Q^1(\mathrm{Re},\mathrm{Out})\vec{\beta}^{\,1}], \quad (3.6)$$

$$\vec{\alpha}^{\,i} = i[Q^{i+1}(\mathrm{Out},\mathrm{Re})\vec{\alpha}^{\,i+1} + Q^{i+1}(\mathrm{Out},\mathrm{Out})\vec{\beta}^{\,i+1}], \quad (3.7)$$

$$\vec{\beta}^{\,i} = -i[Q^{i+1}(\mathrm{Re},\mathrm{Re})\vec{\alpha}^{\,i+1} + Q^{i+1}(\mathrm{Re},\mathrm{Out})\vec{\beta}^{\,i+1}], \quad (3.8)$$

where $i = 1, \ldots, N-1$ and $\vec{\beta}^{\,N} \equiv 0$. The matrices Q^{i+1} are defined in analogy with (2.28), i.e.,

$$Q^{i+1}_{nn'}(\mathrm{Re},\mathrm{Re}) \equiv k_i \int_{S_{i+1}} dS'\hat{n}_{i+1}\cdot\{[\nabla'\times\mathrm{Re}\vec{\psi}_n(k_i\vec{r}\,')]\times\mathrm{Re}\vec{\psi}_{n'}(k_{i+1}\vec{r}\,') + \mu_i\mu_{i+1}^{-1}\mathrm{Re}\vec{\psi}_n(k_i\vec{r}\,')\times[\nabla'\times\mathrm{Re}\vec{\psi}_{n'}(k_{i+1}\vec{r}\,')]\} \quad (3.9)$$

and similarly for the other combinations of Re and Out. The algebraic solution to this set of equations can be written down immediately. It is then convenient to introduce the following systematic notation:

$$\vec{\alpha}^{\,i} \equiv \vec{\alpha}^{\,i,1}, \quad \vec{\beta}^{\,i} \equiv \vec{\alpha}^{\,i,2}, \quad \vec{a} \equiv \vec{\alpha}^{\,0,1}, \quad \vec{f} \equiv \vec{\alpha}^{\,0,2},$$

$$\begin{pmatrix} Q^i(\mathrm{Out},\mathrm{Re}) & Q^i(\mathrm{Out},\mathrm{Out}) \\ Q^i(\mathrm{Re},\mathrm{Re}) & Q^i(\mathrm{Re},\mathrm{Out}) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{Q}^i_{11}, & \mathcal{Q}^i_{12} \\ \mathcal{Q}^i_{21}, & \mathcal{Q}^i_{22} \end{pmatrix} \equiv \mathcal{Q}^i,$$

and if we use an additional vector notation for the two components $\vec{\alpha}^{\,i,1}$ and $\vec{\alpha}^{\,i,2}$, i.e.,

$$\underline{\vec{\alpha}}^{\,i} \equiv \begin{Bmatrix} \vec{\alpha}^{\,i,1} \\ \vec{\alpha}^{\,i,2} \end{Bmatrix},$$

the equations (3.5)–(3.8) may be written

$$\underline{\vec{\alpha}}^{\,i} = \mathcal{Q}^{i+1}\underline{\vec{\alpha}}^{\,i+1}, \quad i = 0, 1, \ldots, N-1 \quad (3.10)$$

i.e.,

$$\underline{\vec{\alpha}}^{\,0} = \mathcal{Q}^1\mathcal{Q}^2\cdots\mathcal{Q}^N\underline{\vec{\alpha}}^{\,N}, \quad (3.11)$$

and since

$$\underline{\vec{\alpha}}^{\,N} \equiv \begin{Bmatrix} \vec{\alpha}^{\,N} \\ 0 \end{Bmatrix},$$

we have

$$\vec{\alpha}^{\,0,1} \equiv \vec{a} = (\mathcal{Q}^1\mathcal{Q}^2\cdots\mathcal{Q}^N)_{11}\vec{\alpha}^{\,N}, \quad (3.12)$$

$$\vec{\alpha}^{\,0,2} \equiv \vec{f} = (\mathcal{Q}^1\mathcal{Q}^2\cdots\mathcal{Q}^N)_{21}\vec{\alpha}^{\,N}, \quad (3.13)$$

i.e., the total T matrix for the N-layered scatterer, denoted $T(1, 2, \ldots, N)$ is

$$T(1,2,\ldots,N) = (\mathcal{Q}^1\mathcal{Q}^2\cdots\mathcal{Q}^N)_{21}(\mathcal{Q}^1\mathcal{Q}^2\cdots\mathcal{Q}^N)_{11}^{-1}. \quad (3.14)$$

However, for numerical applications it is relevant to note that the structure of the equations (3.5)–(3.8) is such that we have a recursion relation for the T matrix itself. The recursion relation is obtained simply by noting that $\vec{\alpha}^{\,i}$ and $\vec{\beta}^{\,i}$ are related by the total T matrix for the $(N-i)$-layered scatterer whose outer surface is S_{i+1}. This T matrix is denoted $T(i+1, \ldots, N)$, i.e., we have $\vec{\beta}^{\,i} = T(i+1, \ldots, N)\vec{\alpha}^{\,i}$, and thus (3.7) and (3.8) can be written

$$\vec{\alpha}^{\,i} = i[Q^{i+1}(\mathrm{Out},\mathrm{Re})\vec{\alpha}^{\,i+1} + Q^{i+1}(\mathrm{Out},\mathrm{Out})T(i+2,\ldots N)\vec{\alpha}^{\,i+1}], \quad (3.15)$$

$$\vec{\beta}^{\,i} = -i[Q^{i+1}(\mathrm{Re},\mathrm{Re})\vec{\alpha}^{\,i+1} + Q^{i+1}(\mathrm{Re},\mathrm{Out})T(i+2,\ldots,N)\vec{\alpha}^{\,i+1}], \quad (3.16)$$

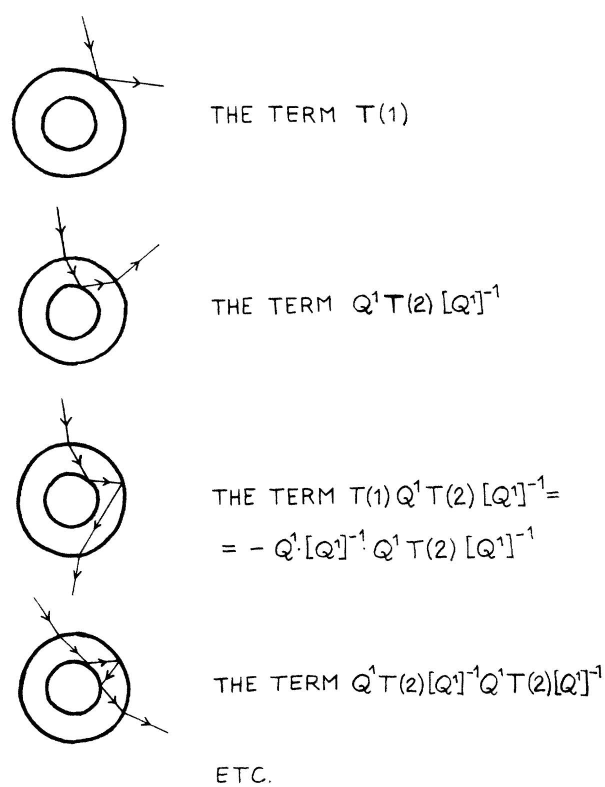

FIG. 2. Multiple-scattering interpretation of Eq. (2.35).

from which we find

$$T(i+1,\ldots,N)=\{T(i+1)-Q^{i+1}(\text{Re, Out})T(i+2,\ldots,N)[Q^{i+1}(\text{Out, Re})]^{-1}\}\times\{1+Q^{i+1}(\text{Out, Out})T(i+2,\ldots,N)[Q^{i+1}(\text{Out, Re})]^{-1}\}^{-1}, \quad (3.17)$$

of which (2.35) is the simplest special case.

IV. THE T MATRIX FOR SEVERAL MULTILAYERED SCATTERERS AND OTHER CONFIGURATIONS

We shall now combine the results of Secs. II and III with those of Ref. 3 and obtain the total T matrix for new classes of scattering configurations. In deriving these new T matrices we shall take some care in referring back to the original equations so as to demonstrate explicitly how the results combine. We consider first the case of several multilayered scatterers. The case of two scatterers each having two layers is generic and from the explicit results for this case the conclusions for the general case are immediate. Thus we shall consider the configuration depicted in Fig. 3. For the surface fields we write ($\hat{n}_{ij}$ is the unit normal vector on S_{ij}, pointing outwards)

$$\hat{n}_{ij}\times\vec{E}_-^{ij}=\sum_n \hat{n}_{ij}\times[\alpha_n^{ij}\,\text{Re}\vec{\psi}_n(k_{ij}\vec{r}''_{ij})+\beta_n^{ij}\vec{\psi}_n(k_{ij}\vec{r}''_{ij})], \quad i,j=1,2 \quad (4.1)$$

and

$$\hat{n}_{ij}\times\vec{H}_-^{ij}=(ik\mu_{ij})^{-1}\sum_n \hat{n}_{ij}\times[\alpha_n^{ij}\nabla''\times\text{Re}\vec{\psi}_n(k_{ij}\vec{r}''_{ij})+\beta_n^{ij}\nabla''\times\vec{\psi}_n(k_{ij}\vec{r}''_{ij})], \quad i,j=1,2 \quad (4.2)$$

where $\beta_n^{i2}=0$, $i=1,2$. We shall now repeat the main steps of the derivations of Sec. III of Ref. 3, with the appropriate modifications. Thus we first apply the Poincaré-Huygens principle to the volume outside S_{11} and S_{21}, with a source term corresponding to the incoming field, and get

$$\left.\begin{matrix}\vec{E}(\vec{r})\\ 0\end{matrix}\right\}=\vec{E}^{\text{inc}}(\vec{r})+\nabla\times\int_{S_{11}+S_{21}}dS'k(\hat{n}\times\vec{E}_+)g(k|\vec{r}-\vec{r}'|)+\nabla\times\left(\nabla\times\int_{S_{11}+S_{21}}dS'i(\hat{n}\times\vec{H}_+)g(k|\vec{r}-\vec{r}'|)\right)$$

$$\text{for}\begin{cases}\vec{r}\text{ outside }S_{11}\text{ and }S_{21},\\ \vec{r}\text{ inside }S_{11}\text{ or }S_{21}.\end{cases} \quad (4.3)$$

By considering $\vec{r}$ outside a sphere with center in 0 and containing S_{11} and S_{21} we get, after (4.1) and (4.2) have been introduced by means of the boundary conditions and after expanding the Green's function as in (2.23) and then comparing the coefficients of $\vec{\psi}_n(k\vec{r})$,

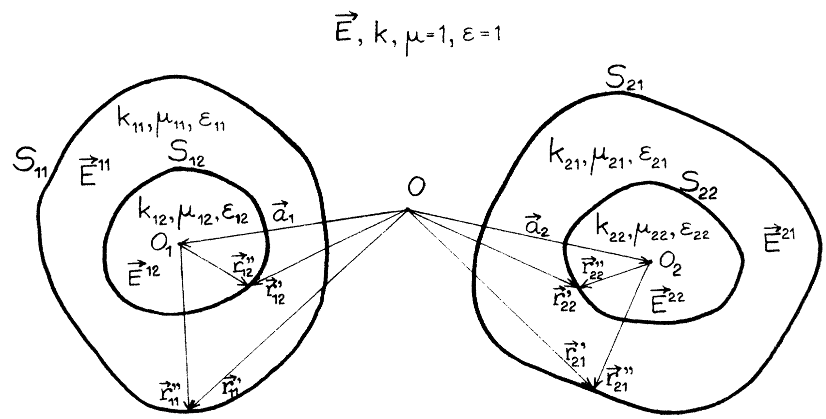

FIG. 3. Geometry and notation for two two-layered scatterers.

$$f_n=(-ik)\sum_{i=1,2}\left[\int_{S_{i1}} dS'\hat{n}_{i1}\cdot\Big(\sum_{n'}\{[\nabla'\times \mathrm{Re}\vec{\psi}_n(k\vec{r}'_{i1})]\times \mathrm{Re}\vec{\psi}_{n'}(k_{i1}\vec{r}''_{i1})+\mu_{i1}^{-1}\mathrm{Re}\vec{\psi}_n(k\vec{r}'_{i1})\times[\nabla''\times \mathrm{Re}\vec{\psi}_{n'}(k_{i1}\vec{r}''_{i1})]\}\alpha^{i1}_{n'} + \sum_{n'}\{[\nabla'\times \mathrm{Re}\vec{\psi}_n(k\vec{r}'_{i1})]\times\vec{\psi}_n(k_{i1}\vec{r}''_{i1})+\mu_{i1}^{-1}\mathrm{Re}\vec{\psi}_n(k\vec{r}'_{i1})\times[\nabla''\times\vec{\psi}_{n'}(k_{i1}\vec{r}''_{i1})]\}\beta^{i1}_{n'}\Big)\right], \qquad (4.4)$$

where $\{f_n\}$ are the expansions coefficients for $\vec{E}^{sc}$ as in (2.1). Here $\vec{r}'_{i1}=\vec{a}_i+\vec{r}''_{i1}$, $i=1,2$, and a translation of the origin of the $\mathrm{Re}\vec{\psi}_n$ functions gives

$$\mathrm{Re}\vec{\psi}_n(k\vec{r}'_{i1})=\mathrm{Re}\vec{\psi}_n(k(\vec{a}_i+\vec{r}''_{i1})) =\sum_{n''}R_{nn''}(\vec{a}_i)\,\mathrm{Re}\vec{\psi}_{n''}(k\vec{r}''_{i1}). \qquad (4.5)$$

The translation matrix $R_{nn''}$ is treated in detail in Ref. 3. It follows that (4.4) may be written

$$\vec{f}=-i\sum_{i=1,2}R(\vec{a}_i)[Q^{i1}(\mathrm{Re},\mathrm{Re})\vec{\alpha}^{i1} +Q^{i1}(\mathrm{Re},\mathrm{Out})\vec{\beta}^{i1}], \qquad (4.6)$$

where the matrices Q^{ij} are defined in complete analogy with (3.9), the integration now being over the surface S_{ij}. Similarly, if we consider $\vec{r}$ inside the inscribed sphere of S_{11}, we find

$$\begin{aligned}\sum_{n'}a_{n'}R_{n'n}(\vec{a}_1)=(ik)\sum_{n'}\Big(&\int_{S_{11}}dS'\hat{n}_{11}\cdot\{[\nabla''\times\vec{\psi}_n(k\vec{r}''_{11})]\times\mathrm{Re}\vec{\psi}_{n'}(k_{11}\vec{r}''_{11})+\mu_{11}^{-1}\vec{\psi}_n(k\vec{r}''_{11})\times[\nabla''\times\mathrm{Re}\vec{\psi}_{n'}(k_{11}\vec{r}''_{11})]\}\alpha^{11}_{n'}\\ &+\int_{S_{11}}dS'\hat{n}_{11}\cdot\{[\nabla''\times\vec{\psi}_n(k\vec{r}''_{11})]\times\vec{\psi}_{n'}(k_{11}\vec{r}''_{11})+\mu_{11}^{-1}\vec{\psi}_n(k\vec{r}''_{11})\times[\nabla''\times\vec{\psi}_{n'}(k_{11}\vec{r}''_{11})]\}\beta^{11}_{n'}\\ &+\int_{S_{21}}dS'\hat{n}_{21}\cdot\{[\nabla''\times\vec{\psi}_n(k(\vec{r}''_{21}+\vec{a}_2-\vec{a}_1))]\times\mathrm{Re}\vec{\psi}_{n'}(k_{21}\vec{r}''_{21})\\ &\qquad+\mu_{21}^{-1}\vec{\psi}_n[k(\vec{r}''_{21}+\vec{a}_2-\vec{a}_1)]\times[\nabla''\times\mathrm{Re}\vec{\psi}_{n'}(k_{21}\vec{r}''_{21})]\}\alpha^{21}_{n'}\\ &+\int_{S_{21}}dS'\hat{n}_{21}\cdot\{[\nabla''\times\vec{\psi}_n(k(\vec{r}''_{21}+\vec{a}_2-\vec{a}_1))]\times\vec{\psi}_{n'}(k_{21}\vec{r}_{21})\\ &\qquad+\mu_{21}^{-1}\vec{\psi}_{n'}(k(\vec{r}''_{21}+\vec{a}_2-\vec{a}_1))\times[\nabla''\times\vec{\psi}_{n'}(k_{21}\vec{r}''_{21})]\}\beta^{21}_{n'}\Big),\end{aligned} \qquad (4.7)$$

where we have used

$$\begin{aligned}\vec{r}-\vec{r}'&\equiv\vec{r}-\vec{r}'_{21}\\ &=\vec{r}_1+\vec{a}_1-\vec{r}'_{21}\\ &=\vec{r}_1-(\vec{r}''_{21}+\vec{a}_2-\vec{a}_1)\end{aligned}$$

in the Green's function in the integrals over S_{21} (thus we have $r_1<|\vec{r}''_{21}+\vec{a}_2-\vec{a}_1|$), and where $\{a_n\}$ are the expansion coefficients for E^{inc} as in (2.5). In order to be able to express (4.7) in terms of Q^{ij} matrices we now assume that the configuration of S_{11} and S_{21} is such that $r''_{21}<|\vec{a}_2-\vec{a}_1|$. When this condition is fulfilled (cf. the discussion in Ref. 3), the change of origin for the $\vec{\psi}_n$ functions leads to the expansion

$$\vec{\psi}_n(k(\vec{r}''_{21}+\vec{a}_2-\vec{a}_1))=\sum_{n''}\sigma_{nn''}(\vec{a}_2-\vec{a}_1) \times\mathrm{Re}\vec{\psi}_{n''}(k\vec{r}''_{21}). \qquad (4.8)$$

The translation matrix $\sigma_{nn''}$ is treated in detail in Ref. 3. In this way we get the equation

$$R^t(\vec{a}_1)\vec{a}=i\{Q^{11}(\mathrm{Out},\mathrm{Re})\vec{\alpha}^{11}+Q^{11}(\mathrm{Out},\mathrm{Out})\vec{\beta}^{11} +\sigma(\vec{a}_2-\vec{a}_1)[Q^{21}(\mathrm{Re},\mathrm{Re})\vec{\alpha}^{21} +Q^{21}(\mathrm{Re},\mathrm{Out})\vec{\beta}^{21}]\}, \qquad (4.9)$$

where R^t denotes the transpose of R. In exactly the same way we obtain, by considering $\vec{r}$ inside the inscribed sphere of S_{21},

$$R^t(\vec{a}_2)\vec{a}=i\{\sigma(\vec{a}_1-\vec{a}_2)[Q^{11}(\mathrm{Re},\mathrm{Re})\vec{\alpha}^{11} +Q^{11}(\mathrm{Re},\mathrm{Out})\vec{\beta}^{11}] +Q^{21}(\mathrm{Out},\mathrm{Re})\vec{\alpha}^{21}+Q^{21}(\mathrm{Out},\mathrm{Out})\vec{\beta}^{21}\}. \qquad (4.10)$$

The equations obtained by invoking the boundary conditions on S_{12} and S_{22} are the same as given before in (2.33) and (2.34), i.e., we now get

$$\vec{\alpha}^{i1}=iQ^{i2}(\mathrm{Out},\mathrm{Re})\vec{\alpha}^{i2},\quad i=1,2 \qquad (4.11)$$

$$\vec{\beta}^{i1}=-iQ^{i2}(\mathrm{Re},\mathrm{Re})\vec{\alpha}^{i2},\quad i=1,2. \qquad (4.12)$$

We introduce

$$\vec{a}^j\equiv i[Q^{j1}(\mathrm{Out},\mathrm{Re})\vec{\alpha}^{j1}+Q^{j1}(\mathrm{Out},\mathrm{Out})\vec{\beta}^{j1}],\quad j=1,2 \qquad (4.13)$$

$$\vec{f}^j\equiv -i[Q^{j1}(\mathrm{Re},\mathrm{Re})\vec{\alpha}^{j1}+Q^{j1}(\mathrm{Re},\mathrm{Out})\vec{\beta}^{j1}],\quad j=1,2. \qquad (4.14)$$

From Sec. II [cf. Eqs. (2.31)–(2.34)] it follows that $\vec{a}^i$ and $\vec{f}^i$ are related by the total T matrix for the two-layered scatterer whose outer surface is S_{i1}. This T matrix will be denoted $T(i1,i2)$, i.e., we have

$$\vec{f}^i = T(i1, i2)\vec{a}^i, \quad i = 1, 2. \tag{4.15}$$

Thus (4.6), (4.9), and (4.10) can be written

$$R^t(\vec{a}_1)\vec{a} = \vec{a}^1 - \sigma(\vec{a}_2 - \vec{a}_1)T(21, 22)\vec{a}^2, \tag{4.16}$$

$$R^t(\vec{a}_2)\vec{a} = -\sigma(\vec{a}_1 - \vec{a}_2)T(11, 12)\vec{a}^1 + \vec{a}^2, \tag{4.17}$$

$$\vec{f} = R(\vec{a}_1)T(11, 12)\vec{a}^1 + R(\vec{a}_2)T(21, 22)\vec{a}^2. \tag{4.18}$$

Equations (4.16)–(4.18) should be compared with Eqs. (3.11), (3.12), and (3.13) of Ref. 3. If the relation $T(i) = -\mathrm{Re}Q^{it} \cdot (Q^{it})^{-1}$, $i = 1, 2$ (in the notation of Ref. 3) is introduced into these equations, they read

$$R^t(\vec{a}_1)\vec{a} = Q^{1t}\vec{\alpha}^1 - \sigma(\vec{a}_2 - \vec{a}_1)T(2)Q^{2t}\vec{\alpha}^2, \tag{4.19}$$

$$R^t(\vec{a}_2)\vec{a} = -\sigma(\vec{a}_1 - \vec{a}_2)T(1)Q^{1t}\vec{\alpha}^1 + Q^{2t}\vec{\alpha}^2, \tag{4.20}$$

$$\vec{f} = R(\vec{a}_1)T(1)Q^{1t}\vec{\alpha}^1 + R(\vec{a}_2)T(2)Q^{2t}\vec{\alpha}^2. \tag{4.21}$$

[Incidentally we may remark here that this way of writing Eqs. (3.11), (3.12), and (3.13) of Ref. 3 shows immediately that the total T matrix for two scatterers can be expressed exclusively in terms of $T(1)$, $T(2)$, and the translation matrices R and σ; this fact is, of course, also obtained from a detailed consideration of the iteration procedure as in Ref. 3.] From a comparison between (4.16)–(4.18) and (4.19)–(4.21) it is clear that the T matrix connecting the $\vec{a}$ and $\vec{f}$ of (4.16)–(4.18) is of the same form as the T matrix connecting the $\vec{a}$ and $\vec{f}$ of (4.19)–(4.21), but with $T(1)$ and $T(2)$ replaced by $T(11, 12)$ and $T(21, 22)$, respectively, i.e., the total T matrix of a configuration of two two-layered scatterers as in Fig. 3, having the individual total T matrices $T(11, 12)$ and $T(21, 22)$ and denoted $T(11, 12; 21, 22)$, is given by[3]

$$\begin{aligned} T(11, 12; 21, 22) = {} & R(\vec{a}_1)\{T(11, 12)[1 - \sigma(\vec{a}_2 - \vec{a}_1)T(21, 22)\sigma(\vec{a}_1 - \vec{a}_2)T(11, 12)]^{-1} \\ & \times [1 + \sigma(\vec{a}_2 - \vec{a}_1)T(21, 22)R(\vec{a}_1 - \vec{a}_2)]\}R(-\vec{a}_1) \\ & + R(\vec{a}_2)\{T(21, 22)[1 - \sigma(\vec{a}_1 - \vec{a}_2)T(11, 12)\sigma(\vec{a}_2 - \vec{a}_1)T(21, 22)]^{-1} \\ & \times [1 + \sigma(\vec{a}_1 - \vec{a}_2)T(11, 12)R(\vec{a}_2 - \vec{a}_1)]\}R(-\vec{a}_2). \end{aligned} \tag{4.22}$$

It is now a straightforward matter to consider the case of an arbitrary number of multilayered scatterers, where the number of layers can be chosen independently for each scatterer, along the same lines and it is easy to see that the same situation arises, i.e., the result is that given in Sec. IV of Ref. 3, the only difference being that the individual T matrices $T(i)$, $i = 1, 2, \ldots, N$ considered there are replaced by the more complicated individual T matrices of Sec. III for the multilayered scatterers.

This result might certainly be said to be an expected one. We emphasize that the result (3.14) of Ref. 3, and thus Eq. (4.22), was obtained under the assumption that each of the scatterers are bounded by one closed outer surface. We have gone into some detail here in showing that the results of Ref. 3 also hold for scatterers having the internal structure considered in Sec. III.

The multilayered scatterers considered so far have consisted of consecutively enclosing surfaces. However, the T-matrix description is also suitable for a situation where one has at some stage a surface which contains several closed surfaces which enclose regions, each of which is characterized by different relative media constants. The generic case is that depicted in Fig. 4. Outside a sphere with center in 0 and enclosing S_2 and S_3 (and possibly in a larger region) the $\vec{E}^1$ and $\vec{H}^1$ fields can be expressed as a linear combination of a regular and an outgoing part and the connection between these parts is given by a T matrix, namely, the T matrix corresponding to the configuration S_2 *and* S_3. Thus we now assume that on S_1 we have the expansion

$$\hat{n}_1 \times \vec{E}^1_- = \sum_n \hat{n}_1 \times [\alpha^1_n \mathrm{Re}\vec{\psi}_n(k_1\vec{r}') + \beta^1_n \vec{\psi}_n(k_1\vec{r}')], \tag{4.23}$$

and thus

$$\hat{n}_1 \times \vec{H}^1_- = (ik\mu_1)^{-1} \sum_n \hat{n}_1 \times [\alpha^1_n \nabla' \times \mathrm{Re}\vec{\psi}_n(k_1\vec{r}') + \beta^1_n \nabla' \times \vec{\psi}_n(k_1\vec{r}')], \tag{4.24}$$

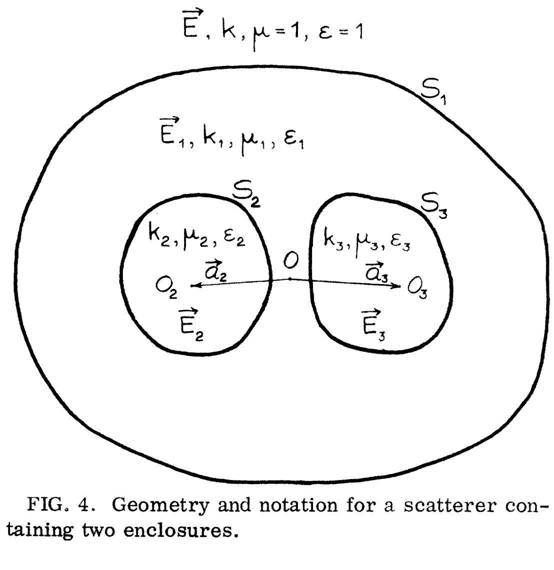

FIG. 4. Geometry and notation for a scatterer containing two enclosures.

where $\vec{\alpha}^1$ and $\vec{\beta}^1$ are connected by the T matrix $T(2;\,3)$ for S_2 and S_3 and where the form of $T(2;\,3)$ is given by Eq. (3.14) of Ref. 3. By invoking the boundary conditions on S_1 we get the equations (2.31) and (2.32) as before. Thus the only difference between (2.31)–(2.34) and the present case is that now $\vec{\alpha}^1$ and $\vec{\beta}^1$ are connected by $T(2;\,3)$ instead of by $T(2)$ as in (2.33) and (2.34), and it follows immediately that the total T matrix of the scatterer depicted in Fig. 4, which will be denoted by $T(1;\,2,3)$, is given by

$$T(1;\,2,3)=\{T(1)-Q^1(\mathrm{Re,Out})T(2;\,3)[Q^1(\mathrm{Out,Re})]^{-1}\}\times\{1+Q^1(\mathrm{Out,Out})T(2;\,3)[Q^1(\mathrm{Out,Re})]^{-1}\}^{-1}\,. \tag{4.25}$$

A case of interest in applications is that of a scatterer having a "displaced" enclosure, i.e., an enclosure situated in such a way that it is most convenient to use different origins for the coordinate systems used for the fields on the outer surface and for those on the enclosure. This case is obtained as a special case of (4.25) by letting, e.g., $S_3\to 0$, in which case $T(2;\,3)\to R(\vec{a}_2)T(2)R(-\vec{a}_2)$.

Taking into account the results of Sec. III and Ref. 3 it is now clear that the above results concerning enclosures generalize in the following way. From Ref. 3 it is clear that the case of S_1 containing an arbitrary number of separate enclosures is obtained by inserting the appropriate T matrix instead of $T(2,3)$ in (4.25). Furthermore, according to the discussion in connection with (4.22), it is clear that each of the enclosures may be multilayered. Here we have considered S_1 as the outer surface, imbedded in a homogeneous medium, but it is again clear that outside S_1 there may again be an arbitrary number of, e.g., consecutive enclosing surfaces. In fact, one can consider any combination of separate and consecutively enclosing surfaces and for each case immediately write down the form of the total T matrix according to the prescriptions given above and in Ref. 3.

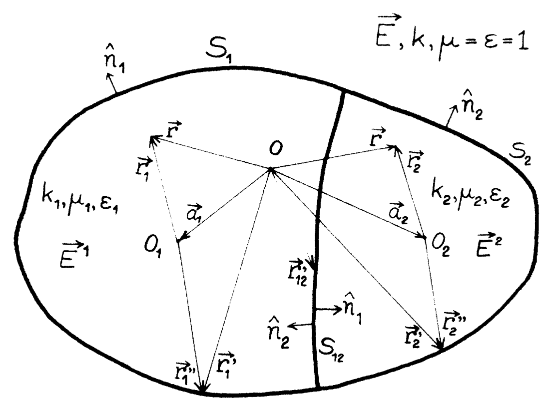

FIG. 5. Geometry and notation for a scatterer consisting of two nonenclosing parts.

Finally, we consider scatterers consisting of several nonenclosing parts, each characterized by arbitrary but constant values of μ and ϵ. The simplest type is that depicted in Fig. 5. A novel feature here is the appearance of edges on the surface enclosing a part with constant μ and ϵ. Edges are allowed by the basic regularity assumptions of the T-matrix formalism of Ref. 2. However, the appearance of edges results in special convergence problems in the numerical treatment, which are usually remedied by rounding them off, as discussed in Refs. 2 and 9. A scatterer of the type depicted in Fig. 5 can be treated by the formalism of Ref. 3 under the appropriate geometrical conditions. To see this we start by using the Poincaré-Huygens principle for the region outside S_1+S_2, and according to (2.9) we then have

$$\left.\begin{matrix}E(\vec{r})\\0\end{matrix}\right\}=\vec{E}^{\rm inc}(\vec{r})+\nabla\times\int_{S_1+S_2}dS'k[\hat{n}\times\vec{E}_+(\vec{r}\,')]G(k|\vec{r}-\vec{r}\,'|)+\nabla\times\left(\nabla\times\int_{S_1+S_2}dS'i[\hat{n}\times\vec{H}_+(\vec{r}\,')]G(k|\vec{r}-\vec{r}\,'|)\right)$$
$$\text{for}\begin{cases}\vec{r}\text{ outside }S_1+S_2\,,\\ \vec{r}\text{ inside }S_1+S_2\end{cases} \tag{4.26}$$

($\hat{n}=\hat{n}_i$ on S_i). The closed surfaces S_i+S_{12}, $i=1,2$ are denoted $\mathcal{S}_i$, and we note that for the outward pointing unit normal vectors $\hat{n}_i$ on $\mathcal{S}_i$ we have $\hat{n}_1=-\hat{n}_2$ on S_{12}. Let 0 in Fig. 5 be the origin for the expansions $\vec{E}^{\rm inc}(\vec{r})=\sum_n a_n\,\mathrm{Re}\vec{\psi}_n(k\vec{r})$ and $\vec{E}^{\rm sc}(\vec{r})=\sum_n f_n\vec{\psi}_n(k\vec{r})$. (Of course, in a particular example it might be more convenient to take 0 to coincide with either 0_1 or 0_2 but the structure of the resulting equations becomes more transparent by a general choice as in Fig. 5; cf. also the discussion in Ref. 3.) By considering $\vec{r}$ outside a sphere with center in 0 and containing S_1+S_2 we find, using (2.23) and comparing the coefficients for $\vec{\psi}_n(k\vec{r})$ (Ref. 2),

$$f_n=k^2\int_{S_1+S_2}dS'\{ik^{-1}(\hat{n}\times\vec{E}_+)\cdot[\nabla'\times\mathrm{Re}\vec{\psi}_n(k\vec{r}\,')]-(\hat{n}\times\vec{H}_+)\cdot\mathrm{Re}\vec{\psi}_n(k\vec{r}\,')\}\,, \tag{4.27}$$

where, according to the boundary conditions, $\hat{n}\times\vec{E}_+=\hat{n}_i\times\vec{E}^i_-$, $\hat{n}\times\vec{H}_+=\hat{n}_i\times\vec{H}^i_-$ on S_i. Furthermore, we have $\hat{n}_1\times\vec{E}^1_- = \hat{n}_1\times\vec{E}^2_-$, $\hat{n}_1\times\vec{H}^1_- = \hat{n}_1\times\vec{H}^2_-$ on S_{12}, i.e., $\hat{n}_1\times\vec{E}^1_- + \hat{n}_2\times\vec{E}^2_- = 0$, $\hat{n}_1\times\vec{H}^1_- + \hat{n}_2\times\vec{H}^2_- = 0$ since $\hat{n}_1 = -\hat{n}_2$ on S_{12}. Thus by adding the relevant combinations of

$$\int_{S_{12}} (\hat{n}_1\times\vec{E}^1_- + \hat{n}_2\times\vec{E}^2_-)\cdot[\nabla'\times \mathrm{Re}\vec{\psi}_n(k\vec{r}\,'_{12})]dS' = \int_{S_{12}} (\hat{n}_1\times\vec{H}^1_- + \hat{n}_2\times\vec{H}^2_-)\cdot \mathrm{Re}\vec{\psi}_n(k\vec{r}\,'_{12})dS' = 0 \quad (4.28)$$

to (4.27), this equation takes the form

$$f_n = k^2 \sum_{i=1,2} \int_{\mathcal{S}_i} dS'\{ik^{-1}(\hat{n}_i\times\vec{E}^i_-)\cdot[\nabla'\times\mathrm{Re}\vec{\psi}_n(k\vec{r}\,')] - (\hat{n}_i\times\vec{H}^i_-)\cdot\mathrm{Re}\vec{\psi}_n(k\vec{r}\,')\} \quad (4.29)$$

($\vec{r}\,' = \vec{r}\,'_i$ on S_i, $\vec{r}\,' = \vec{r}\,'_{12}$ on S_{12}). The expansion of the surface field on $\mathcal{S}_i$ is assumed to be

$$\hat{n}_i\times\vec{E}^i_- = \sum_n \alpha^i_n \hat{n}_i\times\mathrm{Re}\vec{\psi}_n(k_i\vec{r}_i\,''),$$

and thus

$$\hat{n}_i\times\vec{H}^i_- = (ik\mu_i)^{-1}\sum \alpha^i_n\, \hat{n}_i\times[\nabla'\times\mathrm{Re}\vec{\psi}_n(k_i\vec{r}_i\,'')]$$

(here $\vec{r}_i\,'' = \vec{r}_i\,' - \vec{a}_i$ on S_i and $\vec{r}_i\,'' = \vec{r}\,'_{12} - \vec{a}_i$ on S_{12}; cf. Fig. 5). We introduce these expansions into (4.29), write $\vec{r}\,' = \vec{a}_i + \vec{r}_i\,''$ on $\mathcal{S}_i$ and translate according to (4.5). In this way we obtain

$$\vec{f} = -i\sum_{i=1,2} R(\vec{a}_i)Q^i(\mathrm{Re},\mathrm{Re})\vec{\alpha}^{\,i}, \quad (4.30)$$

where Q^i is the Q matrix associated with the closed surface $\mathcal{S}_i$ [cf. (2.16)]. Here we note that Eq. (4.30) is identical to the equation which would be obtained if $\mathcal{S}_{1,2}$ were two separated closed surfaces defining two homogeneous scatterers as in Eq. (3.13) of Ref. 3. By taking $\vec{r}$ inside the inscribed spheres of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, we obtain in a similar way two equations for $\vec{a}$ which are identical to Eqs. (3.11) and (3.12) of Ref. 3, i.e., we get

$$R^t(\vec{a}_1)\vec{a} = i[Q^1(\mathrm{Out},\mathrm{Re})\vec{\alpha}^1 + \sigma(\vec{a}_2-\vec{a}_1)Q^2(\mathrm{Re},\mathrm{Re})\vec{\alpha}^2],$$

$$R^t(\vec{a}_2)\vec{a} = i[\sigma(\vec{a}_1-\vec{a}_2)Q^1(\mathrm{Re},\mathrm{Re})\vec{\alpha}^1 + Q^2(\mathrm{Out},\mathrm{Re})\vec{\alpha}^2].$$

In deriving these equations we have again made the geometrical assumptions which are necessary for (4.8) to hold, i.e., $|\vec{r}\,'| < |\vec{a}_1 - \vec{a}_2|$, where $\vec{r}\,'$ stands for $\vec{r}\,'_1$, $\vec{r}\,'_2$, or $\vec{r}\,'_{12}$ (more details of this derivation are given in Ref. 3). It is clear that a very wide class of scatterers of the kind illustrated in Fig. 5 fulfill these conditions (we recall that the choices of the positions of the coordinate origins 0_1, 0_2, and 0 are largely arbitrary and mainly dictated by convenience). Thus the total T matrix of the scatterer in Fig. 5 is given by an expression analogous to (4.22) with $T(11,12)$ and $T(21,22)$ replaced by the T matrices associated with each of the two parts $\mathcal{S}_1$ and $\mathcal{S}_2$ of the scatterer.

Furthermore, it is now clear that the above conclusions generalize to the case of a scatterer consisting of an arbitrary number of parts, each with arbitrary but constant values of μ and ϵ, provided the obvious generalization of the condition $|\vec{r}\,'| < |\vec{a}_1 - \vec{a}_2|$ is fulfilled for all parts of the scatterer. Thus the total T matrix is then obtained by means of the method described in Sec. IV of Ref. 3.

It may appear as if we have actually complicated the solution of the problem by introducing the vanishing integrals in (4.28) into (4.27). Thereby the physical fact that the tangential fields are continuous across the intersecting surfaces becomes the key to the structure of the equations, but it is not used explicitly later on. However, the above procedure seems to be necessary if we want to extract equations for $\vec{f}$, $\vec{a}$, etc. for a general case.

V. DISCUSSION AND NUMERICAL APPLICATIONS

In the preceding sections we have shown how the T-matrix description of electromagnetic scattering from homogeneous scatterers[2,3] can be generalized to the case of multilayered scatterers. In particular it was shown that the structure of the multiple-scattering equations given in Ref. 3 remains unchanged when the scatterers have the more complicated structure considered in Secs. III and IV. This is true in still more general situations, which will be considered elsewhere. The main observation is that the Poincaré-Huygens principle gives, irrespective of the structure of the scatterers, a set of $N+1$ equations which completely determine the scattering.

Previously, electromagnetic scattering from multilayered scatterers has been considered mainly for a scatterer with concentric spherical layers. The equations for the case of an arbitrary number of such layers are given in Ref. 10, together with several numerical results and references to earlier works. Some work has been done on nonconcentric spherical inclusions in spheres, such as Refs. 11 and 12 for the electromagnetic case and Ref. 13 for the acoustic case. In Ref. 12 the scattering from two nonconcentric plasma cylinders is treated by means of an application of Waterman's matrix formulation.

The formulas given in the previous sections for the total T matrix for the various configurations considered represent exact solutions to the governing equations. In order to obtain numerical results we consider the truncated solutions. A closer study of these truncated solutions is, of

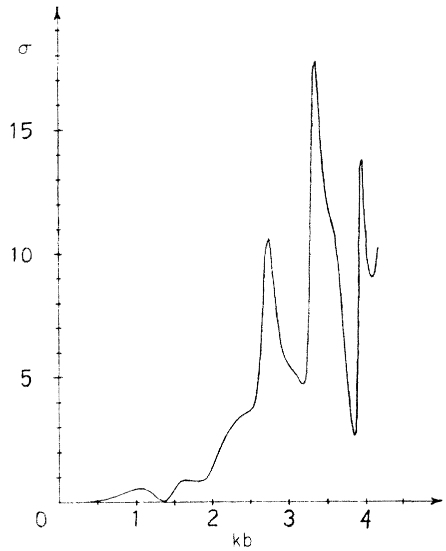

FIG. 6. Normalized backscattering cross section for a homogeneous dielectric sphere with $k_1 = 2k$ and radius b, as a function of kb.

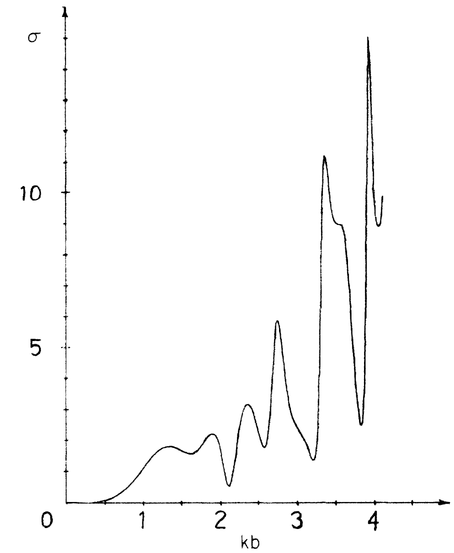

FIG. 7. Normalized backscattering cross section for a dielectric shell, with $k_1 = 2k$ and outer radius b, enclosing a concentric infinitely conducting sphere of radius a, where $a = \frac{1}{3}b$. The cross section is given as a function of kb.

course, of primary interest. Here we restrict ourselves to a consideration of some of the very simplest configurations. Since we primarily want to illustrate the structure of the solutions we take all surfaces involved to be spherical, i.e., all the individual T matrices involved are diagonal. Of course the computer time required increases rapidly if more general individual T matrices are considered. Thus we shall consider the case of a dielectric sphere containing one or two infinitely conducting spheres. The radius of the dielectric sphere is b and that of the infinitely conducting sphere (spheres) is a.

We consider first the case of one infinitely conducting spherical enclosure and we take $b = 3a$ and $k_1 = 2k$. One might then study, e.g., the backscattering cross section as a function of several parameters, such as the distance between the centers of the spheres, the polarization of the incoming wave, the incidence angles, and the wavelength. We content ourselves by illustrating a couple of these dependences (the others could equally well be handled within our formalism and by the computer programs developed). The distance between the centers of the spheres is denoted c and is taken along the positive z direction. Origo is at the center of the outer sphere. The incoming field is taken as a plane wave with the electric vector orthogonal to the z axis. For reference we have first computed the normalized backscattering cross section,

$$\sigma \equiv \lim_{r\to\infty} \frac{4\pi r^2}{\pi b^2} \frac{|\vec{E}^{sc}|^2}{|\vec{E}^{inc}|^2},$$

for a homogeneous dielectric sphere (radius b, $k_1 = 2k$) and for a dielectric sphere with a concentric infinitely conducting spherical enclosure ($b = 3a$, $k_1 = 2k$) as a function of kb up to $kb \approx 4$. These cross sections are given in Figs. 6 and 7, respectively. Figures 8 to 13 show σ as a function of the angle θ' to the positive z axis of the incoming wave vector for three different values kb and for each of these values, for six values of the translation c. For comparison we have also indicated in these figures the corresponding values for a homogeneous dielectric sphere with $k_1 = 2k$ and radius b and for an infinitely conducting sphere of radius a.

Second, we consider a dielectric sphere with $k_1 = 2k$, containing two infinitely conducting spheres situated symmetrically on the z axis and having a distance $2d$ between their centers (i.e., $\vec{a}_2 = -\vec{a}_3$, $|\vec{a}_2| = d$ in Fig. 4). The electric vector of the incoming plane wave is again orthogonal to the z axis. The square F of the absolute value of the backscattering amplitude, defined by

$$F \equiv \lim_{r\to\infty} (kr)^2 \frac{|\vec{E}^{sc}|^2}{|\vec{E}^{inc}|^2},$$

is presented as a function of the angle to the positive z axis of the incoming wave vector in Fig. 14. For comparison, the corresponding diagram for two infinitely conducting spheres, without the

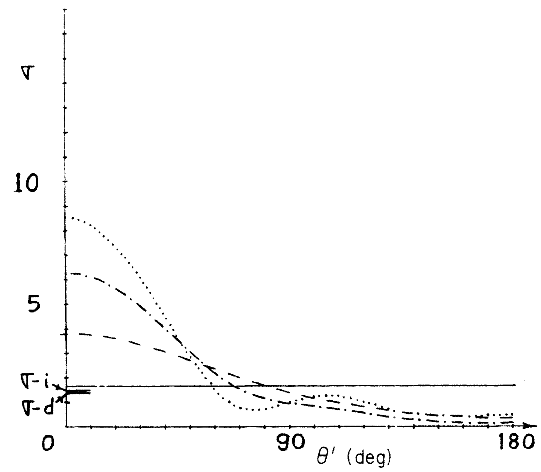

FIG. 8. Normalized backscattering cross section for a dielectric sphere, with $k_1 = 2k$ and radius b, enclosing an infinitely conducting sphere of radius $a = \frac{1}{3}b$, whose center is displaced the amount c along the positive z axis. The electric vector of the incoming field is orthogonal to the z axis and the cross section is given for $kb = 2$ as a function of the angle θ' between the positive z axis and the incoming wave vector for $c/b = 0, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}$, where $c/b = 0$ (solid line), $c/b = \frac{1}{9}$ (dashed line), $c/b = \frac{2}{9}$ (dot-dashed line), $c/b = \frac{3}{9}$ (dotted line). For comparison the backscattering cross sections σ-d and σ-i for a homogeneous dielectric sphere of radius b and an infinitely conducting sphere of radius a, respectively, are also indicated in the figure.

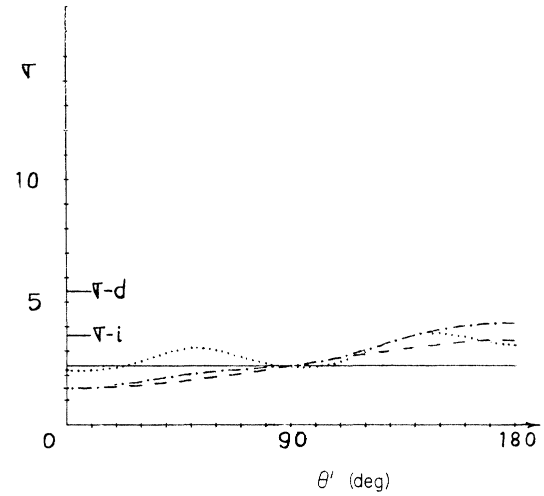

FIG. 10. Same as in Fig. 8, but now with $kb = 3$.

surrounding dielectric sphere, is given in Fig. 15.

Even for the relatively simple scattering configurations which we have considered in our numerical computations there are clearly many more parameter dependences which ought to be studied before one can get anything like a complete picture of the characteristics of these scattering configurations. In the simple case of concentric spheres one generally encounters a resonance-type behavior which can be understood as a result of a wave-guide effect in the dielectric shell (cf., e.g., the discussion in Chap. 5 of Ref. 10). When the enclosed infinitely conducting sphere is nonconcentric this effect is expected to be less marked, and more extensive calculations in this direction would be of interest. As is illustrated in Figs. 7–12, a displacement of the enclosed infinitely conducting sphere can result in a substantial change of the backscattering cross section (up to an order of magnitude, cf. Fig. 9). In Figs.

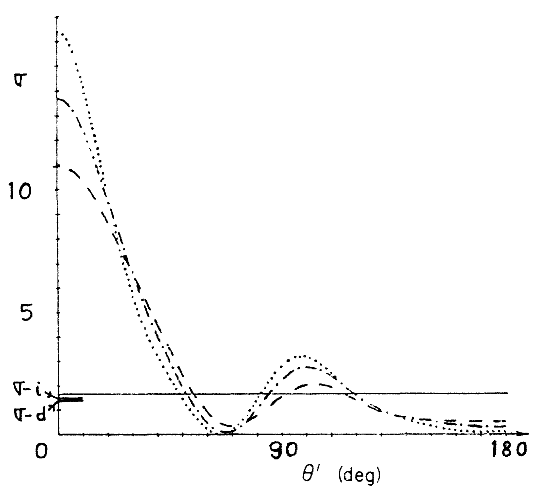

FIG. 9. Same as in Fig. 8, but now for the displacements $c/b = \frac{4}{9}$ (dashed line), $c/b = \frac{5}{9}$ (dot-dashed line), $c/b = \frac{6}{9}$ (dotted line).

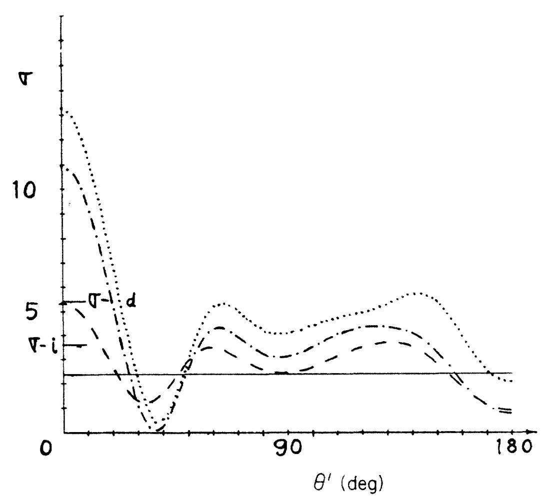

FIG. 11. Same as in Fig. 9, but now with $kb = 3$.

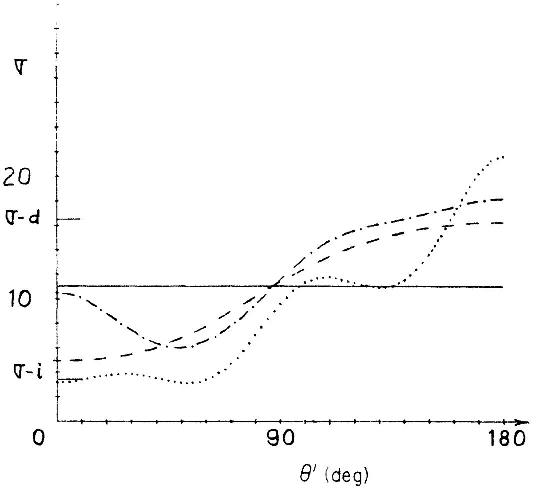

FIG. 12. Same as in Fig. 8, but now with $kb = 3.375$.

8 and 9 we have $kb = 2$, which according to Fig. 7 is close to a minimum for the concentric case, and here the enhancement of the backscattering cross section for small angles is more marked than in Figs. 12 and 13, corresponding to $kb = 3.375$, which according to Fig. 7 is close to a maximum. A remarkable feature of Figs. 15 and 14 is the increase in magnitude of the backscattering amplitude obtained by enclosing the two infinitely conducting spheres in a dielectric sphere. Still, the amplitude is very close to zero in one direction, which is, however, shifted somewhat.

Further details on the numerical aspects of the computations will be given in a separate report.[14]

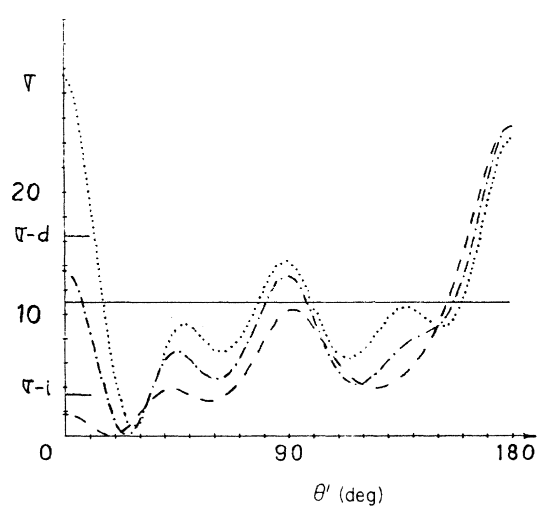

FIG. 13. Same as in Fig. 9, but now with $kb = 3.375$.

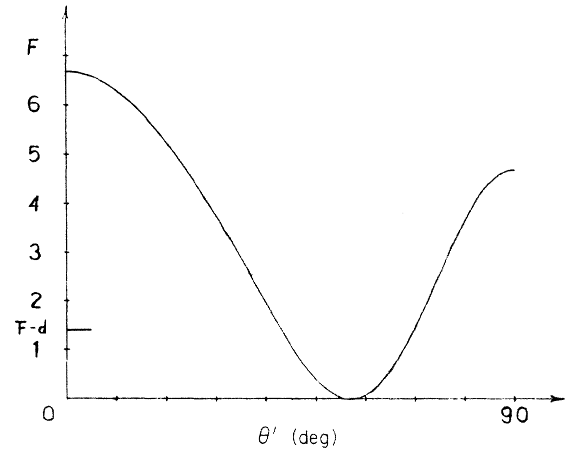

FIG. 14. The square F of the absolute value of the backscattering amplitude for two infinitely conducting spheres as in Fig. 15 symmetrically imbedded in a dielectric sphere of radius $b = 4a$. F is presented as a function of the same angle as in Fig. 15 for $k_1 = 2k$, $kb = 2$ (i.e., $ka = 0.5$). For comparison we have also indicated the corresponding amplitude $F-d$ for a homogeneous dielectric sphere of radius b with $k_1 = 2k$.

The calculations of the T matrix for the two infinitely conducting spheres is done according to Ref. 3. In general, the exact rate of convergence depends on a complicated interplay between the Q, T, and translation matrices, but, of course, convergence is faster the smaller kb is.

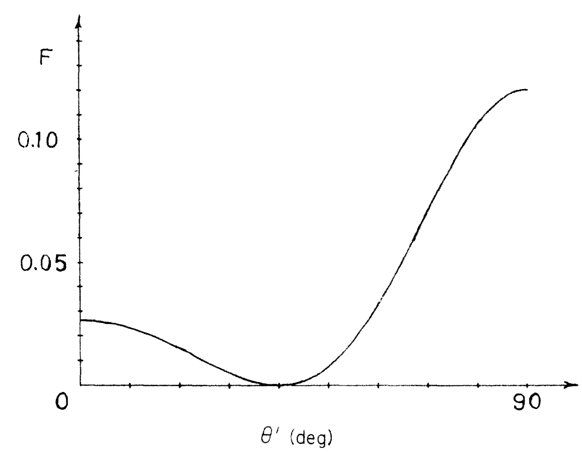

FIG. 15. The square F of the absolute value of the backscattering amplitude for two infinitely conducting spheres with radii a. The spheres are situated on the z axis with a distance $2d = 4a$ between their centers. The electric vector of the incoming field is orthogonal to the z axis and F is represented as a function of the angle θ' between the incoming wave vector and the positive z axis for $ka = 0.5$.

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