PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

New multiperipheral model with identical clusters: Coherent summation over all crisscross graphs

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(Received 11 March 1974)

A new multiperipheral model with identical clusters is suggested. We thereby coherently sum over all crisscross diagrams, which is made possible by our earlier suggested correlation expansion—a rearrangement of the ordinary perturbation expansion. This shows that there are dynamical correlations already in the simple multiperipheral structure. Finally, we demonstrate that the first term in this expansion, which is equal to the eikonal part of our generalized multiperipheral model, describes the bound-state structure and exhibits Regge behavior.

I. INTRODUCTION

In the present paper we suggest a new multiperipheral model where, because of identical clusters, all crisscross diagrams are coherently summed over (Fig. 1). Here we will study only the elastic amplitude, which through unitarity can then be related to the total cross section and inclusive and semi-inclusive particle spectra. In all cases where clusters are identical objects—where we cannot distinguish individuals—we must sum over all possible permutations to get the full amplitude. One naturally asks whether these crisscross diagrams will be of any importance in the high-energy limit, where intuitively everything should tend to a planar strongly ordered graph of longitudinal momenta. The answer is yes since the planarity is only justified for each part of the diagram where a large momentum propagates. Since the input energy is always finite, the number of large-momentum particles must be finite and small, and the rest of the production, because of the limited available energy, must occur in the pionization region. For these small intermediate momenta, which constitute the bulk of the process, planarity and neglect of transverse momenta are not at all justified. The large and small momenta need to be treated together (smooth dissociation), since a cutoff would spoil relativistic invariance and lead to an undesirable cutoff dependence. Therefore we must crisscross over all rungs in the ladder, and treat longitudinal and transverse modes on equal footing for the whole graph. This technique clearly shows that there are dynamical correlations already in the simple multiperipheral structure. The result is given in a correlation expansion, where the first term is the usual eikonal amplitude which involves just kinematical correlations. Finally we show that this eikonal part describes the bound-state structure in the t channel and that these bound states lie on the Regge trajectory obtained from the asymptotic behavior in the s channel.

From a phenomenological point of view the most interesting qualitative conclusions are no doubt obtained most easily from the Chew-Pignotti (CP) multi-Regge-pole type of models.[1-5] However, as argued by Chew and co-workers,[6,7] it is highly questionable whether one can ever create self-consistent Reggeized models. They notice that in the more field-theory-like ABFST (Amati-Bertocchi-Fubini-Stanghellini-Tonin) models,[8-10] on the contrary, one gets output Regge poles, and they conclude that these latter models are on a much better theoretical basis. We agree with this conclusion, but according to our arguments above we cannot find the motivation to discard crisscrosses and transverse momenta.

Until now the multiperipheral configuration has been used in such a way that the dynamical correlation effects are left to be explained by the cluster (fireball) spectrum. Before the insertion of this spectrum the pair correlations are negative ($f_2 < 0$) because of four-momentum conservation, and if these are also neglected one gets no correlations at all ($f_2 = 0$). A more consistent treatment of the crisscrossed multiperipheral ladder provides, as we will see, dynamical correlations essentially different from the above kinematical ones. It will be very interesting to see how many of the positive correlation effects can be accounted for in the multiperipheral ladder itself, before insertion of a remnant fireball effect. We will discuss this competition of the two kinds of dynamical effects under the unitarity restriction at the end of this paper. For a later check of this one should not forget about the possibility of additional links in the multiperipheral model—a certain "dressing" of the crisscross ladder.

We here make a technical study of the generalized infinite ladder (Fig. 1) in a scalar $\varphi^2\phi$ model. For simplicity we associate the field φ with pions with mass m and ϕ with some scalar $\pi\pi$ isobar (because of G parity) with mass μ. The isobar plays the role of a simplified cluster, which can later be replaced by some more realistic low-energy amplitude (Fig. 2) or some effective cluster "field." We will here just study the elastic amplitude to illustrate the technique. This can then be used to derive single-particle inclusive and semi-inclusive spectra, total cross section, etc., via unitarity.

II. THE ELASTIC SCATTERING AMPLITUDE

From the above reasoning we conclude that the dominant contribution to the absorptive part of the elastic amplitude comes from the generalized infinite t ladder (Fig. 1). The wavy lines here represent our naive "clusters." As usual we introduce the invariants

$$t = q_t^2, \quad q_t = p_a + p_{a'}, \tag{2.1}$$

$$s = q_s^2, \quad q_s = p_a - p_b. \tag{2.2}$$

The $(n+1)$th-order amplitude is defined by

$$-iM_{n+1}(s;t) = \frac{(-ig)^{2n+2}}{(n+1)!} \times \int \prod_{l=1}^{n+1} \frac{d^4k_l}{(2\pi)^4} \tilde{\Delta}_F(k_l) I (2\pi)^4 \times \delta^4\left(q_s - \sum_{t=1}^{n+1} k_t\right). \tag{2.3}$$

Here $\Delta_F(k) = i(k^2 - \mu^2 + i\epsilon)^{-1}$ is the usual Feynman propagator of our clusters and I is a product of pion propagators. The δ function is eliminated by integration of the rth momentum, arbitrarily chosen out of all $(n+1)$ quanta. Inserting

$$k_r = q_s - \sum_{t\neq r}^{n+1} k_t, \tag{2.4}$$

we then get

$$\tilde{\Delta}_F(k_r) = \int d^4x\, e^{ik_r\cdot x} \Delta_F(x) = \int d^4x\, \Delta_F(x) \exp\left(iq_s\cdot x - i\sum_{\substack{t=1\\ t\neq r}}^{n+1} k_t\cdot x\right), \tag{2.5}$$

which later will be included in the correlation expansion. By summation over r and all possible permutations we get

$$I_{\text{sym}} = \sum_{r=1}^{n+1} I_{r\,\text{sym}} = \sum_{r=1}^{n+1} \sum_{\text{perm}} I_r. \tag{2.6}$$

The set of all permutations consists of the four sets of permutations D_i ($i = a, b, a', b'$) for each separate prong (Fig. 3) and the remaining factor sets,[11] which include all possible exchanges of vertices between different prongs. In Fig. 3 we have made a schematic picture of what could happen. As mentioned above, the δ function is eliminated via one arbitrarily selected quantum, say the rth. We then divide the quanta into four groups (Fig.3), with respect to position relative to this rth quantum. In Fig. 3 we have also explicitly written out the number of elements in each permutation set. The propagator product $I_{r\,\text{sym}}$ is defined by Feynman rules:

$$I_{r\,\text{sym}} = \sum_{\substack{r_1' r_2'\\ \text{perm}}} \prod_{l=1}^{r-1} A_{j_l}^{a} \prod_{l=r+1}^{n+1} A_{j_l}^{b} \prod_{l=1}^{r'-1} A_{j_l}^{b'} \prod_{l=r'+1}^{n+1} A_{j_l}^{a'}, \tag{2.7}$$

where the propagators A^i now also include the coupling constant

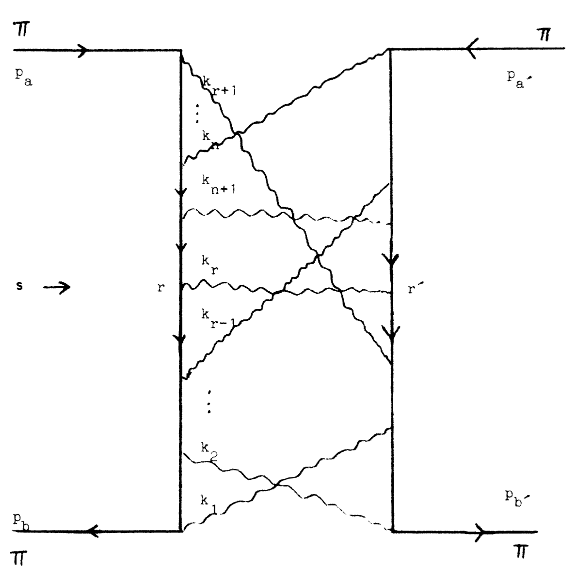

FIG. 1. The $(n+1)$th-order generalized multiperipheral model.

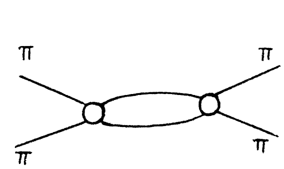

FIG. 2. The low-energy amplitude.

$$A^{i}_{\beta(i)} = \frac{ig}{p_i^2 - m^2 + 2\epsilon_i p_i \cdot K_{\beta(i)} + K_{\beta(i)}^2},$$

$$K_{\beta(i)} = \sum_{t=\alpha(i)}^{\beta(i)} k_t \quad (2.8)$$

and $\alpha(b) = 1$, $\beta(b) = r-1$, $\alpha(a) = r+1$, $\beta(a) = n+1$, $\alpha(b') = 1$, $\beta(b') = r'-1$, $\alpha(a') = r'+1$, and $\beta(a') = n+1$. Here we put all pions on their mass shell $p_i^2 = m^2$. If the sign conventions are chosen as in Fig. 1, then $\epsilon_a = -1$, $\epsilon_b = +1$, $\epsilon_{b'} = -1$, and $\epsilon_{a'} = +1$. However, the form of (2.8) permits any choice of these conventions. The subindices r_1' and r_2' in (2.7) are needed to define which quanta are crossing the rth quantum and which are not. From Fig. 3 it is clear that $r_1' + r_2' = r' - 1$ must apply.

We now apply our previously derived correlation rearrangement to the above ladder expansion.[11] Summing over all possible permutations of quanta attached to the ith particle leg, the corresponding product of propagators can be written as

$$\sum_{\text{perm}} \prod_{l=\alpha(i)}^{\beta(i)} A^{i}_{j_l} = \prod_{l=\alpha(i)}^{\beta(i)} f^{i}_{l} + \sum_{\substack{s,t=\alpha(i)\\ s<t}}^{\beta(i)} \chi^{i}_{st} \prod_{\substack{l=\alpha(i)\\ l\neq s,t}}^{\beta(i)} f^{i}_{l}$$

$$+ \text{ higher correlations}, \quad (2.9)$$

consisting of one "uncorrelated" term which factorizes with respect to all momenta (i.e., can be written as a direct product of current elements, each of which depends on one momentum), plus a second term which factorizes except for an arbitrary pair of momenta, which is called pair correlated, etc. This is proved in the Appendix for the case

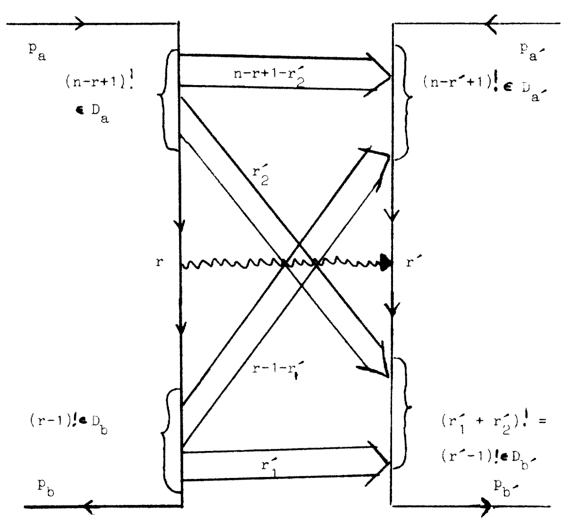

FIG. 3. Schematic picture of the permutation sets relative to the rth quantum, where we explicitly have written out the number of permutations in each separate permutation set.

with an excited throughgoing mass so that $p_i^2 = m^2 + \delta m_i^2$. The involved currents with $\delta m_i^2 \neq 0$ are defined in the Appendix, and in the special case with $\delta m_i^2 = 0$ we have for the uncorrelated currents in (2.9)

$$f_l^i = \frac{ig}{2\epsilon_i p_i \cdot k_l + k_l^2 + i\epsilon} \quad (2.10)$$

and the pair-correlation currents are defined by

$$\chi^{i}_{st} = f^{i}_{s} f^{i}_{t} \frac{-x^{i}_{st}}{1 + x^{i}_{st}}, \qquad x^{i}_{st} = \frac{2k_s \cdot k_t}{2\epsilon_i p_i \cdot (k_s + k_t) + k_s^2 + k_t^2}. \quad (2.11)$$

In passing we notice that the correct triple-correlation current already appears in three attached quanta as the difference between the full triple amplitude and the expression (2.9) for $\alpha(i) = 1$ and $\beta(i) = 3$. This argument could be applied to an arbitrary order, defining the correlation expansion from the ordinary perturbation expansion.[11] However, here we assume that the secondaries in some sense behave like a gas[12–15] with decreasing correlation effects with increasing order of correlation. [This corresponds to short-range-order (SRO) dominance, whereas in general there is no reason to exclude long-range-order corrections.] Then, if the model is at all physically meaningful, this property should also yield the correlations in (2.9), and we therefore drop all correlations higher than pair correlations. We do not here expect that this φ^3 model is a suitable candidate for strong interactions, but it is the simplest laboratory model from which we can learn about more realistic models.

Rewriting all products in (2.7) according to (2.9), summing over the rest of all permutations (all possible ways to link the wavy lines together for $I_{r\,\text{sym}}$) in (2.7) we get[11]

$$I_{r\,\text{sym}} = \prod_{l=1}^{n} J_l J_l'$$

$$+ \sum_{\substack{s,t=1\\ s<t}}^{n} (\chi_{st} J_s' J_t' + J_s J_t \chi_{st}' + \chi_{st}\chi_{st}')$$

$$\times \prod_{\substack{l=1\\ l\neq s,t}}^{n} J_l J_l', \quad (2.12)$$

where the total currents are given by

$$J_l = \sum_i f_l^i, \qquad \chi_{st} = \sum_i \chi^{i}_{st} \quad (2.13)$$

and given similarly for the primed currents. Inserting (2.5) and (2.12) into the integrand in (2.3) we can now exploit the factorization and partial factorization, respectively, and define form-invariant functionals irrespective of which individual quanta are involved.

For the noncorrelated part we define a "scalar" product

$$U(x, s, t) = i \int \frac{d^4k}{(2\pi)^4}\, \tilde{\Delta}_F(k)\, e^{-ik\cdot x} J(k) J'(k) \qquad (2.14)$$

and correspondingly we define a pair-correlation functional

$$P(x, s, t) = i^2 \int \frac{d^4k_s}{(2\pi)^4} \frac{d^4k_t}{(2\pi)^4}\, \tilde{\Delta}_F(k_s)\tilde{\Delta}_F(k_t)\, e^{-i(k_s+k_t)\cdot x} \times (\chi_{st} J_s' J_t' + J_t J_s \chi_{st}' + \chi_{st}\chi_{st}'). \qquad (2.15)$$

For Eq. (2.3) we then obtain

$$-iM_{n+1}(s, t) = \frac{g^2}{(n+1)!} \int d^4x\, e^{iq_s\cdot x}\, \Delta_F(x) \times [(iU)^n + \binom{n}{2} i^2 P (iU)^{n-2}]\,. \qquad (2.16)$$

Summation over n from zero to infinity gives

$$M(s, t) = i g^2 \int d^4x\, e^{iq_s\cdot x}\, \Delta_F(x) \times \left\{ \frac{e^{iU}-1}{iU} + \frac{i^2 P}{2!(iU)^2} \left[(iU-2)e^{iU} + 2\,\frac{e^{iU}-1}{iU} \right] \right\}, \qquad (2.17)$$

where the first term is the usual eikonal result and the second defines dynamical pair correlations. In passing we notice that our method does not depend crucially on a straight-line-path approximation. It is rather a rearrangement of the ordinary perturbation expansion. As we can see from the form of (2.17) the strong coupling constant alone does not determine the strength of the pair correlation, as in a weak-coupling theory,[17] but rather the summed-up factor which multiplies P in (2.17). However, if the strength of the coupling is due to anything but a universal, large coupling constant, as is probably the physical situation, the effect of higher-order correlations might still be negligible.

III. DISCUSSION

As we have seen, we could write the amplitude as the sum of a factorizable and a partially factorizable form, respectively, without introducing any gaps or strong ordering of momenta. Furthermore, because of the form invariance of the correlation functionals irrespective of which quanta were involved, we could sum up infinite orders in the coupling constant. In this way we obtained a closed explicit form for the elastic amplitude (2.17) without neglecting crisscrosses or transverse momenta.

A further positive feature is that in this correlation expansion we can strictly separate kinematical (due to four-momentum conservation) and dynamical correlations. The first term, including just kinematical correlations, is the usual eikonal result and is obtained from the straight-line-path approximation (= c-number sources). The dynamical correlations, defined by the rest of this expansion, thus define the recoil or rather the quantum dynamical effects predicted by the model with operator currents instead of c-number sources. In principle it should therefore now be possible to discriminate among various models through correlation experiments. This would not have been so easy to do with just kinematical correlations (as in all uncorrelated jet models) since these effects are always present.

Because of the form of the currents (2.10), due to the k^2 terms in the denominator, the corresponding "scalar" product has an effective relativistically invariant cutoff automatically built into it. A corresponding infrared damping occurs also in the pair-correlation currents (2.11). From the form of (2.11) we further see that it has an ultraviolet relativistically invariant cutoff built in. If we include the latter, a similar smooth "separation" of pair and triple correlations will also occur, as we saw in Ref. 11. Correspondingly, the triple correlations will be effectively damped in the ultraviolet range by a built-in cutoff. For small k momenta the associated wavelength is long, as in the uncorrelated soft current (2.10), and it is reasonable to work with strictly pointlike particles. For larger k momenta we can resolve more subdetails in the matter, which would imply a certain dispersion in configuration space. It might then not be possible to work with strictly local fields any more; rather, we may have to work with some effective cluster field or low-energy amplitude, since these (instability) effects would probably first appear in the $\pi\pi$-isobar system corresponding to resonances of the cluster.[16] If the isobar system is stable up to momenta typical for pair effects, then rather than going to triple-correlation effects we would find it natural to modify the simple scalar isobar propagator to a more general low-energy amplitude (Fig. 2) or cluster spectrum. This general problem involves higher spin, off-shell effects, and distorted propagators which we are now able to control through the generalizations in Ref. 17. However, first it will be highly interesting to investigate how many of the dynamical correlations are present in the simple multiperipheral structure, i.e., the nonzero P in (2.15). Through unitarity we can then get the total cross section

$$\sigma_T(s) = \frac{1}{s}\,\mathrm{Im}\,M(s;0)\,, \tag{3.1}$$

and similarly we can obtain the partial cross section $\sigma_n(s)$. Because of the explicit form of (2.16), (2.17), by differentiation we can then get the covariant inclusive spectra for one particle, two particles, etc., and semi-inclusive spectra as well.[9,10] The pair correlation, as usual defined by these inclusive (or semi-inclusive) spectra,

$$c_2(s;\vec{k}_1;\vec{k}_2) = \rho_2(s;\vec{k}_1;\vec{k}_2) - \rho_1(s;\vec{k}_1)\rho_1(s;\vec{k}_2)\,, \tag{3.2}$$

describes the correlation effects—of both kinematical and dynamical origin—for an arbitrary two clusters. Before deriving this we could start with a more complete pair correlation (dynamical) as given by (2.15), by the addition of all possible links between this pair. This is discussed in some more detail in Ref. 17 and will not be gone into further here. We think that the pure multiperipheral dynamic *per se,* as discussed above, should be carefully investigated first.

This was just a simple example to illustrate that it is possible to derive a closed expression for the amplitude without too much loss of quantum-dynamical and statistical information in the original model. The information loss in our approach lies in the neglected higher-order correlations. However, we could also include triple correlations without too much complication.[11] A comparison of these effects with the above-discussed excitations of the clusters will show if this is necessary. However, then we must also consider possible additional links outside the simple multiperipheral structure and all radiative corrections (self-energies and vacuum corrections) as well, in order to get the complete correlations. A compressed presentation of this generalized multiperipheral model, together with the results of Refs. 11 and 17, was given in Ref. 18. Finally we notice that the eikonal part of the result possesses Regge behavior and has the correct bound-state structure.[19] In the limit $s=0$ where $p_a = p_b$ and $p_{a'} = p_{b'}$ we have $U = U_0$, where

$$U_0 = \frac{-g^2}{4\pi^2}\int d^4k\, \frac{e^{-ik\cdot x}}{k^2-\mu^2+i\epsilon}\,\delta(2p_a\cdot k)\delta(2p_{b'}\cdot k)\,. \tag{3.3}$$

In the coordinate system where $p_a = (E_a, 0, 0, p)$ and $p_{b'} = (E_{b'}, 0, 0, -p)$ we arrive at

$$U_0 = \frac{g^2}{16\pi^2 p\sqrt{t}}\int d^2k_\perp\, \frac{e^{ik_\perp\cdot x_\perp}}{k_\perp{}^2+\mu^2} = \frac{g^2}{8\pi p\sqrt{t}}\,K_0(\mu x_\perp)\,. \tag{3.4}$$

For high t values, where $q_s \simeq q_{s\perp}$, the first term in (2.17) becomes

$$\begin{aligned} M(s,t,\mu^2) &\simeq -2it\int d^2x_\perp e^{-iq_{s\perp}\cdot x_\perp}(e^{iU_0}-1) \\ &= -2it\int d^2x_\perp x_\perp J_0(q_\perp\cdot x_\perp)(e^{iU_0}-1), \end{aligned} \tag{3.5}$$

where we have used

$$\int dx_0 dx_3 \Delta_F(x,\mu) = (-i/2\pi)K_0(\mu x_\perp)\,.$$

Following the lines of Lévy and Sucher[19] we then find that the high-s behavior $\mu^2 \ll |s| \ll t - 4m^2$ is given by the singularity in U_0 for small $x_\perp{}^2$ values,

$$U_0 \simeq -\frac{g^2}{8\pi p\sqrt{t}}\,\ln|\mu x_\perp|\,. \tag{3.6}$$

Thus for high $s \simeq -q_{s\perp}{}^2$ by insertion of (3.6) in (3.5) we get

$$M(s,t,\mu^2) = \frac{-it}{\mu^2}\left(\frac{-s}{4\mu^2}\right)^{-1+\alpha_0(t)} \frac{\Gamma(-\alpha_0(t)+1)}{\Gamma(\alpha_0(t))}\,, \tag{3.7}$$

where $\alpha_0(t)$ is given by

$$\alpha_0(t) = \frac{ig^2}{16\pi p\sqrt{t}}\,. \tag{3.8}$$

We can also obtain this result in the limit $\mu^2 \to 0$ for rather modest t values and may therefore also use it in the bound-state region

$$\alpha_0(t) = \frac{g^2}{8\pi[(4m^2-t)t]^{1/2}}\,,\quad 0 < t < 4m^2\,. \tag{3.9}$$

Equation (3.7) has poles in t at $-\alpha_0(t)+1 = -n+1$ or rather $\alpha_0(t) = n$, with $n = 1, 2, 3\ldots$. Simultaneously we have the asymptotic Regge behavior in s, and with $\alpha_0(t)-1 = l$, where $l = 0, 1, 2\ldots$, we get precisely the same levels as above. The bound states in the t channel of the amplitude (3.7) are lying on the Regge trajectory obtained from the asymptotic behavior in the s channel.

Naturally the form (3.9) is not a very interesting Regge trajectory from the point of view of strong interactions. However, as we have seen, this trajectory is a direct consequence of the assumption of stable spinless cluster propagators and vertices. A more realistic case will of course involve internal excitations and higher spin effects, which will have direct consequences for the form of the trajectory function. We will not consider the general spin-averaged bootstrap case here, but just notice that with stable spin-1 clusters, instead of inserting the low-energy amplitude itself, we get the trajectory function

$$\alpha_0(t)=\frac{g^2}{8\pi}\frac{2t-4m^2}{[(4m^2-t)t]^{1/2}} . \tag{3.10}$$

The corresponding nonplanar bootstrap model, which is more in line with the original idea in the planar ABFST model, would be an interesting subject for a subsequent work.

ACKNOWLEDGMENTS

I wish to thank Professor M. Jacob, Professor H. B. Nielsen, and Professor H. Satz for interesting discussions.

APPENDIX

For the sake of generality we will here prove the form (2.9) in case of an over-all mass excitation for the two throughgoing particles. In (2.8) this is expressed by insertion of $p_i{}^2-m_i{}^2=\delta M_i{}^2$. For the ith prong with n attached quanta the corresponding part of the amplitude is then given by

$$A_n^i=\sum_{\text{perm}}\frac{1}{y_1^i+\delta m_i{}^2}\,\frac{1}{y_{12}^i\,(1+x_{12}^i)+\delta m_i{}^2}\cdots . \tag{A1}$$

We have here used the same notations as in Ref. 11:

$$y_{12\cdots n}^i=2\epsilon_i p_i\cdot(k_1+k_2+\cdots+k_n)+k_1{}^2+\cdots+k_n{}^2 , \tag{A2}$$

$$x_{12\cdots n}^i=(y_{12\cdots n}^i)^{-1}\,2\sum_{s<t}^{n}k_s\cdot k_t . \tag{A3}$$

However, in order to carry out the factorization and partial factorization respectively with $\delta m_i{}^2\neq 0$

$$y_{12\cdots n}^{i\prime}=y_{12\cdots n}^i+n\delta m_i{}^2 , \tag{A4}$$

$$x_{12\cdots n}^{i\prime}=(y_{12\cdots n}^{i\prime})^{-1}\times\left(2\sum_{\substack{s<t\\1}}^{n}k_s\cdot k_t-(n-1)\delta m_i{}^2\right) . \tag{A5}$$

For $n=2$ we then easily get

$$\begin{aligned}A_2&=\sum_{\text{perm}}\frac{1}{y_1'}\,\frac{1}{y_{12}'(1+x_{12}')}\\&=\left(\frac{1}{y_1'}+\frac{1}{y_2'}\right)\frac{1}{y_{12}'(1+x_{12}')}\\&=\frac{y_1'+y_2'}{y_1'y_2'}\,\frac{1}{y_{12}'(1+x_{12}')}\\&=\frac{1}{y_1'y_2'}\,\frac{1}{(1+x_{12}')} .\end{aligned} \tag{A6}$$

The last step in (A6) (and similar in higher order) crucially depends upon the relation

$$\sum_{t=1}^{n}y_t'=y_{12\cdots n}' , \tag{A7}$$

which cannot be exploited in the unprimed case with the definition (A2), (A3) because $\delta m_i{}^2\neq 0$. For $n=3$ we get

$$\begin{aligned}A_3&=\frac{1}{y_{123}'(1+x_{123}')}\sum_{\text{factor set }(F)}\frac{1}{y_1'y_2'}\,\frac{1}{1+x_{12}'}\\&=\frac{1}{y_{123}'}\sum_F\frac{1}{y_1'y_2'}\,\frac{1}{1+x_{12}'}\\&\quad+\frac{-x_{123}'}{y_{123}'(1+x_{123}')}\sum_F\frac{1}{y_1'y_2'}\,\frac{1}{1+x_{12}'} .\end{aligned} \tag{A8}$$

Here the last term is triple-correlated and we can then write

$$\begin{aligned}A_3&=\left[\frac{1}{y_{123}'}\,\frac{1}{y_1'y_2'y_3'}\sum_F y_3'\left(1+\frac{-x_{12}'}{1+x_{12}'}\right)+\text{higher correlations}\right]\\&=\left(\frac{1}{y_1'y_2'y_3'}+\frac{1}{y_1'y_2'y_3'}\,\frac{1}{y_{123}'}\sum_F\frac{-x_{12}'y_3'}{1+x_{12}'}+\cdots\right)\\&=\frac{1}{y_1'y_2'y_3'}\\&\quad\times\left(1+\sum_{\substack{s<t\\1}}^{3}\frac{-x_{st}}{1+x_{st}'}+\frac{1}{y_{123}'}\sum_F\frac{x_{12}y_{12}}{1+x_{12}}+\cdots\right) ,\end{aligned} \tag{A9}$$

where we have used the relation $y_3'=y_{123}'-y_{12}'$. The last term in (A9) is also triple-correlated through the factor $(y_{123}')^{-1}$. Iterating this procedure for the nth-order amplitude we get

$$A_n=\left[\prod_{l=1}^{n}\frac{1}{y_l'}+\sum_{\substack{s<t\\1}}^{n}\frac{1}{y_s'y_t'}\left(\frac{-x_{st}'}{1+x_{st}'}\right)\prod_{\substack{l\neq s,t\\1}}^{n}\frac{1}{y_l'}+\text{higher correlations}\right] , \tag{A10}$$

which proves (2.9) if we put $\delta m_i{}^2=0$ in (A10).

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