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# Allomorphic corrections to semileptonic processes*

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The notion of allomorphic corrections to $S$-matrix elements for semileptonic processes, which by definition are absent in the lowest nontrivial order of perturbation theory, is introduced and analyzed. The complete fourth-order allomorphic corrections to the leptonic weak currents are calculated in an arbitrary gauge model of weak and electromagnetic interactions, treating the strong interactions (assumed mediated purely by gauge vector gluons) to all orders. These results are used to analyze the induced leptonic pseudoscalar current in charged-pion decay, and it is shown that weak restrictions on particle masses suffice to preserve the $V$-$A$ structure at the leptonic vertex to $O(\alpha G_F)$. A preliminary discussion of the allomorphic corrections to hadronic weak currents is presented in the context of neutral-kaon decay ($K^0 \to \mu^- \mu^+$). It is shown that in theories in which the kaon field dimensionality is two or less, the usual suppression mechanisms involving charmed quarks probably do not suffice to remove induced neutral, strangeness-changing currents in $O(\alpha G_F)$.

## INTRODUCTION

The structure of the higher-order corrections to weak-interaction processes in perturbation theory has been a long-standing problem of particle physics. The difficulties associated with the study of such corrections in previous weak-interaction theories (e.g., the Fermi theory, or the intermediate-vector-boson theory) arose directly from the nonrenormalizable structure of the underlying field theory, so that no sensible, unambiguous treatment of the plethora of infinities arising in higher order was possible. However, with the discovery of spontaneously broken unified gauge theories of weak and electromagnetic processes,[1] and the subsequent proof of renormalizability of such theories,[2] a systematic attack on the problem of higher-order corrections has become possible.

In this regard, the study of higher-order corrections to semileptonic processes seems most fruitful at the present time. From an experimental point of view, such processes offer a phenomenological variety which we do not find in the purely leptonic processes. From a theoretical point of view, semileptonic processes are considerably simpler than the purely hadronic ones, primarily for two reasons. First, in the semileptonic case, we at least understand fully the structure of the "leptonic end" of the diagrams, at which strong corrections are absent. Second, in purely hadronic processes studied to fourth order, the propagators of the weakly interacting bosons (gauge particles, Higgs scalars) can appear both in the same or in two separate independent loops, whereas in semileptonic processes the fourth-order corrections involve at most one loop.

There is by now a considerable literature on the subject of unified gauge models,[3] and in many works the question of higher-order corrections to semileptonic processes has come under scrutiny. However, a study of these corrections in the context of an arbitrary renormalizable gauge model, and treating the strong interactions nonperturbatively using the tools of current algebra and the Wilson operator-product expansion,[4] remains to be done. As Weinberg has shown,[5] such a general attack may be very useful not only in deriving general formulas applicable to arbitrary gauge theories of weak and electromagnetic interactions, but also in deducing general constraints on the strong-interaction field theory from such phenomenological information as the absence of $O(\alpha)$ violations of parity and strangeness in the strong interactions.

In this paper we begin the model-independent study of fourth-order corrections to semileptonic processes. The "allomorphic" fourth-order corrections to the leptonic weak currents are derived and shown to be gauge-independent, unitary, and finite. By "allomorphic" we mean, roughly speaking (see below, Sec. I), those corrections which cannot be absorbed into a renormalization of the zeroth-order parameters of the theory. Our results are applied to a study of charged-pion decay, and in this context it is shown that with only fairly weak restrictions on the particle masses, which in turn depend only slightly on the detailed structure of the strong interactions, dangerous pseudoscalar contributions to the leptonic currents are in fact absent to $O(\alpha G_F)$ in general gauge models. We also present a preliminary (and incomplete) discussion of the allomorphic corrections to the hadronic weak currents, in the context of neutral-kaon decay. Work on the general structure of the hadronic end allomorphic corrections is in progress.

The order of presentation is as follows. In Sec. I, we define the allomorphic corrections to leptonic weak currents. In Sec. II the fourth-order contribution to these corrections arising from the tree diagrams (simple vector or Higgs-scalar exchange) are extracted. Section III contains a discussion of the one-loop diagrams involving two-boson exchange between the leptonic and hadronic ends of the diagram ("box" diagrams). The allomorphic corrections involving hadronic matrix elements of the product of two (hadronic) currents are shown to be separately gauge-independent, unitary, and finite. In Sec. IV, the allomorphic corrections contained in vertex corrections at the leptonic end are extracted. In Sec. V, the contributions exhibited in Secs. II–IV are combined and all gauge-dependent terms, zero-mass Goldstone poles, and infinities are shown to cancel from the allomorphic contributions. Section VI summarizes the results of the preceding and provides a simple prescription for obtaining the correct answer. In Sec. VII, we apply our results to a discussion of charged-pion decay in general gauge theories. Section VIII contains a preliminary discussion of the other types of allomorphic corrections to semileptonic processes, with particular emphasis on the absence of neutral, strangeness-changing to $O(\alpha G_F)$ in the context of kaon decay. A summary of our methods and results is presented in Section IX. Finally, some useful auxiliary material is relegated to Appendixes A–C.

## I. ALLOMORPHIC CORRECTIONS TO LEPTONIC WEAK CURRENTS

In the following, we shall be concerned with the calculation, to fourth order in the coupling constant of the weak and electromagnetic interactions, of a certain class of contributions to arbitrary semileptonic processes. By the latter, we mean all processes involving a *single* lepton line, and arbitrary incoming and outgoing hadrons. As far as the weak and electromagnetic interactions are concerned, the calculation will involve a completely general renormalizable, gauge-invariant Lagrangian containing fermions, Higgs scalars, and vector gauge mesons. We will assume, temporarily, that the strong interactions are mediated by vector gluons[5] and that there are no elementary hadronic scalar fields. The consequences of relaxing this last assumption will be discussed in Secs. VII and VIII.

Consider the structure of the matrix element appearing between the lepton spinors in a general contribution. In the tree approximation, this matrix is a linear combination of Yukawa coupling matrices (see Appendix A) $\Gamma_i$ and the fermion-gauge meson bare vertices $\zeta_\psi\gamma^\mu t_\alpha$. In higher order, we expect contributions of this type to receive infinite contributions arising from coupling-constant renormalizations in the $\Gamma_i$'s and $t_\alpha$'s. Calculation of such higher-order contributions would require a detailed, and complicated, renormalization program. Also, since such contributions would usually be masked by the appearance of similar terms in lowest order, the results would be of limited physical interest.

On the other hand, higher-order contributions to semileptonic processes involving corrections to the leptonic vertices which *cannot* be written as a linear combination of $\Gamma_i$ and $\zeta_\psi\gamma^\mu t_\alpha$ are interesting both physically and from a technical point of view. Such contributions will henceforth be termed "allomorphic" (with the implicit understanding, until Sec. VIII, that we are referring to the leptonic end). They are analogous to the higher-order corrections to natural zeroth-order mass relations which have been much discussed in the literature[6]—here, of course, we are concerned with corrections to physical $S$-matrix elements, rather than propagators. Evidently, the allomorphic corrections to any physical, on-mass-shell $S$-matrix element must separately be gauge-independent, free of unphysical singularities due to zero-mass Goldstone poles ("unitary"), and finite. Naively, one might expect that the renormalization counterterms would affect only non-allomorphic contributions, and could be neglected. It is a remarkable feature of non-Abelian gauge theories that this is not, in fact, the case: Such counterterms are essential to the proof of gauge independence, unitarity, and finiteness. This point is further explicated in Sec. II and Appendix B, where we make the appropriate brief excursion into renormalization theory. Finally, we expect the allomorphic contributions to be of considerable physical interest, since, *by definition,* they are completely absent in the lowest nontrivial order of perturbation theory.

## II. TREE-GRAPH CONTRIBUTIONS; RENORMALIZATION COUNTERTERMS

Since we are calculating to $O(g^4)$, any renormalization counterterms to be included in the calculation of allomorphic contributions must appear as insertions in tree graphs. We shall see that such counterterms can give a gauge-dependent, allomorphic contribution—in the following, we show that the only counterterms relevant are the fermion (i.e., lepton) mass and wave-function renormalization. Before proceeding, the reader may find it useful to review the summary of our notation and conventions presented in Appendix A.

The effective unrenormalized Lagrangian[7] for generating the tree diagrams is evidently

$$\mathcal{L}^u_{\text{tree}} = \mathcal{L}_{\text{quad}}(\phi^u_i, W^u_{\alpha\mu}) + \mathcal{L}_{\text{strong}} - \bar{\psi}^u_l(\zeta_\psi \not{\partial} + m_u)\psi^u_l + S^u_i\phi^u_i + j^u_{\alpha\mu}W^{u\mu}_\alpha . \qquad (2.1)$$

Here $\mathcal{L}_{\text{strong}}$ contains only hadronic fields. The leptonic and hadronic fermions will be distinguished by subscripts $l, h$ respectively. $\mathcal{L}_{\text{quad}}(\phi^u_i, W^u_{\alpha\mu})$ is just the quadratic part of the unrenormalized Lagrangian involving the weakly interacting Higgs scalars and gauge vector mesons. Unrenormalized fields and currents are distinguished by the superscript "$u$." Here the currents

$$\begin{aligned} S^u_i &\equiv -\bar{\psi}^u\Gamma_i\psi^u \\ &= -(\bar{\psi}^u_l\Gamma_i\psi^u_l + \bar{\psi}^u_h\Gamma_i\psi^u_h) \\ &= (S^u_l)_i + (S^u_h)_i , \\ j^u_{\alpha\mu} &\equiv -i\bar{\psi}^u\zeta_\psi\gamma_\mu t_\alpha\psi^u \\ &= -i(\bar{\psi}^u_l\zeta_\psi\gamma_\mu t_\alpha\psi^u_l + \bar{\psi}^u_h\zeta_\psi\gamma_\mu t_\alpha\psi^u_h) \\ &= (j^u_l)_{\alpha\mu} + (j^u_h)_{\alpha\mu} \end{aligned} \qquad (2.2)$$

involve both leptonic and hadronic fermions.

The only renormalizations concerning us are those that break the zeroth-order symmetries of the theory. Coupling-constant renormalizations (i.e., redefinitions of $\Gamma_i$, $t_\alpha$) may be ignored, since the only permissible redefinitions conserve the zeroth-order relations prescribed by the gauge invariance of the theory. For example, suppose the gauge group is simple and the leptons form a single irreducible representation. Then the only allowed renormalization of the $t_\alpha$'s is a redefinition of the single coupling constant $g$ buried in $t_\alpha$, or in other words, an overall *scalar* multiplicative renormalization. This clearly does not give rise to an allomorphic contribution. Similarly, we need not consider renormalizations of the Higgs scalar or gauge vector propagators (cf. Sec. VIII). However, mass and wave-function renormalization of the leptons, if carried out (as we shall) in a fully physical, on-mass-shell manner *(so that all radiative corrections to external lines cancel)* will give rise to *gauge-dependent* allomorphic contributions, provided, as is usually the case, that zeroth-order symmetry-breaking terms appear in the fermion propagator at the one-loop level. To derive the required counterterms, we therefore need make only the following two renormalizations:

$$\begin{aligned} m &= m_u + \delta m , \\ \psi_l &= Z_2^{-1/2}\psi^u_l , \quad Z_2^{1/2} \equiv 1 + z_2 . \end{aligned} \qquad (2.3)$$

To repeat: $\delta m$ and $Z_2$ (both of which are matrices in the internal space of the leptons) are adjusted so that all radiative corrections to external lines cancel. In particular, $\delta m$ *includes* the finite, gauge-independent symmetry-breaking corrections previously calculated by Weinberg.[8] The calculation of $Z_2$ is outlined in Appendix B.

Substituting (2.3) into (2.1), the effective Lagrangian for computing tree graphs becomes

$$\mathcal{L}_{\text{tree}} = \mathcal{L}_R + \mathcal{L}_{\text{CT}} , \qquad (2.4)$$

$$\mathcal{L}_R = \mathcal{L}_{\text{strong}} + \mathcal{L}_{\text{quad}}(\phi^u_i, W^u_{\alpha\mu}) - \bar{\psi}_l(\zeta_\psi\not{\partial} + m)\psi_l + (S^u_h)_i\phi_i + (j^u_h)_{\alpha\mu}W^\mu_\alpha + (S_l)_i\phi_i + (j_l)_{\alpha\mu}W^\mu_\alpha , \qquad (2.5)$$

$$\begin{aligned} \mathcal{L}_{\text{CT}} = &-\bar{\psi}_l[z_2^\dagger(\zeta_\psi\not{\partial} + m) + (\zeta_\psi\not{\partial} + m)z_2]\psi_l \\ &+ \bar{\psi}\delta m\psi - i\bar{\psi}_l(z_2^\dagger\zeta_\psi\gamma^\mu t_\alpha + \zeta_\psi\gamma^\mu t_\alpha z_2)\psi_l W_{\alpha\mu} \\ &- \bar{\psi}_l(z_2^\dagger\Gamma_i + \Gamma_i z_2)\psi_l\phi_i \quad [\text{to } O(g^3)] . \end{aligned} \qquad (2.6)$$

Of course, we need never explicitly consider the strong-interaction renormalizations—the appropriate counterterms are contained implicitly in $\mathcal{L}_{\text{strong}}$ and matrix elements of strong Heisenberg operators are understood to be finite, renormalized quantities.

First, we show that the two tree graphs (Fig. 1) arising from $\mathcal{L}_R$ sum to a gauge-independent $O(g^2)$ contribution, plus an $O(g^4)$ allomorphic, gauge-dependent contribution. Since we are working to $O(g^4)$, we must regard the lepton spinors in the amplitude to involve the physical mass matrix $m$ to $O(g^2)$—the $O(g^2)$ asymmetric corrections to the zeroth-order mass will, as we shall see, give rise to the $O(g^4)$ gauge-dependent contribution mentioned above, plus an implicit gauge-independent allomorphic contribution, also of fourth order.

A short calculation yields the following result for the sum of vector and scalar exchange graphs arising from $\mathcal{L}_R$, to $O(g^4)$:

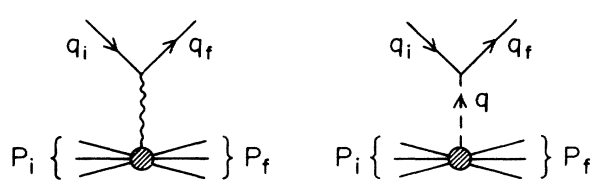


FIG. 1. Tree-graph contributions to semileptonic processes.

$$\delta S_R^{\rm tree} = -i(2\pi)^4\delta(\textstyle\sum P)[\Delta^W_{\alpha\mu\beta}{}^{\nu}(q)\langle j^\mu_\alpha\rangle_{FI}\, i\bar{U}_f\zeta_\psi\gamma^\nu t_\beta U_i + \Delta^\phi_{ij}(q)\langle S_i\rangle_{FI}\bar{U}_f\Gamma_j U_i] \,, \tag{2.7}$$

$$\delta S_R^{\rm tree} \equiv \delta S^{\rm tree}_{R,{\rm GI}} + \delta S^{\rm tree}_{R,{\rm GD}} \,, \tag{2.8}$$

$$\delta S^{\rm tree}_{R,{\rm GI}} = -i(2\pi)^4\delta(\textstyle\sum P)[i(q)_{\alpha\beta}\langle j^\nu_\beta\rangle_{FI}\bar{U}_f\zeta_\psi\gamma_\nu t_\alpha U_i + \hat{\Delta}^\phi_{ij}(q)\langle S_i\rangle_{FI}\bar{U}_f\Gamma_j U_i] \,, \tag{2.9}$$

where we have introduced the convenient abbreviations

$$\delta(\textstyle\sum P) \equiv \delta(P_i + q_i - P_f - q_f) \,,$$

$$\langle j^\mu_\alpha\rangle_{FI} \equiv \langle F|j^\mu_\alpha(0)|I\rangle, \quad \langle S_i\rangle_{FI} \equiv \langle F|S_i(0)|I\rangle \,,$$

$$(q)_{\alpha\beta} \equiv (\zeta_W q^2 + \mu^2)^{-1}{}_{\alpha\beta} \,,$$

$$\hat{\Delta}^\phi_{ij}(q) \equiv (\zeta_\phi q^2 + M^2)^{-1}{}_{ij} + (\theta_\alpha\lambda)_i(\theta_\beta\lambda)_j\frac{(q)_{\alpha\beta}}{q^2} \,.$$

Evidently, $\delta S^{\rm tree}_{R,{\rm GI}}$ is $\xi$-gauge-independent, as it must be, for there are no renormalization counterterms available at $O(g^2)$ to absorb any uncancelled $\xi$ dependence. The allomorphic $O(g^4)$ contribution is found to be of two forms. First, there is an implicit allomorphic contribution which arises from the replacement of the zeroth-order mass matrix in the lepton spinors in (2.9) by the leptonic mass matrix corrected to $O(g^2)$. If these corrected spinors are expanded in terms of spinors involving the zeroth-order mass matrix, then (2.9) is clearly seen to implicitly involve $O(g^4)$ allomorphic contributions.

The second, explicitly gauge-dependent, $O(g^4)$ allomorphic contribution arising from the graphs in Fig. 1 is

$$\delta S^{\rm tree}_{R,{\rm GD}} = i(2\pi)^4\delta(\textstyle\sum P)(\theta_\alpha\lambda)_i\langle S_i\rangle_{FI}[q]_{\alpha\beta}\bar{U}_f \times\gamma_4[t_\beta, \gamma_4\delta m_f]U_i \,. \tag{2.10}$$

Here

$$[q]_{\alpha\beta} \equiv \frac{1}{q^2}[(\xi q^2 + \mu^2)^{-1}{}_{\alpha\beta} - (\zeta_W q^2 + \mu^2)^{-1}{}_{\alpha\beta}] \tag{2.11}$$

and $\delta m_f$ is the finite, $\xi$-independent $O(g^2)$ correction to the lepton mass matrix.[8] We need not be concerned with its explicit form here, but merely note that the only relevant parts of $\delta m_f$ in (2.10) are the contributions from scalar and vector exchange [i.e., $\Sigma_1(p)$ in Appendix B, Eq. (B4)]. The momentum-independent contributions in $\Sigma_2$ (arising in part from tadpoles) involve linear combinations of $\Gamma_i$ matrices, and are hence not allomorphic.

Finally, we calculate the $O(g^4)$ allomorphic contributions arising from the asymmetric renormalization counterterms in $\mathcal{L}_{\rm CT}$. Only the last two terms in $\mathcal{L}_{\rm CT}$ are relevant (the first two are involved only in radiative corrections on the external lepton legs), and we obtain simply

$$\begin{aligned}\delta S^{\rm tree}_{\rm CT} &= (2\pi)^4\delta(\textstyle\sum P)\langle j^\nu_\alpha\rangle_{FI}(q)_{\alpha\beta}\bar{U}_f(z_2^\dagger\zeta_\psi\gamma_\nu t_\beta + \zeta_\psi\gamma_\nu t_\beta z_2)U_i \\ &\quad - (2\pi)^4\delta(\textstyle\sum P)(\theta_\alpha\lambda)_i\langle S_i\rangle_{FI}[q]_{\alpha\beta}\bar{U}_f(z_2^\dagger\zeta_\psi \not{q} t_\beta + \zeta_\psi\not{q} t_\beta z_2)U_i - i(2\pi)^4\delta(\textstyle\sum P)\langle S_i\rangle_{FI}\,\Delta^\phi_{ij}(q)\bar{U}_f(z_2^\dagger\Gamma_j + \Gamma_j z_2)U_i \\ &\equiv \delta S^{\rm tree}_{{\rm CT},\langle j\rangle} + \delta S^{\rm tree}_{{\rm CT},\langle s\rangle} \,.\end{aligned} \tag{2.12}$$

These contributions are clearly gauge-dependent. As we shall see below, the gauge dependence in (2.10) and (2.12) will eventually cancel against gauge-dependent terms arising from one-loop diagrams.

## III. ONE-LOOP CONTRIBUTIONS: "BOX" DIAGRAMS

At the one-loop level, there are four diagrams involving two-boson exchange, as indicated in Fig. 2. Their contributions are

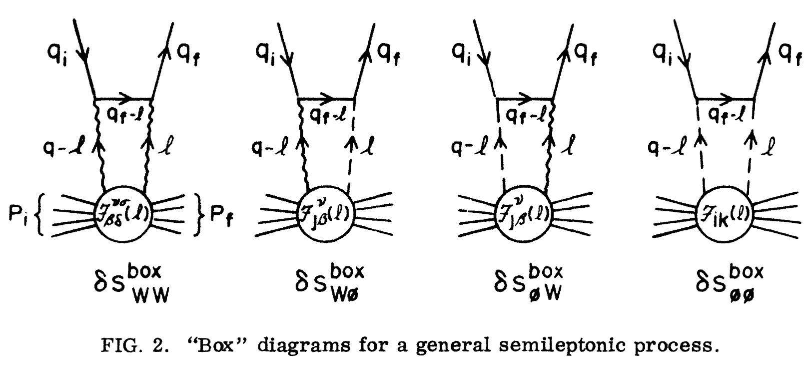


FIG. 2. "Box" diagrams for a general semileptonic process.

$$\delta S^{\rm box}_{WW} \equiv -i\delta(\textstyle\sum P)\displaystyle\int dl\, \mathfrak{F}^{\nu\sigma}_{\beta\delta}(l)\Delta^{W}_{\alpha\mu\beta\nu}(l)\Delta^{W}_{\gamma\rho\delta\sigma}(q-l) \times \bar{U}_f \zeta_\psi \gamma^\mu t_\alpha \Delta^\psi(q_f - l)\zeta_\psi\gamma^\rho t_\gamma U_i, \quad (3.1)$$

$$\delta S^{\rm box}_{W\phi} \equiv -\delta(\textstyle\sum P)\displaystyle\int dl\, \mathfrak{F}^{\nu}_{j\beta}(l)\Delta^{W}_{\alpha\mu\beta\nu}(q-l)\Delta^{\phi}_{jk}(l) \times \bar{U}_f \Gamma_k \Delta^\psi(q_f - l)\zeta_\psi\gamma^\mu t_\alpha U_i\ , \quad (3.2)$$

$$\delta S^{\rm box}_{\phi W} \equiv -\delta(\textstyle\sum P)\displaystyle\int dl\, \mathfrak{F}^{\nu}_{j\beta}(l)\Delta^{\phi}_{jk}(l)\Delta^{W}_{\alpha\mu\beta\nu}(q-l) \times \bar{U}_f \zeta_\psi\gamma^\mu t_\alpha \Delta^\psi(q_i + l)\Gamma_k U_i\ , \quad (3.3)$$

$$\delta S^{\rm box}_{\phi\phi} \equiv i\delta(\textstyle\sum P)\displaystyle\int dl\, \mathfrak{F}_{ik}(l)\Delta^{\phi}_{ij}(l)\Delta^{\phi}_{kl}(q-l) \times \bar{U}_f \Gamma_j \Delta^\psi(q_f - l)\Gamma_l U_i\ , \quad (3.4)$$

where $\delta(\sum P) \equiv \delta(P_f + q_f - P_i - q_i)$. The two-current hadronic matrix elements $\mathfrak{F}^{\nu\sigma}_{\beta\alpha}$, $\mathfrak{F}^{\nu}_{j\beta}$, and $\mathfrak{F}_{ik}$ are defined in Appendix A [Eq. (A18)].

The next step involves substituting the explicit expressions for the propagators (A5) into (3.1)–(3.4), using the relations prescribed by local gauge invariance [in particular, Eqs. (A3), (A10), (A13), (A19), and (A20)], and segregating the resulting contributions according to the structure of the particular hadronic matrix element involved. Recall that all contributions which merely amount to a gauge-covariant renormalization of the lepton-scalar and lepton-vector vertices are dropped. This procedure yields the following terms:

$$\delta S^{\rm box}_{\langle j\rangle} \equiv \langle j^{\nu}_{\epsilon}\rangle_{FI} c_{\beta\delta\epsilon}\,\delta(\textstyle\sum P)\displaystyle\int dl\{\bar{U}_f[\Gamma_j\Delta^\psi(q_f-l)\zeta_\psi\gamma_\nu t_\gamma + \zeta_\psi\gamma_\nu t_\gamma\Delta^\psi(q_i+l)\Gamma_j]U_i(\theta_\alpha\lambda)_j(q-l)_{\gamma\delta} + i\bar{U}_f\Gamma_j\Delta^\psi(q_f-l)\Gamma_k U_i(\theta_\alpha\lambda)_j(\theta_\gamma\lambda)_k l_\nu[q-l]_{\gamma\delta}\}[l]_{\alpha\beta}\ , \quad (3.5)$$

$$\delta S^{\rm box}_{\langle S\rangle} \equiv \langle S_i\rangle_{FI}\,\delta(\textstyle\sum P)\displaystyle\int dl\{\bar{U}_f\Gamma_j\Delta^\psi(q_f-l)\Gamma_k U_i(\theta_\alpha\lambda)_j(\theta_\gamma\lambda)_k(\theta_\beta\theta_\delta\lambda)_i[q-l]_{\gamma\delta} + \bar{U}_f[\Gamma_m\Delta^\psi(q_f-l)\Gamma_k + \Gamma_k\Delta^\psi(q_i+l)\Gamma_m]U_i(\theta_\alpha\lambda)_m(\theta_\beta)_{ji}\Delta^{\phi}_{kj}(q-l)\}[l]_{\alpha\beta}\ . \quad (3.6)$$

As there are other one-loop graphs leading to contributions involving a single-current matrix element at the hadronic end, we should not expect $\delta S^{\rm box}_{\langle j\rangle}$ or $\delta S^{\rm box}_{\langle S\rangle}$ to be gauge-independent. A glance at (3.5) and (3.6) confirms this expectation. However, two-current hadronic matrix elements occur only in $\delta S^{\rm box}$, so that their contributions from the box graphs should be separately $\xi$-independent. A detailed calculation confirms that the requisite cancellations do indeed take place, and we are left with

$$\delta S^{\rm box}_{\langle jj\rangle} \equiv -i\delta(\textstyle\sum P)\displaystyle\int dl\, \mathfrak{F}^{\mu\nu}_{\alpha\beta}(l)(l)_{\alpha\gamma}(q-l)_{\beta\delta} \times \bar{U}_f\zeta_\psi\gamma_\mu t_\gamma\Delta^\psi(q_f-l)\zeta_\psi\gamma_\nu t_\delta U_i, \quad (3.7)$$

$$\delta S^{\rm box}_{\langle jS\rangle} \equiv -\delta(\textstyle\sum P)\displaystyle\int dl\, \mathfrak{F}^{\nu}_{i\beta}(l)\hat{\Delta}^{\phi}_{ij}(l)(q-l)_{\alpha\beta} \times \bar{U}_f(\Gamma_j\Delta^\psi(q_f-l)\zeta_\psi\gamma_\nu t_\alpha + \zeta_\psi\gamma_\nu t_\alpha\Delta^\psi(q_i+l)\Gamma_j)U_i\ , \quad (3.8)$$

$$\delta S^{\rm box}_{\langle SS\rangle} \equiv i\delta(\textstyle\sum P)\displaystyle\int dl\, \mathfrak{F}_{ij}(l)\hat{\Delta}^{\phi}_{ik}(l)\hat{\Delta}^{\phi}_{jp}(q-l) \times \bar{U}_f\Gamma_k\Delta^\psi(q_f-l)\Gamma_p U_i\ . \quad (3.9)$$

In addition to being explicitly $\xi$-independent, these contributions are "unitary"—they involve propagators with poles only at masses of physical particles. In particular, the "gauge-independent, unitary" scalar propagator $\hat{\Delta}^{\phi}_{ij}(q)$ has no poles at $q^2=0$, corresponding to Goldstone bosons. We

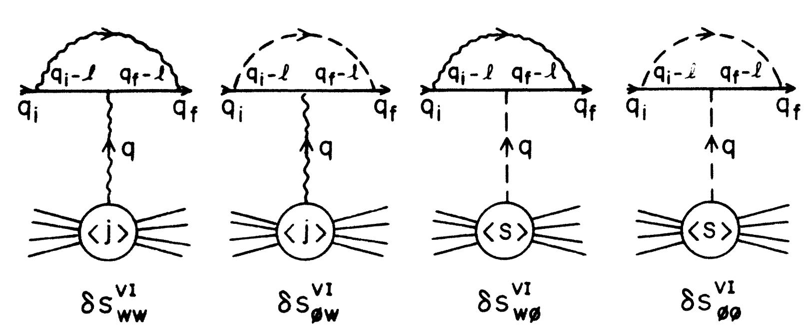


FIG. 3. Vertex-type corrections (first variety) to semileptonic processes.

conclude this section by noting that $\delta S^{\text{box}}_{\langle jj\rangle}$, $\delta S^{\text{box}}_{\langle jS\rangle}$, $\delta S^{\text{box}}_{\langle SS\rangle}$ are evidently finite.

## IV. ONE-LOOP CONTRIBUTIONS: "VERTEX" DIAGRAMS

The remaining one-loop graphs giving rise to allomorphic contributions are just the one-loop radiative corrections to the lepton-scalar and lepton-vector vertices. They fall into two classes: those (Fig. 3) in which only lepton-scalar and lepton-vector vertices appear (such terms will henceforth be designated by superscript VI), and those (Fig. 4) in which there is a trilinear boson vertex (designated by VII). At this point the analysis becomes somewhat technical—a summary of our results is presented in Sec. VI. Explicitly, the contributions of the graphs of the first variety are found to be

$$\delta S^{\text{VI}}_{WW} = i\langle j^{\mu}_{\alpha}\rangle_{FI}\Delta^{W}_{\alpha\mu\beta\nu}(q)\delta(\textstyle\sum P)\displaystyle\int dl\,\Delta^{W}_{\gamma\rho\delta\sigma}(l)\overline{U}_f\zeta_\psi\gamma^\rho t_\gamma\Delta^\psi(q_f-l)\zeta_\psi\gamma^\nu t_\beta\Delta^\psi(q_i-l)\zeta_\psi\gamma^\sigma t_\delta U_i\ , \tag{4.1}$$

$$\delta S^{\text{VI}}_{\phi W} = -i\langle j^{\mu}_{\alpha}\rangle_{FI}\Delta^{W}_{\alpha\mu\beta\nu}(q)\delta(\textstyle\sum P)\displaystyle\int dl\,\Delta^{\phi}_{ij}(l)\overline{U}_f\Gamma_i\Delta^\psi(q_f-l)\zeta_\psi\gamma^\nu t_\beta\Delta^\psi(q_i-l)\Gamma_j U_i\ , \tag{4.2}$$

$$\delta S^{\text{VI}}_{W\phi} = \langle S_i\rangle_{FI}\Delta^{\phi}_{ij}(q)\delta(\textstyle\sum P)\displaystyle\int dl\,\Delta^{W}_{\alpha\mu\beta\nu}(l)\overline{U}_f\zeta_\psi\gamma^\mu t_\alpha\Delta^\psi(q_f-l)\Gamma_j\Delta^\psi(q_i-l)\zeta_\psi\gamma^\nu t_\beta U_i\ , \tag{4.3}$$

$$\delta S^{\text{VI}}_{\phi\phi} = -\langle S_i\rangle_{FI}\Delta^{\phi}_{ij}(q)\delta(\textstyle\sum P)\displaystyle\int dl\,\Delta^{\phi}_{kl}(l)\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\Gamma_j\Delta^\psi(q_i-l)\Gamma_l U_i\,. \tag{4.4}$$

As before, the analysis proceeds by substituting the explicit expressions for the propagators into the above equations, and examining for cancellations of $\xi$-dependent terms, employing the arsenal of relations exhibited in Appendix A. One finds a number of such cancellations, but the final result (as we should expect) is still gauge-dependent. Once again, we segregate the contributions according to the nature of the hadronic matrix element involved. It is also convenient to separate out at this stage some obviously *gauge-independent* contributions. Thus, we write

$$\begin{aligned}\delta S^{\text{VI}} &= \delta S^{\text{VI}}_{WW} + \delta S^{\text{VI}}_{W\phi} + \delta S^{\text{VI}}_{\phi W} + \delta S^{\text{VI}}_{\phi\phi}\\ &\equiv \delta S^{\text{VI}}_{\langle j\rangle,\text{GI}} + \delta S^{\text{VI}}_{\langle j\rangle,\text{GD}} + \delta S^{\text{VI}}_{\langle S\rangle,\text{GI}} + \delta S^{\text{VI}}_{\langle S\rangle,\text{GD}}\end{aligned} \tag{4.5}$$

(the subscripts GI and GD refer to "gauge-independent" and "gauge-dependent", respectively). One finds

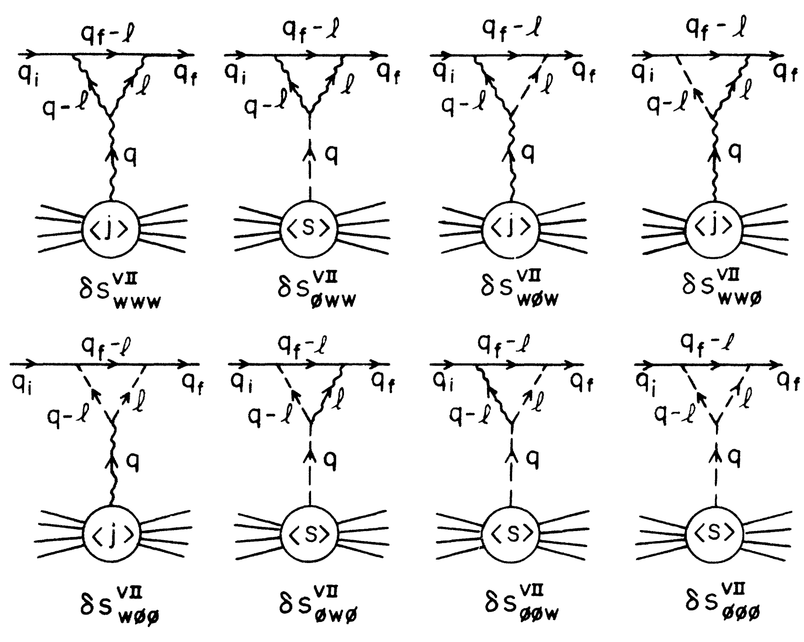


FIG. 4. Vertex-type corrections (second variety) to semileptonic processes.

$$\delta S^{\rm VI}_{\langle j\rangle,{\rm GI}} = i\langle j^{\mu}_{\alpha}\rangle_{FI}(q)_{\alpha\beta}\delta(\textstyle\sum P)\displaystyle\int dl[(l)_{\gamma\delta}\bar U_f\zeta_\psi\gamma_\nu t_\gamma\Delta^\psi(q_f-l)\zeta_\psi\gamma^\mu t_\beta\Delta^\psi(q_i-l)\zeta_\psi\gamma^\nu t_\delta U_i$$
$$-\hat\Delta^\phi_{kl}(l)\bar U_f\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi\gamma^\mu t_\beta\Delta^\psi(q_i-l)\Gamma_l U_i]\ , \tag{4.6}$$

$$\delta S^{\rm VI}_{\langle S\rangle,{\rm GI}} = \langle S_i\rangle_{FI}\delta(\textstyle\sum P)\displaystyle\Big\{\hat\Delta^\phi_{ij}(q)\int dl[(l)_{\alpha\beta}\bar U_f\zeta_\psi\gamma^\mu t_\alpha\Delta^\psi(q_f-l)\Gamma_j\Delta^\psi(q_i-l)\zeta_\psi\gamma_\mu t_\beta U_i$$
$$-\hat\Delta^\phi_{kl}(l)\bar U_f\Gamma_k\Delta^\psi(q_f-l)\Gamma_j\Delta^\psi(q_i-l)\Gamma_l U_i]$$
$$+(\theta_\alpha\lambda)_i\frac{(q)_{\alpha\beta}}{q^2}\int dl[(l)_{\gamma\delta}\bar U_f(\zeta_\psi\gamma^\mu t_\gamma t_\beta\Delta^\psi(q_i-l)\zeta_\psi\gamma_\mu t_\delta-\zeta_\psi\gamma^\mu t_\gamma\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_\beta t_\delta)U_i$$
$$-\hat\Delta^\phi_{kl}(l)\bar U_f(\Gamma_k t_\beta\Delta^\psi(q_i-l)\Gamma_l-\Gamma_k\Delta^\psi(q_f-l)\bar t_\beta\Gamma_l)U_i]\Big\}\ , \tag{4.7}$$

$$\delta S^{\rm VI}_{\langle j\rangle,{\rm GD}} = -i\langle j_{\alpha\mu}\rangle_{FI}(q)_{\alpha\beta}\delta(\textstyle\sum P)\displaystyle\int dl[l]_{\gamma\delta}\bar U_f(\zeta_\psi\gamma^\mu t_\gamma t_\beta t_\delta+\zeta_\psi\gamma^\mu t_\gamma t_\beta\Delta^\psi(q_i-l)\Gamma_k(\theta_\delta\lambda)_k$$
$$-(\theta_\gamma\lambda)_k\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi\gamma^\mu t_\beta t_\delta)U_i\ , \tag{4.8}$$

$$\delta S^{\rm VI}_{\langle S\rangle,{\rm GD}} = \langle S_i\rangle_{FI}\delta(\textstyle\sum P)\displaystyle\int dl\Big\{-\Delta^\phi_{ij}(q)[l]_{\alpha\beta}\bar U_f(\bar t_\alpha\Gamma_j t_\beta+\bar t_\alpha\Gamma_j\Delta^\psi(q_i-l)\Gamma_k(\theta_\beta\lambda)_k-(\theta_\alpha\lambda)_k\Gamma_k\Delta^\psi(q_f-l)\Gamma_j t_\beta)U_i$$
$$+[q]_{\alpha\beta}[l]_{\gamma\delta}(\theta_\alpha\lambda)_i\bar U_f(\gamma_4[t_\gamma t_\beta t_\delta,\gamma_4 m]+i\zeta_\psi\not q t_\gamma t_\beta\Delta^\psi(q_i-l)\Gamma_k(\theta_\delta\lambda)_k$$
$$-(\theta_\gamma\lambda)_k\Gamma_k\Delta^\psi(q_f-l)i\zeta_\psi\not q t_\beta t_\delta)U_i$$
$$+\frac{(\xi q^2+\mu^2)^{-1}{}_{\alpha\beta}}{q^2}(\theta_\alpha\lambda)_i$$
$$\times[(l)_{\gamma\delta}\bar U_f(\zeta_\psi\gamma^\mu t_\gamma\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_\beta t_\delta-\zeta_\psi\gamma^\mu t_\gamma t_\beta\Delta^\psi(q_i-l)\zeta_\psi\gamma_\mu t_\delta)U_i$$
$$-\hat\Delta^\phi_{kl}(l)\bar U_f(\Gamma_k\Delta^\psi(q_f-l)\bar t_\beta\Gamma_l-\Gamma_k t_\beta\Delta^\psi(q_i-l)\Gamma_l)U_i]\Big\}\ . \tag{4.9}$$

Next, we turn to the contributions arising from the second variety of vertex diagrams (Fig. 4). The calculations are tedious, though straightforward; we merely summarize the results here. Taking the diagram involving three internal vector propagators first:

$$\delta S^{\rm VII}_{WWW} = -i(\zeta_W)_{\alpha\beta}c_{\beta\gamma\delta}\delta(\textstyle\sum P)\displaystyle\int dl\{(q_\mu\Delta^W_{\alpha\nu\alpha'\mu'}(q)-q_\nu\Delta^W_{\alpha\mu\alpha'\mu'}(q))\Delta^{W\mu}_{\gamma}{}_{\beta'\nu'}(l)\Delta^{W\nu}_{\delta}{}_{\gamma'\rho'}(q-l)\langle j^{\mu'}_{\alpha'}\rangle_{FI}$$
$$\times\bar U_f\zeta_\psi\gamma^{\nu'}t_{\beta'}\Delta^\psi(q_f-l)\zeta_\psi\gamma^{\rho'}t_{\gamma'}U_i$$
$$-(l_\mu\Delta^W_{\alpha\nu\alpha'\mu'}(l)-l_\nu\Delta^W_{\alpha\mu\alpha'\mu'}(l))\Delta^{W\mu}_{\gamma}{}_{\beta'\nu'}(q)\Delta^{W\nu}_{\delta}{}_{\gamma'\rho'}(q-l)$$
$$\times\langle j^{\nu'}_{\beta'}\rangle_{FI}\bar U_f\zeta_\psi\gamma^{\mu'}t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma^{\rho'}t_{\gamma'}U_i$$
$$+((l-q)_\mu\Delta^W_{\alpha\nu\alpha'\mu'}(q-l)-(l-q)_\nu\Delta^W_{\alpha\mu\alpha'\mu'}(q-l))\Delta^{W\mu}_{\gamma}{}_{\beta'\nu'}(q)\Delta^{W\nu}_{\delta}{}_{\gamma'\rho'}(l)$$
$$\times\langle j^{\nu'}_{\beta'}\rangle_{FI}\bar U_f\zeta_\psi\gamma^{\rho'}t_{\gamma'}\Delta^\psi(q_f-l)\zeta_\psi\gamma^{\mu'}t_{\alpha'}U_i\}$$
$$\equiv\delta S^{\rm VII}_{WWW;\langle S\rangle,{\rm GI}}+\delta S^{\rm VII}_{WWW;\langle S\rangle,{\rm GD}}+\delta S^{\rm VII}_{WWW;\langle j\rangle,{\rm GI}}+\delta S^{\rm VII}_{WWW;\langle j\rangle,{\rm GD}}\ , \tag{4.10}$$

where, discarding as usual terms which merely renormalize the bare scalar and vector lepton vertices, one finds

$$\delta S^{\rm VII}_{WWW;\langle S\rangle,{\rm GI}} = i(\zeta_W)_{\alpha\beta}c_{\beta\gamma\delta}(q)_{\beta'\gamma}(\theta_{\beta'}\lambda)_i\langle S_i\rangle_{FI}\delta(\textstyle\sum P)\displaystyle\int dl(l)_{\alpha\alpha'}(q-l)_{\delta\gamma'}\bar U_f\zeta_\psi\gamma^\mu t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_{\gamma'}U_i\ , \tag{4.11}$$

$$
\begin{aligned}
\delta S^{\rm VII}_{WWW;\,\langle S\rangle,\,{\rm GD}} = i(\zeta_W)_{\alpha\beta}c_{\beta\gamma\delta}(\theta_\beta,\lambda)_i\langle S_i\rangle_{FI}\delta(\textstyle\sum P)\displaystyle\int dl\,\{ & (q^2-2q\cdot l)[q]_{\beta'\gamma}(l)_{\alpha\alpha'}(q-l)_{\delta\gamma'} \\
& \times \overline{U}_f\zeta_\psi\gamma^\mu t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_{\gamma'}U_i \\
& + i(l)_{\alpha\alpha'}[l^2[q]_{\beta'\gamma}[q-l]_{\delta\gamma'}+[q]_{\beta'\gamma}(q-l)_{\delta\gamma'}+(q)_{\beta'\gamma}[q-l]_{\delta\gamma'}] \\
& \times \overline{U}_f\zeta_\psi \not{q}\, t_{\alpha'}\Delta^\psi(q_f-l)\Gamma_k(\theta_\gamma,\lambda)_k U_i \\
& - i(q-l)_{\delta\gamma'}[(q-l)^2[q]_{\beta'\gamma}[l]_{\alpha\alpha'}+[q]_{\beta'\gamma}(l)_{\alpha\alpha'}+(q)_{\beta'\gamma}[l]_{\alpha\alpha'}] \\
& \times \overline{U}_f(\theta_\alpha,\lambda)_j\Gamma_j\Delta^\psi(q_f-l)\zeta_\psi\not{q}\, t_{\gamma'}U_i \\
& + [[l]_{\alpha\alpha'}(q-l)_{\delta\gamma'}(q\cdot(q-l)[q]_{\beta'\gamma}+(q)_{\beta'\gamma}) \\
& \quad + q\cdot l[q-l]_{\delta\gamma'}([l]_{\alpha\alpha'}(q)_{\beta'\gamma}-(l)_{\alpha\alpha'}[q]_{\beta'\gamma})] \\
& \quad \times(\theta_\alpha,\lambda)_j(\theta_\gamma,\lambda)_k\overline{U}_f\Gamma_j\Delta^\psi(q_f-l)\Gamma_k U_i\}\ , \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
\delta S^{\rm VII}_{WWW;\,\langle j\rangle,\,{\rm GI}} = {} & i(\zeta_W)_{\alpha\beta}c_{\beta\gamma\delta}(q)_{\beta'\gamma}\langle j^{\nu'}_{\beta'}\rangle_{FI}\delta(\textstyle\sum P) \\
& \times\int dl\,(l)_{\alpha\alpha'}(q-l)_{\delta\gamma'}\overline{U}_f(2l_{\nu'}\zeta_\psi\gamma^\mu t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_{\gamma'}+2\zeta_\psi\not{q}\, t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma_{\nu'}t_{\gamma'} \\
& \qquad -2\zeta_\psi\gamma_{\nu'}t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\not{q}\, t_{\gamma'}+i(\theta_\alpha,\lambda)_j\Gamma_j\Delta^\psi(q_f-l)\zeta_\psi\gamma_{\nu'}t_{\gamma'} \\
& \qquad -i\zeta_\psi\gamma_{\nu'}t_{\alpha'}\Delta^\psi(q_f-l)(\theta_{\gamma'}\lambda)_k\Gamma_k)U_i\ , \qquad (4.13)
\end{aligned}
$$

$$
\begin{aligned}
\delta S^{\rm VII}_{WWW;\,\langle j\rangle,\,{\rm GD}} = {} & i(\zeta_W)_{\alpha\beta}c_{\beta\gamma\delta}(q)_{\beta'\gamma}\langle j^{\nu'}_{\beta'}\rangle_{FI}\delta(\textstyle\sum P) \\
& \times\int dl\,\{i(l^2-2q\cdot l)[l]_{\alpha\alpha'}(q-l)_{\delta\gamma'}(\theta_\alpha,\lambda)_j\overline{U}_f\Gamma_j\Delta^\psi(q_f-l)\zeta_\psi\gamma_{\nu'}t_{\gamma'}U_i \\
& \quad + i(q^2-l^2)(l)_{\alpha\alpha'}[q-l]_{\delta\gamma'}(\theta_\gamma,\lambda)_k\overline{U}_f\zeta_\psi\gamma_{\nu'}t_{\alpha'}\Delta^\psi(q_f-l)\Gamma_k U_i \\
& \quad + l_{\nu'}(q^2[l]_{\alpha\alpha'}[q-l]_{\delta\gamma'}+[l]_{\alpha\alpha'}(q-l)_{\delta\gamma'}+(l)_{\alpha\alpha'}[q-l]_{\delta\gamma'})(\theta_\alpha,\lambda)_j(\theta_{\gamma'}\lambda)_k \\
& \quad \times\overline{U}_f\Gamma_j\Delta^\psi(q_f-l)\Gamma_k U_i\}\ . \qquad (4.14)
\end{aligned}
$$

Again, in deriving (4.11)–(4.14), liberal use is made of the relations imposed by local gauge invariance, especially Eqs. (A13) and (A19).

The graphs containing two internal vector lines and a single internal scalar line give the explicit contributions

$$
\begin{aligned}
\delta S^{\rm VII}_{\phi WW} = {} & \langle S_i\rangle_{FI}\Delta^\phi_{ij}(q)(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j\delta(\textstyle\sum P) \\
& \times\int dl\,\Delta^W_{\alpha\mu\alpha'\mu'}(l)\Delta^{W\mu}_{\beta}{}_{\beta'\nu'}(q-l) \\
& \times\overline{U}_f\zeta_\psi\gamma^{\mu'}t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma^{\nu'}t_{\beta'}U_i\ , \qquad (4.15)
\end{aligned}
$$

$$
\begin{aligned}
\delta S^{\rm VII}_{W\phi W} = {} & -i\langle j^{\mu'}_{\alpha'}\rangle_{FI}\Delta^W_{\alpha'\mu'\alpha\mu}(q)(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j\delta(\textstyle\sum P) \\
& \times\int dl\,\Delta^\phi_{jk}(l)\Delta^{W\mu}_{\beta}{}_{\beta'\nu'}(q-l) \\
& \times\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi\gamma^{\nu'}t_{\beta'}U_i\ . \qquad (4.16)
\end{aligned}
$$

$$
\begin{aligned}
\delta S^{\rm VII}_{WW\phi} = {} & -i\langle j^{\mu'}_{\alpha'}\rangle_{FI}\Delta^W_{\alpha'\mu'\alpha\mu}(q)(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j\delta(\textstyle\sum P) \\
& \times\int dl\,\Delta^\phi_{jk}(q-l)\Delta^{W\mu}_{\beta}{}_{\beta'\nu'}(l) \\
& \times\overline{U}_f\zeta_\psi\gamma^{\nu'}t_{\beta'}\Delta^\psi(q_f-l)\Gamma_k U_i\ , \qquad (4.17)
\end{aligned}
$$

while those graphs involving two internal scalar lines and a single internal vector line are found to contribute

$$
\begin{aligned}
\delta S^{\rm VII}_{W\phi\phi} = {} & \langle j^\nu_\beta\rangle_{FI}\Delta^W_{\beta\nu\alpha\mu}(q)(\zeta_\phi\theta_\alpha)_{ij}\delta(\textstyle\sum P) \\
& \times\int dl(q-2l)^\mu\Delta^\phi_{jm}(q-l)\Delta^\phi_{ik}(l) \\
& \times\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\Gamma_m U_i\ , \qquad (4.18)
\end{aligned}
$$

$$
\begin{aligned}
\delta S^{\rm VII}_{\phi W\phi} = {} & i\langle S_i\rangle_{FI}\Delta^\phi_{ik}(q)(\zeta_\phi\theta_\alpha)_{kj}\delta(\textstyle\sum P) \\
& \times\int dl(2q-l)^\mu\Delta^W_{\alpha\mu\beta\nu}(l)\Delta^\phi_{jm}(q-l) \\
& \times\overline{U}_f\zeta_\psi\gamma^\nu t_\beta\Delta^\psi(q_f-l)\Gamma_m U_i\ , \qquad (4.19)
\end{aligned}
$$

$$\delta S^{\rm VII}_{\phi\phi W} = i\langle S_i\rangle_{FI}\,\Delta^{\phi}_{ik}(q)(\zeta_\phi\theta_\alpha)_{kj}\delta(\textstyle\sum P) \times \int dl(q+l)^\mu \Delta^{W}_{\alpha\mu\beta\nu}(q-l)\Delta^{\phi}_{jm}(l) \times \overline{U}_f\Gamma_m\Delta^{\psi}(q_f-l)\zeta_\psi\gamma^\nu t_\beta U_i\ . \tag{4.20}$$

Substantial cancellations of $\xi$ dependence occur *between* these two sets of contributions, so it is convenient to define

$$\begin{aligned}\delta S^{\rm VII(\phi,W)} &\equiv \delta S^{\rm VII}_{\phi WW}+\delta S^{\rm VII}_{W\phi W}+\delta S^{\rm VII}_{WW\phi}+\delta S^{\rm VII}_{W\phi\phi} + \delta S^{\rm VII}_{\phi W\phi}+\delta S^{\rm VII}_{\phi\phi W}\\ &=\delta S^{\rm VII(\phi,W)}_{\langle j\rangle,{\rm GI}}+\delta S^{\rm VII(\phi,W)}_{\langle j\rangle,{\rm GD}} +\delta S^{\rm VII(\phi,W)}_{\langle S\rangle,{\rm GI}}+\delta S^{\rm VII(\phi,W)}_{\langle S\rangle,{\rm GD}}\ .\end{aligned} \tag{4.21}$$

Again, dropping all terms which are manifestly *not* allomorphic, we find after a somewhat tedious calculation

$$\begin{aligned}\delta S^{\rm VII(\phi,W)}_{\langle j\rangle,{\rm GI}} &= \langle j^\mu_{\alpha'}\rangle_{FI}(q)_{\alpha'\alpha}\delta(\textstyle\sum P)\\ &\times\int dl\{-i(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j(\zeta_\phi(q-l)^2+M^2)^{-1}{}_{jk}(l)_{\beta\beta'}\overline{U}_f\zeta_\psi\gamma_\mu t_{\beta'}\Delta^\psi(q_f-l)\Gamma_k U_i\\ &\quad -i(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j(\zeta_\phi l^2+M^2)^{-1}{}_{jk}(q-l)_{\beta\beta'}\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_{\beta'}U_i\\ &\quad + l_\mu(\theta_\alpha\zeta_\phi)_{jl}[\hat\Delta^\phi_{lm}(l)(\zeta_\phi(q-l)^2+M^2)^{-1}{}_{jk}-\hat\Delta^\phi_{kl}(q-l)(\zeta_\phi l^2+M^2)^{-1}{}_{jm}]\,\overline{U}_f\Gamma_m\Delta^\psi(q_f-l)\Gamma_k U_i\}\ ,\end{aligned} \tag{4.22}$$

$$\begin{aligned}\delta S^{\rm VII(\phi,W)}_{\langle S\rangle,{\rm GI}} &= \langle S_i\rangle_{FI}\,\delta(\textstyle\sum P)\displaystyle\int dl\{(\zeta_\phi q^2+M^2)^{-1}{}_{ij}(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j(l)_{\alpha\alpha'}(q-l)_{\beta\beta'}\overline{U}_f\zeta_\psi\gamma^\mu t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_{\beta'}U_i\\ &\quad +(\zeta_\phi q^2+M^2)^{-1}{}_{ij}\,i(\theta_\beta\zeta_\phi)_{jl}(q-l)_{\beta\beta'}\hat\Delta^\phi_{kl}(l)\,\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi \not{l}\, t_{\beta'}U_i\\ &\quad +(\zeta_\phi q^2+M^2)^{-1}{}_{ij}\,i(\theta_\beta\zeta_\phi)_{jl}(l)_{\beta\beta'}\hat\Delta^\phi_{kl}(q-l)\,\overline{U}_f\zeta_\psi(\not{q}-\not{l})t_{\beta'}\Delta^\psi(q_f-l)\Gamma_k U_i\\ &\quad -\hat\Delta^\phi_{il}(q)\cdot i(\theta_\beta\zeta_\phi)_{jl}(\zeta_\phi l^2+M^2)^{-1}{}_{kj}(q-l)_{\beta\beta'}\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi\not{q}\, t_{\beta'}U_i\\ &\quad -\hat\Delta^\phi_{il}(q)\cdot i(\theta_\beta\zeta_\phi)_{jl}(\zeta_\phi(q-l)^2+M^2)^{-1}{}_{kj}(l)_{\beta\beta'}\overline{U}_f\zeta_\psi\not{q}\, t_{\beta'}\Delta^\psi(q_f-l)\Gamma_k U_i\\ &\quad -(q)_{\beta\beta'}(\theta_\beta\zeta_\phi)_{jl}(\theta_{\beta'}\lambda)_i\hat\Delta^\phi_{kl}(q-l)(\zeta_\phi l^2+M^2)^{-1}{}_{mj}\,\overline{U}_f\Gamma_m\Delta^\psi(q_f-l)\Gamma_k U_i\}\ ,\end{aligned} \tag{4.23}$$

$$\begin{aligned}\delta S^{\rm VII(\phi,W)}_{\langle j\rangle,{\rm GD}} &= \langle j^\mu_{\alpha'}\rangle_{FI}(q)_{\alpha'\alpha}\delta(\textstyle\sum P)\\ &\times\int dl\Bigg\{-i(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j(l)_{\beta\beta'}\frac{(\xi(q-l)^2+\mu^2)^{-1}{}_{\gamma\gamma'}}{(q-l)^2}(\theta_\gamma\lambda)_k(\theta_{\gamma'}\lambda)_j\,\overline{U}_f\zeta_\psi\gamma_\mu t_{\beta'}\Delta^\psi(q_f-l)\Gamma_k U_i\\ &\quad -i(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j(q-l)_{\beta\beta'}\frac{(\xi l^2+\mu^2)^{-1}{}_{\gamma\gamma'}}{l^2}(\theta_\gamma\lambda)_k(\theta_{\gamma'}\lambda)_j\,\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_{\beta'}U_i\\ &\quad + l_\mu(\theta_\beta\zeta_\phi\theta_\alpha\lambda)_j((\theta_{\beta'}\lambda)_k[q-l]_{\beta\beta'}\Delta^\phi_{mj}(l)-(\theta_{\beta'}\lambda)_m[l]_{\beta\beta'}\Delta^\phi_{kj}(q-l))\ \overline{U}_f\Gamma_m\Delta^\psi(q_f-l)\Gamma_k U_i\\ &\quad + l_\mu(\theta_\alpha\zeta_\phi\theta_\beta\lambda)_j\left((\theta_{\beta'}\lambda)_m\hat\Delta^\phi_{jk}(q-l)\frac{(\xi l^2+\mu^2)^{-1}{}_{\beta\beta'}}{l^2}(\theta_{\beta'}\lambda)_k\hat\Delta^\phi_{mj}(l)\frac{(\xi(q-l)^2+\mu^2)^{-1}{}_{\beta\beta'}}{(q-l)^2}\right)\\ &\quad\times\overline{U}_f\Gamma_m\Delta^\psi(q_f-l)\Gamma_k U_i\Bigg\}\ ,\end{aligned} \tag{4.24}$$

$$\delta S^{\mathrm{VII}(\phi,W)}_{\langle S\rangle,\mathrm{GD}}=\langle S_i\rangle_{FI}\delta(\textstyle\sum P)$$

$$\times\int dl\Big\{(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j(l)_{\alpha\alpha'}(q-l)_{\beta\beta'}\frac{(\xi q^2+\mu^2)^{-1}{}_{\gamma\gamma'}}{q^2}(\theta_\gamma\lambda)_i(\theta_{\gamma'}\lambda)_j\overline{U}_f\zeta_\psi\gamma^\mu t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_{\beta'}U_i$$

$$+i(q-l)_{\alpha\alpha'}\Big[(\theta_\beta\zeta_\phi\theta_\alpha\lambda)_j([q]_{\beta\beta'}\Delta^\phi_{kj}(l)(\theta_{\beta'}\lambda)_i-[l]_{\beta\beta'}\Delta^\phi_{ij}(q)(\theta_{\beta'}\lambda)_k)$$

$$+(\theta_\alpha\zeta_\phi\theta_\beta\lambda)_j\Big(\hat{\Delta}^\phi_{ij}(q)\frac{(\xi l^2+\mu^2)^{-1}{}_{\beta\beta'}}{l^2}(\theta_{\beta'}\lambda)_k-\hat{\Delta}^\phi_{jk}(l)\frac{(\xi q^2+\mu^2)^{-1}{}_{\beta\beta'}}{q^2}(\theta_{\beta'}\lambda)_i\Big)\Big]$$

$$\times\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi\not{q}t_{\alpha'}U_i$$

$$+i(l)_{\alpha\alpha'}\Big[(\theta_\beta\zeta_\phi\theta_\alpha\lambda)_j([q]_{\beta\beta'}\Delta^\phi_{kj}(q-l)(\theta_{\beta'}\lambda)_i-[q-l]_{\beta\beta'}\Delta^\phi_{ij}(q)(\theta_{\beta'}\lambda)_k)$$

$$+(\theta_\alpha\zeta_\phi\theta_\beta\lambda)_j\Big(\frac{(\xi(q-l)^2+\mu^2)^{-1}{}_{\beta\beta'}}{(q-l)^2}\hat{\Delta}^\phi_{ij}(q)(\theta_{\beta'}\lambda)_k-\frac{(\xi q^2+\mu^2)^{-1}{}_{\beta\beta'}}{q^2}\hat{\Delta}^\phi_{kj}(q-l)(\theta_{\beta'}\lambda)_i\Big)\Big]$$

$$\times\overline{U}_f\zeta_\psi\not{q}t_{\alpha'}\Delta^\psi(q_f-l)\Gamma_kU_i$$

$$+\Big[(\theta_\beta\zeta_\phi\theta_\alpha\lambda)_j(l)_{\alpha\alpha'}[q-l]_{\beta\beta'}\Delta^\phi_{ij}(q)(\theta_{\alpha'}\lambda)_m(\theta_{\beta'}\lambda)_k$$

$$+(\theta_\beta\zeta_\phi\theta_\alpha\lambda)_j(q-l)_{\alpha\alpha'}[l]_{\beta\beta'}\Delta^\phi_{ij}(q)(\theta_{\alpha'}\lambda)_k(\theta_{\beta'}\lambda)_m$$

$$+(\theta_\beta\zeta_\phi)_{jl}\hat{\Delta}^\phi_{lm}(l)\Delta^\phi_{ij}(q)[q-l]_{\beta\beta'}l\cdot(q-l)(\theta_{\beta'}\lambda)_k$$

$$-(\theta_\beta\zeta_\phi)_{jl}\hat{\Delta}^\phi_{lm}(l)(\xi q^2+\mu^2)^{-1}{}_{\gamma\gamma'}(1/q^2)(\theta_\gamma\lambda)_i(\theta_{\gamma'}\lambda)_j(q-l)_{\beta\beta'}(\theta_{\beta'}\lambda)_k$$

$$+(\theta_\beta\zeta_\phi)_{jl}\hat{\Delta}^\phi_{lk}(q-l)\Delta^\phi_{ij}(q)[l]_{\beta\beta'}l\cdot(q-l)(\theta_{\beta'}\lambda)_m$$

$$-(\theta_\beta\zeta_\phi)_{jl}\hat{\Delta}^\phi_{lk}(q-l)(\xi q^2+\mu^2)^{-1}{}_{\gamma\gamma'}(1/q^2)(\theta_\gamma\lambda)_i(\theta_{\gamma'}\lambda)_j(l)_{\beta\beta'}(\theta_{\beta'}\lambda)_m$$

$$-(\theta_\beta\zeta_\phi)_{jl}\hat{\Delta}^\phi_{il}(q)\Delta^\phi_{kj}(q-l)[l]_{\beta\beta'}q\cdot l(\theta_{\beta'}\lambda)_m$$

$$+(\theta_\beta\zeta_\phi\theta_\alpha\lambda)_j(q)_{\alpha\alpha'}(\theta_{\alpha'}\lambda)_i\Delta^\phi_{mj}(l)[q-l]_{\beta\beta'}(\theta_{\beta'}\lambda)_k$$

$$-(\theta_\beta\zeta_\phi)_{jl}\hat{\Delta}^\phi_{il}(q)\Delta^\phi_{mj}(l)[q-l]_{\beta\beta'}q\cdot(q-l)(\theta_{\beta'}\lambda)_k$$

$$-(\theta_\beta\zeta_\phi)_{jl}\hat{\Delta}^\phi_{lm}(l)\Delta^\phi_{kj}(q-l)[q]_{\beta\beta'}q\cdot l(\theta_{\beta'}\lambda)_i$$

$$-(\theta_\beta\zeta_\phi)_{jl}\hat{\Delta}^\phi_{lk}(q-l)\Delta^\phi_{mj}(l)[q]_{\beta\beta'}q\cdot(q-l)(\theta_{\beta'}\lambda)_i$$

$$-(\theta_\beta\zeta_\phi)_{jl}\hat{\Delta}^\phi_{lk}(q-l)\frac{(\xi l^2+\mu^2)^{-1}{}_{\gamma\gamma'}}{l^2}(\theta_\gamma\lambda)_m(\theta_{\gamma'}\lambda)_j(q)_{\beta\beta'}(\theta_{\beta'}\lambda)_i\Big]\overline{U}_f\Gamma_m\Delta^\psi(q_f-l)\Gamma_kU_i\Big\}\,. \tag{4.25}$$

Finally we investigate the contribution of the graph involving three internal scalar lines, given explicitly by

$$\delta S^{\mathrm{VII}}_{\phi\phi\phi}=-\langle S_i\rangle_{FI}f_{i'j'k'}\delta(\textstyle\sum P)\int dl\,\Delta^\phi_{ii'}(q)\Delta^\phi_{jj'}(l)\Delta^\phi_{kk'}(q-l)\overline{U}_f\Gamma_j\Delta^\psi(q_f-l)\Gamma_kU_i$$

$$\equiv\delta S^{\mathrm{VII}}_{\phi\phi\phi,\mathrm{GI}}+\delta S^{\mathrm{VII}}_{\phi\phi\phi,\mathrm{GD}}\;, \tag{4.26}$$

where

$$\delta S^{\mathrm{VII}}_{\phi\phi\phi,\mathrm{GI}}=-\langle S_i\rangle_{FI}f_{i'j'k'}\delta(\textstyle\sum P)\int dl\,\hat{\Delta}^\phi_{ii'}(q)\hat{\Delta}^\phi_{jj'}(l)\hat{\Delta}^\phi_{kk'}(q-l)\overline{U}_f\Gamma_j\Delta^\psi(q_f-l)\Gamma_kU_i\;, \tag{4.27}$$

$$\delta S^{\mathrm{VII}}_{\phi\phi\phi,\mathrm{GD}}=\langle S_i\rangle_{FI}\delta(\textstyle\sum P)\int dl\,\overline{U}_f\Gamma_j\Delta^\psi(q_f-l)\Gamma_kU_i\{[M^2,\theta_\alpha]_{j'k'}(\theta_{\alpha'}\lambda)_i[q]_{\alpha\alpha'}\hat{\Delta}^\phi_{jj'}(l)\hat{\Delta}^\phi_{kk'}(q-l)$$

$$+[M^2,\theta_\beta]_{i'k'}(\theta_{\beta'}\lambda)_j[l]_{\beta\beta'}\hat{\Delta}^\phi_{ii'}(q)\hat{\Delta}^\phi_{kk'}(q-l)$$

$$+[M^2,\theta_\gamma]_{i'j'}(\theta_{\gamma'}\lambda)_k[q-l]_{\gamma\gamma'}\hat{\Delta}^\phi_{ii'}(q)\hat{\Delta}^\phi_{jj'}(l)$$

$$+(M^2\theta_\alpha\theta_\beta\lambda)_{k'}(\theta_{\alpha'}\lambda)_i(\theta_{\beta'}\lambda)_j[q]_{\alpha\alpha'}[l]_{\beta\beta'}\hat{\Delta}^\phi_{kk'}(q-l)$$

$$+(M^2\theta_\alpha\theta_\gamma\lambda)_{j'}(\theta_{\alpha'}\lambda)_i(\theta_{\gamma'}\lambda)_k[q]_{\alpha\alpha'}[q-l]_{\gamma\gamma'}\hat{\Delta}^\phi_{jj'}(l)$$

$$+(M^2\theta_\beta\theta_\gamma)_{i'}(\theta_{\beta'}\lambda)_j(\theta_{\gamma'}\lambda)_k[l]_{\beta\beta'}[q-l]_{\gamma\gamma'}\hat{\Delta}^\phi_{ii'}(q)\}\;. \tag{4.28}$$

In deriving (4.27) and (4.28), we use the Glashow-Weinberg relations (A14) to eliminate the trilinear scalar coupling matrix $f_{i'j'k'}$ from the gauge-dependent terms. Note that $\delta S^{\rm VII}_{\phi\phi\phi;{\rm GI}}$ is separately gauge-independent, unitary, and finite.

## V. GAUGE INDEPENDENCE, UNITARITY, AND FINITENESS

We have already seen [Eqs. (3.7)–(3.9)] that the allomorphic contributions involving two-current hadronic matrix elements arising from the box graphs are separately gauge-independent, unitary, and finite. In this section we verify that the same is true of the contributions involving the matrix elements $\langle j^\mu_\alpha\rangle_{FI}$, $\langle S_i\rangle_{FI}$.

First, consider the $\xi$-dependent terms involving the hadronic matrix element of the weak vector current, $\langle j^\mu_\alpha\rangle_{FI}$. We regroup these terms according to the structure of the *leptonic* matrix element. Thus

$$\delta S^{\rm tree}_{{\rm CT},\langle j\rangle}+\delta S^{\rm box}_{\langle j\rangle}+\delta S^{\rm VI}_{\langle j\rangle,{\rm GD}}+\delta S^{\rm VII}_{WWW;\langle j\rangle,{\rm GD}}+\delta S^{{\rm VII}(\phi,W)}_{\langle j\rangle,{\rm GD}}\equiv\sum_{i=1}^{5}\mathcal{J}_i\,. \quad (5.1)$$

(i) $\mathcal{J}_1$ consists of all terms in the sum (5.1) in which the lepton propagator has been cancelled so that the resulting leptonic matrix element is independent of the loop momentum $l$. Namely,

$$\mathcal{J}_1\equiv\delta\Big(\sum P\Big)\int dl\Big(-i\langle j_{\alpha\mu}\rangle_{FI}(q)_{\alpha\beta}[l]_{\gamma\delta}\overline{U}_f\zeta_\psi\gamma^\mu t_\gamma t_\beta t_\delta U_i+\tfrac{1}{2}i\langle j_{\alpha\mu}\rangle_{FI}(q)_{\alpha\beta}[l]_{\gamma\delta}\times\overline{U}_f\zeta_\psi\gamma^\mu\{t_\beta,t_\gamma t_\delta\}U_i\Big)\,. \quad (5.2)$$

The first term in $\mathcal{J}_1$ arises from $\delta S^{\rm VI}_{\langle j\rangle,{\rm GD}}$ [Eq. (4.8)] and the second arises from $\delta S^{\rm tree}_{{\rm CT},\langle j\rangle}$ [Eq. (2.12)]. One readily sees that $\mathcal{J}_1$ contains no allomorphic contributions, thus providing us with our first example of the role of renormalization counterterms in cancelling the gauge dependence of one-loop allomorphic corrections.

(ii) $\mathcal{J}_2$ is defined to contain all the terms involving the structure $\overline{U}_f\Gamma_j\Delta^\psi(q_f-l)\zeta_\psi\gamma_{\nu'}t_\gamma U_i$. After a brief calculation employing the usual arsenal of gauge-covariant interrelationships, we find the gauge-independent result

$$\mathcal{J}_2=-i\langle j^{\nu'}_{\beta'}\rangle_{FI}(\lambda,\zeta_\phi\{\theta_\gamma,\theta_\delta\}\theta_{\alpha'}\lambda)(\theta_\alpha\lambda)_i(q)_{\beta'\gamma}\delta\Big(\sum P\Big)\times\int dl(q-l)_{\delta\gamma'}\frac{(l)_{\alpha\alpha'}}{l^2}\overline{U}_f\Gamma_j\Delta^\psi(q_f-l)\zeta_\psi\gamma_{\nu'}t_{\gamma'}U_i\,. \quad (5.3)$$

(iii) Similarly, we find for contributions of the form $\overline{U}_f\zeta_\psi\gamma_{\nu'}t_{\alpha'}\Delta^\psi\Gamma_jU_i$

$$\mathcal{J}_3=-i\langle j^{\nu'}_{\beta'}\rangle_{FI}(\lambda,\zeta_\phi\{\theta_\alpha,\theta_\gamma\}\theta_\delta\lambda)(\theta_{\gamma'}\lambda)_j(q)_{\beta'\gamma}\delta\Big(\sum P\Big)\times\int dl\,\frac{(q-l)_{\delta\gamma'}}{(q-l)^2}(l)_{\alpha\alpha'}\overline{U}_f\zeta_\psi\gamma_{\nu'}t_{\alpha'}\Delta^\psi(q_f-l)\Gamma_jU_i\,. \quad (5.4)$$

(iv) The contributions involving the leptonic matrix element $\overline{U}_f\Gamma_i\Delta^\psi(q_f-l)\Gamma_kU_i$ are found to sum to the gauge-invariant result

$$\mathcal{J}_4=\langle j^\nu_{\alpha'}\rangle_{FI}(q)_{\alpha\alpha'}(\theta_\alpha\zeta_\phi\theta_\beta\lambda)_j\delta\Big(\sum P\Big)\times\int dl\,l_\nu\Big[(\theta_{\beta'}\lambda)_i\hat\Delta^\phi_{jk}(q-l)\frac{(l)_{\beta\beta'}}{l^2}-(\theta_{\beta'}\lambda)_k\hat\Delta^\phi_{ij}(l)\frac{(q-l)_{\beta\beta'}}{(q-l)^2}\Big]\times\overline{U}_f\Gamma_i\Delta^\psi(q_f-l)\Gamma_kU_i\,. \quad (5.5)$$

(v) Finally, we are left with the gauge-invariant part of $\delta S^{\rm tree}_{{\rm CT},\langle j\rangle}$

$$\mathcal{J}_5=(2\pi)^4\langle j^\nu_\alpha\rangle_{FI}\delta\Big(\sum P\Big)(q)_{\alpha\beta}\times\overline{U}_f(z^\dagger_{2,{\rm GI}}\zeta_\psi\gamma_\nu t_\beta+\zeta_\psi\gamma_\nu t_\beta z_{2,{\rm GI}})U_i\,. \quad (5.6)$$

The calculation of $z_{2,{\rm GI}}$ is discussed in Appendix B.

This concludes the proof of gauge invariance of the allomorphic terms involving hadronic matrix elements of the weak vector current. Next, we show that such terms sum to a unitary, finite result. Summoning the explicitly gauge-invariant contributions, which so far have patiently waited in the wings, we obtain the complete contribution involving $\langle j^\mu_\alpha\rangle_{FI}$:

$$\delta S^{\langle j\rangle}=\delta S^{\rm VI}_{\langle j\rangle,{\rm GI}}+\delta S^{\rm VII}_{WWW;\langle j\rangle,{\rm GI}}+\delta S^{{\rm VII}(\phi,W)}_{\langle j\rangle,{\rm GI}}+\sum_{i=2}^{5}\mathcal{J}_i\,. \quad (5.7)$$

The first two terms in (5.7) are readily seen to separately satisfy unitarity since they involve only the "unitary propagators" $(k)_{\alpha\beta}$, $\hat\Delta^\phi_{ij}(k)$, from which unphysical zero-mass Goldstone poles are absent:

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$$\delta S^{\rm VI}_{\langle j\rangle,{\rm GI}}=i\langle j^\mu_\alpha\rangle_{FI}(q)_{\alpha\beta}\delta\Big(\sum P\Big)\int dl\{(l)_{\gamma\delta}\overline{U}_f\zeta_\psi\gamma^\nu t_\gamma\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_\beta\Delta^\psi(q_i-l)\zeta_\psi\gamma_\nu t_\delta U_i-\Delta^\phi_{kl}(l)\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_\beta\Delta^\psi(q_i-l)\Gamma_lU_i\}\,, \quad (5.8)$$

$$\delta S^{\rm VII}_{WWW;\langle j\rangle,\rm GI}=i(\zeta_W)_{\alpha\beta}c_{\beta\gamma\delta}(q)_{\beta'\gamma}\langle j^{\nu'}_{\beta'}\rangle_{FI}\delta(\sum P)$$
$$\times\int dl\,(l)_{\alpha\alpha'}(q-l)_{\delta\gamma'}\bar U_f\{2l_{\nu'}\zeta_\psi\gamma^\mu t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_{\gamma'}+2\zeta_\psi \not{q} t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma_{\nu'}t_{\gamma'}$$
$$-2\zeta_\psi\gamma_{\nu'}t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\not{q}t_{\gamma'}+i(\theta_{\alpha'}\lambda)_j\Gamma_j\Delta^\psi(q_f-l)\zeta_\psi\gamma_{\nu'}t_{\gamma'}$$
$$-i\zeta_\psi\gamma_{\nu'}t_{\alpha'}\Delta^\psi(q_f-l)(\theta_{\gamma'}\lambda)_k\Gamma_k\}U_i\,. \tag{5.9}$$

The last two terms in (5.7) combine to give a unitary result:

$$\delta S^{\rm VII\,(\phi,W)}_{\langle j\rangle,\rm GI}+\sum_{i=2}^{5}\mathcal{J}_i=(2\pi)^4\langle j^\nu_\alpha\rangle_{FI}(q)_{\alpha\beta}\delta(\sum P)\bar U_f(\tilde z^\dagger_{2,\rm GI}\zeta_\psi\gamma_\nu t_\beta+\zeta_\psi\gamma_\nu t_\beta z_{2,\rm GI})U_i$$
$$+\langle j^\mu_{\alpha'}\rangle_{FI}(q)_{\alpha\alpha'}\delta(\sum P)\int dl\big[-i(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j\hat\Delta^\phi_{jk}(q-l)(l)_{\beta\beta'}\bar U_f\zeta_\psi\gamma_\mu t_{\beta'}\Delta^\psi(q_f-l)\Gamma_k U_i$$
$$-i(\zeta_\phi\{\theta_\alpha,\theta_\beta\}\lambda)_j\hat\Delta^\phi_{jk}(l)(q-l)_{\beta\beta'}\bar U_f\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_{\beta'}U_i$$
$$+2l_\mu(\theta_\alpha\zeta_\phi)_{jl}\Delta^\phi_{kl}(l)\hat\Delta^\phi_{mj}(q-l)\bar U_f\Gamma_k\Delta^\psi(q_f-l)\Gamma_m U_i\big]\,. \tag{5.10}$$

The contributions (5.8), (5.9), and (5.10) taken separately involve ultraviolet divergences, which are all logarithmic. Nevertheless, $\delta S^{\langle j\rangle}$ is finite. To see this, it is sufficient to replace the various propagators in the loop integrals by their asymptotic limits as $l\to\infty$:

$$\Delta^\psi(q_f-l),\ \Delta^\psi(q_i-l)\to\frac{i\not{l}}{l^2}\zeta_\psi{}^{-1}\,,$$
$$(l)_{\alpha\beta},\ (q-l)_{\alpha\beta}\to(\zeta_W)^{-1}{}_{\alpha\beta}\frac{1}{l^2}\,, \tag{5.11}$$
$$\hat\Delta^\phi_{jk}(l),\ \hat\Delta^\phi_{jk}(q-l)\to(\zeta_\phi)^{-1}{}_{jk}\frac{1}{l^2}\,.$$

Making the replacements (5.11), and using (B11) to extract the divergent part of $z_{2,\rm GI}$ we find for the asymptotic behavior of the various integrands

$$\delta S^{\rm VI}_{\langle j\rangle,\rm GI}\to -i\langle j^\mu_\alpha\rangle_{FI}(q)_{\alpha\beta}\delta(\sum P)$$
$$\times\int\frac{dl}{(l^2)^2}\{(\zeta_W)^{-1}{}_{\gamma\delta}\bar U_f\zeta_\psi\tilde l_\gamma\gamma_\mu t_\beta t_\delta U_i$$
$$+\tfrac12(\zeta_\phi)^{-1}{}_{kl}\bar U_f\Gamma_k\gamma^\mu\tilde l_\beta\zeta_\psi{}^{-1}\Gamma_l U_i\}$$
$$\equiv\mathcal{K}_1\,, \tag{5.12}$$

$$\delta S^{\rm VII}_{WWW;\langle j\rangle,\rm GI}\to c_{\alpha'\gamma\delta}(q)_{\beta'\gamma}\langle j^{\nu'}_{\beta'}\rangle_{FI}(\zeta_W)^{-1}{}_{\delta\gamma'}\delta(\sum P)$$
$$\times\int\frac{dl}{(l^2)^2}\bar U_f\zeta_\psi\tilde t_{\alpha'}\gamma_{\nu'}t_{\gamma'}U_i$$
$$\equiv\mathcal{K}_2\,, \tag{5.13}$$

$$\delta S^{\rm VII\,(\phi,W)}_{\langle j\rangle,\rm GI}+\sum_{i=2}^{5}\mathcal{J}_i\to\tfrac12 i\langle j^\mu_{\alpha'}\rangle_{FI}(q)_{\alpha\alpha'}(\theta_\alpha)_{jk}(\zeta_\phi)^{-1}{}_{mj}\,\delta(\sum P)\int\frac{dl}{(l^2)^2}\bar U_f\Gamma_k\gamma_\mu\zeta_\psi{}^{-1}\Gamma_m U_i$$
$$+i\langle j^\mu_\alpha\rangle_{FI}(q)_{\alpha\beta}\delta(\sum P)\int\frac{dl}{(l^2)^2}\bar U_f(\tilde t_\gamma\tilde t_\delta(\zeta_W)^{-1}{}_{\gamma\delta}\zeta_\psi\gamma_\mu t_\beta+\tfrac12\Gamma_l\zeta_\psi{}^{-1}\tilde\Gamma_k(\zeta_\phi)^{-1}{}_{kl}\gamma_\mu t_\beta)U_i$$
$$\equiv\mathcal{K}_3\,. \tag{5.14}$$

In calculating $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3$, we have of course ignored all manifestly finite terms. The proof of finiteness is concluded by noting simply

$$\mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3=0\,. \tag{5.15}$$

There remain now only the contributions involving the hadronic matrix element of the weak scalar current $\langle S_i\rangle_{FI}$. We first prove, in complete analogy to the preceding, that these contributions sum to a gauge-invariant quantity. It is convenient to calculate first the quantity

$$\mathcal{S}_0 \equiv \delta S^{\,\text{VI}}_{\langle S\rangle,\text{GD}} + \delta S^{\text{tree}}_{R,\text{GD}} + \delta S^{\text{tree}}_{\text{CT},\langle S\rangle}$$
$$= -i(2\pi)^4(\theta_\alpha\lambda)_i\langle S_i\rangle_{FI}\delta(\textstyle\sum P)\frac{(q)_{\alpha\beta}}{q^2}\bar U_f\gamma_4[t_\beta,\gamma_4\delta m]U_i - i(2\pi)^4\langle S_i\rangle_{FI}(\zeta_\phi q^2+M^2)^{-1}{}_{ij}\bar U_f(\bar z^{\dagger}_{2,\text{GI}}\Gamma_j+\Gamma_j z_{2,\text{GI}})U_i$$
$$+(2\pi)^4\langle S_i\rangle_{FI}(\theta_\alpha\lambda)_i\frac{(q)_{\alpha\beta}}{q^2}\bar U_f(\bar z^{\dagger}_{2,\text{GI}}\zeta_\psi \not q t_\beta+\zeta_\psi\not q t_\beta z_{2,\text{GI}})U_i$$
$$+\langle S_i\rangle_{FI}\delta(\textstyle\sum P)$$
$$\times\int dl\Big\{-\Delta^{\phi}_{ij}(q)[l]_{\alpha\beta}\bar U_f(\gamma_4[t_\alpha,\gamma_4\Gamma_j]\Delta^{\psi}(q_i-l)\gamma_4[t_\beta,\gamma_4 m]+\gamma_4[t_\alpha,\gamma_4 m]\Delta^{\psi}(q_f-l)\gamma_4[t_\beta,\gamma_4\Gamma_j])U_i$$
$$+[q]_{\alpha\beta}[l]_{\gamma\delta}(\theta_\alpha\lambda)_i\bar U_f(i\zeta_\psi\not q[t_\gamma,t_\beta]\Delta^{\psi}(q_i-l)\gamma_4[t_\delta,\gamma_4 m]-\gamma_4[t_\gamma,\gamma_4 m]\Delta^{\psi}(q_f-l)i\zeta_\psi\not q[t_\beta,t_\delta])U_i$$
$$+\frac{(\xi q^2+\mu^2)^{-1}{}_{\alpha\beta}}{q^2}(\theta_\alpha\lambda)_i(l)_{\gamma\delta}\bar U_f(\zeta_\psi\gamma^\mu t_\gamma\Delta^{\psi}(q_f-l)\zeta_\psi\gamma_\mu[t_\beta,t_\delta]+\zeta_\psi\gamma^\mu[t_\beta,t_\gamma]\Delta^{\psi}(q_i-l)\zeta_\psi\gamma_\mu t_\delta)U_i$$
$$-\frac{(\xi q^2+\mu^2)^{-1}{}_{\alpha\beta}}{q^2}(\theta_\alpha\lambda)_i\hat\Delta^{\phi}_{kl}(l)\bar U_f(\Gamma_k\Delta^{\psi}(q_f-l)\gamma_4[t_\beta,\gamma_4\Gamma_l]+\gamma_4[t_\beta,\gamma_4\Gamma_k]\Delta^{\psi}(q_i-l)\Gamma_l)U_i\Big\}. \tag{5.16}$$

A glance at the bracketed terms in (5.16) shows that the tree-graph counterterms have reduced the leptonic matrix elements in the vertex graphs of the type in Fig. 3 (involving *three* coupling matrice between the spinors) to structures which appear in the remaining one-loop contributions (namely, those involving only *two* coupling matrices between the lepton spinors). We are now in a position to examine for cancellation of gauge-dependent quantities. We define

$$\mathcal{S}_0+\delta S^{\,\text{box}}_{\langle S\rangle}+\delta S^{\text{VII}}_{WWW;\langle S\rangle,\text{GD}}+\delta S^{\text{VII}(\phi,W)}_{\langle S\rangle,\text{GD}}+\delta S^{\text{VII}}_{\phi\phi\phi,\text{GD}}\equiv\sum_{i=1}^{5}\mathcal{S}_i. \tag{5.17}$$

(i) In $\mathcal{S}_1$ we group all terms involving a leptonic matrix element of the form $\bar U_f\zeta_\psi\gamma^\mu t_{\alpha'}\Delta^{\psi}(q_f-l)\times\zeta_\psi\gamma_\mu t_{\gamma'}U_i$. A straightforward calculation yields the gauge-invariant result

$$\mathcal{S}_1\equiv -i\langle S_i\rangle_{FI}(\zeta_W)_{\alpha\beta}c_{\beta\gamma\delta}(\theta_{\beta'}\lambda)_i\frac{(q)_{\beta'\gamma}}{q^2}\delta(\textstyle\sum P)$$
$$\times\int dl(q^2-2q\cdot l)(l)_{\alpha\alpha'}(q-l)_{\delta\gamma'}$$
$$\times\bar U_f\zeta_\psi\gamma^\mu t_{\alpha'}\Delta^{\psi}(q_f-l)\zeta_\psi\gamma_\mu t_{\gamma'}U_i\,. \tag{5.18}$$

(ii) Next, we group terms involving the structure $\bar U_f\zeta_\psi\not q t_{\alpha'}\Delta^{\psi}(q_f-l)\Gamma_k U_i$. The result is again gauge-independent:

$$\mathcal{S}_2\equiv i\langle S_i\rangle_{FI}(\theta_\alpha\zeta_\phi\theta_\beta\lambda)_j\delta(\textstyle\sum P)$$
$$\times\int dl(l)_{\alpha\alpha'}\bar U_f\zeta_\psi\not q t_{\alpha'}\Delta^{\psi}(q_f-l)\Gamma_k U_i\Big\{(\theta_{\beta'}\lambda)_k\hat\Delta^{\phi}_{ij}(q)\frac{(q-l)_{\beta\beta'}}{(q-l)^2}-(\theta_{\beta'}\lambda)_i\hat\Delta^{\phi}_{kj}(q-l)\frac{(q)_{\beta\beta'}}{q^2}\Big\}\,. \tag{5.19}$$

(iii) Similarly, we find

$$\mathcal{S}_3\equiv i\langle S_i\rangle_{FI}(\theta_\alpha\zeta_\phi\theta_\beta\lambda)_j\delta(\textstyle\sum P)$$
$$\times\int dl(q-l)_{\alpha\alpha'}\bar U_f\Gamma_k\Delta^{\psi}(q_f-l)\zeta_\psi\not q t_{\alpha'}U_i\Big\{(\theta_{\beta'}\lambda)_k\hat\Delta^{\phi}_{ij}(q)\frac{(l)_{\beta\beta'}}{l^2}-(\theta_{\beta'}\lambda)_i\hat\Delta^{\phi}_{kj}(l)\frac{(q)_{\beta\beta'}}{q^2}\Big\}\,. \tag{5.20}$$

(iv) The terms involving leptonic matrix elements of the form $\bar U_f\Gamma_m\Delta^{\psi}(q_f-l)\Gamma_k U_i$ are depressingly complicated; nevertheless, after much algebraic manipulation, we find the remarkably simple, and *gauge-invariant*, result:

$$\mathcal{S}_4\equiv\langle S_i\rangle_{FI}\delta(\textstyle\sum P)\int dl\,\bar U_f\Gamma_m\Delta^{\psi}(q_f-l)\Gamma_k U_i\Big\{(\theta_\gamma\zeta_\phi\theta_\beta\lambda)_l(\theta_{\beta'}\lambda)_i\Big[(q)_{\beta'\gamma}\frac{(l)_{\beta\gamma'}}{l^2}(\theta_{\gamma'}\lambda)_m\hat\Delta^{\phi}_{kl}(q-l)$$
$$+\frac{(q)_{\beta\beta'}}{q^2}(l)_{\gamma\gamma'}(\theta_{\gamma'}\lambda)_m\hat\Delta^{\phi}_{kl}(q-l)$$
$$+\frac{(q)_{\beta\beta'}}{q^2}\hat\Delta^{\phi}_{lm}(l)(\theta_{\gamma'}\lambda)_k(q-l)_{\gamma\gamma'}\Big]$$
$$-\frac{(q)_{\beta\beta'}}{q^2}(\theta_{\beta'}\lambda)_i((\theta_\beta)_{kl}\hat\Delta^{\phi}_{ml}(l)+(\theta_\beta)_{ml}\hat\Delta^{\phi}_{kl}(q-l))\Big\}\,. \tag{5.21}$$

(v) Finally, we are left with the gauge-invariant part of $\mathcal{S}_0$:

$$\mathcal{S}_5 \equiv -i(2\pi)^4\langle S_i\rangle_{FI}\,\delta(\textstyle\sum P)\Big((\theta_\alpha\lambda)_i\,\frac{(q)_{\alpha\beta}}{q^2}\,\overline{U}_f\,\gamma_4[t_\beta,\gamma_4\delta m]U_i+(\zeta_\phi q^2+M^2)^{-1}{}_{ij}\overline{U}_f(\tilde{z}^\dagger_{2,\mathrm{GI}}\Gamma_j+\Gamma_j z_{2,\mathrm{GI}})U_i\Big)$$
$$+(2\pi)^4\langle S_i\rangle_{FI}\,\delta(\textstyle\sum P)(\theta_\alpha\lambda)_i\,\frac{(q)_{\alpha\beta}}{q^2}\,\overline{U}_f(\tilde{z}^\dagger_{2,\mathrm{GI}}\zeta_\psi\not{q}t_\beta+\zeta_\psi\not{q}t_\beta z_{2,\mathrm{GI}})U_i\ . \quad (5.22)$$

Having verified the gauge independence of these contributions, and hence of the entire $O(g^4)$ allomorphic corrections, we now turn to the question of unitarity and finiteness of the terms involving $\langle S_i\rangle_{FI}$. Adding in the explicitly gauge-invariant contributions, the entire contribution of this form is given by

$$\delta S^{\langle S\rangle}=\delta S^{\mathrm{VI}}_{\langle S\rangle,\mathrm{GI}}+\delta S^{\mathrm{VII}}_{WWW;\langle S\rangle,\mathrm{GI}}+\delta S^{\mathrm{VII}(\phi,W)}_{\langle S\rangle,\mathrm{GI}}+\delta S^{\mathrm{VII}}_{\phi\phi\phi,\mathrm{GI}}+\sum_{i=1}^{5}\mathcal{S}_i\ . \quad (5.23)$$

After some algebra, we find that the unphysical zero-mass Goldstone poles in the separate terms in (5.23) cancel exactly, and we are left with the following manifestly gauge-invariant, unitary expression for $\delta S^{\langle S\rangle}$:

$$\begin{aligned}\delta S^{\langle S\rangle}=\langle S_i\rangle_{FI}\,\delta(\textstyle\sum P)\Big\{&-i(2\pi)^4\hat{\Delta}^\phi_{ij}(q)\overline{U}_f(\tilde{z}^\dagger_{2,\mathrm{GI}}\Gamma_j+\Gamma_j z_{2,\mathrm{GI}})U_i\\
&+\int dl\,[\,\hat{\Delta}^\phi_{ij}(q)((l)_{\alpha\beta}\overline{U}_f\zeta_\psi\gamma^\mu t_\alpha\Delta^\psi(q_f-l)\Gamma_j\Delta^\psi(q_i-l)\zeta_\psi\gamma_\mu t_\beta U_i\\
&\qquad-\hat{\Delta}^\phi_{kl}(l)\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\Gamma_j\Delta^\psi(q_i-l)\Gamma_l U_i)\\
&\qquad+2(\zeta_\phi\theta_\beta\theta_\alpha\lambda)_j\hat{\Delta}^\phi_{ij}(q)(l)_{\alpha\alpha'}(q-l)_{\beta\beta'}\overline{U}_f\zeta_\psi\gamma^\mu t_{\alpha'}\Delta^\psi(q_f-l)\zeta_\psi\gamma_\mu t_{\beta'}U_i\\
&\qquad+2i(\theta_\alpha\zeta_\phi)_{jl}\hat{\Delta}^\phi_{ij}(q)((q-l)_{\alpha\alpha'}\hat{\Delta}^\phi_{kl}(l)\overline{U}_f\Gamma_k\Delta^\psi(q_f-l)\zeta_\psi\not{q}t_{\alpha'}U_i\\
&\qquad\qquad+(l)_{\alpha\alpha'}\hat{\Delta}^\phi_{kl}(q-l)\overline{U}_f\zeta_\psi\not{q}t_{\alpha'}\Delta^\psi(q_f-l)\Gamma_k U_i)\\
&\qquad-f_{i'j'k'}\hat{\Delta}^\phi_{ii'}(q)\hat{\Delta}^\phi_{jj'}(l)\hat{\Delta}^\phi_{kk'}(q-l)\overline{U}_f\Gamma_j\Delta^\psi(q_f-l)\Gamma_k U_i\\
&\qquad-(\theta_\alpha\zeta_\phi)_{ml}\overline{U}_f\Gamma_j\Delta^\psi(q_f-l)\Gamma_k U_i((\theta_{\alpha'}\lambda)_i(q)_{\alpha\alpha'}\hat{\Delta}^\phi_{kl}(q-l)\hat{\Delta}^\phi_{mj}(l)\\
&\qquad\qquad+(\theta_{\alpha'}\lambda)_j(l)_{\alpha\alpha'}\hat{\Delta}^\phi_{kl}(q-l)\hat{\Delta}^\phi_{im}(q)\\
&\qquad\qquad+(\theta_{\alpha'}\lambda)_k(q-l)_{\alpha\alpha'}\hat{\Delta}^\phi_{jl}(l)\hat{\Delta}^\phi_{im}(q))]\Big\}\ . \end{aligned} \quad (5.24)$$

We conclude this section by noting that the allomorphic component of (5.24) is in fact finite. The divergent terms in (5.24) involve (asymptotically as $l\to\infty$) leptonic matrix elements with the structure

$$\overline{U}_f\delta\Gamma_j U_i\equiv\overline{U}_f\big(a(\zeta_W)^{-1}{}_{\alpha\beta}\Gamma_j t_\alpha t_\beta+a(\zeta_W)^{-1}{}_{\alpha\beta}\tilde{t}_\alpha\tilde{t}_\beta\Gamma_j+b(\zeta_\phi)^{-1}{}_{kl}\Gamma_j\tilde{\zeta}_\psi{}^{-1}\tilde{\Gamma}_k\zeta_\psi{}^{-1}\Gamma_l$$
$$+b(\zeta_\phi)^{-1}{}_{kl}\Gamma_l\tilde{\zeta}_\psi{}^{-1}\tilde{\Gamma}_k\zeta_\psi{}^{-1}\Gamma_j+c(\zeta_W)^{-1}{}_{\alpha\beta}\tilde{t}_\alpha\Gamma_j t_\beta+d(\zeta_\phi)^{-1}{}_{kl}\Gamma_k\tilde{\zeta}_\psi{}^{-1}\tilde{\Gamma}_j\zeta_\psi{}^{-1}\Gamma_l\big)U_i\ . \quad (5.25)$$

However, $\delta\Gamma_j$ merely constitutes a gauge-covariant renormalization of $\Gamma_j$. In fact, using (A3), (A4), and (A13), one can readily show that $\delta\Gamma_j$ satisfies exactly the same zeroth-order constraints as $\Gamma_j$:

$$\gamma_4[t_\alpha,\gamma_4\delta\Gamma_j]=(\theta_\alpha)_{kj}\Gamma_k\ , \quad (5.26)$$

$$\delta\Gamma_j{}^\dagger=\delta\tilde{\Gamma}_j\ . \quad (5.27)$$

## VI. SUMMARY OF RESULTS

Before proceeding to the more detailed study and applications of our results, it may be useful to provide a simple prescription for generating the correct answer. The fourth-order (leptonic) allomorphic contributions have been seen to arise from basically three sources:

(a) One source is the "implicit" allomorphic correction obtained by using the lepton mass matrix *corrected to* $O(g^2)$ in the tree-graph contribution (2.9). In other words, the leptonic spinors appearing in (2.9) must be defined with leptonic masses *including* any one-loop corrections to zeroth-order symmetries. These corrections have already been calculated in general models by Weinberg[8] (see also Appendix B).

(b) Another source is a contribution arising from the asymmetric wave-function renormalization of the leptons in a *physical, on-mass-shell* renormalization scheme:

$$\delta S_{\rm CT} \equiv (2\pi)^4 \langle j^{\nu}_{\alpha} \rangle_{FI} (q)_{\alpha\beta} \delta(\textstyle\sum P) \overline{U}_f (\tilde{z}^{\dagger}_{2,\rm GI} \zeta_\psi \gamma_\nu t_\beta + \zeta_\psi \gamma_\nu t_\beta z_{2,\rm GI}) U_i$$
$$- i (2\pi)^4 \langle S_i \rangle_{FI} \hat{\Delta}^{\phi}_{ij}(q) \delta(\textstyle\sum P) \overline{U}_f (\tilde{z}^{\dagger}_{2,\rm GI} \Gamma_j + \Gamma_j z_{2,\rm GI}) U_i . \quad (6.1)$$

Here $(q)_{\alpha\beta}$, $\hat{\Delta}^{\phi}_{ij}(q)$ are basically the gauge vector and Higgs-scalar propagators in a "generalized Feynman gauge" characterized by the replacement $\xi \to \zeta_W$:

$$(q)_{\alpha\beta} \equiv (\zeta_W q^2 + \mu^2)^{-1}{}_{\alpha\beta},$$
$$\hat{\Delta}^{\phi}_{ij}(q) \equiv (\zeta_\phi q^2 + M^2)^{-1}{}_{ij} + \frac{(q)_{\alpha\beta}}{q^2} (\theta_\alpha \lambda)_i (\theta_\beta \lambda)_j . \quad (6.2)$$

$z_{2,\rm GI}$ is the gauge-independent wave-function renormalization constant introduced in Sec. II. The logarithmically divergent part of $z_{2,\rm GI}$ is explicitly calculated in Appendix B.

(c) A third source is a contribution obtained simply by evaluating the one-loop graphs depicted in Figs. 2, 3, and 4 in the generalized Feynman gauge (6.2). This contribution may be expressed as

$$\delta S_{\text{one loop}} \equiv \delta S^{\rm box} + \delta S^{\langle j \rangle} + \delta S^{\langle S \rangle} - \delta S_{\rm CT} , \quad (6.3)$$

where

$$\delta S^{\rm box} \equiv \delta S^{\rm box}_{\langle jj \rangle} + \delta S^{\rm box}_{\langle jS \rangle} + \delta S^{\rm box}_{\langle SS \rangle} , \quad (6.4)$$

with $S^{\rm box}_{\langle jj \rangle}$, $S^{\rm box}_{\langle jS \rangle}$, $S^{\rm box}_{\langle SS \rangle}$ given in (3.7)–(3.9), and

$$\delta S^{\langle j \rangle} \equiv \delta S^{\langle j \rangle}_1 + \delta S^{\langle j \rangle}_2 + \delta S^{\langle j \rangle}_3 , \quad (6.5)$$

where $\delta S^{\langle j \rangle}_{1,2,3}$ are given as the right-hand sides of equations (5.8)–(5.10), respectively. Finally, $\delta S^{\langle S \rangle}$ is given explicitly in (5.24). $\delta S_{\text{one loop}}$ is clearly gauge-independent and unitary, since $\xi$ dependence and unphysical Goldstone poles are both absent from the propagators (6.2). However, as pointed out in Sec. V, $\delta S_{\text{one loop}}$ still contains infinite allomorphic terms, which have been shown to cancel exactly against similar terms in $\delta S_{\rm CT}$.

The enormous utility of the general $\xi$-gauge calculation presented above is emphasized by the appearance of the contribution (6.1). We are automatically alerted to the necessity for including any missing terms (provided, of course, these are themselves gauge-dependent) by the failure of all $\xi$-dependent terms to cancel. The cancellation of zero-mass Goldstone poles from among the various contributions provides yet another useful check on the calculation.

Our results may be obtained, therefore, by writing down, in the generalized Feynman gauge, all terms which could lead to allomorphic corrections. Of course, we could use any other gauge, but the unitarity of the result is most clearly manifest in this gauge, which seems to appear "magically" at the end of the calculations.

If one adds, for example, elementary hadronic scalars to the theory, then there may be additional terms in the interaction Lagrangian leading to additional contributions—the new scalar fields may, for example, contribute to the weak currents. However, the contributions we have calculated will still be present, and since the asymptotic behavior of two-current hadronic matrix elements is very sensitive to the presence of elementary strongly interacting scalar fields,[5] they can be used to test for the possibility of inconsistency of theories containing such fields with the phenomenological data. Such a test will be the major object of Secs. VII and VIII.

## VII. INDUCED PSEUDOSCALAR EFFECTS: CHARGED-PION DECAY

As an illustrative application of the general results derived above for the leptonic end allomorphic corrections, we investigate the general nature of the constraints placed on possible gauge theories of weak and electromagnetic interactions by the charged-pion-decay branching ratio $\Gamma_{\pi\to\mu\nu}/\Gamma_{\pi\to e\nu}$. This quantity, as a result of the very small mass of the electron, is extremely sensitive to admixtures of pseudoscalar interactions in the lowest-order $V-A$ leptonic currents. In fact, one easily sees that an effective pseudoscalar current of $O(\alpha G_F)$ would lead to a correction of $O(1)$ to the branching ratio. The phenomenal agreement of the $V-A$ theory with experiment [up to inevitable corrections of $O(\alpha)$ to the branching ratio, partially electromagnetic in origin] then requires the elimination of any induced leptonic pseudoscalar current in $O(\alpha G_F)$.

The constraints derived will depend slightly on the assumed theory of strong interactions. Nevertheless, it will be seen that gauge theories tend readily to preserve the lowest-order $V-A$ structure of the leptonic weak currents, given only some very loose and plausible restrictions on particle masses. Even the presence in the theory of an elementary pion does not lead to insuperable difficulties—it will be seen that an additional constraint on vector-meson masses suffices to make such theories "safe." It may be mentioned here that the situation is quite different in the case of the hadronic end allomorphic corrections, a preliminary discussion of which is presented in the following section. Such corrections are apparently much more sensitive to the presence in the theory of elementary, strongly interacting scalar fields.

Before proceeding, we note that dangerous pseudoscalar effects are assumed already absent in lowest order; either by virtue of the structure of Yukawa couplings [e.g., in simple theories such as the original Weinberg model,[1] or the Georgi-Glashow O(3) model,[9] the charged Higgs scalars are unphysical] or by removing all physical scalars to very high mass.

The contributions involving the single-current hadronic matrix elements $\langle j^{\mu}_{\alpha}\rangle_{FI}, \langle S_i\rangle_{FI}$ are discussed in detail in Appendix C. The condition for effective pseudoscalar contributions to pion decay of $O(\alpha G_F)$ to arise in $\delta S^{\langle j\rangle}$ is found there to be

$$m_l q \approx m_W^2 . \tag{7.1}$$

Here $m_l$ is a typical (heavy) lepton mass, corresponding to a particle with which the electron mixes, and $q$ is a typical lepton momentum. In theories in which the electron mixes neither with the muon nor with some other heavy lepton, $m_l$ must be taken as the electron mass in (7.1) in estimating the induced pseudoscalar effect in $\pi^- \to e^-\bar{\nu}_e$, which leads to an $O(\alpha)$ correction to the branching ratio. But theories containing such mixing effects are interesting from the point of view of computing the electron mass,[9] and (7.1) shows that they lead to no dangerous terms in $\delta S^{\langle j\rangle}$. In fact, (7.1) implies (with $q \approx 0.1$ GeV for pion decay) dangerous contributions only when $m_l \gtrsim 10^4$ GeV, if we take $m_W \gtrsim 30$ GeV.

The story is rather similar for the $\delta S^{\langle S\rangle}$ contributions. Here (taking for definiteness $m_l \lesssim m_\phi \lesssim m_W$, with $m_\phi$ a typical Higgs-scalar mass) the leading induced pseudoscalar contribution becomes $O(\alpha G_F)$ when

$$m_l m_h \approx m_\phi^2, \tag{7.2}$$

for theories in which the pion couples to a charged physical Higgs scalar, and [cf. Eq. (C18)] when

$$m_l^3 m_h \approx m_\phi^2 m_W^2 \tag{7.2$'$}$$

otherwise. In any case, theories in which both the leptonic and hadronic fermions are considerably less massive than the weakly interacting scalar and vector particles are automatically "safe."

We will now study the contributions involving hadronic matrix elements of the product of two currents—these arise from the "box" diagrams of Fig. 2. To even estimate the magnitude of the induced leptonic pseudoscalar current here, we need to make some assumption concerning the strong interactions. We will consider three imaginable situations:

(i) The pion is elementary (or couples to an elementary field, which amounts to the same thing). This statement, of course, only has an objective meaning if anomalous dimensions are small (less than unity, say), as we shall assume.

(ii) The pion, although not itself elementary, has the quantum numbers of a bound state of two elementary hadronic scalars.

(iii) The operator (constructed solely from hadronic fields) of *least dimension* coupling the pion to the vacuum is $\bar{\psi}\gamma_5\psi$. A sufficient, though not necessary, condition for this situation to obtain is simply that the theory contains no elementary hadronic scalars.

It will be assumed in the following that there are no superheavy fermions and that the Higgs and gauge mesons are of comparable mass. Also, contributions in which photons participate lead only to an $O(\alpha)$ correction to the branching ratio (photons do not mix the electron with either the muon or some other heavy lepton) and may be ignored. As a result, the only diagram which can conceivably lead to an $O(\alpha G_F)$ allomorphic effect[10] is the two-vector-exchange one [cf. Eq. (3.7)]:

$$\delta S'_{\langle jj\rangle} = -i\delta\left(\sum P\right)\int dl\, \mathfrak{F}^{\mu\nu}_{\alpha\beta}(l)(l)'_{\alpha\gamma}(q-l)'_{\beta\delta} \times \bar{U}_f \gamma_\mu t_\gamma \Delta^{\psi}(q_f - l)\gamma_\nu t_\delta U_i , \tag{7.3}$$

where the prime on $(l)'_{\alpha\gamma}, (q-l)'_{\beta\delta}$ indicates that these propagators are projected onto the subspace of massive mesons ($\zeta_\psi, \zeta_\phi, \zeta_W$ are assumed unity for simplicity). We may immediately conclude that models without massive neutral currents [e.g., the Georgi-Glashow O(3) model] are automatically safe [subject only to the constraints implied by (7.1) and (7.2)], since in such models $\delta S'_{\langle jj\rangle}$ generates only $\Delta Q = 0, \pm 2$ currents corresponding to $W^+W^-, W^+W^+, W^-W^-$ exchange. For the rest of this section, we will therefore assume that we are dealing with models which *do* contain heavy neutral gauge mesons.

The effective pseudoscalar contribution to $\delta S'_{\langle jj\rangle}$ is of order

$$\delta S'_{\langle jj\rangle} \sim \alpha m_l \int dl \frac{1}{l^2 + m_W^2}\frac{1}{(q-l)^2 + m_W^2} \times \frac{1}{(q_f - l)^2 + m_l^2} k_{\alpha\beta,\mu\nu}\mathfrak{F}^{\mu\nu}_{\alpha\beta}(l), \tag{7.4}$$

where $k_{\alpha\beta,\mu\nu}$ is some dimensionless tensor of order unity. If $\mathfrak{F}^{\mu\nu}_{\alpha\beta}(l)$ is asymptotically constant as $l \to \infty$ (in the deep Euclidean sense), $\delta S'_{\langle jj\rangle} \sim \alpha G_F$, whereas if $\mathfrak{F}^{\mu\nu}_{\alpha\beta}(l) \lesssim O(1/l)$ in this limit, $\delta S'_{\langle jj\rangle} \sim G_F^2$, since both factors of $m_W^2$ serve to suppress the contribution (recall that $\delta S'_{\langle jj\rangle}$ is by definition a *purely weak* contribution). We will now employ the Wilson operator product expansion[4,5] (OPE) to analyze the asymptotic behavior of $\mathcal{F}^{\mu\nu}_{\alpha\beta}(l)$ in the three situations envisaged above.

(i) If the pion couples to elementary scalar field field(s) $\Phi_p$, the leading term in the OPE for $\mathcal{F}^{\mu\nu}_{\alpha\beta}(l)$ is

$$\mathcal{F}^{\mu\nu}_{\alpha\beta}(l) \sim \alpha m_h\left(a_{\alpha\beta,p}(l^2)g^{\mu\nu} + b_{\alpha\beta,p}(l^2)\frac{l^\mu l^\nu}{l^2}\right) \times \langle 0|\Phi^R_p(0)|\pi\rangle\,. \tag{7.5}$$

Here $m_h$ is a typical quark mass and $a_{\alpha\beta,p}(l^2)$, $b_{\alpha\beta,p}(l^2)$ are dimensionless coefficient functions which, in the spirit of our assumption of smallness of anomalous dimensions, will be assumed to be asymptotically constant. The factor of $m_h$ arises because all the couplings in the hadronic part of the Lagrangian which are not invariant under $\psi\to\gamma_5\psi$, $\Phi_p\to-\Phi_p$ (such as the quark-mass terms, and trilinear couplings in the hadronic scalars) involve such a factor. In the symmetric limit in which such couplings vanish, the single scalar $\langle 0|\Phi^R_p(0)|\pi\rangle$ contributions to the OPE also must disappear, since the currents $j^\mu_\alpha$ are even under the transformation cited above, while $\Phi^R_p$ is odd.

Inserting (7.5) into (7.4), we find a contribution of order

$$\delta S^{\langle jj\rangle} \sim \alpha^2\frac{m_h m_l}{m_W{}^2}\langle 0|\Phi^R|\pi\rangle\,. \tag{7.6}$$

From (7.6) one sees that theories in which (a) the pion is elementary, (b) there exist heavy neutral mesons, and (c) the electron mixes with the muon or some other massive lepton ($m_l \geq m_\mu$) potentially generate an unacceptably large induced leptonic pseudoscalar effect of $O(\alpha G_F)$ in pion decay. However, we immediately see from (7.6) that this problem is cured simply by introducing a superstrong symmetry breaking in the vector-meson mass matrix so that the gauge mesons responsible for the mixing are removed to the superheavy regime ($m_W \gg$ mass of meson mediating $\Delta S=0$, $\Delta Q=1$ weak current).

(ii) If the pion is a bound state of two elementary hadronic scalars, the situation is similar to the above. Now one finds

$$\mathcal{F}^{\mu\nu}_{\alpha\beta}(l) \sim \alpha\left(a_{\alpha\beta,N}(l^2)g^{\mu\nu} + b_{\alpha\beta,N}(l^2)\frac{l^\mu l^\nu}{l^2}\right) \times \langle 0|X^{(N)}_{pq}\Phi^R_p(0)\Phi^R_q(0)|\pi\rangle\,, \tag{7.7}$$

so that $\mathcal{F}^{\mu\nu}_{\alpha\beta}(l)$ is still asymptotically constant and the same conclusions obtain as in (i).

(iii) If there are no elementary hadronic scalars, or more generally, if the lowest-dimensionality operator linking the single pion state to the vacuum is $\bar\psi\gamma_5\psi$, the asymptotic behavior of $\mathcal{F}^{\mu\nu}_{\alpha\beta}(l)$ is given by

$$\mathcal{F}^{\mu\nu}_{\alpha\beta}(l) \sim \alpha\left\langle 0\left|\bar\psi\left(a_{\alpha\beta}(l^2)\frac{l_\rho}{l^2}g^{\mu\nu} + b_{\alpha\beta}(l^2)\frac{l^\mu}{l^2}g^\nu_\rho - b_{\beta\alpha}(l^2)\frac{l^\nu}{l^2}g^\mu_\rho + c_{\alpha\beta}(l^2)\frac{l^\mu l^\nu l_\rho}{(l^2)^2} + \epsilon^{\mu\nu}{}_{\sigma\rho}\,d_{\alpha\beta}(l^2)\frac{l^\sigma}{l^2}\right)\gamma^\rho\psi\right|\pi\right\rangle\,. \tag{7.8}$$

The scalar, tensor, and pseudoscalar contributions [by a chirality argument analogous to that given in (i) above] involve an extra factor of $m_h$, reducing the asymptotic behavior to $O(1/l^2)$, rather than $O(l_\rho/l^2)$ as above. Since only vector or axial-vector currents are involved in (7.8) [both in $\mathcal{F}^{\mu\nu}_{\alpha\beta}(l)$ *and* on the right-hand side], there are in fact *no anomalous dimensions* associated with any of the coefficient functions $a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}$. These are *rigorously constant*[11] as $l^2\to\infty$. Inserting (7.8) into (7.4), we find

$$\delta S^{\langle jj\rangle} \sim \alpha^2\frac{q m_l}{(m_W{}^2)^2}\langle 0|\bar\psi\gamma_5\psi|\pi\rangle \sim G_F{}^2 q m_l\langle 0|\bar\psi\gamma_5\psi|\pi\rangle\,. \tag{7.9}$$

This contribution is of similar magnitude to those arising in $\delta S^{\langle j\rangle}$, and is already quite small enough without any additional assumption of superheavy gauge mesons.

To summarize, the presence of elementary hadronic fields coupled to the pion (either linearly or quadratically) places additional, but fairly weak, constraints on the structure of the weak interactions. Clearly, in a wide range of gauge theories, the lowest order $V-A$ structure of the leptonic weak currents is preserved to at least $O(\alpha G_F)$, irrespective of the structure of the strong interactions.

## VIII. HADRONIC ALLOMORPHIC CORRECTIONS: NEUTRAL-KAON DECAY

The fourth-order allomorphic corrections to general semileptonic processes may conveniently be placed in three categories. First, there are the leptonic end allomorphic corrections, which are obtained by computing that part of the corrected leptonic vertex which does not arise from insertions of the "lowest-order" currents $j^\mu_\alpha$, $S_i$. The general theory of such corrections has been presented in Secs. I–VI.

Second, there are the contributions which arise from corrections to the intermediate vector or scalar meson propagators. These corrections (to fourth order) evidently leave the zeroth-order structure at both the leptonic and hadronic vertices unaltered. The allomorphic part of such contributions is obtained by computing the one-loop corrections to the vector and scalar propagators, ignoring terms which can be absorbed in a redefinition of the zeroth-order parameters of the theory. Note that (if we imagine all the Higgs-scalar particles made sufficiently massive) such contributions need not be considered in calculating induced leptonic pseudoscalar corrections. We will not consider further such "propagator allomorphic" effects.

Finally, there are the hadronic end allomorphic corrections—these are just the corrections to the hadronic matrix element which cannot be written as a linear combination of the lowest-order matrix elements $\langle F|j_\alpha^\mu|I\rangle$, of $\langle F|S_i|I\rangle$ of the hadronic weak currents. As a specific example, one frequently constructs theories in which, for particular hadronic states $|I\rangle$, $|F\rangle$, and some specific hadronic current $j_0^\mu$, $\langle F|j_0^\mu|I\rangle$ vanishes exactly, to all orders in the strong interactions (see below). Any nonvanishing value (in higher order of the *weak* interactions) for the coupling of the gauge meson associated with $j_0^\mu$ to the states $|I\rangle$, $|F\rangle$ would then correspond to a hadronic allomorphic correction.

The complete discussion of the hadronic end allomorphic corrections will be presented in a future publication. In the following, we analyze the hadronic end allomorphic terms arising from the "box" diagrams solely. The particular hadronic allomorphic terms to be examined are those corresponding to the appearance of neutral strangeness-changing currents ($\Delta Q = 0$, $\Delta S = 1$) to $O(\alpha G_F)$ in neutral-kaon decay. The decay $K_L \to \mu^+\mu^-$ is suppressed relative to $K^+ \to \mu^+\nu_\mu$ by a factor $\lesssim 10^{-8} \sim O(\alpha^4)$. Consequently, neutral strangeness-changing currents are absent to $O(\alpha G_F)$, although they may occur to $O(\alpha^2 G_F)$ (compare pion decay). Of course, unless some specific suppression mechanism is operative, one expects to find such contributions. We will therefore be concerned with the general circumstances under which such a suppression mechanism can be successful. In particular, we will confine our attention to the mechanism suggested by Glashow *et al.*[12] (henceforth referred to as GIM), and, as mentioned above, we will only investigate the neutral strangeness-changing currents generated in the two-boson exchange diagrams, ignoring weak corrections to the strong vertex. It will appear that the GIM mechanism is probably insufficient in theories in which the kaon field dimensionality is two or less, but suffices to remove dangerous contributions from the two-boson exchange graphs for a kaon field of dimensionality three ("fermion-antifermion bound state"). The failure appears unavoidable in theories in which the kaon is regarded as elementary.

The GIM mechanism is implemented by introducing an additional quark (denoted $\mathcal{P}'$) in such a way that whenever the strong interactions are exactly invariant under an O(2) symmetry between the $\mathcal{P}'$ quark and the usual proton quark $\mathcal{P}$, then $\mathcal{P}$ and $\mathcal{P}'$ may be redefined in such a way as to effectively eliminate the Cabibbo angle. The result is the vanishing of the neutral two-current matrix element between states of nonequal strangeness, to all orders. Since the additional quantum number associated with the new quark $\mathcal{P}'$ has not been detected experimentally, it is extremely likely that $\mathcal{P}$ and $\mathcal{P}'$ differ considerably in mass, if indeed the latter exists. As a result we expect that the two-current hadronic matrix element $\langle 0|j^\mu_{\Delta Q=+1}(x)j^\nu_{\Delta Q=-1}(0)|K_L\rangle$ no longer vanishes, but will be proportional to $\Delta m_q \equiv m_{\mathcal{P}'} - m_{\mathcal{P}}$. The extraction of a factor of the quark mass difference lowers the asymptotic behavior of the Fourier transform of the above matrix element. Our task is to examine the circumstances under which such a suppression of asymptotic behavior suffices to remove the specter of an induced neutral strangeness-changing current of $O(\alpha G_F)$.

In analogy to our study of pion decay, we consider three possible situations:

(i) The kaon is elementary. Assuming as usual that the effects of scalar exchange are minimized by making the Higgs mesons massive ($m_\phi \sim m_W$) and avoiding superheavy fermions, only the two-vector exchange diagram can conceivably yield an $O(\alpha G_F)$ contribution. Application of the OPE yields, in analogy to (7.5),

$$\mathcal{F}^{\mu\nu}_{\alpha\beta}(l) \sim \alpha\Delta m_q\left(a_{\alpha\beta,p}(l^2)g^{\mu\nu} + b_{\alpha\beta,p}(l^2)\frac{l^\mu l^\nu}{l^2}\right) \times\langle 0|\Phi_p^R(0)|K_L^0\rangle \tag{8.1}$$

so the purely weak contribution arising from $W^+W^-$ exchange is too large:

$$\delta S'_{\langle jj\rangle} \sim \alpha G_F q_K \Delta m_q \quad (q_K \sim \text{kaon momentum}). \tag{8.2}$$

(ii) In theories in which the kaon behaves like a bound state of two elementary scalar fields, the GIM mechanism succeeds in suppressing those contributions in which the weak currents hook onto fermions. The OPE yields in this case

$$\mathcal{F}^{\mu\nu}_{\alpha\beta}(l) \sim \alpha \frac{m_h \Delta m_q}{l^2}\left(a_{\alpha\beta,N}(l^2) g^{\mu\nu} + b_{\alpha\beta,N}(l^2)\frac{l^\mu l^\nu}{l^2}\right) \times \langle 0| X^{(N)}_{pp'} \Phi^R_p(0)\Phi^R_{p'}(0)|K^0_L\rangle,$$

and the suppression of asymptotic behavior leads to an $O(G_F{}^2)$ contribution. However, there are still contributions in which the weak currents hook onto a *scalar* line. Since theories with elementary hadronic scalars may violate[3] natural conservation of strangeness in the *strong* interactions (i.e., apart from *ad hoc* choices of the parameters in the Lagrangian) there is the danger of $O(\alpha^2)$ and $O(\alpha G_F)$ contributions arising both from purely weak and from electromagnetic effects, as indicated in Fig. 5.

(iii) Finally, we consider theories in which the kaon behaves effectively as a fermion-antifermion bound state. Consider for definiteness a theory in which the strong interactions are mediated by gauge vector gluons, and in which the gauge symmetry of the strong interactions commutes with that of the weak interactions ("colored quark" model). Contributions involving photon exchange are suppressed beyond $O(\alpha^2 G_F)$ in such theories, as such contributions necessarily involve a neutral, strangeness-changing weak interaction followed by (two-) photon exchange.

Finally, we note that the lowest operator in the OPE for $\mathcal{F}^{\mu\nu}_{\alpha\beta}(l)$ is $\bar{\psi}X\psi$, so that $\mathcal{F}^{\mu\nu}_{\alpha\beta}(l)$ is asymptotically of order $\Delta m_q/l^2$. Both factors of vector meson mass serve to suppress the purely weak contribution, and $\delta S'_{\langle jj \rangle}$ is now of $O(G_F{}^2)$.

Of course, one must also investigate the hadronic allomorphic corrections to the strong vertex, as these might conceivably lead to unsuppressed $O(\alpha G_F)$ neutral strangeness-changing effects. Work on this problem is in progress. However, it seems clear that, unless some special additional suppression mechanism is at work, the suggestion of Glashow *et al.* does not suffice to remove induced $\Delta S = 1$, $\Delta Q = 0$ currents to $O(\alpha G_F)$ in theories in which the dimensionality of the kaon is two or less.

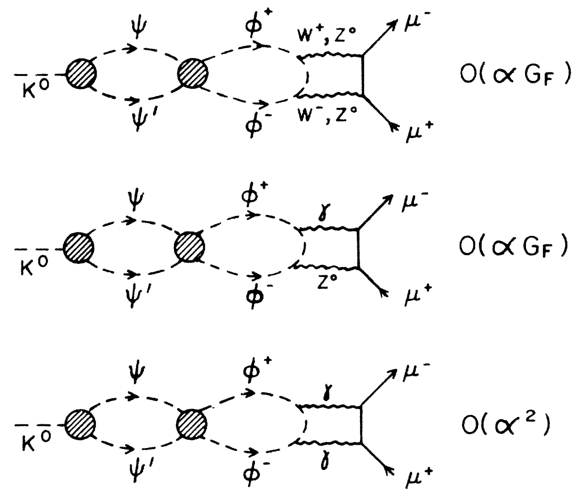


FIG. 5. Contributions to kaon decay in type (ii) theories.

## IX. SUMMARY AND CONCLUSIONS

In this paper we have presented what may be regarded as the first step in the formulation of a general theory of higher-order corrections to semileptonic processes, namely, the evaluation of fourth-order allomorphic corrections to the leptonic matrix element. We also discussed briefly the other types of allomorphic corrections ("intermediate propagator" and "hadronic end"), a complete treatment of which is reserved for a future publication.

The general machinery developed was used to investigate the conditions for the absence of (a) an induced leptonic pseudoscalar current to $O(\alpha G_F)$ in charged-pion decay, and (b) an induced neutral strangeness-changing current to $O(\alpha G_F)$ in neutral-kaon decay. Here, the use of the Wilson operator product expansion was crucial in allowing us to derive rigorous and general order-of-magnitude estimates for the various contributions. The basic conclusion seems to be that the presence of elementary hadronic scalars, although possibly placing additional (and plausible) constraints on the masses of the weak gauge vector and Higgs mesons, does not necessarily lead to an induced $O(\alpha G_F)$ pseudoscalar contribution to pion decay. Theories without elementary hadronic scalars preserve the lowest-order $V-A$ structure of the leptonic weak interactions with only very weak restrictions on particle masses. On the other hand, the absence of neutral strangeness-changing currents to $O(\alpha G_F)$, as ensured by the Glashow-Iliopoulos-Maiani mechanism, seems to depend crucially on the usual interpretation of the pseudoscalar mesons as fermion-antifermion bound states.

A detailed application of the formalism developed above will perhaps have to wait until a greater variety of phenomenological information becomes available, especially concerning the as-yet-unseen weakly interacting Higgs-scalar particles and heavy vector bosons (which presently must be admitted to exist in a quasimythical realm). However, it is clear from the discussion above that many theoretical constraints of a general nature can probably be obtained from a consideration of the phenomenological information already at our disposal.

## ACKNOWLEDGMENTS

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## APPENDIX A: NOTATION, CONVENTIONS, AND OTHER PRELIMINARIES

As in Ref. 5, our metric and Dirac conventions are

$$\begin{aligned} &g_{00}=-1=-g_{ii} \quad (i=1,2,3), \\ &\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu} \quad (\mu,\nu=0,1,2,3), \\ &\gamma_4\equiv -i\gamma_0, \quad \gamma_5\equiv\gamma_1\gamma_2\gamma_3\gamma_4, \\ &\gamma_4\gamma_\mu^\dagger\gamma_4=-\gamma_\mu . \end{aligned} \tag{A1}$$

We deal throughout with a general model of the type espoused by Weinberg,[5] containing weakly interacting fermions, scalars, and vector gauge particles, and strongly interacting vector gluons. The strong and weak gauge groups are assumed to commute.

Leaving out the strong gluon kinetic terms (irrelevant as we work throughout in a "strong Heisenberg" representation), the quadratic part of the Lagrangian is (after shift)

$$\begin{aligned} \mathfrak{L}_{\text{quad}} =& -\tfrac{1}{4}(\partial_\mu W_{\alpha\nu}-\partial_\nu W_{\alpha\mu})(\zeta_W)_{\alpha\beta}(\partial^\mu W^\nu_\beta-\partial^\nu W^\mu_\beta)-\tfrac{1}{2}(\zeta_\phi)_{ij}\,\partial_\mu\phi_i\,\partial^\mu\phi_j-\tfrac{1}{2}\mu^2{}_{\alpha\beta}W_{\alpha\mu}W^\mu_\beta-\bar\psi(\zeta_\psi\partial\!\!\!/+m)\psi-\tfrac{1}{2}M^2{}_{ij}\,\phi_i\,\phi_j \\ &-\tfrac{1}{2}\xi(\partial_\mu W^\mu_\alpha)(\partial_\nu W^\nu_\alpha)+(1/2\xi)(\zeta_\phi\theta_\alpha\lambda)_i(\zeta_\phi\theta_\alpha\lambda)_j\,\phi_i\,\phi_j-\partial_\mu\omega^*_\alpha\partial^\mu\omega_\alpha-(1/\xi)\mu^2{}_{\alpha\beta}\omega^*_\alpha\omega_\beta , \end{aligned} \tag{A2}$$

where $\psi$, $W_{\alpha\mu}$, $\phi_i$, $\omega_\alpha$ represent the fermion, weak gauge vector, shifted scalar, and ghost fields, respectively. The usual constraints of Hermiticity and gauge and Lorentz invariance imply the following relations:

$$[\zeta_W,\tau_\alpha]=[\zeta_\psi,\tilde t_\alpha]=[\zeta_\phi,\theta_\alpha]=0, \tag{A3}$$

where $\tau_\alpha$, $t_\alpha$, $\theta_\alpha$ generate the weak gauge group in the adjoint, fermion, and scalar representations, respectively. For any matrix $X$, $\tilde X\equiv\gamma_4X\gamma_4$.

$$\begin{aligned} &t^\dagger_\alpha=t_\alpha , \\ &\theta^\dagger_\alpha=\theta_\alpha=-\theta^*_\alpha , \\ &\zeta^\dagger_\psi=\zeta_\psi, \quad \zeta^\dagger_W=\zeta^T_W=\zeta_W, \quad \zeta^\dagger_\phi=\zeta^T_\phi=\zeta_\phi . \end{aligned} \tag{A4}$$

The matrices $t_\alpha$, $\zeta_\psi$ may contain $\gamma_5$'s, but we assume that the zeroth-order mass matrix $m$ has been diagonalized to remove $\gamma_5$'s.

The propagators are found to be [with $\mu^2{}_{\alpha\beta}\equiv(\lambda,\zeta_\phi\theta_\alpha\theta_\beta\lambda)$]

$$\begin{aligned} &\Delta^W_{\alpha\mu,\beta\nu}(l)=g_{\mu\nu}(l)_{\alpha\beta}+l_\mu l_\nu[l]_{\alpha\beta}, \\ &\Delta^\phi_{ij}(l)=(\zeta_\phi l^2+M^2)^{-1}{}_{ij} \\ &\qquad\qquad +\frac{1}{l^2}(\xi l^2+\mu^2)^{-1}{}_{\alpha\beta}(\theta_\alpha\lambda)_i(\theta_\beta\lambda)_j, \\ &\Delta^\psi(l)=(i\zeta_\psi l\!\!\!/+m)^{-1}, \\ &\Delta^\omega_{\alpha\beta}(l)=\xi(\xi l^2+\mu^2)^{-1}{}_{\alpha\beta} , \end{aligned} \tag{A5}$$

where $(l)_{\alpha\beta}$, $[l]_{\alpha\beta}$ are defined as follows:

$$\begin{aligned} &(l)_{\alpha\beta}\equiv(\zeta_W l^2+\mu^2)^{-1}{}_{\alpha\beta} , \\ &[l]_{\alpha\beta}\equiv\frac{1}{l^2}\left((\xi l^2+\mu^2)^{-1}{}_{\alpha\beta}-(l)_{\alpha\beta}\right) . \end{aligned} \tag{A6}$$

It is also convenient to define a "gauge-independent, unitary" scalar propagator

$$\begin{aligned} \hat\Delta^\phi_{ij}(l)&\equiv(\zeta_\phi l^2+M^2)^{-1}{}_{ij}+\frac{(l)_{\alpha\beta}}{l^2}(\theta_\alpha\lambda)_i(\theta_\beta\lambda)_j \\ &=\Delta^\phi_{ij}(l)-[l]_{\alpha\beta}(\theta_\alpha\lambda)_i(\theta_\beta\lambda)_j \\ &=[(\zeta_\phi l^2+M^2)^{-1}\Pi]_{ij} \\ &\quad-\mathrm{Tr}[(\zeta_W l^2+\mu^2)^{-1}\zeta_W(\mu^2)^{-1}\Omega_{ji}] . \end{aligned} \tag{A7}$$

In the last identity, $\Pi$ is the projection operator onto the subspace of physical, massive scalars:

$$\begin{aligned} &\Pi_{ij}=\delta_{ij}+(\mu^2)^{-1}{}_{\alpha\beta}(\theta_\alpha\lambda)_i(\zeta_\phi\theta_\beta\lambda)_j , \\ &(\Omega_{ij})_{\alpha\beta}=(\theta_\alpha\lambda)_i(\theta_\beta\lambda)_j . \end{aligned} \tag{A8}$$

In our calculations, we use fermion spinors normalized, for free fields $\psi$, by

$$\begin{aligned} &\langle 0|\psi(x)|q_i\rangle=e^{iq_i\cdot x}U_i(q_i), \\ &\langle q_f|\bar\psi(x)|0\rangle=\bar U_f(q_f)e^{-iq_f\cdot x} \end{aligned}$$

(suppressing spin and internal labels) for initial and final fermion states. [These spinors contain factors of $(2\pi)^{-3/2}(m/E)^{1/2}\zeta_\psi^{-1/2}$ relative to conventionally normalized spinors.] The Dirac equation in momentum space is

$$\bar U_f(q_f)(i\zeta_\psi q\!\!\!/_f+m)=(i\zeta_\psi q\!\!\!/_i+m)U_i(q_i)=0 . \tag{A10}$$

The weak interaction Lagrangian is given by

$$\mathcal{L}_W^{\text{int}}=\tfrac{1}{2}(\zeta_W)_{\alpha\beta}c_{\beta\gamma\delta}(\partial_\mu W_{\alpha\nu}-\partial_\nu W_{\alpha\mu})W^\mu_\gamma W^\nu_\delta-\tfrac{1}{4}(\zeta_W)_{\alpha\beta}c_{\alpha\gamma\delta}c_{\beta\epsilon\eta}W^\gamma_\mu W^\delta_\nu W^\mu_\epsilon W^\mu_\eta-i(\zeta_\phi\theta_\alpha)_{ij}(\partial_\mu\phi_i)\phi_j W^\mu_\alpha-(\theta_\beta\zeta_\phi\theta_\alpha\lambda)_i\phi_i W_{\alpha\mu}W^\mu_\beta$$

$$-\tfrac{1}{2}(\theta_\beta\zeta_\phi\theta_\alpha)_{ij}\phi_i\phi_j W^\mu_\alpha W_{\beta\mu}-\frac{1}{3!}f_{ijk}\phi_i\phi_j\phi_k-\frac{1}{4!}f_{ijkl}\phi_i\phi_j\phi_k\phi_l-\partial_\mu\omega^*_\alpha c_{\alpha\beta\gamma}\omega_\beta W^\mu_\gamma$$

$$-\frac{1}{\xi}\omega^*_\alpha\omega_\beta(\theta_\beta\zeta_\phi\theta_\alpha\lambda)_i\phi_i+j^\mu_\alpha W_{\alpha\mu}+S_i\phi_i\ , \tag{A11}$$

where the vector and scalar currents linking the fermion and boson systems are

$$j_{\alpha\mu}(x)\equiv -i\bar\psi(x)\zeta_\psi\gamma_\mu t_\alpha\psi(x)\,,$$
$$S_i(x)\equiv-\bar\psi(x)\Gamma_i\psi(x)\,. \tag{A12}$$

Finally, as shown by Weinberg,[8] local gauge invariance leads to the bevy of relations (all of which will be crucial for verifying the $\xi$ independence of our results)

$$[t_\alpha,\gamma_4\Gamma_j]=(\theta_\alpha)_{ij}\gamma_4\Gamma_i\ ,$$
$$[t_\alpha,\gamma_4 m]=(\theta_\alpha\lambda)_i\gamma_4\Gamma_i\ ; \tag{A13}$$

$$f_{ijk}(\theta_\alpha\lambda)_i=[\theta_\alpha,M^2]_{jk}\ ,$$
$$f_{ijk}(\theta_\alpha\lambda)_i(\theta_\beta\lambda)_j=-(M^2\theta_\alpha\theta_\beta\lambda)_k\ , \tag{A14}$$
$$f_{ijk}(\theta_\alpha\lambda)_i(\theta_\beta\lambda)_j(\theta_\gamma\lambda)_k=0\,.$$

Finally, we summarize the current-algebraic results which we shall need. Here it will be assumed that the strong interactions are mediated purely by gauge vector gluons. It is then at least plausible that Schwinger terms in the time-space canonical commutation relations (CCR's) are $c$ numbers and may be dropped in computing $S$-matrix elements. Finally, to ensure renormalizability, we must insist on the cancellation of triangle anomalies. The relevant CCR's are then[5]

$$[j^0_\alpha(\vec{x},t),j^\nu_\beta(\vec{y},t)]=-i\delta^3(\vec{x}-\vec{y})c_{\alpha\beta\gamma}j^\nu_\gamma(\vec{x},t)+c\text{-number Schwinger terms}\,, \tag{A15}$$

$$[j^0_\alpha(\vec{x},t),S_i(\vec{y},t)]=\delta^3(\vec{x}-\vec{y})(\theta_\alpha)_{ij}S_j(\vec{x},t)\,. \tag{A16}$$

Also, the equations of motion (in the strong Heisenberg representation) imply

$$\partial_\mu j^\mu_\alpha(x)=-i(\theta_\alpha\lambda)_i S_i(x)\,. \tag{A17}$$

Let $|I\rangle$, $|F\rangle$ be arbitrary hadron states of total four-momentum $P_i$, $P_f$, respectively. We define (all fields in strong Heisenberg representation)

$$\langle j^\mu_\alpha\rangle_{FI}\equiv\langle F|j^\mu_\alpha(0)|I\rangle\,,$$
$$\langle S_i\rangle_{FI}\equiv\langle F|S_i(0)|I\rangle\,,$$
$$\mathfrak{F}^{\mu\nu}_{\alpha\beta}(l)\equiv\int dx\,e^{-ilx}\langle F|T\{j^\mu_\alpha(x)j^\nu_\beta(0)\}|I\rangle\,,$$
$$\mathfrak{F}^\mu_{i\alpha}(l)\equiv\int dx\,e^{-ilx}\langle F|T\{S_i(x)j^\mu_\alpha(0)\}|I\rangle\,, \tag{A18}$$
$$\mathfrak{F}^\mu_{\alpha i}(l)\equiv\int dx\,e^{-ilx}\langle F|T\{j^\mu_\alpha(x)S_i(0)\}|I\rangle\ ,$$
$$\mathfrak{F}_{ij}(l)\equiv\int dx\,e^{-ilx}\langle F|T\{S_i(x)S_j(0)\}|I\rangle\ .$$

Using (A15)–(A17) one obtains ($q\equiv P_i-P_f$)

$$q_\mu\langle j^\mu_\alpha\rangle_{FI}=-(\theta_\alpha\lambda)_i\langle S_i\rangle_{FI}\,; \tag{A19}$$

$$l_\mu\mathfrak{F}^{\mu\nu}_{\alpha\beta}(l)=-(c_{\alpha\beta\gamma}\langle j^\nu_\gamma\rangle_{FI}+(\theta_\alpha\lambda)_i\mathfrak{F}^\nu_{i\beta}(l))\,,$$
$$(q-l)_\nu\mathfrak{F}^{\mu\nu}_{\alpha\beta}(l)=c_{\alpha\beta\gamma}\langle j^\mu_\gamma\rangle_{FI}-(\theta_\beta\lambda)_i\mathfrak{F}^\mu_{\alpha i}(l)\,,$$
$$(q-l)_\mu\mathfrak{F}^\mu_{i\alpha}(l)=-i(\theta_\alpha)_{ij}\langle S_j\rangle_{FI}-(\theta_\alpha\lambda)_j\mathfrak{F}_{ji}(q-l)\,, \tag{A20}$$
$$l_\mu(q-l)_\nu\mathfrak{F}^{\mu\nu}_{\alpha\beta}(l)=c_{\alpha\beta\gamma}l_\mu\langle j^\mu_\gamma\rangle_{FI}-i(\theta_\alpha\theta_\beta\lambda)_j\langle S_j\rangle_{FI}+(\theta_\alpha\lambda)_i(\theta_\beta\lambda)_j\mathfrak{F}_{ij}(l)\,;$$

$$\mathfrak{F}^{\mu\nu}_{\alpha\beta}(l)=\mathfrak{F}^{\nu\mu}_{\beta\alpha}(q-l)\,,$$
$$\mathfrak{F}^\mu_{i\alpha}(l)=\mathfrak{F}^\mu_{\alpha i}(q-l)\,, \tag{A21}$$
$$\mathfrak{F}_{ij}(l)=\mathfrak{F}_{ji}(q-l)\,.$$

## APPENDIX B: RENORMALIZATION OF THE FERMION PROPAGATOR

In this appendix, we discuss the fermion (i.e., lepton) wave-function renormalization in a completely physical, on-mass-shell renormalization scheme defined by the absence of external-fermion-leg radiative corrections. In other words if $Z_2^{1/2}\equiv 1+z_2$ is the wave-function renormalization matrix for the fermions, $\delta m$ is the mass counterterm, and $\Sigma(p)$ is the fermion self-energy (all computed to one loop), we choose $z_2$ and $\delta m$ by requiring

$$\lim_{p\to p_i}\frac{1}{i\zeta_\psi \not p+m}\left(\Sigma(p)+\delta m-(i\zeta_\psi \not p+m)z_2-\bar z_2^+(i\zeta_\psi\not p+m)\right)\times U(p_i)=0\,. \quad \text{(B1)}$$

On general grounds the self-energy $\Sigma(p)$ can be shown to take the form

$$\Sigma(p)=\tilde A^+(p^2)(i\zeta_\psi\not p+m)+(i\zeta_\psi\not p+m)A(p^2)+B(p^2)\,, \quad \text{(B2)}$$

where $A(p^2)$ and $B(p^2)$, in Dirac space, involve the identity and possibly $\gamma_5$.

Repeating Weinberg's calculation of Ref. 8, but being careful to keep terms involving $(i\zeta_\psi\not p+m)$ on the extreme right or left, we find that the self-energy can be written as follows:

$$\Sigma(p)=\Sigma_1(p)+\Sigma_2(p)+\Sigma_3(p)\,, \quad \text{(B3)}$$

$$\Sigma_1(p)\equiv\frac{i}{(2\pi)^4}\int dl\left(\zeta_\psi\gamma^\mu t_\alpha\Delta^\psi(p-l)\zeta_\psi\gamma_\mu t_\beta(l)_{\alpha\beta}-\Gamma_i\Delta^\psi(p-l)\Gamma_j\hat\Delta^\phi_{ij}(l)\right), \quad \text{(B4)}$$

$$\Sigma_2(p)\equiv\Gamma_i M^{-2}{}_{ij}\frac{\partial V_1}{\partial\lambda_j}+\frac{i}{2(2\pi)^4}\gamma_4[t_\beta,[t_\alpha,\gamma_4 m]]\int dl\frac{(l)_{\alpha\beta}}{l^2}, \quad \text{(B5)}$$

$$\Sigma_3(p)\equiv\frac{i}{(2\pi)^4}\Big\{(i\zeta_\psi\not p+m)\int dl[l]_{\alpha\beta}\left(\tfrac{1}{2}t_\alpha t_\beta+t_\alpha\Delta^\psi(p-l)\gamma_4[t_\beta,\gamma_4 m]\right)+\int dl[l]_{\alpha\beta}\left(\tfrac{1}{2}\bar t_\beta\bar t_\alpha+\gamma_4[\gamma_4 m,t_\beta]\Delta^\psi(p-l)\bar t_\alpha\right)(i\zeta_\psi\not p+m)\Big\}\,. \quad \text{(B6)}$$

$\Sigma_1(p)$ arises basically from vector and scalar emission-reabsorption diagrams, and is separately gauge-independent. It is momentum-dependent and hence will be involved in the calculation of $z_2$. In the following, we will denote the contribution of $\Sigma_1(p)$ to the wave-function renormalization by $z_{2,\mathrm{GI}}$. $\Sigma_2(p)$ is, of course, momentum-independent, and contributes solely to mass renormalization. In calculating $\Sigma_3(p)$, a term has been neglected involving factors of $(i\zeta_\psi\not p+m)$ on both the right and left: Such a term is evidently, from (B1), irrelevant to the determination of either $\delta m$ or $z_2$. $\Sigma_3(p)$ is clearly gauge-dependent; its contribution to $z_2$ can be directly read off from (B6) as

$$z_{2,\mathrm{GD}}=\frac{i}{(2\pi)^4}\int dl[l]_{\alpha\beta}\times\left(\tfrac{1}{2}t_\alpha t_\beta+t_\alpha\Delta^\psi(\text{“}p\text{”}-l)\gamma_4[t_\beta,\gamma_4 m]\right). \quad \text{(B7)}$$

The "$p$" in (B7) is to be interpreted as the momentum of the spinor on which $z_{2,\mathrm{GD}}$ acts. To justify (B7), we note that

$$\lim_{p\to p_i}\frac{1}{i\zeta_\psi\not p+m}\left\{\bar z^+(i\zeta_\psi\not p+m)+(i\zeta_\psi\not p+m)z-\tilde A^+(p^2)(i\zeta_\psi\not p+m)-(i\zeta_\psi\not p+m)A(p^2)\right\}\times U(p_i)=0 \quad \text{(B8)}$$

is satisfied automatically for any given $A(p^2)$ by defining

$$zU(p_i)\equiv A(p_i^{\,2})U(p_i)\,. \quad \text{(B9)}$$

Finally, it remains to calculate $z_{2,\mathrm{GI}}$. Writing

$$\Sigma_1(p)=(i\zeta_\psi p+m)A_{\mathrm{GI}}(p^2)+\tilde A^+_{\mathrm{GI}}(p^2)(i\zeta_\psi\not p+m)+B_{\mathrm{GI}}(p^2) \quad \text{(B10)}$$

a short calculation yields

$$i\zeta_\psi\not p A_{\mathrm{GI}}(p^2)=\frac{i}{(2\pi)^4}\int dl\, i\zeta_\psi(\not p-\not l)\{t_\alpha D(p-l)t_\beta(l)_{\alpha\beta}+\tfrac{1}{2}\tilde\Gamma_j'\tilde D(p-l)\Gamma_k'\hat\Delta^\phi_{jk}(l)\}\,, \quad \text{(B11)}$$

$$B_{\mathrm{GI}}(p^2)=\frac{i}{2(2\pi)^4}\int dl\{4\bar t_\alpha(mD(p-l)+\tilde D^+(p-l)m)t_\beta(l)_{\alpha\beta}-\tilde\Gamma_i'^+(m\tilde D(p-l)+D^+(p-l)m)\Gamma_j'\hat\Delta^\phi_{ij}(l)\}-\tilde A^+_{\mathrm{GI}}(p^2)m-mA_{\mathrm{GI}}(p^2)\,, \quad \text{(B12)}$$

where we have introduced a modified Yukawa coupling matrix $\Gamma_k'\equiv\zeta_\psi^{-1}\Gamma_k$ and a denominator matrix $D(p-l)\equiv\left((p-l)^2+\bar\zeta_\psi^{-1}m\zeta_\psi^{-1}m\right)^{-1}$. The gauge-invariant wave-function renormalization thus arises from two sources:

$$z_{2,\mathrm{GI}}\equiv z_{2,D}+z_{2,F}\,. \quad \text{(B13)}$$

From (B8), (B9), and (B10) the logarithmically divergent contribution to $z_{2,\mathrm{GI}}$ is given directly as

$$z_{2,D}=A_{\mathrm{GI}}(``p^2") . \tag{B14}$$

In the evaluation of $A_{\mathrm{GI}}(p^2)$ it is convenient[13] to diagonalize the boson mass matrices. Introducing the appropriate transformation matrices

$$\begin{aligned} f(\overline{\mu}^2)_{\alpha\beta}&=\sum_n f(\overline{\mu}_n{}^2)C_{\alpha n}C_{\beta n} ,\\ f(\overline{M}^2)_{ij}&=\sum_N f(\overline{M}_N{}^2)O_{iN}O_{jN} , \end{aligned} \tag{B15}$$

with

$$\overline{\mu}^2\equiv \zeta_W{}^{-1/2}\mu^2\zeta_W{}^{-1/2}=(\overline{\mu}^2)^T ,$$

$$\overline{M}^2\equiv \zeta_\phi{}^{-1/2}M^2\zeta_\phi{}^{-1/2}=(\overline{M}^2)^T ,$$

and using the explicitly unitary form (A7) for $\Delta^\phi_{jk}(l)$, we may recast (B11) in the following form:

$$\begin{aligned} i\zeta_\psi \not{p} A_{\mathrm{GI}}(p^2)=\frac{i}{(2\pi)^4}\int dl\, i\zeta_\psi(\not{p}-\not{l})\Big\{ &\sum_n t_n D(p-l)t_n(l^2+\overline{\mu}_n{}^2)^{-1}+\tfrac{1}{2}\sum_{N,\overline{M}_N\neq 0}\tilde{\Gamma}_N\tilde{D}(p-l)\Gamma_N(l^2+\overline{M}_N{}^2)^{-1}\\ &-\tfrac{1}{2}\sum_{n,\overline{\mu}_n\neq 0}\tilde{\zeta}_\psi{}^{-1}[t_n,\gamma_4 m]D(p-l)\tilde{\zeta}_\psi{}^{-1}[t_n,\gamma_4 m]\frac{1}{\overline{\mu}_n{}^2(l^2+\overline{\mu}_n{}^2)}\Big\} , \end{aligned} \tag{B16}$$

where

$$t_n\equiv C_{\alpha n}(\zeta_W{}^{-1/2})_{\alpha\beta}t_\beta , \tag{B17}$$

$$\Gamma_N\equiv O_{iN}(\zeta_\phi{}^{-1/2})_{ij}\Gamma'_j . \tag{B18}$$

(B16) will be evaluated using the dimensional cutoff of 't Hooft and Veltman.[14] The calculation yields the following Feynman parameter integral for $z_{2,D}$:

$$\begin{aligned} z_{2,D}=-\frac{\pi^{\kappa/2}\Gamma(2-\frac{1}{2}\kappa)}{(2\pi)^4}\int_0^1 dx(1-x)\Big\{&\sum_n t_n(``p^2"x(1-x)+x\overline{m}^2+(1-x)\overline{\mu}_n{}^2)^{\kappa/2-2}t_n\\ &+\tfrac{1}{2}\sum_{N,\overline{M}_N\neq 0}\tilde{\Gamma}_N(``p^2"x(1-x)+x\overline{m}^2+(1-x)\overline{M}_N{}^2)^{\kappa/2-2}\Gamma_N\\ &-\tfrac{1}{2}\sum_{n,\overline{\mu}_n\neq 0}\tilde{\zeta}_\psi{}^{-1}[t_n,\gamma_4 m](``p^2"x(1-x)+x\overline{m}^2+(1-x)\overline{\mu}_n{}^2)^{\kappa/2-2}\,\tilde{\zeta}_\psi{}^{-1}[t_n,\gamma_4 m]\frac{1}{\overline{\mu}_n{}^2}\Big\} , \end{aligned} \tag{B19}$$

where $\kappa$ is the continued dimensionality of the Feynman integral in (B16), and $\overline{m}^2\equiv\tilde{\zeta}_\psi{}^{-1}m\zeta_\psi{}^{-1}m$.

$z_{2,F}$ arises from the on-mass-shell derivative of $B_{\mathrm{GI}}(p^2)$. The electromagnetic contribution can be directly calculated in the usual way, as the charge generator commutes with the fermion mass matrix. However, the calculation of the nonelectromagnetic contribution, $z^{\mathrm{wk}}_{2,F}$, for a general gauge theory is complicated by the noncommutativity of the various matrices involved. This computation is best performed in the context of specific models. We merely note here that the contribution of $z_{2,F}$ to the allomorphic $S$-matrix component is separately gauge-independent, unitary, and *finite*—the last property following from the fact that $z_{2,F}$ arises as a momentum derivative of the logarithmically divergent $B_{\mathrm{GI}}(p^2)$. Finally, the following order of magnitude estimate holds for the nonelectromagnetic part of $z_{2,F}$ ($m_l, m_\phi$ are typical lepton and Higgs-scalar masses, respectively):

$$z^{\mathrm{wk}}_{2,F}\sim O\left(\max\left(G_F m_l{}^2, G_F\frac{m_l{}^4}{m_\phi{}^2}\right)\right) . \tag{B20}$$

In particular, for $m_l{}^2\lesssim m_\phi{}^2$, there can be no $O(\alpha)$ *purely weak* contributions to $z_{2,F}$, as there certainly are to $z_{2,D}$.

## APPENDIX C: EVALUATION OF $\delta S^{(S)}, \delta S^{(j)}$

In this appendix, we reduce the allomorphic contributions involving single-current hadronic matrix elements to integrals over Feynman parameters, at which juncture model-dependent specifications and approximations are most conveniently applied. We define

$$\begin{aligned} \delta S^{(S)}+\delta S^{(j)}\equiv \delta(\textstyle\sum P)(\langle j^\mu_\alpha\rangle_{FI}(q)_{\alpha\beta}J_{\beta\mu} &+\langle S_i\rangle_{FI}\hat{\Delta}^\phi_{ij}(q)\mathcal{S}_j\\ &+\langle S_i\rangle_{FI}(\theta_\alpha\lambda)_i(q)_{\alpha\beta}\mathcal{S}_\beta) . \end{aligned} \tag{C1}$$

Our object will be to calculate $J_{\beta\mu}$, $\mathcal{S}_j$, $\mathcal{S}_\alpha$. We will assume, for algebraic simplicity, that the invariant matrices $\zeta_\psi=\zeta_\phi=\zeta_W=1$, and that (with no loss of generality) any $\gamma_5$ factors in the fermion mass matrix have been removed by redefining the fermion fields.

It is very convenient to work in a representation in which the various mass matrices are diagonal:

$$f(\mu^2)_{\alpha\beta}=\sum_n f(\mu_n{}^2)C_{\alpha n}C_{\beta n}\,,$$
$$f(M^2)_{ij}=\sum_N f(M_N{}^2)O_{iN}O_{jN}\,, \qquad \text{(C2)}$$
$$f(m^2)_{pq}=\sum_K f(m_K{}^2)\mathfrak{U}_{pK}\mathfrak{U}^*_{qK}\,,$$

with $\mu^2$, $M^2$, and $m^2$ the vector, Higgs-scalar, and fermion squared-mass matrices, respectively.

The external spinors correspond to eigenstates of fermion mass, so

$$(U_i)_p=\mathfrak{U}_{pI}u(q_i)\,,\quad m_{qp}\mathfrak{U}_{pI}=m_I\mathfrak{U}_{qI}\,,$$
$$(\overline{U}_f)_p=\overline{u}(q_f)\mathfrak{U}^*_{pF}\,,\quad m_{qp}\mathfrak{U}_{pF}=m_F\mathfrak{U}_{qF}\,. \qquad \text{(C3)}$$

Matrix elements involving antileptons can, of course, be obtained immediately from crossing relations. We will also need the various coupling matrices in this representation:

$$t_n\equiv C_{\alpha n}t_\alpha,\quad \tau_\alpha\equiv\mathfrak{U}^\dagger t_\alpha\mathfrak{U},\quad \tau_n\equiv\mathfrak{U}^\dagger t_n\mathfrak{U}\,,$$
$$\Gamma_N\equiv O_{iN}\Gamma_i\,,\quad \chi_i\equiv\mathfrak{U}^\dagger\Gamma_i\mathfrak{U},\quad \chi_N\equiv\mathfrak{U}^\dagger\Gamma_N\mathfrak{U}\,. \qquad \text{(C4)}$$

The calculation is best performed using a dimensional cutoff[14]: Particularly useful are the formulas

$$[I,I^\mu,I^{\mu\nu}]\equiv\int d^\kappa l\,[1,l^\mu,l^\mu l^\nu](l^2+m_0{}^2)^{-1}((q_1-l)^2+m_1^2)^{-1}((q_2-l)^2+m_2{}^2)^{-1}$$
$$=i\pi^{\kappa/2}\int_0^1 dx_1\int_0^{x_1}dx_2 D(x_1,x_2;q_1,q_2;m_0,m_1,m_2)^{\kappa/2-3}$$
$$\times[\Gamma(3-\tfrac{1}{2}\kappa),\Gamma(3-\tfrac{1}{2}\kappa)((x_1-x_2)q_1^\mu+(1-x_1)q_2^\mu)\,,$$
$$\Gamma(3-\tfrac{1}{2}\kappa)((x_1-x_2)q_1^\mu+(1-x_1)q_2^\mu)((x_1-x_2)q_1^\nu+(1-x_1)q_2^\nu)$$
$$+\Gamma(2-\tfrac{1}{2}\kappa)\tfrac{1}{2}g^{\mu\nu}D(x_1,x_2;q_1,q_2;m_0,m_1,m_2)]\,, \qquad \text{(C5)}$$

where $D(x_1x_2;q_1q_2;m_0m_1m_2)$ is the denominator function

$$D(x_1,x_2;q_1,q_2;m_0,m_1,m_2)\equiv x_2m_0{}^2+(x_1-x_2)(q_1{}^2+m_1{}^2)+(1-x_1)(q_2{}^2+m_2{}^2)-((x_2-x_1)q_1+(x_1-1)q_2)^2\,. \qquad \text{(C6)}$$

The calculation is straightforward—in stating the results we employ a summation convention on repeated indices, modified by the instruction to sum only over massive vector mesons $\mu_n\neq 0$ whenever factors of $(\mu_n{}^2)^{-1}$ appear; similarly, the sums involving definite-mass Higgs mesons omit the unphysical Goldstone particles. Note also that the mass scale employed in defining the arguments of logarithms is arbitrary, since the logarithmic infinities in $\delta S^{\langle S\rangle}+\delta S^{\langle j\rangle}$, as shown in Sec. V, are not allomorphic. We find

$$J_{\beta\mu}=(J_1)_{\beta\mu}+(J_2)_{\beta\mu}+(J_3)_{\beta\mu}+(J_4)_{\beta\mu}\,. \qquad \text{(C7)}$$

First, the explicitly finite contribution from wavefunction renormalization is given by (cf. Appendix B)

$$(J_1)_{\beta\mu}=(2\pi)^4\overline{U}_f(\bar{z}^\dagger_{2,F}\zeta_\psi\gamma_\mu t_\beta+\zeta_\psi\gamma_\mu t_\beta z_{2,F})U_i\,. \qquad \text{(C8)}$$

The other contributions are

$$(J_2)_{\beta\mu}=\tfrac{1}{2}\pi^2\int_0^1 dx(1-x)\overline{u}(q_f)\Big\{2\gamma_\mu(\tau_n)_{FJ}(\tau_n\tau_\beta)_{JI}\ln(q_f{}^2x(1-x)+xm_J{}^2+(1-x)\mu_n{}^2)$$
$$+2\gamma_\mu(\tau_\beta\tau_n)_{FJ}(\tau_n)_{JI}\ln(q_i{}^2x(1-x)+xm_J{}^2+(1-x)\mu_n{}^2)$$
$$+(\chi_N)_{FJ}\gamma_\mu(\chi_N\tau_\beta)_{JI}\ln(q_f{}^2x(1-x)+xm_J{}^2+(1-x)M_N{}^2)$$
$$+\gamma_\mu(\tau_\beta\tilde{\chi}_N)_{FJ}(\chi_N)_{JI}\ln(q_i{}^2x(1-x)+xm_J{}^2+(1-x)M_N{}^2)$$
$$-\frac{(\theta_n\lambda)_i(\theta_n\lambda)_j}{\mu_n{}^2}[(\chi_i)_{FJ}\gamma_\mu(\chi_j\tau_\beta)_{JI}\ln(q_f{}^2x(1-x)+xm_J{}^2+(1-x)\mu_n{}^2)$$
$$+\gamma_\mu(\tau_\beta\tilde{\chi}_i)_{FJ}(\chi_j)_{JI}\ln(q_i{}^2x(1-x)+xm_J{}^2+(1-x)\mu_n{}^2)]\Big\}u(q_i)\,, \qquad \text{(C9)}$$

$$
\begin{aligned}
(J_3)_{\beta\mu} = -\pi^2 \int_0^1 dx_1 \int_0^{x_1} dx_2 \bar{u}(q_f)\Big[ & 2\ln D(x_1, x_2; q_f, q_i; \mu_n, m_J, m_L)\gamma_\mu(\tau_n)_{FJ}(\tau_\beta)_{JL}(\tau_n)_{LI} \\
& + 2ic_{\beta'\beta\delta'}C_{\beta' n}C_{\delta' m}\ln D(x_1, x_2; q_f, q; \mu_n, m_J, \mu_m)(\tilde{\tau}_n)_{FJ}\gamma_\mu(\tau_m)_{JI} \\
& + \ln D(x_1, x_2; q_f, q_i; M_N, m_J, m_L)(\chi_N)_{FJ}\gamma_\mu(\tilde{\tau}_\beta)_{JL}(\chi_N)_{LI} \\
& - \ln D(x_1, x_2; q_f, q_i; \mu_n, m_J, m_L)\frac{(\theta_n\lambda)_k(\theta_n\lambda)_l}{\mu_n{}^2}(\chi_k)_{FJ}\gamma_\mu(\tilde{\tau}_\beta)_{JL}(\chi_l)_{LI} \\
& - \ln D(x_1, x_2; q_f, q; M_N, m_J, M_P)(O^T\theta_\beta O)_{PN}(\chi_N)_{FJ}\gamma_\mu(\chi_P)_{JI} \\
& - \ln D(x_1, x_2; q_f, q; M_N, m_J, \mu_n)(O^T\theta_\beta\theta_n\lambda)_N\frac{(\theta_n\lambda)_l}{\mu_n{}^2}(\chi_N)_{FJ}\gamma_\mu(\chi_l)_{JI} \\
& + \ln D(x_1, x_2; q_f, q; \mu_n, m_J, M_P)(O^T\theta_\beta\theta_n\lambda)_P\frac{(\theta_n\lambda)_k}{\mu_n{}^2}(\chi_k)_{FJ}\gamma_\mu(\chi_P)_{JI} \\
& + \ln D(x_1, x_2; q_f, q; \mu_n, m_J, \mu_p)\frac{(\theta_n\lambda)_k(\theta_p\lambda)_l}{\mu_n{}^2\mu_p{}^2}(\lambda, \theta_p\theta_\beta\theta_n\lambda)(\chi_k)_{FJ}\gamma_\mu(\chi_l)_{JI}\Big]u(q_i)
\end{aligned}
\tag{C10}
$$

[in (C10), $O^T$ refers to the transpose of the orthogonal matrix $O$],

$$
\begin{aligned}
(J_4)_{\beta\mu} \equiv \pi^2 \int_0^1 dx_1 \int_0^{x_1} dx_2 \bar{u}(q_f)\Big\{ & -2D(x_1, x_2; q_f, q_i; \mu_n, m_J, m_L)^{-1}(\tilde{\tau}_n)_{FJ} \\
& \times[(\not{Q}(q_f, q_i) - \not{q})\gamma_\mu\not{Q}(q_f, q_i)(\tau_\beta)_{JL} + 2im_J(\tau_\beta)_{JL}(q_\mu - Q_\mu(q_f, q_i)) \\
& \quad - 2i(\tilde{\tau}_\beta)_{JL}m_L Q_\mu(q_f, q_i) - \gamma_\mu m_J(\tilde{\tau}_\beta)_{JL}m_L](\tau_n)_{LI} \\
& - 4c_{\beta'\beta\delta'}C_{\beta' n}C_{\delta' m}D(x_1, x_2; q_f, q; \mu_n, m_J, \mu_m)^{-1}(q_{f_\mu} - Q_\mu(q_f, q_i))(\tilde{\tau}_n)_{FJ} \\
& \times(i\not{Q}(q_f, q_i) + 2m_J)(\tau_m)_{JI} \\
& + D(x_1, x_2; q_f, q_i; M_N, m_J, m_L)^{-1}(\chi_N)_{FJ}(-i\not{Q}(q_f, q_i) + m_J)\gamma_\mu(\tau_\beta)_{JL} \\
& \times[-i(\not{Q}(q_f, q_i) - \not{q}) + m_L](\chi_N)_{LI} \\
& - D(x_1, x_2; q_f, q_i; \mu_n, m_J, m_L)^{-1}\frac{(\theta_n\lambda)_k(\theta_n\lambda)_l}{\mu_n{}^2}(\chi_k)_{FJ}(-i\not{Q}(q_f, q_i) + m_J)\gamma_\mu(\tau_\beta)_{JL} \\
& \times[-i(\not{Q}(q_f, q_i) - \not{q}) + m_L](\chi_l)_{LI} \\
& + 2iD(x_1, x_2; q_f, q; M_N, m_J, M_P)^{-1}(O^T\theta_\beta O)_{PN}(q_{f_\mu} - Q_\mu(q_f, q))(\chi_N)_{FJ} \\
& \times(-i\not{Q}(q_f, q) + m_J)(\chi_P)_{JI} \\
& + 2iD(x_1, x_2; q_f, q; M_N, m_J, \mu_n)^{-1}(O^T\theta_\beta\theta_n\lambda)_N\frac{(\theta_n\lambda)_l}{\mu_n{}^2}(q_{f_\mu} - Q_\mu(q_f, q))(\chi_N)_{FJ} \\
& \times(-i\not{Q}(q_f, q) + m_J)(\chi_l)_{JI} \\
& - 2iD(x_1, x_2; q_f, q; \mu_n, m_J, M_P)^{-1}(O^T\theta_\beta\theta_n\lambda)_P\frac{(\theta_n\lambda)_k}{\mu_n{}^2}(q_{f_\mu} - Q_\mu(q_f, q))(\chi_k)_{FJ} \\
& \times(-i\not{Q}(q_f, q) + m_J)(\chi_P)_{JI} \\
& - 2iD(x_1, x_2; q_f, q; \mu_n, m_J, M_P)^{-1}(\lambda, \theta_p\theta_\beta\theta_n\lambda)\frac{(\theta_n\lambda)_k(\theta_p\lambda)_l}{\mu_n{}^2\mu_p{}^2}(q_{f_\mu} - Q_\mu(q_f, q))(\chi_k)_{FJ} \\
& \times(-i\not{Q}(q_f, q) + m_J)(\chi_l)_{JI} \\
& + c_{\beta\gamma\delta}C_{\gamma n}C_{\delta p}D(x_1, x_2; q_f, q; \mu_n, m_J, \mu_p)^{-1} \\
& \times[2\not{q}(\tau_n)_{FJ}(-i\not{Q}(q_f, q) + m_J)\gamma_\mu(\tau_p)_{JI} - 2\gamma_\mu(\tau_n)_{FJ}(-i\not{Q}(q_f, q) + m_J)\not{q}(\tau_p)_{JI} \\
& \quad + i(\theta_n\lambda)_j(\chi_j)_{FJ}(-i\not{Q}(q_f, q) + m_J)\gamma_\mu(\tau_p)_{JI} \\
& \quad - i\gamma_\mu(\tau_n)_{FJ}(-i\not{Q}(q_f, q) + m_J)(\theta_p\lambda)_k(\chi_k)_{JI}] \\
& + (\{\theta_\beta, \theta_n\}\lambda)_N D(x_1, x_2; q_f, q; \mu_n, m_J, M_N)^{-1}\gamma_\mu(\tau_n)_{FJ}(-i\not{Q}(q_f, q) + m_J)(\chi_N)_{JI} \\
& + (\{\theta_\beta, \theta_n\}\lambda)_N D(x_1, x_2; q_i, q; \mu_n, m_J, M_N)^{-1}(\chi_N)_{FJ}(-i\not{Q}(q_i, q) + m_J)\gamma_\mu(\tau_n)_{JI}\Big\}u(q_i).
\end{aligned}
\tag{C11}
$$

In (C11) we have defined the four-vector quantity

$$Q(x_1, x_2; q_1, q_2) \equiv q_1(1-x_1+x_2)+q_2(x_1-1)$$
$$\equiv Q(q_1, q_2)\,. \qquad \text{(C12)}$$

We proceed similarly with the contributions involving the hadronic matrix element $\langle S_i\rangle_{FI}$, namely

$$\mathcal{S}_i \equiv (\mathcal{S}_1)_i + (\mathcal{S}_2)_i + (\mathcal{S}_3)_i + (\mathcal{S}_4)_i\,. \qquad \text{(C13)}$$

Again, we first have the contributions arising from the convergent part of the wave-function renormalization:

$$(\mathcal{S}_1)_i = -i\,(2\pi)^4 \overline{U}_f\,(\tilde{z}^{\dagger}_{2,F}\Gamma_j + \Gamma_j z_{2,F})U_i\,. \qquad \text{(C14)}$$

Next, the terms from $\delta S_{\rm CT}$ yield

$$\begin{aligned}(\mathcal{S}_2)_j = -\tfrac{1}{2}i\pi^2\int_0^1 dx(1-x)\overline{u}(q_f)\\
\times\Big\{&2(\tilde{\tau}_n)_{FJ}(\tilde{\tau}_n\chi_j)\ln(q_f{}^2x(1-x)+xm_J{}^2+(1-x)\mu_n{}^2)\\
&+2(\chi_j\tau_n)_{FJ}(\tau_n)_{JI}\ln(q_i{}^2x(1-x)+xm_J{}^2+(1-x)\mu_n{}^2)\\
&+(\chi_N)_{FJ}(\tilde{\chi}_N\chi_j)_{JI}\ln(q_f{}^2x(1-x)+xm_J{}^2+(1-x)M_N^2)\\
&+(\chi_j\tilde{\chi}_N)_{FJ}(\chi_N)_{JI}\ln(q_i{}^2x(1-x)+xm_J{}^2+(1-x)M_N^2)\\
&-\frac{(\theta_n\lambda)_i(\theta_n\lambda)_k}{\mu_n{}^2}\,[(\chi_j\tilde{\chi}_i)_{FJ}(\chi_k)_{JI}\ln(q_i{}^2x(1-x)+xm_J{}^2+(1-x)\mu_n{}^2)\\
&\qquad+(\chi_i)_{FJ}(\tilde{\chi}_k\chi_j)_{JI}\ln(q_f{}^2x(1-x)+xm_J{}^2+(1-x)\mu_n{}^2)]\Big\}u(q_i)\,, \qquad \text{(C15)}\end{aligned}$$

while the remaining logarithmic terms are

$$\begin{aligned}(\mathcal{S}_3)_j = 2i\pi^2\int_0^1 dx_1\int_0^{x_1} dx_2\,\overline{u}(q_f)\Big\{&4\ln D(x_1,x_2;q_f,q_i;\mu_n,m_J,m_L)(\tilde{\tau}_n)_{FJ}(\chi_j)_{JL}(\tau_n)_{LI}\\
&-\ln D(x_1,x_2;q_f,q_i;M_N,m_J,m_L)(\chi_N)_{FJ}(\tilde{\chi}_j)_{JL}(\chi_N)_{LI}\\
&+\ln D(x_1,x_2;q_f,q_i;\mu_n,m_J,m_L)\frac{(\theta_n\lambda)_k(\theta_n\lambda)_l}{\mu_n{}^2}(\chi_k)_{FJ}(\tilde{\chi}_j)_{JL}(\chi_l)_{LI}\Big\}u(q_i) \qquad \text{(C16)}\end{aligned}$$

[here again, the (uniform) mass scale used to define the logarithms is arbitrary, as far as the allomorphic contributions are concerned],

$$\begin{aligned}(\mathcal{S}_4)_j = i\pi^2\int_0^1 dx_1\int_0^{x_1}dx_2\,\overline{u}(q_f)\Big\{&D(x_1,x_2;q_f,q_i;\mu_n,m_J,m_L)^{-1}(\tilde{\tau}_n)_{FJ}\\
&\times[-4Q(q_f,q_i)\cdot(Q(q_f,q_i)-q)(\chi_j)_{JL}+4m_J(\tilde{\chi}_j)_{JL}m_L\\
&\quad+2i\,\not{Q}(q_f,q_i)(\tilde{\chi}_j)_{JL}m_L+2im_J(\tilde{\chi}_j)_{JL}(\not{Q}(q_f,q_i)-\not{q})](\tau_n)_{LI}\\
&-D(x_1,x_2;q_f,q_i;M_N,m_J,m_L)^{-1}(\chi_N)_{FJ}(-i\not{Q}(q_f,q_i)+m_J)(\chi_j)_{JL}\\
&\quad\times[i(\not{q}-\not{Q}(q_f,q_i))+m_L](\chi_N)_{LI}\\
&+D(x_1,x_2;q_f,q_i;\mu_n,m_J,m_L)^{-1}\frac{(\theta_n\lambda)_k(\theta_n\lambda)_l}{\mu_n{}^2}(\chi_k)_{FJ}(-i\not{Q}(q_f,q_i)+m_J)(\chi_j)_{JL}\\
&\quad\times[i(\not{q}-\not{Q}(q_f,q_i))+m_L](\chi_l)_{LI}\\
&+4(\theta_p\theta_n\lambda)_jD(x_1,x_2;q_f,q;\mu_n,m_J,\mu_p)^{-1}(\tilde{\tau}_n)_{FJ}(i\not{Q}(q_f,q)+2m_J)(\tau_p)_{JI}\\
&+2i(\theta_nO)_{jN}D(x_1,x_2;q_f,q;M_N,m_J,\mu_n)^{-1}(\chi_N)_{FJ}(-i\not{Q}(q_f,q)+m_J)\not{q}(\tau_n)_{JI}\\
&-2i(\theta_n)_{jl}\frac{(\theta_p\lambda)_k\theta_p\lambda)_l}{\mu_p{}^2}D(x_1,x_2;q_f,q;\mu_p,m_J,\mu_n)^{-1}(\chi_k)_{FJ}(-i\not{Q}(q_f,q)+m_J)\not{q}(\tau_n)_{JI}\\
&+2i(\theta_nO)_{jN}D(x_1,x_2;q_f,q;\mu_n,m_J,M_N)^{-1}\not{q}(\tau_n)_{FJ}(-i\not{Q}(q_f,q)+m_J)(\chi_N)_{JI}\\
&-2i(\theta_n)_{jk}\frac{(\theta_p\lambda)_k(\theta_p\lambda)_{k'}}{\mu_p{}^2}D(x_1,x_2;q_f,q;\mu_n,m_J,\mu_p)^{-1}\not{q}(\tau_n)_{FJ}\\
&\quad\times(-i\not{Q}(q_f,q)+m_J)(\chi_{k'})_{JI}\end{aligned}$$

$$
\begin{aligned}
&-f_{jj'k}O_{j'N}O_{kP}D(x_1,x_2;q_f,q;M_N,m_J,M_P)^{-1}(\chi_N)_{FJ}(-i\not{Q}(q_f,q)+m_J)(\chi_P)_{JI}\\
&+f_{jj'k}O_{kP}\frac{(\theta_n\lambda)_{i'}(\theta_n\lambda)_{j'}}{\mu_n^2}D(x_1,x_2;q_f,q;\mu_n,m_J,M_P)^{-1}(\chi_{i'})_{FJ}\\
&\quad\times(-i\not{Q}(q_f,q)+m_J)(\chi_P)_{JI}\\
&+f_{jj'k}O_{j'N}\frac{(\theta_p\lambda)_k(\theta_p\lambda)_{k'}}{\mu_p^2}D(x_1,x_2;q_f,q;M_N,m_J,\mu_p)^{-1}(\chi_N)_{FJ}\\
&\quad\times(-i\not{Q}(q_f,q)+m_J)(\chi_{k'})_{JI}\\
&-f_{jj'k}\frac{(\theta_n\lambda)_{i'}(\theta_n\lambda)_{j'}(\theta_p\lambda)_k(\theta_p\lambda)_{k'}}{\mu_n^2\mu_p^2}D(x_1,x_2;q_f,q;\mu_n,m_J,\mu_p)^{-1}(\chi_{i'})_{FJ}\\
&\quad\times(-i\not{Q}(q_f,q)+m_J)(\chi_{k'})_{JI}\\
&-(O^T\theta_n\lambda)_N(\theta_nO)_{jP}D(x_1,x_2;q_f,q;\mu_n,m_J,M_P)^{-1}(\chi_N)_{FJ}(-i\not{Q}(q_f,q)+m_J)(\chi_P)_{JI}\\
&+(O^T\theta_n\lambda)_N(\theta_n)_{jk}\frac{(\theta_p\lambda)_k(\theta_p\lambda)_{k'}}{\mu_p^2}D(x_1,x_2;q_f,q;\mu_n,m_J,\mu_p)^{-1}(\chi_N)_{FJ}\\
&\quad\times(-i\not{Q}(q_f,q)+m_J)(\chi_{k'})_{JI}\\
&-(O^T\theta_p\lambda)_P(\theta_pO)_{jN}D(x_1,x_2;q_f,q;M_N,m_J,\mu_p)^{-1}(\chi_N)_{FJ}(-i\not{Q}(q_f,q)+m_J)(\chi_P)_{JI}\\
&+(O^T\theta_p\lambda)_P(\theta_p)_{jj'}\frac{(\theta_n\lambda)_{i'}(\theta_n\lambda)_{j'}}{\mu_n^2}D(x_1,x_2;q_f,q;\mu_n,m_J,\mu_p)^{-1}(\chi_j)_{FJ}\\
&\quad\times(-i\not{Q}(q_f,q)+m_J)(\chi_P)_{JI}\Big\}u(q_i)\,. \qquad\qquad (\text{C17})
\end{aligned}
$$

Finally, we find for $\mathcal{S}_\alpha$

$$
\begin{aligned}
\mathcal{S}_\alpha=-i\pi^2\int_0^1dx_1\int_0^{x_1}dx_2\bar{u}(q_f)\Big\{&(O^T\theta_\alpha O)_{NP}D(x_1,x_2;q_f,q;M_N,m_J,M_P)^{-1}(\chi_N)_{FJ}(-i\not{Q}(q_f,q)+m_J)(\chi_P)_{JI}\\
&-(\theta_\alpha O)_{j'P}\frac{(\theta_n\lambda)_j(\theta_n\lambda)_{j'}}{\mu_n^2}D(x_1,x_2;q_f,q;\mu_n,m_J,M_P)^{-1}(\chi_j)_{FJ}\\
&\quad\times(-i\not{Q}(q_f,q)+m_J)(\chi_P)_{JI}\\
&-(O^T\theta_\alpha)_{Nk'}\frac{(\theta_p\lambda)_k(\theta_p\lambda)_{k'}}{\mu_p^2}D(x_1,x_2;q_f,q;M_N,m_J,\mu_p)^{-1}(\chi_N)_{FJ}\\
&\quad\times(-i\not{Q}(q_f,q)+m_J)(\chi_k)_{JI}\\
&+\frac{(\theta_n\lambda)_j(\theta_n\lambda)_{j'}(\theta_p\lambda)_k(\theta_p\lambda)_{k'}}{\mu_n^2\mu_p^2}(\theta_\alpha)_{j'k'}D(x_1,x_2;q_f,q;\mu_n,m_J,\mu_p)^{-1}(\chi_j)_{FJ}\\
&\quad\times(-i\not{Q}(q_f,q)+m_J)(\chi_k)_{JI}\Big\}u(q_i)\,. \qquad\qquad (\text{C18})
\end{aligned}
$$

In estimating the magnitudes of the above contributions, one may use (with some care) the following relations:

$$
\begin{aligned}
&\chi_i,\Gamma_i\sim e\frac{m_l}{m_W}\,,\quad \langle S_i\rangle_{FI}\sim e\frac{m_h}{m_W}\,,\\
&\tau_\alpha,t_\alpha\sim e,\quad \langle j^\mu_\alpha\rangle_{FI}\sim e, \qquad\qquad (\text{C19})\\
&f_{ijk}\sim e\frac{m_\phi^2}{m_W}\,,
\end{aligned}
$$

where

$m_l$ ~typical lepton mass,

$m_h$ ~typical hadronic fermion (quark) mass, (C20)

$m_W$ ~typical (massive) gauge vector mass,

$m_\phi$ ~typical Higgs-scalar mass.

Using (C19) and (C20), together with the estimate (ignoring logarithms), valid for any reasonably continuous $f(x_1,x_2)$,

$$
\begin{aligned}
&\int_0^1dx_1\int_0^{x_1}dx_2f(x_1,x_2)D(x_1,x_2;q_1,q_2;m_0,m_1,m_2)^{-1}\\
&\quad\sim(\max(q_1^2,q_1\cdot q_2,q_2^2,m_0^2,m_1^2,m_2^2))^{-1}, \qquad (\text{C21})
\end{aligned}
$$

one easily obtains the conditions (7.1), (7.2) for $O(\alpha G_F)$ pseudoscalar contributions in pion decay.

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[10]In type (i) theories there may appear to be the possibility of an $O(\alpha^2)$ contribution from $\delta S_{\langle jS\rangle}$. This is not, however, allomorphic, involving a linear combination of Yukawa coupling matrices at the leptonic end. Nevertheless, such a term might conceivably lead to difficulties in theories in which the absence of pseudoscalar effects in tree approximation is enforced by constraints on Higgs meson masses.

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