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Asymptotic freedom and Goldstone realization of chiral symmetry*

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The feasibility of Goldstone realization of weakly probed chiral symmetries is examined in the case that strong interactions are described by an asymptotically free theory of zero-bare-mass quarks and gauge vector gluons. This investigation is restricted to finding solutions to the homogeneous Bethe-Salpeter equation for the symmetry-breaking part G of the quark propagator S, $G(p)\gamma_5 \propto \{S^{-1}(p), \gamma_5\}$, in the limit $p^2 \to -\infty$. Renormalization-group techniques are extremely useful in this limit, and are used extensively. Their naive application implies the leading asymptotic behavior $G(p) \sim (\ln p)^{-A}$, where A is a calculable positive constant. More importantly, it is shown that the Bethe-Salpeter kernel is well approximated by the ladder graph alone, with the effective coupling $g^2(p) \sim (\ln p)^{-1}$, *when the strong interactions are asymptotically free*. Two solutions are found for G. The asymptotically dominant one, $G_+(p) \sim (\ln p)^{-A}$, is just what was predicted by straightforward renormalization-group analysis, and does *not* correspond to Goldstone realization of the symmetry. The other solution has much softer asymptotic behavior, $G_-(p) \sim p^{-2}(\ln p)^A$. That this solution actually corresponds to the Goldstone mode is established by relating it, through the axial-vector Ward identity, to the Goldstone-boson–quark–antiquark vertex function, whose large-momentum limit is analyzed via the Wilson operator-product expansion.

I. INTRODUCTION AND SYNOPSIS

In this paper we bring together two important ideas of particle theory to determine if they are mutually compatible. The first, and most venerable, of these was proposed by Nambu and Jona-Lasinio[1] to explain the fact that the pion is *almost* massless and the axial-vector current $j_{5\lambda}$ to which it couples *almost* conserved. Their observation, based on analogy with the Bardeen-Cooper-Schrieffer theory of superconductivity,[2] was that even though a Lagrangian may be chiral-invariant, $\partial^\lambda j_{5\lambda}=0$, this symmetry will not be reflected algebraically in S-matrix elements if it is not a symmetry of the ground state. Moreover, this dynamical, or spontaneous, "breakdown" of chiral symmetry is necessarily accompanied by the presence of massless pseudoscalar mesons, now commonly referred to as Goldstone bosons.[3] The symmetry is not really broken,[4] of course; the axial-vector current is exactly conserved, and the pion remains massless through all perturbative orders of the Lagrangian.

One of the stumbling blocks to this idea—how to explain the *small, nonzero* mass of the pion in a natural way—was removed by Weinberg[5,6] in the course of his work on unified gauge theories of weak and electromagnetic interactions.[7] His explanation is that, while the chiral currents are exactly Goldstone-conserved in the strong interactions, the chiral symmetry is broken in an *explicit* and *calculable* way by the weak and electromagnetic interactions. This can happen if the *only* elementary hadrons participating in the weak interactions are fermion "quarks," with the mass of the proton- and neutron-type quarks arising entirely from Goldstone realization of chiral symmetry by the *strong* interactions. The strong dynamics simultaneously generates the massless pions as bound states of quark-antiquark pairs. The explicit chiral-symmetry breaking by the weak interactions produces a calculable divergence, $\partial^\lambda j_{5\lambda}$, which is nominally of order $\alpha \cong \frac{1}{137}$. Consequently, the pion mass is determined, via current-algebraic techniques, by a formula equating $(F_\pi m_\pi)^2$ to $O(\alpha)$ times an integral over spectral functions of weak-interaction currents.[8]

In a way, this very appealing explanation for the finite pion mass puts the cart before the horse. At our present level of knowledge, this proposal will work *only* if the strong interactions of the quarks are described by a renormalizable field theory involving the coupling of quarks to *vector* gluons which are *neutral* under the weak gauge group. The important—and to a large extent, unanswered—question is: Are such interactions capable of dynamically generating the quark masses and bound-state Goldstone bosons? This question has been addressed in Abelian vector-gluon theories.[9] While an affirmative answer seems quite plausible, the approximations involved are difficult to justify and, in our view, the question is still open. (More will be said about these approximations later).

This brings us to the second important idea alluded to, namely, the asymptotically free theories of quarks interacting strongly with the Yang-Mills quanta of an unbroken, non-Abelian gauge symmetry. These theories were proposed[10] initially to explain the scaling observed in deepinelastic electroproduction.[11] Subsequently, Weinberg[6] made a good case for asymptotic freedom when he showed that the weak corrections of order α to natural symmetries of such strong interactions do not depend importantly on details of the interaction. This is because these weak corrections are determined by the short-distance behavior of the coefficient functions in a Wilson operator-product expansion[12] of two weak currents, and this behavior is essentially canonical when strong interactions are asymptotically free.

In this paper we investigate the central problem which remains, namely, whether Goldstone realization of chiral symmetry is possible in an asymptotically free model of strong interactions. The Nambu–Jona-Lasinio–Goldstone hypothesis seems to be the only way to understand the nature of the pion, and as similar phenomena occur throughout many-body physics,[2] this idea must be regarded as standing on much firmer ground than does asymptotic freedom. Therefore, such theories cannot be regarded as complete descriptions of even just the field-theoretic manifestations of strong interactions unless they possess Goldstone solutions.

In passing, we remark that we are not concerned here with the possibility[13] of dynamical breakdown of the *strong* gauge symmetry—which would be a mechanism for raising the masses of the gluons while maintaining asymptotic freedom.[14] Attractive as this possibility may be, it seems more important to us that the "weak" chiral symmetries be dynamically broken and that the pion emerge as a Goldstone boson.[15] Of course, if no sensible hadronic S matrix exists unless gluon masses are raised, the question of the pion's existence can become moot.

The existence of Goldstone solutions to realistic theories is never established directly. Rather, one relies on a self-consistency argument which makes heavy use of (i) the Ward identity relating the proper axial-vector vertex function, $\Gamma_{5\lambda}$, and the fermion propagator, S (see Ref. 16):

$$q^{\lambda}\Gamma_{5\lambda}(p,p+q) = -\gamma_5 S^{-1}(p+q) - S^{-1}(p)\gamma_5\,, \tag{1.1}$$

and (ii) the integral equation for $\Gamma_{5\lambda}$:

$$\Gamma_{5\lambda}(p,p+q) = \gamma_\lambda\gamma_5 + i^2\int\frac{d^4k}{(2\pi)^4}\,[S(k)\Gamma_{5\lambda}(k,k+q)S(k+q)] \times K(p,k,q)\,, \tag{1.2}$$

where K is the fermion-antifermion ($\psi\bar{\psi}$) scattering kernel. [See Figs. 1 and 2 for pictorial representations of Eq. (1.2) and the kernel.]

This simple argument proceeds along the following lines: From Eq. (1.1), the fermion can have a nonzero mass if and only if $\Gamma_{5\lambda}$ has a pseudoscalar pole at $q^2=0$, whose residue is $F_\pi\mathcal{P}(p,p+q)\gamma_5$. Here, $\mathcal{P}(p,p+q)\gamma_5$ is the proper pseudoscalar-fermion-antifermion ($P\psi\bar{\psi}$) vertex function, and F_π is the pseudoscalar "decay constant." This pole in $\Gamma_{5\lambda}$ is attributed to a pole at $q^2=0$ in the one-particle-irreducible $\psi\bar{\psi}$ scattering amplitude T:

$$T(p,k,q) = K(p,k,q) + i^2\int KSST\,, \tag{1.3}$$

$$[T]_{\text{pole}} = \mathcal{P}(k+q,k)\gamma_5\,\frac{i}{q^2}\,\mathcal{P}(p,p+q)\gamma_5\,. \tag{1.4}$$

Equations (1.2)–(1.4) imply that $\mathcal{P}$ is the solution of the homogeneous Bethe-Salpeter equation[17]:

$$\mathcal{P}(p,p+q)\gamma_5 = i^2\int[S(k)\mathcal{P}(k,k+q)\gamma_5 S(k+q)]K(p,k,q) \qquad (q^2=0)\,. \tag{1.5}$$

Finally, Goldstone-realized chiral symmetry is considered to be a viable alternative for the theory if there exists a nontrivial solution to (1.5), satisfying the Bethe-Salpeter normalization condition,[18]

$$\begin{aligned}2iq_\mu = {} & i^2\int d^4p\,\mathrm{Tr}\left\{S(p)\mathcal{P}(p,p+q)\gamma_5 S(p+q)\left[\frac{\partial}{\partial q^\mu}S^{-1}(p+q)S^{-1}(p)\right]S(p+q)\mathcal{P}(p+q,p)\gamma_5 S(p)\right\}\\ & + \int d^4p\,d^4k\,\mathrm{Tr}\left\{S(k)\mathcal{P}(k,k+q)\gamma_5 S(k+q)\left[\frac{\partial}{\partial q^\mu}K(p,k,q)\right]S(p+q)\mathcal{P}(p+q,p)\gamma_5 S(p)\right\} \quad (q^2=0)\,.\end{aligned} \tag{1.6}$$

Modulo questions regarding the physical significance of their results, this program has been carried out successfully by Nambu and Jona-Lasinio (in a nonrenormalizable model in four dimensions)[1] and, recently, by Gross and Neveu (in a renormalizable, two-dimensional version of the same model).[19] An important feature of both models is that the scattering amplitude T is shown to have a pseudoscalar pole at $q^2=0$ *in the same approximation* for the kernel in which Eq. (1.5) is solved.

In the more realistic, gluon theories, however, the integral equation (1.5) is so complicated that

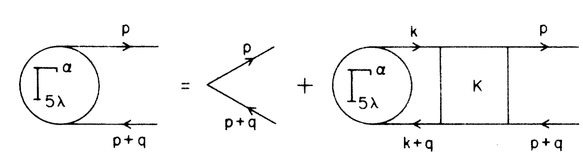

FIG. 1. Graphical representation of the integral Eq. (2.6) for the (unrenormalized) axial-vector vertex function $\Gamma^{\alpha}_{5\lambda}(p,p+q)$.

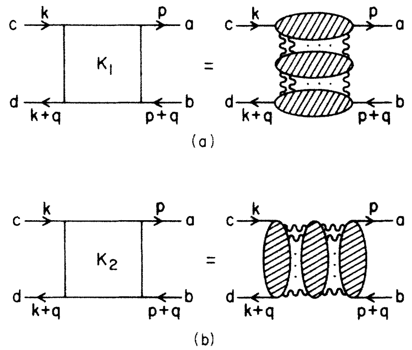

FIG. 2. Graphical representation of the components (a) K_1 and (b) K_2 of the color-singlet, two-fermion irreducible, scattering kernel $K_{ac,db}(p,k,q)$. Letters a, b, c, d refer to external fermion spinor components; fermion lines are straight and vector-gluon lines are wavy; shaded blobs represent all possible subgraphs consistent with the definition of K.

one resorts to a drastic approximation scheme, as follows[9,20]: First, the coupled equations (1.5) (for each invariant function in $\mathcal{P}$) are replaced by the single equation

$$P(p^2)\gamma_5 \equiv \lim_{q\to 0} \mathcal{P}(p, p+q)\gamma_5$$
$$= i^2 \int [S(k)P(k^2)\gamma_5 S(k)] K(p, k, 0)\,. \quad (1.7a)$$

Equivalently, the Ward identity (1.1) implies

$$\{\gamma_5, S^{-1}(p)\} = i^2 \int \{\gamma_5, S(k)\} K(p, k, 0)\,. \quad (1.7b)$$

This is still very complicated, so the second stage consists of seeking a solution to (1.7) in the limit $p^2 \to -\infty$, where it is *assumed* that the ladder approximation to the kernel is valid. By the ladder approximation, we mean the single-gluon exchange graph, in which lowest-order expressions are used for both the gluon propagator and the gluon-fermion vertex function.

This scheme, whose spirit we also follow, is open to two obvious criticisms. The first is that a solution to (1.7) (also satisfying the normalization condition) does not imply that (1.5) has consistent solutions, nor does it guarantee that T has a pole at $q^2 = 0$. Here, we have not improved on past work, and we take as a working assumption that it suffices to examine the question of solutions to (1.7).

The second criticism is that it is not at all obvious that the ladder approximation is justified, especially when $p^2 \to -\infty$. Most of this paper is devoted to the technicalities of establishing its validity, *when the fermion-gluon interactions are asymptotically free*. In the remainder of this section, we motivate the arguments for proving this, and summarize our principal conclusions regarding spontaneous breakdown.

The question of the ladder approximation is best answered via the techniques of the renormalization group,[21] which we use throughout much of this paper. When $p^2 \to -\infty$, the integral in (1.7) gets its major contribution from large values of the integration momentum k. Suppose we replace the momenta p and k in that equation by κp and κk, and scale κ to infinity. According to the renormalization group, the behavior of the kernel as a function of asymptotic momenta, κp and κk, and the renormalized fermion-gluon coupling constant (call it g_R) is governed by its behavior as a function of finite momenta, p and k, and of the so-called effective coupling constant, $g(\kappa, g_R)$. Therefore, a perturbative scheme such as the ladder approximation is meaningless *unless the effective coupling is small*. That this is the case has never been established in the quark-massive-gluon model. On the other hand, $g(\kappa) \sim (b \ln\kappa)^{-1}$ as $\kappa \to \infty$ in an asymptotically free gauge theory, so here there is hope of justifying the ladder approximation. (Here b is constant determined from group theory.)

In Sec. II we propose a model of massless quarks and gluons, define and discuss the Green's functions of interest, and describe how they are to be renormalized. Our renormalization procedure follows closely that of Weinberg[22] and 't Hooft,[23] modified slightly to account for the assumption that quarks with zero bare mass have nonzero, dynamically generated physical mass. All this is a necessary prelude to Sec. III, in which the renormalization-group analysis of K and S is carried out. Special attention is paid here to the fact that the external momenta, $(p, k, 0)$, in the kernel are exceptional.

The principal results emerging from this analysis are as follows[24]:

(i) The *leading* asymptotic behavior of the symmetry-breaking part G of the fermion propagator is (henceforth, we deal with G, proportional to P by virtue of the Ward identity)

$$G(\kappa p, g_R)\gamma_5 \propto \{\gamma_5, S^{-1}(\kappa p, g_R)\}$$
$$\underset{\kappa\to\infty}{\sim} (\ln\kappa)^{-A} G(p, g(\kappa))\,, \quad (1.8)$$

where A is a positive constant determined by the structure of the gauge group and the representation assignments of the fermions. Therefore, we expect the asymptotically dominant solution to the Bethe-Salpeter Eq. (1.7) to have this logarithmic behavior at large momentum.

(ii) As $\kappa \to \infty$, the kernel is given by the ladder approximation with g_R replaced by $g(\kappa)$:

$$K(\kappa p, \kappa k, 0; g_R) \underset{\kappa\to\infty}{\sim} \kappa^{-2} K(p, k, 0; g(\kappa))$$
$$= 3 i^3 g^2(\kappa) C[\kappa(k-p)]^{-2} + O(g^4(\kappa)/\kappa^2 \sim (\kappa \ln\kappa)^{-2}), \tag{1.9}$$

where C is another group-theoretical constant. We emphasize that Eq. (1.9) is not merely the asymptotic form of just the ladder graph, but of the entire part of the kernel which contributes in Eq. (1.7).

The Bethe-Salpeter Eq. (1.7) is solved in Sec. IV, using the scaling trick $p, k \to \kappa p, \kappa k (\kappa \to \infty)$, and the kernel (1.9). The expected errors made in this approximation are such that terms of order $p^{-2}(\ln p)^B$ in $G(p)$ are left undetermined. When written as a differential equation, (1.7) has *two* possible solutions,

$$G_+(p) \underset{p\to\infty}{\sim} a_+ (\ln p)^{-A} \tag{1.10}$$

and

$$G_-(p) \underset{p\to\infty}{\sim} a_- p^{-2} (\ln p)^A, \tag{1.11}$$

where $a_\pm$ are constants to be determined by the normalization condition (1.6). The constant A is the same as that in Eq. (1.8). As far as one can tell from just solving the Bethe-Salpeter equation, the error in Eq. (1.11) is of the same order as the solution itself.

The asymptotic behavior of the dominant solution, G_+, is exactly that predicted by straightforward renormalization-group analysis, which is insensitive to the fact that G is to be associated with Goldstone realization of the chiral symmetry. Indeed, G_+ is the answer one gets when the quark's bare mass is nonzero. To decide on the appropriate *Goldstone* solution, the bound-state nature of the Goldstone boson has to be taken into account. This is done using the Wilson operator-product expansion to determine the behavior of the vertex function $\mathcal{P}(p, p+q)$ in the limit $p \to \infty$, $(p+q)^2/p^2$ fixed. This leads, through the Ward identity, $G(p) \propto \lim_{q\to 0} \mathcal{P}(p, p+q)$, to precisely the solution (1.11). Moreover, the Wilson analysis implies that corrections to (1.11) are down by powers of p, not just powers of $\ln p$. Thus, insofar as definite conclusions may be drawn from arguments valid only at large momenta, chiral symmetry in the Goldstone mode seems a viable possibility for an asymptotically free theory of the strong interactions. It goes without saying that analysis at low momenta is needed to firmly establish this possibility.

II. SPECIFICATION OF THE MODEL AND RENORMALIZATION PRESCRIPTION

A. The model

We consider a model in which the strong interactions of zero-bare-mass fermion "quarks" (ψ) are mediated by a set of "colored" vector gluons (G_μ^i) which are the gauge bosons of a strong symmetry group G_s. For definiteness, we shall assume $G_s = \mathrm{SU}(N)$, and that the quarks consist of n multiplets of G_s, each one transforming as the fundamental representation (N) of SU(N). Writing (superscript T denotes transpose)

$$\psi = \begin{pmatrix} \mathcal{P}_1 & \mathcal{P}_2 & \cdots & \mathcal{P}_N \\ \mathfrak{N}_1 & \mathfrak{N}_2 & \cdots & \mathfrak{N}_N \\ \vdots & & & \end{pmatrix} = (\psi_1 \;\; \psi_2 \;\; \cdots \;\; \psi_N),$$
$$\psi^T = \begin{bmatrix} \mathcal{P}_1 & \mathfrak{N}_1 & \cdots \\ \mathcal{P}_2 & \mathfrak{N}_2 & \cdots \\ \vdots & & \\ \mathcal{P}_N & \mathfrak{N}_N & \cdots \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix}, \tag{2.1}$$

the Lagrangian describing these strong interactions is

$$\mathcal{L} = -\tfrac{1}{4} G_{\mu\nu} G^{\mu\nu} + \mathrm{Tr}[\bar{\psi}^T i\gamma^\mu(\partial_\mu - ig\tfrac{1}{2}\tau\cdot G_\mu)\psi^T]$$
$$+ \text{gauge-fixing terms} + \text{ghost terms}. \tag{2.2}$$

Here, g is the unrenormalized coupling constant of the theory, and τ_i $(i = 1, \ldots, N^2-1)$ are the matrix representatives for (N), normalized to $\mathrm{Tr}(\tau_i\tau_j) = 2\delta_{ij}$.

The theory described by Lagrangian (2.2) is formally invariant under the "weak" symmetry group $G_w = \mathrm{U}(n)\times\mathrm{U}(n)$, with transformations generated by the color-singlet currents

$$j_{5\lambda}^\alpha = \mathrm{Tr}(\bar\psi\gamma_\lambda\gamma_5\tfrac{1}{2}\lambda_\alpha\psi) = \sum_{r=1}^N \bar\psi_r\gamma_\lambda\gamma_5\tfrac{1}{2}\lambda_\alpha\psi_r, \tag{2.3a}$$
$$j_\lambda^\alpha = \mathrm{Tr}(\bar\psi\gamma_\lambda\tfrac{1}{2}\lambda_\alpha\psi), \tag{2.3b}$$
$$j_{5\lambda} = \mathrm{Tr}(\bar\psi\gamma_\lambda\gamma_5\psi), \tag{2.3c}$$
$$j_\lambda = \mathrm{Tr}(\bar\psi\gamma_\lambda\psi). \tag{2.3d}$$

Here, λ_α $(\alpha = 1, \ldots, n^2-1)$ are the matrix representatives of the representation (n) in SU(n). We shall investigate the possibility that the axial-vector currents are conserved in the manner of Nambu, Jona-Lasinio, and Goldstone.

Actually, because of the familiar triangle anomaly,[25] the U(1) axial-vector current, $j_{5\lambda}$, is not conserved. The anomaly causes difficulties in the Ward identity for $j_{5\lambda}$ because it is the only axial-vector current which can communicate with two-gluon channels. This does not invalidate the Goldstone theorem for $j_{5\lambda}$, however, since the anomaly vanishes at zero momentum. In principle, therefore, it seems we are stuck with n^2 Goldstone pseudoscalars instead of the physically more desirable n^2-1. In the context of a quark-gluon model, we see two ways of avoiding this problem.

First, it is possible that the last pseudoscalar does not couple to physical (on shell) quantities, in the same way that Goldstone bosons are absorbed via the Higgs mechanism in models with elementary scalar fields.[7] Here, one might envisage the existence of a bound-state axial-vector meson (with the quantum numbers of $j_{5\lambda}$) whose mass is raised from zero by this process.[26] We should add that the existence of this bound state is not mandatory (and, in any case, extremely difficult to prove). Recalling the original arguments of Nambu and Jona-Lasinio,[1] conservation of $j_{5\lambda}$ is consistent with nonzero fermion mass and *no* massless pseudoscalar if the fermion's axial-vector charge form factor vanishes at zero momentum transfer. The Higgs mechanism is just one natural way of accomplishing this feat.

A second possibility is that $j_{5\lambda}$ is anomalously nonconserved in the sense of Johnson, Baker, and Willey.[27] This will happen if the homogeneous integral equation for the vertex function of $\partial^\lambda j_{5\lambda}$ has a nontrivial solution. Langacker and Pagels have argued recently[28] that this *must* happen, simply because of the presence of the triangle anomaly.

Neither of these possibilities should alter the conservation laws and Ward identities for the (anomaly-free) SU(n) axial-vector currents, $j^\alpha_{5\lambda}$. Nor are they in conflict with any available data. Accordingly, we assume solutions to the theory with no Goldstone pole in $j_{5\lambda}$, and from now on consider only the conserved currents $j^\alpha_{5\lambda}$.

B. Renormalization

We now describe our procedure for renormalizing the Green's functions of this theory. We shall be interested especially in the fermion propagator and the axial-vector vertex function, their integral equations and the Ward identity connecting them. For simplicity, we always work in the Landau gauge, which is invariant under renormalization; hence, gauge dependencies of quantities defined in this section and the next may be ignored.

In a theory described by the Lagrangian (2.2), with an appropriately defined cutoff, the fermion propagator satisfies the integral equation

$$\begin{aligned} S^{-1}(p) &\equiv \not{p}-\Sigma(p) \\ &= \not{p} - ig^2 \int \frac{d^4k}{(2\pi)^4}\, \Delta^{i\mu,j\nu}(k)\Gamma^i_\mu(p,p-k) \\ &\qquad \times S(p-k)\gamma_\nu \tfrac{1}{2}\tau_j\,, \end{aligned} \tag{2.4}$$

where $\Delta^{i\mu,j\nu}$ is the gluon propagator, and Γ^i_μ the gluon-fermion vertex function. The axial-vector vertex function is defined by

$$i^2 S(p)\Gamma^\alpha_{5\lambda}(p,p+q)S(p+q) = \frac{1}{N}\int d^4x\, d^4y\, e^{i[p\cdot x-(p+q)\cdot y]}\left\langle 0 \left| T\left(\sum_1^N \psi_r(x) j^\alpha_{5\lambda}(0)\bar{\psi}_r(y)\right)\right| 0\right\rangle\,, \tag{2.5}$$

and it satisfies the integral equation (a, b, c, d refer to spinor indices)

$$[\Gamma^\alpha_{5\lambda}(p,p+q)]_{ab} = (\gamma_\lambda\gamma_5)_{ab}\tfrac{1}{2}\lambda_\alpha + i^2\int\frac{d^4k}{(2\pi)^4}\,[S(k)\Gamma^\alpha_{5\lambda}(k,k+q)S(k+q)]_{cd}K_{ac,db}(p,k,q)\,. \tag{2.6}$$

This integral equation is depicted in Fig. 1, and the 2-fermion irreducible kernel K in Fig. 2.

This kernel describes fermion-antifermion scattering, with the pair transforming as a color singlet in the channel in which the total momentum is q (the q channel). As shown in Fig. 2, this kernel may be decomposed into two pieces: one (K_1) which cannot be cut in the q channel without cutting at least two fermion lines, while the other (K_2) may be separated by cutting only (two or more) gluon lines. In equations,

$$\begin{aligned} K_{ac,db} &= (K_1)_{ac,db} + (K_2)_{ac,db} \\ &= (K_1)_{ac}(K_1')_{db} + (K_2)_{ab}(K_2')_{dc}\,. \end{aligned} \tag{2.7}$$

Only K_1 can contribute to the integral equation (2.6), because gluons are neutral under the weak group SU(n)$\times$SU(n). This fact will be useful to us in the renormalization-group analysis of Sec. III. [Note that *both* K_1 and K_2 contribute to the similar integral equation for the U(1) axial-vector current $j_{5\lambda}$, whose triangle anomaly is associated with K_2.]

For our later discussion of large-momentum behavior of propagator and kernel, it is very convenient to adopt the zero-mass renormalization procedure (ZMRP) of Weinberg[22] and 't Hooft.[23] These two versions of ZMRP are essentially equivalent,[29] differing only in whether the theory is to be regularized by means of a gauge-invariant momentum cutoff (Λ),[30] or by dimensional regularization.[31] Our remarks will be couched in the language of a cutoff.

According to ZMRP, wave-function and charge renormalization constants are determined by specifying the value of the appropriate Green's functions when all external momenta p_i are fixed at a *nonexceptional* Euclidean point, with $p_i^2 = -\mu^2$, and *with all bare masses*, m_0, *set equal to zero*. Renormalization constants are then functions of the ratio Λ/μ and the unrenormalized coupling g; similarly, the renormalized coupling $g_R = g_R(\Lambda/\mu, g)$. For example, S^{-1} and $\Gamma^\alpha_{5\lambda}$ are to be renormalized by multiplying them by constants Z_2 and Z_A, respectively, defined by

$$Z_2^{-1}\left(\frac{\Lambda}{\mu}, g\right) = \text{coefficient of } \not p \text{ in } S^{-1}(p, \Lambda, m_0, g)\big|^{m_0=0}_{p^2=-\mu^2}\ , \tag{2.8a}$$

$$Z_A^{-1}\left(\frac{\Lambda}{\mu}, g\right) = \text{coefficient of } \gamma_\lambda\gamma_5\tfrac{1}{2}\lambda_\alpha \text{ in } \Gamma^\alpha_{5\lambda}(p, p+q; \Lambda, m_0, g)\big|^{m_0=0}_{p_i^2=-\mu^2}\ . \tag{2.8b}$$

The kernel in Eq. (2.6) is renormalized by multiplication with Z_2^{-2}.

It is also necessary to provide for mass renormalization in a theory with $m_0 \neq 0$. Weinberg does this by introducing a "renormalized" mass (which is *not* the position of the pole in S),

$$m_R = Z_m^{-1}\left(\frac{\Lambda}{\mu}, g\right) m_0\,, \tag{2.9}$$

where Z_m is the constant which renormalizes the mass operator $\bar\psi\psi$. To sum up, then, renormalized Green's functions Γ_R are obtained by expressing m_0 and g in terms of m_R and g_R, and multiplying the unrenormalized function by the appropriate Z:

$$\Gamma_R(p_i, \mu, m_R, g_R) = Z_\Gamma\left(\frac{\Lambda}{\mu}, g\right)\Gamma(p_i, \Lambda, m_0, g)\,, \tag{2.10}$$

where Z_Γ is determined by the nature of the external lines.

One obvious advantage of ZMRP is that it preserves the chiral symmetry manifest in the Lagrangian (2.2); that is, $m_R = 0$ if $m_0 = 0$. However, if we presume that this symmetry is only Goldstone-realized, the procedure will have to be modified somewhat if we are to learn anything from the renormalization group about such solutions to the theory. Specifically, we would like to define a *nonzero* mass, m_R, in terms of which a broken-symmetry solution for the propagator may be written

$$S^{-1}(p, \mu, m_R, g_R) = \not p F(p, \mu, m_R, g_R) - m_R G(p, \mu, m_R, g_R)\,. \tag{2.11}$$

Renormalization-group techniques then will be used to analyze the large-p limit of the chiral-symmetry breaking function G.

Proceeding much as one does in Johnson-Baker-Willey electrodynamics,[27] we introduce a cutoff-dependent bare mass, $m_0(\Lambda)$, into the theory and continue to define m_R by Eq. (2.9). This breaks chiral symmetry in the cutoff theory, but if $m_0 \to 0$ "rapidly enough" as $\Lambda \to \infty$, the renormalized Ward identities will be symmetric again. Moreover, if $Z_m \to 0$ at the same rate as m_0, it is possible to have a cutoff-independent $m_R \neq 0$, which we can use to parameterize the Goldstone solutions.

This procedure seems clumsy. In practical discussions of spontaneous symmetry breaking however, it is always necessary to assume the existence of nonzero masses (for example), while demanding that such assumptions and their consequences be self-consistent. For now, there is no other way to deal with quantities which vanish in every perturbative order of realistic field theories.

To argue that one can have $m_R \neq 0$, even though $m_0(\Lambda \to \infty) \to 0$, we demand that m_R be cutoff-independent. Then, from Eq. (2.9),

$$0 = \Lambda\frac{d}{d\Lambda} m_R = m_R\left[\Lambda\frac{d}{d\Lambda}\ln m_0(\Lambda) - \Lambda\frac{d}{d\Lambda}\ln Z_m\left(\frac{\Lambda}{\mu}, g\right)\right]. \tag{2.12}$$

If we identify

$$\Lambda\frac{d}{d\Lambda}\ln Z_m \cong \Lambda\frac{\partial}{\partial\Lambda}\ln Z_m\Big|_{g,\mu\ \text{fixed}} \equiv -\gamma_m(g_R)\,, \tag{2.13}$$

where γ_m is the anomalous dimension of the mass operator, then we can have a nontrivial solution to Eq. (2.12) with the desired features—provided γ_m is positive. Equation (2.13) is approximately true when g_R is small, and then $\gamma_m > 0$ in the theories under discussion (see Sec. III).

To see that it is possible to have symmetric Ward identities even when $m_R \neq 0$, we examine the identity satisfied by the axial-vector vertex in the cutoff theory,[32]

$$q^\lambda\Gamma^\alpha_{5\lambda}(p, p+q; \Lambda, m_0, g) = -2m_0(\Lambda)\Gamma^\alpha_5(p, p+q; \Lambda, m_0, g) - \gamma_5\tfrac{1}{2}\lambda_\alpha S^{-1}(p+q, \Lambda, m_0, g) - S^{-1}(p, \Lambda, m_0, g)\gamma_5\tfrac{1}{2}\lambda_\alpha\,. \tag{2.14}$$

Here, $2m_0\Gamma^\alpha_5$ is the vertex function for the divergence $\partial^\lambda j^\alpha_{5\lambda}$,

$$2i^3 m_0(\Lambda)S(p)\Gamma_5^\alpha(p,p+q)S(p+q)=\frac{1}{N}\int e^{i[p\cdot x-(p+q)\cdot y]}\left\langle 0\left|T\left(\sum_1^N \psi_r(x)\partial^\lambda j_{5\lambda}^\alpha(0)\bar{\psi}_r(y)\right)\right|0\right\rangle , \tag{2.15}$$

and it satisfies the integral equation

$$m_0(\Lambda)\Gamma_5^\alpha(p,p+q)=m_0(\Lambda)\gamma_5\tfrac{1}{2}\lambda_\alpha + i^2\int[S\,m_0(\Lambda)\Gamma_5^\alpha S]K\,. \tag{2.16}$$

Letting Z_D be the renormalization constant appropriate to Γ_5^α, and denoting ZMRP-renormalized quantities by a tilde, we rewrite Eq. (2.14) as

$$q^\lambda\tilde{\Gamma}_{5\lambda}^\alpha=-2m_0(\Lambda)Z_A Z_D^{-1}\tilde{\Gamma}_5^\alpha - Z_A Z_2^{-1}(\gamma_5\tfrac{1}{2}\lambda_\alpha\tilde{S}^{-1}+\tilde{S}^{-1}\gamma_5\tfrac{1}{2}\lambda_\alpha)\,. \tag{2.17}$$

Going onto the fermion mass shell (if this has meaning in an unbroken gauge theory), we see from (2.17) that $m_0(\Lambda)Z_A/Z_D$ is Λ-independent, hence Z_A/Z_2 is also. It follows that $\bar{m}\tilde{\Gamma}_5^\alpha=Z_A m_0\Gamma_5^\alpha=(Z_A m_0/Z_D)\tilde{\Gamma}_5^\alpha$ is independent of cutoff, and satisfies

$$\bar{m}\tilde{\Gamma}_5^\alpha=Z_A m_0(\Lambda)\gamma_5\tfrac{1}{2}\lambda_\alpha+i^2\int[\tilde{S}\bar{m}\tilde{\Gamma}_5^\alpha\tilde{S}]\tilde{K}\,. \tag{2.18}$$

We have already argued that $m_0\to 0$ as $\Lambda\to\infty$. In the Landau gauge, Z_2 and Z_A are finite through second order in g, so $Z_A m_0(\Lambda)$ vanishes in the limit of infinite cutoff—at least for small coupling. We conjecture that this is a general result. Then $\bar{m}\tilde{\Gamma}_5^\alpha$ satisfies a homogeneous integral equation, the trivial solution to which yields the renormalized Ward identity

$$q^\lambda\tilde{\Gamma}_{5\lambda}^\alpha(p,p+q) = -\left(\frac{Z_A}{Z_2}\right)[\gamma_5\tfrac{1}{2}\lambda_\alpha\tilde{S}^{-1}(p+q)+\tilde{S}^{-1}(p)\tfrac{1}{2}\lambda_\alpha\gamma_5]\,. \tag{2.19}$$

For nonvanishing m_R, this trivial solution to (2.18) corresponds to Goldstone realization of chiral symmetry, manifested by the appearance of a pole at $q^2=0$ in $\tilde{\Gamma}_{5\lambda}^\alpha$.

This establishes the consistency of our procedure for renormalizing the Lagrangian (2.2) while maintaining chiral-symmetric Ward identities—at least for small coupling and in the Landau gauge. To do better might require a program like that of Johnson, Baker, and Willey[27]—in particular, showing that to each order in perturbation theory, one may find a gauge in which $m_0(\Lambda\to\infty)\to 0$, $m_R\neq 0$, and Eq. (2.19) is correct. Such a program is beyond the scope of this paper, and, in any case, we believe that it is unnecessary. If we had not introduced m_R, we would get the Ward identity (2.19) in any gauge. We view m_R as a parameter to be used in place of the proposed physical mass m, especially convenient for the renormalization-group analysis in Sec. III.

Finally, we record the renormalized version of some of the standard formulas associated with Goldstone chiral symmetries. Assuming that $SU(n)\times SU(n)$ spontaneously breaks down to (algebraic) $SU(n)$, we have (dropping the tildes from now on)

$$\lim_{q\to 0} q^\lambda\Gamma_{5\lambda}^\alpha(p,p+q)=-\left(\frac{Z_A}{Z_2}\right)\{\gamma_5,S^{-1}(p)\}\tfrac{1}{2}\lambda_\alpha \equiv\left(\frac{Z_A}{Z_2}\right)2m_R G(p,\mu,m_R,g_R)\tfrac{1}{2}\lambda_\alpha\,. \tag{2.20}$$

This corresponds to a bound-state pseudoscalar pole at $q^2=0$ in $\Gamma_{5\lambda}^\alpha$, the bound state transforming according to the adjoint representation of $SU(n)$ and as a color singlet:

$$[\Gamma_{5\lambda}^\alpha(p,p+q)]_{\text{pole}}\cong\frac{q_\lambda}{q^2}F_\pi\mathcal{P}(p,p+q)\gamma_5\tfrac{1}{2}\lambda_\alpha\,. \tag{2.21}$$

Here, $\mathcal{P}(p,p+q)\gamma_5$ is the Goldstone-boson–fermion–antifermion vertex function, and F_π the usual "decay constant."

As discussed in the Introduction, the Goldstone pole really exists only if the homogeneous Bethe-Salpeter equation for $\mathcal{P}$ has a normalizable solution [Eqs. (1.5) and (1.6)]. The technical difficulties of establishing this are so great that we attack a simpler problem instead. Namely, we combine Eqs. (2.20) and (2.21) to get the Goldberger-Treiman relation,

$$F_\pi P(p)\equiv\lim_{q\to 0}F_\pi\mathcal{P}(p,p+q)=2m_R\left(\frac{Z_A}{Z_2}\right)G(p)\,, \tag{2.22}$$

and seek normalizable solutions to the Bethe-Salpeter equation for P, or what is the same thing, for G. The existence of such solutions is necessary, though not sufficient, for the validity of Eqs. (2.20) and (2.21).

III. RENORMALIZATION-GROUP ANALYSIS

The problem before us now is to find a way of solving the Bethe-Salpeter equation for the mass function G:

$$2m_R G(p,\mu,m_R,g_R)(\gamma_5)_{ab}=\int\frac{d^4k}{(2\pi)^4}\{\gamma_5,S(k,\mu,m_R,g_R)\}_{cd}K_{ac,db}(p,k,0;\mu,m_R,g_R)\,. \tag{3.1}$$

It is quite out of the question to solve (3.1) for arbitrary momentum, so we follow the standard approach[9,20] of attempting a solution for large, spacelike p ($-p^2 \gg m_R^2$). Our method will be to analyze the large-momentum behavior of S and K via renormalization-group techniques. We shall show that the ladder approximation for K is valid, *by virtue of the asymptotic freedom of the quark-gluon interaction*. This is to be contrasted with the unjustified assumption of the ladder approximation in the case that quark-gluon interactions are *not* asymptotically free.

In addition to preserving the symmetries of the theory, an important advantage of ZMRP is that it leads to renormalization-group equations[22,23] which are more powerful than those derived from more conventional renormalization prescriptions.[33] The new equations are valid and may be solved for *all* momenta (though solutions are not always directly useful), and they permit systematic study of the effect of mass (m_R) insertions. These features will be helpful to us in our study of Eq. (3.1) because (i) determination of the asymptotic behavior of the kernel is complicated by the fact that it is evaluated at exceptional momenta ($q=0$), and (ii) asymptotic behavior of G is obtained by mass insertion in the propagator S.

Before addressing the problem of exceptional momenta, let us review the renormalization-group equations for S and K. For example, the equations for the propagator are derived[22] from the observation that

$$Z_2^{-1}\left(\frac{\Lambda}{\mu}, g\right) S^{-1}(p, \mu, m_R, g_R) = Z_2^{-1}(\not p F - m_R G)$$

is independent of μ. With κ a dimensionless parameter used to scale the momentum p, the renormalization-group equations for F and G turn out to be

$$\left\{\kappa \frac{\partial}{\partial \kappa} - \beta(g_R)\frac{\partial}{\partial g_R} + [1+\gamma_m(g_R)] m_R \frac{\partial}{\partial m_R} + \gamma(g_R)\right\} F(\kappa p, \mu, m_R, g_R) = 0 \tag{3.2}$$

and

$$\left\{\kappa \frac{\partial}{\partial \kappa} - \beta(g_R)\frac{\partial}{\partial g_R} + [1+\gamma_m(g_R)] m_R \frac{\partial}{\partial m_R} + \gamma(g_R) + \gamma_m(g_R)\right\} G(\kappa p, \mu, m_R, g_R) = 0\,. \tag{3.3}$$

The functions β, γ_m, and γ in Eqs. (3.2) and (3.3) are defined by

$$\beta(g_R) = \mu \frac{\partial}{\partial \mu} g_R(\Lambda/\mu, g)\Big|_{\Lambda, g}, \tag{3.4a}$$

$$\gamma_m(g_R) = \mu \frac{\partial}{\partial \mu} \ln Z_m(\Lambda/\mu, g)\Big|_{\Lambda, g}, \tag{3.4b}$$

$$\gamma(g_R) = \mu \frac{\partial}{\partial \mu} \ln Z_2(\Lambda/\mu, g)\Big|_{\Lambda, g}. \tag{3.4c}$$

For an asymptotically free theory based on the gauge group SU(N), with fermions assigned to n fundamental representations (N), these functions are given in the Landau gauge, for small g_R, by[13]

$$\beta(g_R) \cong -\frac{b}{2} g_R^3, \quad b = \frac{1}{24\pi^2}[11 C_2(\mathrm{SU}(N)) - 4n\, T((N))] = \frac{1}{24\pi^2}(11N - 2n) \tag{3.5a}$$

$$\gamma_m(g_R) \cong c g_R^2, \quad c = \frac{3}{8\pi^2} C_2((N)) = \frac{3}{8\pi^2}\left(\frac{N^2-1}{2N}\right) \tag{3.5b}$$

$$\gamma(g_R) \cong f g_R^4. \tag{3.5c}$$

Note that γ_m is positive, as was necessary for $m_0(\Lambda) \to 0$ as $\Lambda \to \infty$.

Equations (3.2) and (3.3) are solved by first introducing the κ-dependent effective coupling constant and mass, defined by

$$\kappa \frac{dg(\kappa, g_R)}{d\kappa} = \beta(g(\kappa, g_R)), \quad g(1, g_R) = g_R \tag{3.6a}$$

and

$$\kappa \frac{dm(\kappa, m_R)}{d\kappa} = -[1 + \gamma_m(g(\kappa, g_R))] m(\kappa, m_R),$$

$$m(1, m_R) = m_R\,. \tag{3.7a}$$

From (3.5a) and (3.6a), it follows that

$$g^2(\kappa) \underset{\kappa\to\infty}{\sim} (b\ln\kappa)^{-1} + O\left(\frac{\ln\ln\kappa}{(\ln\kappa)^2}\right). \tag{3.6b}$$

Also, Eq. (3.7a) has the solution

$$m(\kappa) = \frac{m_R}{\kappa} \exp\left[-\int_{g_R}^{g(\kappa)} \frac{dx\,\gamma_m(x)}{\beta(x)}\right]$$

$$\underset{\kappa\to\infty}{\sim} \frac{m_R}{\kappa} (\ln\kappa)^{-c/b}. \tag{3.7b}$$

In terms of $g(\kappa)$ and $m(\kappa)$, the desired solutions to Eqs. (3.2) and (3.3) are (valid for *all* κ)

$$F(\kappa p, \mu, m_R, g_R) = \exp\left[-\int_{g_R}^{g(\kappa)} \frac{dx\,\gamma(x)}{\beta(x)}\right] \times F(p, \mu, m(\kappa), g(\kappa)), \tag{3.8}$$

and

$$G(\kappa p, \mu, m_R, g_R) = \exp\left[-\int_{g_R}^{g(\kappa)} \frac{dx}{\beta(x)}[\gamma(x) + \gamma_m(x)]\right] \times G(p, \mu, m(\kappa), g(\kappa)). \tag{3.9}$$

The same analysis can be carried out on the kernel appearing in Eq. (3.1), even though it is only that part of the kernel depicted in Fig. 2(a), and it is evaluated at exceptional momenta. This may be seen, for example, directly from Eq. (3.1), which we rewrite as

$$G(\kappa p, \mu, m_R, g_R) = \kappa^4 \int \frac{d^4k}{(2\pi)^4} \frac{G(\kappa k, \ldots, g_R) K(\kappa p, \kappa k, 0; \mu, m_R, g_R)}{\kappa^2 k^2 F^2(\kappa k, \ldots, g_R) - m_R^2 G^2(\kappa k, \ldots, g_R)}. \tag{3.10}$$

In Eq. (3.10), we have introduced

$$K(p, k, q; \mu, m_R, g_R) = \tfrac{1}{4}(\gamma_5)_{cd} K_{ac,db}(p, k, q, \ldots)(\gamma_5)_{ba} = \tfrac{1}{4}\,\mathrm{Tr}(\gamma_5 K_1 \gamma_5 K_1'), \tag{3.11}$$

the second equality following from Eq. (2.7). Application of Eqs. (3.2) and (3.3) to both sides of (3.10) yields the renormalization-group equation for the kernel:

$$\left\{\kappa\frac{\partial}{\partial\kappa} - \beta(g_R)\frac{\partial}{\partial g_R} + [1+\gamma_m(g_R)]m_R\frac{\partial}{\partial m_R} + 2\gamma(g_R) + 2\right\} K(\kappa p, \kappa k, 0; \ldots) = 0, \tag{3.12}$$

with the solution

$$K(\kappa p, \kappa k, 0; \mu, m_R, g_R) = \kappa^{-2} \exp\left[-2\int_{g_R}^{g(\kappa)} \frac{dx\,\gamma(x)}{\beta(x)}\right] K(p, k, 0; \mu, m(\kappa), g(\kappa)). \tag{3.13}$$

From Eqs. (3.7)–(3.13), it follows that $G(p, \mu, m(\kappa), g(\kappa))$ satisfies the same integral equation that $G(p, \mu, m_R, g_R)$ does:

$$G(\kappa p, \mu, m_R, g_R) = \exp\left[-\int_{g_R}^{g(\kappa)} \frac{dx}{\beta(x)}(\gamma+\gamma_m)\right] G(p, \mu, m(\kappa), g(\kappa))$$

$$= \exp\left[-\int_{g_R}^{g(\kappa)} \frac{dx}{\beta(x)}(\gamma+\gamma_m)\right] \int \frac{d^4k}{(2\pi)^4} \frac{G(k, \mu, m(\kappa), g(\kappa)) K(p, k, 0; \mu, m(\kappa), g(\kappa))}{k^2 F^2(k, \ldots, g(\kappa)) - m^2(\kappa) G^2(k, \ldots, g(\kappa))}. \tag{3.14}$$

Because we have used the ZMRP, we know that one-particle irreducible Green's functions $\Gamma(p, \mu, m_R, g_R)$ are twice, though generally not thrice, differentiable with respect to m_R at zero mass, *so long as we stay away from exceptional momenta.*[22] The reason for the caveat is that, for nonexceptional momenta, the momentum flowing into a given closed loop almost never vanishes, so that only one (fully dressed) internal fermion line in each loop can have zero momentum at a time. For nonexceptional momenta, therefore, the strongest singularity of Γ in m_R is given generally by

$$\Gamma \underset{m_R\to 0}{\sim} \sum_{n\ge 0} a_n \left[\int_{k^2\sim 0} d^4k(\not k - m_R)^{-1}\right]^n$$

$$\sim m_R^3 \ln m_R^2.$$

According to this, we expect the m-even functions F and G to behave as $m(\kappa)\to 0$ as

$$F(p, \mu, m(\kappa), g(\kappa)) = F(p, \mu, 0, g(\kappa)) + \tfrac{1}{2}m^2(\kappa)\left[\frac{\partial^2 F}{\partial m^2}\right]_{m=0} + O(m^4(\kappa)\ln m^2(\kappa)) \tag{3.15a}$$

and

$$G(p, \mu, m(\kappa), g(\kappa)) = G(p, \mu, 0, g(\kappa)) + O(m^2(\kappa)\ln m^2(\kappa)). \tag{3.15b}$$

The same sort of statement may be made for the kernel when it is evaluated at nonexceptional momenta p, k, q; since K is an even function of m [see Eq. (3.11)],

$$K(p,k,q;\mu,m(\kappa),g(\kappa))=K(p,k,q;\mu,0,g(\kappa))+\tfrac{1}{2}m^2(\kappa)\left[\frac{\partial^2K}{\partial m^2}\right]_{m=0}+O(m^4(\kappa)\ln m^2(\kappa))\,. \tag{3.16}$$

The crucial question is this: Is it true that, even for $q=0$,

$$K(p,k,0;\mu,m(\kappa),g(\kappa))=K(p,k,0;\mu,0,g(\kappa))+O(m^2(\kappa))\,, \tag{3.17}$$

or does the singularity in m get promoted to one of the form $\ln m^2(\kappa)\sim\ln[\kappa(\ln\kappa)^{+c/b}]$? If Eq. (3.17) is valid, then we can usefully approximate the kernel in Eq. (3.14) by setting $m=0$ and expanding in powers of $g^2(\kappa)\sim(\ln\kappa)^{-1}$. Moreover, we can then hope to solve the Bethe-Salpeter equation (3.14).

We now argue that the expansion (3.17) is correct. Formally, we see that this must be so if the solution $G(\ldots m(\kappa))$ of Eq. (3.14) is to be expandable according to Eq. (3.15b). G cannot be less singular in m than K is. A more direct argument is based on the analytic structure of the kernel.

The dangerous situation that "more than one internal fermion line in a loop has zero momentum at a time" arises when the set of momenta entering and leaving that loop is exceptional, for all possible values of any integration momenta that may be entering or leaving that loop. This means that we need consider only those loops for which the total momentum entering from the right is q (see Fig. 3, for example). In the kernel K, such loops contain some portion of both fermion lines ac and db. Then, in order that the momenta of two or more fermion lines (in the same loop) simultaneously vanish, these momenta must become equal when $q\to 0$. This cannot happen because the kernel is two-fermion irreducible in the q channel. Thus, we can expand K according to Eq. (3.17), safe in the knowledge that neglected mass terms are down by a factor of order κ^2 and cannot invalidate an expansion in powers of $g^2(\kappa)$.

IV. EXISTENCE OF GOLDSTONE SOLUTIONS AND DISCUSSION

A. Solving the integral equation for G

The problem of solving the Bethe-Salpeter equation (3.1), for $\kappa p\to\infty$, now has been reduced to finding solutions of

$$G(p,\mu,0,g(\kappa))=\int\frac{d^4k}{(2\pi)^4}\,\frac{G(k,\mu,0,g(\kappa))}{k^2F^2(k,\mu,0,g(\kappa))}\times K(p,k,0;\mu,0,g(\kappa))\,, \tag{4.1}$$

in terms of which $G(\kappa p)$ is given, asymptotically, by Eq. (3.9) as

$$G(\kappa p,\mu,m_R,g_R)\cong[g^2(\kappa)]^{c/b}[G(p,\mu,0,g(\kappa))+O(m^2(\kappa)\ln m^2(\kappa))]\,. \tag{4.2}$$

If a solution exists to Eq. (4.1), consistency with (4.2) demands that it behave as

$$G(p,\ldots,g(\kappa))\underset{\kappa\to\infty}{\sim}\left(\frac{\ln\kappa}{\ln\kappa p}\right)^{c/b}\,. \tag{4.3}$$

In fact, we shall obtain two solutions to (4.1), with the asymptotically dominant one behaving in just this way.

Taking advantage of asymptotic freedom, the analysis of Sec. III permits us to approximate the kernel by

$$K_{ac,db}(p,k,0;\mu,0,g(\kappa))=i^3g^2(\kappa)C_2((N))(\gamma^\lambda)_{ac}\left[\frac{(k-p)_\lambda(k-p)_\nu/(k-p)^2-g_{\lambda\nu}}{(k-p)^2+i\epsilon}\right](\gamma^\nu)_{db}+O(g^4(\kappa))\,, \tag{4.4}$$

or, from Eq. (3.11),

$$K(p,k,0;\mu,0,g(\kappa))=3i^3g^2(\kappa)C_2((N))\,\frac{1}{(k-p)^2}+O(g^4(\kappa))\,. \tag{4.5}$$

These expressions, valid when κ is large, are calculated in the Landau gauge, for the case of fermions assigned to n fundamental representations (N) of SU(N); the Casimir operator, $C_2((N))=(N^2-1)/2N$, for (N) was already given in Eq. (3.5b). We emphasize once again that this "improved" ladder approximation faithfully represents the complete (relevant) kernel, and not merely the asymptotic limit of the ladder graph.

Our final approximation will be to write

$$F^2(k,\mu,0,g(\kappa))=1+O(g^2(\kappa)) \tag{4.6}$$

for all values of momentum k. This is consistent with the asymptotic behavior of $F(\kappa k,g_R)$ deduced from Eqs. (3.8) and (3.5c).

The integral equation (4.1) at last becomes (letting $p/\mu\to p$ and $k/\mu\to k$)

$$G(p,g(\kappa))=-3ig^2(\kappa)C_2((N))\int\frac{d^4k}{(2\pi)^4}\,\frac{G(k,g(\kappa))}{k^2(k-p)^2}\,. \tag{4.7}$$

The solution to this equation, together with Eq. (4.2), will be a good approximation to Eq. (3.14) for $G(\kappa p, g_R)$ so long as

$$m_R/\kappa p \sim m(\kappa)/p \ll 1 . \tag{4.8}$$

Errors made by using asymptotic forms in (3.14) even when the integration momentum is small, $k \lesssim m(\kappa)$, will be of order $m^2(\kappa)/p^2$ when (4.8) is true.

Equation (4.7) has been investigated by many authors.[34] One method is to rewrite it in Euclidean form via a Wick rotation, after which it may be converted into the differential equation

$$\left[p^4\left(\frac{d}{dp^2}\right)^2 + 4p^2\frac{d}{dp^2} + 2 + \frac{3g^2(\kappa)C_2((N))}{16\pi^2}\right] \times p^{-2}G(p, g(\kappa)) = 0 . \tag{4.9}$$

The solutions of this equation are subject to the boundary conditions[35]

$$\lim_{p^2\to\infty}\left(p^2\frac{d}{dp^2}+1\right)G = 0 , \tag{4.10a}$$

$$\lim_{p^2\to 0} p^4\frac{d}{dp^2}\,G = 0 . \tag{4.10b}$$

Define

$$\nu = +\left[1 - 3g^2(\kappa)C_2((N))/4\pi^2\right]^{1/2} \cong 1 - c/b\ln\kappa , \tag{4.11}$$

where, from Eqs. (3.5), the ratio c/b is

$$\frac{c}{b} = \frac{9(N^2-1)}{2N(11N-2n)} . \tag{4.12}$$

Then, the solutions to (4.9) are

$$G_+(p, g(\kappa)) = a_+p^{-1+\nu} \underset{\kappa\to\infty}{\cong} a_+\left(\frac{\ln\kappa}{\ln\kappa p}\right)^{c/b} \tag{4.13}$$

and

$$G_-(p, g(\kappa)) = a_-p^{-1-\nu} \underset{\kappa\to\infty}{\cong} \frac{a_-}{p^2}\left(\frac{\ln\kappa p}{\ln\kappa}\right)^{c/b} . \tag{4.14}$$

The coefficients $a_\pm$ may be functions of $g(\kappa)$; whether they are zero or not is partly determined by the Bethe-Salpeter normalization condition (discussed below). Both solutions satisfy the boundary conditions (4.10) for large, but finite κ. This is to be contrasted with the more familiar case,[34] in which the boundary conditions fix the solution of (4.9) uniquely.

B. Discussion of solutions

Let us consider the solutions $G_\pm$ in turn. Combining Eqs. (4.2) and (4.13) gives

$$\begin{aligned} G_+(\kappa p, \mu, m_R, g_R) &\cong (g^2(\kappa))^{c/b}\left[a_+p^{-1+\nu} + O(m^2(\kappa)\ln m^2(\kappa))\right] \\ &\cong a_+(\ln\kappa p)^{-c/b} + O(\kappa^{-2}(\ln\kappa)^B) \end{aligned} \tag{4.15}$$

Here, B is an undetermined constant. This is the asymptotically dominant solution to Eq. (3.1), and we see that its behavior as $\kappa p\to\infty$ is just what we could have concluded immediately from the renormalization-group analysis of Sec. III. The reason for this, of course, is that the calculation of the (+) solution from the improved ladder approximation, (4.4) to (4.7), exactly corresponds to the calculation of the anomalous dimension γ_m in Eq. (3.5b).

Turning next to the (−) solution, we combine Eqs. (4.14) and (4.2) to get

$$\begin{aligned} G_-(\kappa p, \mu, m_R, g_R) &\cong (g^2(\kappa))^{c/b}\left[a_-p^{-1-\nu} + O(m^2(\kappa)\ln m^2(\kappa))\right] \\ &\cong \frac{a_-}{p^2}\left[\frac{\ln\kappa p}{(\ln\kappa)^2}\right]^{c/b} + O(\kappa^{-2}(\ln\kappa)^B) , \end{aligned} \tag{4.16}$$

where B is as in Eq. (4.15). Since this expression can be a function of κ and p only through their product, a_- must have the asymptotic form (up to a constant)

$$a_- \sim (\ln\kappa)^{2c/b}\kappa^{-2} . \tag{4.17a}$$

This can be rewritten as a function of $g(\kappa, g_R)$:

$$a_-(g(\kappa, g_R)) \sim (g^2(\kappa))^{-2c/b}\exp\left[-2/bg^2(\kappa)\right] . \tag{4.17b}$$

Since $g(\kappa, g_R)$ satisfies[13]

$$\left[\kappa\frac{\partial}{\partial\kappa} - \beta(g_R)\frac{\partial}{\partial g_R}\right]g(\kappa, g_R) = 0 , \tag{4.18}$$

the solution (4.16), with a_- given by (4.17b), is consistent with the renormalization-group equation (3.3) for $G(\kappa p, g_R)$.

Accordingly, G_- will be written as

$$G_-(p, \mu, m_R, g_R) \underset{\kappa = p/m_R\to\infty}{\sim} \kappa^{-2}(\ln\kappa)^{c/b} + O(\kappa^{-2}(\ln\kappa)^B) . \tag{4.19}$$

So far as we can tell from the Bethe-Salpeter equation alone, the error in G_- is of the same order as the solution itself.

As remarked in the Introduction, $G_\pm$ are subject to the Bethe-Salpeter normalization condition if they are to be Goldstone solutions. In the limit $q\to 0$, Eq. (1.6) becomes[36]

$$\begin{aligned} 2i\frac{F_\pi Z_2}{m_R Z_A}q_\mu = {} & i^2\int d^4p\,\mathrm{Tr}\left\{S(p)G(p)\gamma_5S(p)\left[\frac{\partial}{\partial q^\mu}S^{-1}(p+q)S^{-1}(p)\right]S(p)G(p)\gamma_5S(p)\right\} \\ & + \int d^4p\,d^4k\,\mathrm{Tr}\left\{S(k)G(k)\gamma_5S(k)\left[\frac{\partial}{\partial q^\mu}K(p,k,q)\right]S(p)G(p)\gamma_5S(p)\right\} . \end{aligned} \tag{4.20}$$

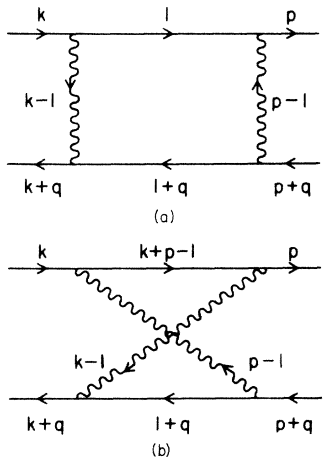

FIG. 3. Examples of fermion-antifermion scattering loop graphs in which the total momentum entering the loop is q. l is an integration momentum. Graphs, such as (a), whose singularity in m becomes stronger in the limit $q\to 0$, do not contribute to K. (b) Does contribute to K, and is not troublesome when $q\to 0$.

For the (+) solution, these integrals converge only if G_+^2 falls off faster $(\ln p)^{-1}$, that is, only if

$$\frac{c}{b}=\frac{9(N^2-1)}{2N(11N-2n)}>\tfrac{1}{2}\ \ [(+)\text{ solution}]\ . \tag{4.21}$$

We forego discussion of this restriction for reasons that will soon be apparent. There is no such restriction on the (−) solution, Eq. (4.19): The normalization integrals will converge so long as G_- is less singular than p^{-2} as $p\to 0$.

Now we must decide which solution, (4.15) or (4.19), is the correct one, i.e., which actually represents chiral symmetry realized in the Goldstone mode. As noted, the asymptotic behavior $G_+(p)\sim(\ln p)^{-c/b}$ exactly corresponds to what we learn from a straightforward renormalization-group analysis, and does not differ from the case in which quarks have nonzero bare mass and $G(p)$ does not have purely dynamical origin. This conflicts with the intuition that functions such as G ought to have softer asymptotic behavior when the symmetry breaking is dynamical than when it is not. On the basis of this intuition, we expect that G_- is the correct solution, and we now show that this is so.

What is missing from the simple analysis of G in Sec. III is the assumption that G arises solely from dynamics and is proportional, through the Ward identity, to the bound-state vertex function $\mathcal{P}(p,p+q)$ (at $q=0$). Writing

$$iS(p)\mathcal{P}(p,p+q)\gamma_5 S(p+q)\delta_{\alpha\beta} = \frac{Z_A}{Z_2}\int d^4 e^{ip\cdot x}\langle\pi_\alpha(q)|T[\mathrm{tr}(\psi(x)\bar\psi(0)\lambda_\beta)]|0\rangle\,, \tag{4.22}$$

we may learn the asymptotic behavior of $\mathcal{P}$, hence of G, by taking the limit $|p^2|\to\infty$, $(p+q)^2/p^2\to 1$. The Wilson operator-product expansion[12] is appropriate in this limit, and one readily deduces

$$iS(p)\mathcal{P}(p,p+q)\gamma_5 S(p+q)\delta_{\alpha\beta} \sim U(p,\mu,m_R,g_R)\langle\pi_\alpha(q)|\bar\psi(0)\gamma_5\tfrac{1}{2}\lambda_\beta\psi(0)|0\rangle\,. \tag{4.23}$$

That the Goldstone boson π_α is a bound state is accounted for by the fact that the dominant (renormalized) operator, $\bar\psi\gamma_5\lambda_\beta\psi$, has canonical dimension three, and not one. Nonleading terms in (4.23) correspond to operators with canonical dimension four or more, and are expected to be down by approximately a factor of p relative to the exhibited leading term.

Standard renormalization-group analysis (see, e.g., Ref. 22) of the Wilson coefficient function U gives

$$U(\kappa p,\mu,m_R,g_R)=\kappa^{-4}\exp\left\{\int_{g_R}^{g(\kappa)}\frac{dx}{\beta(x)}[\gamma_{\bar\psi\gamma_5\lambda_\beta\psi}(x)+\gamma(x)]\right\} \times U(p,\mu,m(\kappa),g(\kappa))\,, \tag{4.24a}$$

while, from Sec. III, we know that

$$S^{-1}(\kappa p,\mu,m_R,g_R)=\kappa\exp\left[-\int_{g_R}^{g(\kappa)}\frac{dx\,\gamma(x)}{\beta(x)}\right] \times S^{-1}(p,\mu,m(\kappa),g(\kappa))\,. \tag{4.24b}$$

Finally, the facts [deduced from Eq. (2.17)] that $m_0(\Lambda)Z_A/Z_D=m_R(Z_m Z_A/Z_D)$ and Z_A/Z_2 are cutoff-independent imply that

$$\gamma_{\bar\psi\gamma_5\lambda_\beta\psi}\equiv\mu\frac{\partial}{\partial\mu}\ln Z_D=\gamma+\gamma_m\,. \tag{4.25}$$

It follows that

$$\begin{aligned}G(p,\mu,m_R,g_R)&\propto\lim_{q\to 0}\mathcal{P}(p,p+q)\\ &\underset{\kappa=p/m_R\to\infty}{\sim}\kappa^{-2}\exp\left[\int_{g_R}^{g(\kappa)}\frac{dx\,\gamma_m(x)}{\beta(x)}\right]\\ &\approx\kappa^{-2}(\ln\kappa)^{c/b}\,,\end{aligned} \tag{4.26}$$

which is precisely the G_- solution. Moreover, we now know that the error in this solution is $O(\kappa^{-3}(\ln\kappa)^B)$.

To sum up: We have shown that asymptotically free theories of quark-gluon interactions meet certain conditions necessary for the realization of chiral symmetry in the Nambu–Jona-Lasinio–Goldstone mode. Namely, the Bethe-Salpeter equation for the symmetry-breaking part, $G(p)$, of the quark propagator has a nontrivial, normalizable solution in the limit $p\to\infty$. The renormalization group was particularly useful for this analysis, permitting us to establish the ladder approximation for the Bethe-Salpeter kernel (in the large-p limit), and to decide on the appropriate solution to the integral equation through the joint use of the Ward identity and the Wilson expansion for the vertex function $\mathcal{P}$.

We emphasize again that our results are not sufficient to demonstrate the existence of Goldstone solutions. This question can only be settled by analysis of the low-momentum region of the Bethe-Salpeter equation. The reason, of course, is that bound-state formation and dynamical symmetry breaking are essentially low-momenta phenomena.

Finally, we would like to draw attention to the intriguing fact that, when written as

$$G(p,\mu,m_R,g_R)\underset{\kappa=p/m_R\to\infty}{\sim}(g^2(\kappa))^{-c/b}\exp[-2/bg^2(\kappa)]\,, \tag{4.27}$$

G has an essential singularity in the effective coupling. We have recently shown that any dynamically generated mass also has such an essential singularity.[37] In particular, if a mass m arises dynamically in an asymptotically free theory, $m\propto\exp(-1/bg_R^2)$. This nonanalytic behavior in the coupling is just a statement of the impossibility of perturbative calculation of purely dynamical quantities. The energy gap in a superconductor has similar nonanalytic behavior in the electron-electron coupling,[2] thereby strengthening the Nambu–Jona-Lasinio proposal that the near masslessness of the pion and superconductivity are intimately related phenomena.

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