Lagrangian and Hamiltonian formulation of relativistic particle mechanics

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It is shown that any (relativistic) action-at-a-distance theory formulated in terms of a (relativistically invariant) Fokker action, which is invariant under any change of the world-line parameter, also can be given a Lagrangian formulation. The parametrization invariance makes it possible to introduce a common parameter which also can be identified with time. The resulting Lagrangians are nonlocal in time and therefore the applicability of the action principle requires a generalized variation method which is developed here. The resulting generators are expected to generate the corresponding variations in a Poisson-bracket sense, i.e., a Hamiltonian formulation of the theory is expected to exist. However, in order to check this it is necessary to (formally) solve the equations of motion, as one needs Poisson brackets for unequal time due to the nonlocality of the generators. But since the equations of motion themselves are nonlocal in time, the Newtonian initial data will not yield a unique solution. Thus, in order to retain the Newtonian degrees of freedom one needs a selection principle as a subsidiary condition. Such a principle has been proposed which states that one has to choose the solution which has a nonrelativistic limit. In the case of a general vector interaction the class of solutions which is obtained by an iterative method (which starts from straight lines) is considered. These solutions are uniquely determined by the Newtonian initial data, but they are applicable only to scattering processes. By choosing the asymptotic straight lines as canonical variables it is explicitly shown that the Poincaré generators fulfill the Lie algebra of the Poincaré group and that the physical positions transform as Lorentz vectors. It is also shown that the physical positions cannot be chosen as canonical variables. Quantization is discussed and a general argument is put forth which states that the above solutions cannot be quantized by imposing the canonical commutation relations on the canonical variables.

I. INTRODUCTION

Newton's theory of gravity (1686) is the prototype of an action-at-a-distance theory. In this theory the physical masses are acting and reacting with equal amount and at the same instant of time even over large distances.[1] This is the basic picture of nonrelativistic particle mechanics, which also constitutes our most consistent physical theory up to date. However, with the introduction of relativity, simultaneity is no longer an invariant concept. It is therefore generally believed that relativity requires a field theory, i.e., a field-mediated interaction, and that no relativistic mechanics of the action-at-a-distance type is possible. However, this belief is badly founded as one has already shown that action at a distance is possible in relativistic theories.

Classical electrodynamics of point charges as given by Maxwell and Lorentz is the prototype of a theory with field-mediated interaction; the interaction between the charges is there mediated by the electromagnetic field. Many efforts to put this theory on a Newtonian form failed in the beginning. However, Gauss seems to have been thinking in the right direction at a very early stage according to a letter to Weber in 1845.[2] The keyword is namely noninstantaneous action-at-a-distance instead of the instantaneous one as in Newton's theory of gravity. By noninstantaneous action is meant that the action between the charges should propagate with a finite velocity. However, it took very long before action-at-a-distance electrodynamics became a reality. But as soon as Liénard and Wiechert[3] had given the expression for the radiation from an accelerated charge, it became possible to describe the interaction between two charges with a direct interaction by the elimination of the electromagnetic field.[4] Fokker[5] showed that such an interaction could be derived from an action principle if the advanced as well as the retarded solutions of Maxwell's equations were included. The theory was then finally completed by Wheeler and Feynman[6] with their absorber theory of radiation.

In contrast to the electrodynamics of point charges according to Maxwell and Lorentz, the Wheeler-Feynman theory is quite consistent. The former is plagued with the infinite self-energies, while the latter is finite with no self-energies at all. Dirac's[7] consistent but *ad hoc* procedure to eliminate the infinite part of the self-fields in the Maxwell-Lorentz theory becomes exact in the Wheeler-Feynman theory.[6] In fact if one tries to make the Maxwell-Lorentz theory consistent (finite), one seems to be forced to introduce a direct interaction between the charges in some way or another.[8] The new features which the Wheeler-Feynman theory brings in are mainly philosophical in nature. But these new concepts are indeed revolutionary:

(i) There does not exist any electromagnetic field as an independent entity with degrees of freedom of its own.

(ii) A charge does not act upon itself (and consequently has no infinite self-energy).

(iii) Radiation can only be discussed in presence of an absorber; radiation occurs when the absorber is accelerated by the source, and the radiation reaction occurs when in turn the source is accelerated by the absorber due to the equality of action and reaction.

(iv) The concept of causality of conventional classical theories is violated as the motion of a charge is determined by the past and *future* behavior of the other charges (by the timesymmetry of the theory). However, this noncausality is limited in its influence to a very short time interval of the order of e^2/mc^3, and it manifests itself in, e.g., the preacceleration already noticed by Dirac[6,7] (and much earlier in the nonrelativistic case of radiative reaction[9]). In fact this noncausality is not experimentally detectable as its appearance is exactly on the level where one has to take quantum phenomena[10] into account. This feature might therefore be considered in favor of the theory as the latter already on a prequantized stage points to a breakdown of classical concepts for short distances.

Though the Wheeler-Feynman theory seems quite satisfactory from a logical point of view, it is not particularly useful in practical examples. This is because of their condition of complete absorption which demands us to insert an absorber or to include all charges of the universe (besides that, we have to assume that there is no radiation at the boundary of the universe or, stated within the action-at-a-distance language, we have to assume that there is no interaction at infinity in the Euclidean space). Hence for any few-body problem for which the action-at-a-distance electrodynamics could be useful, the Wheeler-Feynman absorption condition is not fulfilled. The interpretation[6] in terms of the Maxwell-Lorentz-Dirac theory for such a system is that apart from the particles in the system we also have an external free field which does not affect the system as a whole but changes the forces on the individual particles. Since there seems to be some physics in the few-body problems (see, e.g., the bound-state solutions below) one could perhaps take the following physical standpoint: Elementary systems should be described by action-at-a-distance electrodynamics and the Dirac theory is only statistically (approximately?) satisfied for macroscopic systems (i.e., for many-body problems).

Now the appearance of the action-at-a-distance electrodynamics made it possible to construct other examples of relativistic particle theories. In fact any field-mediated interaction can be cast into an action-at-a-distance form[11,12] irrespective of the spin and mass of the field involved. Such an action-at-a-distance theory is said to have an analogous (but not equivalent) field theory. One may go even further and construct theories[13,14] with no field-theoretic analog at all. One may even thereby introduce a finite self-interaction[15] (but there are problems with mass renormalization[16]). Hence, there exists a great variety of possible relativistic mechanical theories of interacting point particles. They are all characterized by the following properties: (a) They are derivable from a Fokker action. (b) This Fokker action is parametrization-invariant, i.e., the equations of motion do not depend on which parameter one chooses to describe the world line of a particle.

One of their attractive features is that they allow for the description of bound states. The two-body problem is, however, not reducible to a one-body problem as in the nonrelativistic case; therefore one has difficulties in solving the equations of motion and consequently not much more than circular solutions[17-20] have been found so far. The same is true for the three-body problem.[21] Anyway, the results obtained are in agreement with similar results from, e.g., the Bethe-Salpeter equation.[18]

On the classical level the above models seem to be quite satisfactory. Thus, if one also could quantize them in a satisfactory way, they would certainly play an important part in modern physics. They contain in fact several good properties to be of interest even in the case of strong interaction.[12] However, this quantization problem has not been solved so far.[22] However, encouraging examples exist where one has performed a Bohr quantization of the circular solutions found in the two-body problem.[17,18,23] The energy levels are found to agree, e.g., with the corresponding ones obtained from the conventional treatment of a Dirac field in an external potential.[24] This opens, therefore, the possibility of treating, e.g., the hydrogen atom as a true two-body problem. What remains, however, is the development of a general quantization scheme. A natural requirement of such a scheme is that it must coincide with the nonrelativistic quantization scheme in the nonrelativistic limit. The easiest way to satisfy this requirement would be to construct a Lagrangian and Hamiltonian formulation of the classical theory and to perform a canonical-like quantization of this theory. So far one has, however, not been able to put the classical theory on a canonical basis. The main object of the present paper is, therefore, as a first step towards a satisfactory quantization, to show that the classical theory in fact can be put on a Lagrangian and Hamiltonian basis. This will be done in an explicit manner.

The Lagrangian and, in particular, the Hamiltonian formulation of relativistic particle mechanics has been treated extensively in the past. But one has always considered strict instantaneous action-at-a-distance and not a noninstantaneous one of the above type. In the limit of no interaction they coincide, of course, and for free relativistic particles there are essentially two possible formulations; the covariant and the noncovariant one.[16] However, one encounters severe difficulties when one wants to introduce interaction in the instantaneous case. In the beginning[25] one was satisfied with a canonical representation of the inhomogeneous Lorentz group without making sure that the positions transformed correctly under this group. Relativistic invariance of the theory was therefore not at all ensured as such a representation can be constructed in an obviously nonrelativistic theory.[26] In fact, a theory which allows a canonical representation of the inhomogeneous Lorentz group and in which the physical positions transform as the space part of a four-vector cannot contain any interaction according to the remarkable no-go theorem of Currie, Jordan, and Sudarshan.[27] However, one has realized that one can avoid this theorem if one chooses a canonical coordinate which is not the physical position.[28-30] That this is possible has also been shown before.[31] Hence, there is still some hope within the instantaneous scheme. Covariance conditions[32] have also been given independently by Currie[33] and Hill,[34] and various examples which fulfill these conditions have been constructed.[35] However, no physically completely satisfactory example has been given so far,[36] and in this paper we shall not consider this possibility.

II. THE LAGRANGIAN AND HAMILTONIAN FORMULATION OF FREE RELATIVISTIC PARTICLE MECHANICS

In this section we shall consider the Lagrangian and Hamiltonian formalism of N free relativistic particles. Although no new results will be presented, we shall exhibit this case carefully here because when we later introduce interaction between the particles, the Lagrangian formalism will be developed along the same lines and the Hamiltonian case will, in the present paper, even turn out to be based upon the free Hamiltonian formalism. What we in particular want to emphasize is the importance of the parameter invariance of the action integral: how it reduces the degrees of freedom, and how it allows for transitions between multitime and single-time formulations, and between covariant and noncovariant formulations.

For the description of N free relativistic particles one usually starts from the action

$$W_0 = -\sum_{a=1}^{N} m_a c^2 \int d\tau_a \ , \tag{2.1}$$

where τ_a is the proper time of particle a. The action is thus proportional to the sum of the lengths of the world lines of the particles multiplied by their respective masses. Notice that

$$\begin{aligned} c\,d\tau_a &= (dx_a^\mu dx_{a\mu})^{1/2} \\ &= \left(\frac{dx_a^\mu}{dt_a}\frac{dx_{a\mu}}{dt_a}\right)^{1/2} dt_a \ , \end{aligned}$$

where t_a is another parameter along the world line. The action (2.1) may thus also be written as

$$W_0 = -\sum_{a=1}^{N} m_a c \int dt_a [\dot{x}_a^\mu(t_a)\dot{x}_{a\mu}(t_a)]^{1/2} \ . \tag{2.2}$$

The principle of least action states now that the particles follow the shortest paths. We get in fact the equations of motion

$$\begin{aligned} m_a c \frac{d}{dt_a}\left(\frac{\dot{x}_a^\mu(t_a)}{[\dot{x}_a^\mu(t_a)\dot{x}_{a\mu}(t_a)]^{1/2}}\right) &= m_a \ddot{x}_a^\mu(\tau_a) \\ &= 0, \\ a = 1, \dots, N\ , \quad \mu &= 0, 1, 2, 3\ . \end{aligned} \tag{2.3}$$

The action (2.2) is constructed in such a way that $\dot{x}_a^\mu(\tau_a)\dot{x}_{a\mu}(\tau_a) = c^2$ will be satisfied identically.

An important property of the action (2.2) is that it is invariant under any change of the world line parameters t_a as long as there is a one-to-one correspondence between t_a and τ_a. This allows us in fact to introduce a common parameter t which is connected to the proper times by monotonic (increasing) functions f_a, i.e., $t = f_a(t_a)$ for $a = 1, \dots, N$. By use of this common parameter we may define the following Lagrangian:

$$L_0(t) \equiv -\sum_{a=1}^{N} m_a c [\dot{x}_a^\mu(t)\dot{x}_{a\mu}(t)]^{1/2} \ . \tag{2.4}$$

The action (2.2) may now be written as

$$W_0 = \int dt\, L_0(t) \ . \tag{2.5}$$

[The principle of least action yields (2.3) with the particular choices $t_a = t$, $a = 1, \dots, N$.]

Now the action (2.5) is furthermore independent of the choice of parameter t, which also can be said to be due to the fact that L_0 is homogeneous in $\dot{x}_a^\mu$ of first degree, i.e.,

$$\frac{\partial L_0(t)}{\partial \dot{x}_a^\mu(t)} \dot{x}_a^\mu(t) = L_0(t), \quad a = 1, \dots, N\ . \tag{2.6}$$

Owing to this homogeneity property, only $3N$ of the $4N$ equations of motion are independent. This may be seen as follows:

$$\dot{x}_a^\mu\left(\frac{\partial L_0}{\partial x_a^\mu}-\frac{d}{dt}\frac{\partial L_0}{\partial \dot{x}_a^\mu}\right)=\frac{dL_0}{dt}-\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{x}_a^\mu}\dot{x}_a^\mu\right)\equiv 0\ .$$

In this paper we shall always choose the equations of motion of the space coordinates to be the independent ones.

In order to be able to derive conserved quantities, we introduce the finite action

$$W_{21}=\int_{t_1}^{t_2}dt\,L_0(t) \tag{2.7}$$

and the action principle

$$\delta W_{21}=F(t_1)-F(t_2)\ , \tag{2.8}$$

where $F(t)$ is the generator of the infinitesimal variation δ.

A functional variation δ_0 of W_{21} yields

$$\delta_0 W_{21}=\int_{t_1}^{t_2}dt\sum_{a=1}^{N}m_ac\frac{d}{dt}\left(\frac{\dot{x}_a^\mu}{(\dot{x}_{a\mu}\dot{x}_a^\mu)^{1/2}}\right)\delta_0x_{a\mu}-\int_{t_1}^{t_2}dt\frac{d}{dt}\left(\sum_{a=1}^{N}m_ac\frac{\dot{x}_a^\mu\delta_0x_{a\mu}}{(\dot{x}_a^\mu\dot{x}_{a\mu})^{1/2}}\right)\ . \tag{2.9}$$

The action principle (2.7) gives now the equations of motion (2.3) and the infinitesimal generator

$$F(t)=\sum_{a=1}^{N}m_ac\frac{\dot{x}_a^\mu\delta_0x_{a\mu}}{(\dot{x}_a^\mu\dot{x}_{a\mu})^{1/2}}\ . \tag{2.10}$$

If one also varies the parameter t ($t\to t+\delta t$), then one gets an additional term in (2.8) due to the end-point variation:

$$\int_{t_1}^{t_2}\delta(dt)L_0(t)=\int_{t_1}^{t_2}dt\frac{d}{dt}\left[\delta t L_0(t)\right]\ . \tag{2.11}$$

The action principle (2.7) yields thus the following generator for general variations:

$$F(t)=\sum_{a=1}^{N}\frac{m_ac}{(\dot{x}_a^\mu\dot{x}_{a\mu})^{1/2}}(\dot{x}_a^\nu\delta_0x_{a\nu}+\dot{x}_a^\nu\dot{x}_{a\nu}\delta t)\ . \tag{2.12}$$

If W_0 is invariant under an infinitesimal transformation ($x_a^\mu\to x_a^\mu+\delta_0x_a^\mu$, $t\to t+\delta t$), then we have that $F(t)$ is conserved, i.e., $dF(t)/dt=0$ (Noether's theorem[37]).

An infinitesimal parameter transformation $t\to t+\lambda(t)$ (λ an infinitesimal function) induces a functional variation $\delta_0x_a^\mu=-\lambda\dot{x}_a^\mu$ of the space-time coordinates of the particles. [Let $x_a'^\mu$ be the world line with respect to the new parameter $t+\lambda$, i.e., $x_a'^\mu(t+\lambda)=x_a^\mu(t)$. This implies that $x_a'^\mu(t)=x_a^\mu(t-\lambda)=x_a^\mu(t)-\lambda\dot{x}_a^\mu(t)$ infinitesimally. Thus in order to leave the world lines of the particles intact one has to add to these the functional variation $\delta_0x_a^\mu=-\lambda x_a^\mu$). Thus, $t\to t+\lambda(t)$ yields the infinitesimal generator

$$F(t)\equiv 0\ . \tag{2.13}$$

This is again a consequence of the (local) parameter invariance due to the homogeneity property (2.6). The property (2.13) will later allow us to identify t with time.

An infinitesimal space-time translation $x_a^\mu\to x_a^\mu+\epsilon^\mu$, $a=1,\ldots,N$, yields the generator $F(t)=\epsilon_\mu P^\mu(t)$ where P^μ is the energy-momentum vector, which by (2.12) is given by

$$P^\mu(t)=\sum_{a=1}^{N}\frac{m_ac}{(\dot{x}_a^\nu\dot{x}_{a\nu})^{1/2}}\dot{x}_a^\mu(t)\ , \tag{2.14}$$

which is conserved [cf. (2.3)].

An infinitesimal homogeneous Lorentz transformation $x_a^\mu\to x_a^\mu+\epsilon^{\mu\nu}x_{a\nu}$, $\epsilon^{\mu\nu}=-\epsilon^{\nu\mu}$, $a=1,\ldots,N$, yields the generator $F(t)=-\frac{1}{2}\epsilon_{\mu\nu}J^{\mu\nu}$, where $J^{\mu\nu}(t)$ is the angular momentum tensor, which by (2.12) is given by the expression

$$J^{\mu\nu}(t)=\sum_{a=1}^{N}\frac{m_ac}{(\dot{x}_a^\mu\dot{x}_{a\mu})^{1/2}}(\dot{x}_a^\nu x_a^\mu-\dot{x}_a^\mu x_a^\nu)=-J^{\nu\mu}(t)\ , \tag{2.15}$$

which is conserved.

A covariant formulation with (2.14) and (2.15) is in principle possible, but there is no natural invariant parameter t at our disposal. If the particles are parametrized by means of their proper times, (2.14) and (2.15) are turned into multitime quantities. However, a natural noncovariant formulation is possible if we identify t with time (the zero components of the particles' space-time coordinates divided by c), i.e.,

$$ct=x_1^0(t)=x_2^0(t)=\cdots=x_N^0(t)\ . \tag{2.16}$$

That this identification is possible can be seen as follows: As we are free to parametrize a particle's world line by any parameter which can be expressed by a monotonic increasing function of the particle's proper time τ_a, we are also allowed to take the particular choice $ct_a=n_\mu x_a^\mu(\tau_a)$, where n^μ is a fixed timelike unit vector pointing in the positive time direction. In terms of this parameter the particle's position fulfill $ct_a=n_\mu x_a^\mu(t_a)$, and if the same choice of parameter is made for all particles (which is a further restriction) we have in particular $ct=n_\mu x_a^\mu(t)$, $a=1,\ldots,N$. This relation is just (2.16) for the particle choice $n^\mu=(1,0,0,0)$, the only case we will consider in this paper. Equation (2.13) tells us that the identification (2.16) does not affect the form of the generators.

The Lagrangian (2.1) now becomes

$$L_0(t)=-\sum_{a=1}^{N}\frac{m_ac^2}{\gamma_a(t)}\ , \tag{2.17}$$

where

$$\gamma_a(t) \equiv \left(1 - \frac{\dot{x}_a^i(t)\dot{x}_a^i(t)}{c^2}\right)^{-1/2} \tag{2.18}$$

(repeated latin indices are summed over from 1 to 3). The Hamiltonian is given by

$$H(t) = cP^0(t) = \sum_{a=1}^{N} m_a c^2 \gamma_a(t) , \tag{2.19}$$

and the total linear momentum by

$$P^i(t) = \sum_{a=1}^{N} m_a \gamma_a(t) \dot{x}_a^i(t), \quad i = 1, 2, 3 . \tag{2.20}$$

The total angular momentum is then given by

$$J^{ij}(t) = \sum_{a=1}^{N} m_a \gamma_a [\dot{x}_a^j x_a^i(t) - \dot{x}_a^i x_a^j(t)] , \tag{2.21}$$

and the generators of Lorentz transformations are

$$K^i(t) = \frac{J^{i0}(t)}{c} = \sum_{a=1}^{N} m_a \gamma_a (x_a^i - \dot{x}_a^i t) . \tag{2.22}$$

Turning to the Hamiltonian formulation, one has to introduce generalized momenta defined by

$$p_a^i \equiv \frac{\partial L_0}{\partial \dot{x}_a^i} = m_a \gamma_a \dot{x}_a^i , \tag{2.23}$$

which may be solved for $\dot{x}_a^i$:

$$\dot{x}_a^i = \frac{p_a^i c}{(m_a^2 c^2 + p_a^k p_a^k)^{1/2}} . \tag{2.24}$$

H may now be written in terms of p_a^i:

$$H = \sum_{a=1}^{N} c(m_a^2 c^2 + p_a^i p_a^i)^{1/2} . \tag{2.25}$$

Notice also that $H = \sum_{a=1}^{N} p_a^i \dot{x}_a^i - L_0$, which by (2.23) tells us that we have the usual Legendre transformation.

Now all dynamical functions can be expressed in terms of the canonical variables x_a^i and p_a^i (and possibly an explicit dependence on time). We have, e.g.,

$$P^i(t) = \sum_{a=1}^{N} p_a^i, \quad i = 1, 2, 3 \tag{2.26}$$

and

$$J^{ij}(t) = \sum_{a=1}^{N} (p_a^j x_a^i - p_a^i x_a^j) = -J^{ji}(t) , \tag{2.27}$$

which also may be written as an axial vector (ϵ^{ijk} = the antisymmetric three-tensor):

$$J^i(t) = \tfrac{1}{2}\epsilon^{ijk} J^{jk} = \sum_{a=1}^{N} \epsilon^{ijk} x_a^j p_a^k, \quad i = 1, 2, 3 \tag{2.28}$$

and

$$K^i(t) = \sum_{a=1}^{N} \left(\frac{1}{c} (m_a^2 c^2 + p_a^k p_a^k)^{1/2} x_a^i - p_a^i t \right) , \quad i = 1, 2, 3 . \tag{2.29}$$

Introducing the Poisson bracket between any two dynamical functions A and B,

$$[A, B] \equiv \sum_{a=1}^{N} \left(\frac{\partial A}{\partial x_a^i(t)} \frac{\partial B}{\partial p_a^i(t)} - \frac{\partial A}{\partial p_a^i(t)} \frac{\partial B}{\partial x_a^i(t)} \right) , \tag{2.30}$$

we get the following important relations:

$$[x_a^i, P^j] = \delta^{ij} , \tag{2.31a}$$

$$[x_a^i, J^j] = \epsilon^{ijk} x_a^k , \tag{2.31b}$$

$$[x_a^i, H] = \dot{x}_a^i , \tag{2.31c}$$

$$[x_a^i, K^j] = \frac{1}{c^2} \dot{x}_a^i x_a^j - \delta^{ij} t , \tag{2.31d}$$

and

$$[P^i, H] = 0 , \tag{2.32a}$$

$$[P^i, P^j] = 0 , \tag{2.32b}$$

$$[J^i, H] = 0 , \tag{2.32c}$$

$$[K^i, H] = P^i , \tag{2.32d}$$

$$[J^i, J^j] = \epsilon^{ijk} J^k , \tag{2.32e}$$

$$[P^i, J^j] = \epsilon^{ijk} P^k , \tag{2.32f}$$

$$[J^i, K^j] = \epsilon^{ijk} K^k , \tag{2.32g}$$

$$[K^i, K^j] = -\frac{1}{c^2} \epsilon^{ijk} J^k , \tag{2.32h}$$

$$[K^i, P^j] = \frac{1}{c^2} \delta^{ij} H , \tag{2.32i}$$

where (2.31) tells us that the physical positions are transformed as Lorentz vectors under the inhomogeneous Lorentz group, and where (2.32) is just the Lie algebra of this group.

Thus, starting from the relativistically invariant action (2.1) we have derived a Lagrangian and Hamiltonian formalism in complete analogy with classical nonrelativistic analytic dynamics. How close the presented formulation is with the corresponding nonrelativistic formulation is seen by taking the nonrelativistic limit. One finds

$$[K^i(t), K^j(t)] = 0 , \quad [K^i(t), P^j(t)] = \delta^{ij} \sum_{a=1}^{N} m_a , \tag{2.33}$$

$$[x_a^i, K^j] = -\delta^{ij} t , \tag{2.34}$$

where (2.33) together with (2.32a)–(2.32g) constitutes the extended (by the neutral element

$\sum_{a=1}^{N} m_a$) Galilean group, and where (2.34) together with (2.31a)–(2.31c) tells us that the positions of the particles now are transformed correctly under the Galilean group.

III. THE LAGRANGIAN FORMALISM OF INTERACTING RELATIVISTIC PARTICLES

In this section we shall introduce interaction between the N relativistic particles which were considered to be free in Sec. II. We shall then only consider interaction of the noninstantaneous action-at-a-distance type. All field-mediated interactions are excluded.

We shall in fact only consider the following class of relativistically invariant action integrals[38] (for the construction of more general relativistically invariant action integrals, see Ref. 39):

$$W^r = W_0 + W_I^r \ , \tag{3.1}$$

where W_0 is the free action (2.5) and W_I^r is given by

$$W_I^r = -\tfrac{1}{2}\sum_{a,b=1}^{N} g_a g_b \int\int dt''dt' w_{ab}^r(t'', t') \ , \tag{3.2}$$

where g_a, $a=1,\ldots,N$, are real constants (coupling constants) and

$$\begin{aligned} w_{ab}^r(t'', t') = G_{ab}[\dot{x}_a^{\mu}(t'')\dot{x}_{b\mu}(t')]^r \\ \times [\dot{x}_a^{\nu}(t'')\dot{x}_{a\nu}(t'')]^{(1-r)/2} \\ \times [\dot{x}_b^{\rho}(t')\dot{x}_{b\rho}(t')]^{(1-r)/2} \ , \end{aligned} \tag{3.3}$$

where in turn G_{ab} is a function of

$$[x_a^{\mu}(t'') - x_b^{\mu}(t')][x_{a\mu}(t'') - x_{b\mu}(t')]$$

and which is furthermore assumed to be sufficiently nice to justify the following manipulations. r is a number whose value characterizes the interaction: $r=0$ yields scalar interaction (gravitational theories), $r=1$ yields vector interaction (most important; contains, e.g., the Wheeler-Feynman theory), and $r\geq 2$ yields tensor interactions of different orders. ($r=2$ contains a homogeneous version of Whitehead's theory of gravitation.[40,41])

We note that w_{ab}^r is homogeneous in $\dot{x}_a^{\mu}(t'')$ and $\dot{x}_b^{\mu}(t')$ of the first degree, i.e.,

$$\begin{aligned} \frac{\partial w_{ab}^r}{\partial \dot{x}_a^{\mu}}\dot{x}_a^{\mu} = \frac{\partial w_{ab}^r}{\partial \dot{x}_b^{\mu}}\dot{x}_b^{\mu} \\ = w_{ab}^r, \quad a, b = 1,\ldots,N \ . \end{aligned} \tag{3.4}$$

This implies that the action (3.2) is parameter-invariant, i.e., (3.2) is invariant under the replacement $t' \to f(t')$, $t'' \to g(t'')$ for monotonic increasing functions f and g. [For more general homogeneous interactions interactions than (3.3), see, e.g., Ref. 12.]

The condition (3.4) is usually imposed to ensure $\dot{x}_a^{\mu}\dot{x}_{a\mu} \equiv c^2$ (Ref. 12) when t is equal to the proper time of particle a. It is also a sufficient condition for a satisfactory nonrelativistic limit.[20] Here it is imposed as a necessary condition for retaining the noncovariant formulation of the theory.

The most general Lagrangian whose integral is the action (3.2) is given by the following expression:

$$L_{I,\alpha}^r(t) = -\tfrac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N} g_a g_b \int d\xi\, w_{ab}^r(t-\alpha\xi,\, t+\xi-\alpha\xi) \ , \tag{3.5}$$

where we have introduced an overall parameter t by $t=\alpha t' + \beta t''$ and $\alpha+\beta=1$, where α and β are real parameters. This parameter choice is motivated by the requirement that $t' \to t'+a$ and $t'' \to t''+a$ shall imply $t \to t+a$, where a is a constant. Thus, $t=\alpha(t'-t'')+t''$, which, by a change of variables $\xi = t'-t''$, becomes $t=\alpha\xi+t''$ or $t'=t+\xi-\alpha\xi$ and $t'=t-\alpha\xi$. α is an arbitrary real parameter. In fact an integration of (3.5) yields the action (3.2) for any value of α [which may even have different values in every term of (3.5)]. However, we have no reason to choose a particular value of α, i.e., we are not able to define a unique Lagrangian, although the equivalent choices $\alpha=0$ and $\alpha=1$ may in a way be said to be fundamental as $L_{I,0}^r(t) = L_{I,1}^r(t)$ is partly local in time. That the Lagrangians (3.5) contain integrals is just an expression of the fact that the action (3.2) only contains *noninstantaneous* action-at-a-distance theories.

The next step is to apply the action principle (2.8) to the total Lagrangian

$$L_{\alpha}^r(t) = L_0(t) + L_{I,\alpha}^r(t) \ , \tag{3.6}$$

where L_0 is given by (2.1). But since $L_{I,\alpha}^r$ is nonlocal in t, this is not a straightforward matter as the conventional variation method is no longer applicable. Therefore a new variation technique has to be developed and this is done in Appendix A, where general formulas also are derived.

The equations of motion may be derived by use of the formulas (A16) and (A23) or, alternatively, by putting the functional derivative of the action $W=\int dt\, L_{\alpha}^r(t)$ equal to zero. One finds (with $c=1$ here and in what follows)

$$\begin{aligned} m_a \frac{d}{dt}\left(\frac{\dot{x}_a^{\mu}(t)}{(\dot{x}_a^{\nu}\dot{x}_{a\nu})^{1/2}}\right) = g_a \sum_{b=1}^{N} g_b \int d\xi \Bigg\{ -\frac{d}{dt}\left[G_{ab}(\dot{x}_a^{\nu}\dot{x}_{b\nu})^r(\dot{x}_a^{\,2})^{(1-r)/2}(\dot{x}_b^{\,2})^{(1-r)/2}\left(r\frac{\dot{x}_b^{\mu}}{(\dot{x}_a^{\rho}\dot{x}_{b\rho})} + (1-r)\frac{\dot{x}_a^{\mu}}{(\dot{x}_a^{\,2})}\right)\right] \\ + \frac{\partial G_{ab}}{\partial x_{a\mu}}(\dot{x}_a^{\nu}\dot{x}_{b\nu})^r(\dot{x}_a^{\,2})^{(1-r)/2}(\dot{x}_b^{\,2})^{(1-r)/2}\Bigg\} \ , \end{aligned} \tag{3.7}$$

where x_a^μ and x_b^μ have the arguments t and $t+\xi$, respectively. Equation (3.7) is independent of the parameter α, as it should be. (The total action is unique.)

Equation (3.7) may be rewritten in a more explicit form as

$$\left\{m_a g^{\mu\nu} - g_a \sum_{b=1}^{N} g_b(r-1)\int dt' G_{ab}(\dot{x}_b{}^2)^{1/2}(u_a^\rho u_{b\rho})^{r-2}[(u_a^\rho u_{b\rho})^2 g^{\mu\nu} - r u_b^\mu u_b^\nu + r(u_a^\rho u_{b\rho})u_a^\mu u_b^\nu]\right\}\dot{u}_{a\nu}$$
$$= -2g_a \sum_{b=1}^{N} g_b \int dt' G'_{ab}(\dot{x}_a{}^2\dot{x}_b{}^2)^{1/2}(x_{a\nu}-x_{b\nu})(u_a^\rho u_{b\rho})^{r-1}\{u_a^\nu u_{b\rho}[r g^{\rho\mu} + (1-r)u_a^\mu u_a^\rho] - g^{\nu\mu}u_a^\rho u_{b\rho}\}\ , \quad (3.8)$$

where $u_a^\mu \equiv \dot{x}_a^\mu(\dot{x}_a{}^2)^{-1/2}$ are the four-velocities and x_a^μ and x_b^μ have the arguments t and t', respectively. G'_{ab} denotes the derivative of G_{ab} with respect to $(x_a - x_b)^2$.

Equation (3.8) simplifies for $r=0$ and in particular for $r=1$. These two cases will therefore be considered separately in the sequel. Equation (3.8) yields

$$\left(m_a + g_a \sum_{b=1}^{N} g_b \int dt' G_{ab}(\dot{x}_b{}^2)^{1/2}\right)\frac{d}{dt}\left(\frac{\dot{x}_a^\mu}{(\dot{x}_a{}^2)^{1/2}}\right)$$
$$= -2g_a\left(\frac{\dot{x}_a^\mu \dot{x}_a^\nu}{\dot{x}_a{}^2} - g^{\mu\nu}\right)\sum_{b=1}^{N} g_b \int dt' G_{ab}'(\dot{x}_a{}^2\dot{x}_b{}^2)^{1/2} \times (x_{a\nu}-x_{b\nu}) \quad (3.9)$$

for scalar interaction [$r=0$ in (3.8)] and

$$m_a \frac{d}{dt}\left(\frac{\dot{x}_a^\mu}{(\dot{x}_a{}^2)^{1/2}}\right)$$
$$= -2g_a \dot{x}_{a\nu}\sum_{b=1}^{N} g_b \int dt' G_{ab}'[\dot{x}_b^\mu(x_a^\nu - x_b^\nu) - \dot{x}_b^\nu(x_a^\mu - x_b^\mu)] \quad (3.10)$$

for vector interaction [$r=1$ in (3.8)].

The infinitesimal generator of a general variation of the action (3.1) may be found by means of the formulas (A14), (A22′), and (A24):

$$F_\alpha^r(t) = \sum_{a=1}^{N}\frac{m_a}{(\dot{x}_a{}^2)^{1/2}}\dot{x}_a^\mu \delta_0 x_{a\mu}(t) + \sum_{a,b=1}^{N}\tfrac{1}{2}\int d\xi\, F_{ab,\alpha}^r(t,\xi) - L_\alpha^r(t)\delta t\ , \quad (3.11)$$

where

$$F_{ab,\alpha}^r(t,\xi) = \int dt'\Delta(t-t', -\tfrac{1}{2}\alpha\xi)\left[f_{ab\mu}^r(t'-\tfrac{1}{2}\alpha\xi, t'+\xi-\tfrac{1}{2}\alpha\xi) + g_{ab\mu}^r(t'-\tfrac{1}{2}\alpha\xi, t'+\xi-\tfrac{1}{2}\alpha\xi)\frac{d}{dt'}\right]\delta_0 x_a^\mu(t'-\tfrac{1}{2}\alpha\xi)$$
$$+\int dt'\Delta(t-t', \tfrac{1}{2}\xi-\tfrac{1}{2}\alpha\xi)\left[f_{ab\mu}^r(t'+\tfrac{1}{2}\xi-\tfrac{1}{2}\alpha\xi, t'-\tfrac{1}{2}\xi-\tfrac{1}{2}\alpha\xi) + g_{ab\mu}^r(t'+\tfrac{1}{2}\xi-\tfrac{1}{2}\alpha\xi, t'-\tfrac{1}{2}\xi-\tfrac{1}{2}\alpha\xi)\frac{d}{dt'}\right]$$
$$\times\delta_0 x_a^\mu(t'+\tfrac{1}{2}\xi-\tfrac{1}{2}\alpha\xi)$$
$$+g_{ab\mu}^r(t,t+\xi)\delta_0 x_a^\mu(t) + g_{ab\mu}^r(t,t-\xi)\delta_0 x_a^\mu(t) + (a \leftrightarrow b)\ , \quad (3.12)$$

where in turn

$$\Delta(t,\lambda) \equiv \theta(t+\lambda) - \theta(t-\lambda) \quad (3.13)$$

and

$$f_{ab\mu}^r(t,t') \equiv \tfrac{1}{2}g_a g_b \frac{\partial w_{ab}^r(t,t')}{\partial x_a^\mu(t)}$$
$$= g_a g_b G_{ab}'(x_{a\mu} - x_{b\mu})(\dot{x}_a^\nu \dot{x}_{b\nu})^r \times (\dot{x}_a{}^2)^{(1-r)/2}(\dot{x}_b{}^2)^{(1-r)/2} \quad (3.14)$$

and

$$g_{ab\mu}^r(t,t') \equiv \tfrac{1}{2}g_a g_b \frac{\partial w_{ab}^r(t,t')}{\partial \dot{x}_a^\mu(t)}$$
$$= \tfrac{1}{2}g_a g_b G_{ab}(\dot{x}_b{}^2)^{(1-r)/2} \times (\dot{x}_a{}^2)^{-(1+r)/2}(\dot{x}_a^\nu \dot{x}_{b\nu})^{r-1} \times [r\dot{x}_{b\mu}\dot{x}_a{}^2 + (1-r)\dot{x}_{a\mu}(\dot{x}_a^\nu \dot{x}_{b\nu})]\ . \quad (3.15)$$

The arguments of particle a and b are t and t', respectively.

Consider first an infinitesimal parameter transformation $t \to t + \lambda(t)$ [$\lambda(t)$ is an infinitesimal function] which induces the functional variation $\delta_0 x_a^\mu = -\lambda\dot{x}_a^\mu$. Equation (3.11) yields here after some calculations

$$F_\alpha^r(t) \equiv 0 \quad \text{for all } r \text{ and } \alpha\ . \quad (3.16)$$

This is due to the (local) parameter invariance of the total action, or equivalently the homogeneity properties (2.4) and (3.4) [cf. (2.13)].

Consider next an infinitesimal space-time translation $x_a^\mu \to x_a^\mu + \epsilon^\mu$, $a=1,\ldots,N$, where ϵ^μ is an infinitesimal constant four-vector. $\delta_0 x_a^\mu = \epsilon^\mu$ ($a=1,\ldots,N$) and $\delta t = 0$ in (3.11) yields the generator $F_a^r(t) = \epsilon^\mu P_\mu^r(t)$, where

$$P^r_\mu(t) = \sum_{a=1}^{N} \frac{m_a}{(\dot{x}_a{}^2)^{1/2}} \dot{x}_{a\mu} + \sum_{a,b=1}^{N} \int d\xi \left[\int dt' \Delta(t-t', \tfrac{1}{2}\xi) f^r_{ab\mu}(t'+\tfrac{1}{2}\xi, t'-\tfrac{1}{2}\xi) + g^r_{ab\mu}(t, t+\xi) + g^r_{ba\mu}(t, t+\xi) \right] \quad (3.17)$$

is the energy-momentum vector. Notice that $p^r_\mu(t)$ is independent of the parameter α.

An infinitesimal homogeneous Lorentz transformation $x^\mu_a \to x^\mu_a + \epsilon^{\mu\nu} x_{a\nu}$, $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$, $a = 1, \ldots, N$, yields in turn, by inserting $\delta_0 x^\mu_a = \epsilon^{\mu\nu} x_{a\nu}$, $\delta t = 0$, in (3.11) the generator $F^r_\alpha(t) = -\frac{1}{2}\epsilon^{\mu\nu} J^r_{\mu\nu}(t)$, where

$$\begin{aligned} J^r_{\mu\nu}(t) = & \sum_{a=1}^{N} \frac{m_a}{(\dot{x}_a{}^2)^{1/2}} (\dot{x}_{a\nu} x_{a\mu} - \dot{x}_{a\mu} x_{a\nu}) \\ & + \sum_{a,b=1}^{N} \int d\xi \Big(\int dt' \Delta(t-t', \tfrac{1}{2}\xi) \{ A^r_{ab}(t'-\tfrac{1}{2}\xi, t'+\tfrac{1}{2}\xi)[x_{a\mu}(t'-\tfrac{1}{2}\xi) x_{b\nu}(t'+\tfrac{1}{2}\xi) - x_{a\nu}(t'-\tfrac{1}{2}\xi) x_{b\mu}(t'+\tfrac{1}{2}\xi)] \\ & \qquad + r B^r_{ab}(t'-\tfrac{1}{2}\xi, t'+\tfrac{1}{2}\xi)[\dot{x}_{a\nu}(t'-\tfrac{1}{2}\xi)\dot{x}_{b\mu}(t'+\tfrac{1}{2}\xi) - \dot{x}_{a\mu}(t'-\tfrac{1}{2}\xi)\dot{x}_{b\nu}(t'+\tfrac{1}{2}\xi)] \} \\ & \qquad + r B^r_{ab}(t, t+\xi)[x_{a\mu}(t)\dot{x}_{b\nu}(t+\xi) - \dot{x}_{b\mu}(t+\xi) x_{a\nu}(t)] + r B^r_{ab}(t+\xi, t)[x_{b\mu}(t)\dot{x}_{a\nu}(t+\xi) - x_{b\nu}(t)\dot{x}_{a\mu}(t+\xi)] \\ & \qquad + (r-1) B^r_{ab}(t, t+\xi) \frac{\dot{x}^\rho_a(t)\dot{x}_{b\rho}(t+\xi)}{\dot{x}_a{}^2(t)} [\dot{x}_{a\mu}(t) x_{a\nu}(t) - \dot{x}_{a\nu}(t) x_{a\mu}(t)] \\ & \qquad + (r-1) B^r_{ab}(t+\xi, t) \frac{\dot{x}^\rho_a(t+\xi)\dot{x}_{b\rho}(t)}{\dot{x}_b{}^2(t)} [\dot{x}_{b\mu}(t) x_{b\nu}(t) - \dot{x}_{b\nu}(t) x_{b\mu}(t)] \Big) , \end{aligned} \quad (3.18)$$

where

$$A^r_{ab}(t, t') \equiv g_a g_b G_{ab}{}'(\dot{x}^\nu_a \dot{x}_{b\nu})^r \times (\dot{x}_a{}^2)^{(1-r)/2} (\dot{x}_b{}^2)^{(1-r)/2}$$

and

$$B^r_{ab}(t, t') \equiv \tfrac{1}{2} g_a g_b G_{ab} (\dot{x}^\nu_a \dot{x}_{b\nu})^{r-1} \times (\dot{x}_a{}^2)^{(1-r)/2} (\dot{x}_b{}^2)^{(1-r)/2} .$$

The arguments of particle a and b are t and t', respectively.

Equation (3.18) is the angular momentum tensor and is like the energy-momentum vector (3.17) independent of α. Thus, the nonuniqueness of the Lagrangian (3.6), as represented by the parameter α, is neither reflected in the equations of motion (3.7) nor in the conserved quantities (3.17) and (3.18).

In the particular case of scalar interaction ($r=0$) we have the conserved quantities

$$P^\mu(t) = \sum_{a=1}^{N} \frac{M_a(t)}{[\dot{x}_a{}^2(t)]^{1/2}} \dot{x}^\mu_a(t) + \sum_{a,b=1}^{N} g_a g_b \int d\xi \int_t^{t+\xi} dt' G_{ab}{}'[x^\mu_a(t') - x^\mu_b(t'-\xi)][\dot{x}_a{}^2(t')\dot{x}_b{}^2(t'-\xi)]^{1/2} , \quad (3.19)$$

$$J^{\mu\nu}(t) = \sum_{a=1}^{N} \frac{M_a(t)}{[\dot{x}_a{}^2(t)]^{1/2}} [\dot{x}^\nu_a x^\mu_a(t) - \dot{x}^\mu_a x^\nu_a(t)] + \sum_{a,b=1}^{N} g_a g_b \int d\xi \int_t^{t+\xi} dt' G_{ab}(\dot{x}^\rho_a \dot{x}_{b\rho})[x^\mu_a(t'-\xi) x^\nu_b(t') - x^\nu_a(t'-\xi) x^\mu_b(t')] , \quad (3.20)$$

where

$$M_a(t) \equiv m_a + g_a \sum_{b=1}^{N} g_b \int d\xi\, G_{ab}[\dot{x}_b{}^2(t+\xi)]^{1/2} \quad (3.21)$$

(particle a has the argument t) and in the case of vector interaction ($r=1$) we have

$$\begin{aligned} P^\mu(t) = & \sum_{a=1}^{N} \frac{m_a}{[\dot{x}_a{}^2(t)]^{1/2}} \dot{x}^\mu_a(t) + \sum_{a,b=1}^{N} g_a g_b \int d\xi\, G_{ab} \dot{x}^\mu_b(t+\xi) \\ & + \sum_{a,b=1}^{N} g_a g_b \int d\xi \int_t^{t+\xi} dt' G_{ab}{}'[x^\mu_a(t') - x^\mu_b(t'-\xi)] \dot{x}^\nu_a(t')\dot{x}_{b\nu}(t'-\xi), \end{aligned} \quad (3.22)$$

$$\begin{aligned} J^{\mu\nu}(t) = & \sum_{a=1}^{N} \frac{m_a}{[\dot{x}_a{}^2(t)]^{1/2}} [\dot{x}^\nu_a x^\mu_a(t) - \dot{x}^\mu_a x^\nu_a(t)] \\ & + \sum_{a,b=1}^{N} g_a g_b \Big(\int d\xi\, G_{ab}[x^\mu_a(t)\dot{x}^\nu_b(t+\xi) - \dot{x}^\mu_b(t+\xi) x^\nu_a(t)] \\ & \qquad + \int d\xi \int_t^{t+\xi} dt' \{ G_{ab}{}' \dot{x}^\rho_a \dot{x}_{b\rho}[x^\mu_a(t'-\xi) x^\nu_b(t') - x^\nu_a(t'-\xi) x^\mu_b(t')] \\ & \qquad\qquad + \tfrac{1}{2} G_{ab}[\dot{x}^\nu_a(t'-\xi)\dot{x}^\mu_b(t') - \dot{x}^\mu_a(t'-\xi)\dot{x}^\nu_b(t')] \} \Big) . \end{aligned} \quad (3.23)$$

(The arguments of particle a and b in factors where they are not written out are always the same as in those factors which are multiplying them and whose arguments are explicitly written out.)

The difference between these results and earlier results may be described as follows: Earlier, one let the world lines of the different particles depend on different parameters; e.g., the free Lagrangian (2.4) would thereby be transformed into a multitime Lagrangian, $L_0(t_1, \ldots, t_N)$. When one then derived conserved quantities by a direct integration of the equations of motion or by application of the finite Fokker action principle,[20,41] one obtained multitime quantities like $P^\mu(t_1, \ldots, t_N)$ and $J^{\mu\nu}(t_1, \ldots, t_N)$. After (or before) this derivation, one identified the parameters t_a, $a = 1, \ldots, N$, with the proper-times of the particles. But then one can no longer escape a multitime formulation, and a Lagrangian and Hamiltonian formulation is no longer possible. If one instead would identify the parameters t_a, $a = 1, \ldots, N$, with one common parameter t, i.e., $t_a = t$, $a = 1, \ldots, N$, then the conserved quantities P^μ and $J^{\mu\nu}$ would reduce to the ones obtained in this paper. But it should be noted that such an identification is not reconcilable with a description in terms of proper-times.

What we have achieved so far is thus a single parameter description of action-at-a-distance theories. However, we have not yet specified the parameter t or identified it with some well-known quantity. One could, e.g., identify it with the proper-time of one of the particles, which however would be a very unnatural choice. Instead we think that the possible choices given in Sec. II are the only ones worth considering, and since we here intend to construct a theory in complete analogy with classical nonrelativistic mechanics, the most natural identification is in fact the following one:

$$t = x_1^0(t) = x_2^0(t) = \cdots = x_N^0(t) , \tag{3.24}$$

which is the same as (2.16) and which is justified exactly by the same reasons which were given in Sec. II. [Because of (3.16) the generators are not affected by (3.24).]

By means of (3.24) we get the following equations of motion in the case of vector interaction [see (3.10)]:

$$\begin{aligned} m_a \frac{d}{dt}(\gamma_a \dot{x}_a^i) &= g_a \sum_{b=1}^N g_b \dot{x}_a^k \int d\xi \left(\dot{x}_b^k(t+\xi)\frac{\partial G_{ab}}{\partial x_a^i(t)} - \dot{x}_b^i(t+\xi)\frac{\partial G_{ab}}{\partial x_a^k(t)} \right) \\ &\quad + g_a \sum_{b=1}^N g_b \int d\xi \left(2G_{ab}'\xi \dot{x}_b^i(t+\xi) - \frac{\partial G_{ab}}{\partial x_a^i(t)} \right) , \\ &\qquad a = 1, \ldots, N \end{aligned} \tag{3.25}$$

for $\mu = i$ and

$$\begin{aligned} m_a \frac{d}{dt}\gamma_a &= g_a \sum_{b=1}^N g_b \dot{x}_a^k \int d\xi \left(2G_{ab}'\xi \dot{x}_b^k(t+\xi) - \frac{\partial G_{ab}}{\partial x_a^k} \right) , \\ &\qquad a = 1, \ldots, N \end{aligned} \tag{3.26}$$

for $\mu = 0$, where γ_a is defined by (2.18). [G_{ab} has the argument $\xi^2 - (\vec{x}_a - \vec{x}_b)^2$.]

Because of our choice of a homogeneous action, (3.26) is not independent of (3.25). Given (3.25) it is in fact not difficult to show that (3.26) is automatically satisfied. Equation (3.26) is therefore a redundant equation which may be discarded. The conserved quantities associated with (3.25) are

$$\begin{aligned} H(t) \equiv P^0(t) &= \sum_{a=1}^N m_a\gamma_a + \sum_{a,b=1}^N g_a g_b \int d\xi\, G_{ab} \\ &\quad + \sum_{a,b=1}^N g_a g_b \int d\xi \int_0^\xi d\eta\, G_{ab}'\xi[1 - \dot{x}_a^i(t+\eta)\dot{x}_b^i(t+\eta-\xi)] \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} P^i(t) &= \sum_{a=1}^N m_a\gamma_a \dot{x}_a^i(t) + \sum_{a,b=1}^N g_a g_b \int d\xi\, G_{ab}\dot{x}_b^i(t+\xi) \\ &\quad - \tfrac{1}{2}\sum_{a,b=1}^N g_a g_b \int d\xi \int_0^\xi d\eta \frac{\partial G_{ab}}{\partial x_a^i(t+\eta)} [1 - \dot{x}_a^k(t+\eta)\dot{x}_b^k(t+\eta-\xi)] . \end{aligned} \tag{3.28}$$

These are derived by just putting (3.24) into (3.22). From (3.23) we have furthermore

$$\begin{aligned} J^{ij}(t) &= \sum_{a=1}^N m_a\gamma_a(\dot{x}_a^j x_a^i - \dot{x}_a^i x_a^j) + \sum_{a,b=1}^N g_a g_b \int d\xi\, G_{ab}[x_a^i(t)\dot{x}_b^j(t+\xi) - \dot{x}_b^i(t+\xi)x_a^j(t)] \\ &\quad + \sum_{a,b=1}^N g_a g_b \int d\xi \int_0^\xi d\eta \{ G_{ab}'(1 - \dot{x}_a^k\dot{x}_b^k)[x_a^i(t+\eta-\xi)x_b^j(t+\eta) - x_a^j(t+\eta-\xi)x_b^i(t+\eta)] \\ &\qquad + \tfrac{1}{2}G_{ab}[\dot{x}_a^j(t+\eta-\xi)\dot{x}_b^i(t+\eta) - \dot{x}_a^i(t+\eta-\xi)\dot{x}_b^j(t+\eta)]\} \end{aligned} \tag{3.29}$$

and

$$J^{i0}(t)=\sum_{a=1}^{N} m_a\gamma_a(x_a^i-\dot{x}_a^i t)+\sum_{a,b=1}^{N} g_a g_b \int d\xi\, G_{ab}[x_a^i(t)-\dot{x}_b^i(t+\xi)t]$$
$$+\sum_{a,b=1}^{N} g_a g_b \int d\xi \int_0^{\xi} d\eta\{G_{ab}'(1-\dot{x}_a^k\dot{x}_b^k)[x_a^i(t+\eta-\xi)(t+\eta)-(t+\eta-\xi)x_b^i(t+\eta)]$$
$$+\tfrac{1}{2}G_{ab}[\dot{x}_b^i(t+\eta)-\dot{x}_a^i(t+\eta-\xi)]\}\ . \tag{3.30}$$

Notice that H, P^i, and J^{ij} have no explicit time dependence as G_{ab} has no such explicit dependence.

Expressions corresponding to (3.25)–(3.30) for scalar and other interactions are derived in a similar fashion, i.e., by inserting (3.24) into (3.9), (3.19), and (3.20) or into (3.8), (3.17), and (3.18).

Finally we would like to remark that we could, of course, have made the identification (3.24) already in the Lagrangian (3.6) and then derived the conserved quantities (3.27)–(3.30) by use of the generalized action principle. For vector interaction, such a Lagrangian looks like

$$L_\alpha(t)=-\sum_{a=1}^{N}\frac{m_a}{\gamma_a(t)}$$
$$-\tfrac{1}{2}\sum_{a,b=1}^{N} g_a g_b \int d\xi\, G_{ab}$$
$$\times[1-\dot{x}_a^i(t-\alpha\xi)\dot{x}_b^i(t+\xi-\alpha\xi)]\ . \tag{3.31}$$

A similar Lagrangian, Taylor-expanded at the instant t, has to some extent also been considered before.[28,42]

IV. THE HAMILTONIAN FORMALISM OF INTERACTING RELATIVISTIC PARTICLES

Since we have shown that one can apply the full action principle to relativistic particle mechanics we strongly suspect that the derived conserved quantities also fulfill the right generator properties (2.31) and (2.32) in some Poisson-bracket sense. Peierls,[43] e.g., once proposed a method of defining Poisson brackets directly from a Lagrangian formulation and maybe his method is applicable here. (It has been applied to field theories with nonlocal interaction,[44] which are structurally similar to relativistic particle theories.) Anyhow, Pauli,[31] who also has considered this problem, seems to have believed that there always exist Poisson brackets fulfilling (2.31) and (2.32). He investigated in particular the question as to whether there also exist canonical variables in terms of which these Poisson brackets could be defined in the usual fashion. His answer was positive if only the equations of motion fulfill the Newtonian Cauchy problem, i.e., the particle's position and velocity at a particular time must uniquely determine its whole world line. This requirement is due to the fact that for a system with N particles the Poisson brackets are defined in terms of 6 N-independent variables given at a particular instant of time [see (2.30)].

Now the equations of motion for noninstantaneous action-at-a-distance theories are in their general forms functional-differential equations [notice that (3.25) and (3.26) are nonlocal in time], and even if one usually restricts the class of functions in the interaction terms such that they reduce to integrodifferential or difference-differential equations as, e.g., in action-at-a-distance electrodynamics, the above initial-value condition will not be satisfied.[19,45] This is a problem which we will come across in the following manner: As our intention in this section is to check explicitly the generator properties (2.31) and (2.32) we need Poisson brackets for unequal times due to the fact that the Lagrangian and therefore also the generators are nonlocal in time. Consequently, we have to solve the equations of motion, and from the above it then follows that there will exist many possible solutions for a given set of Newtonian Cauchy data. The question is now whether the relations (2.31) and (2.32) are fulfilled for all these solutions and, if this is the case, is it then in terms of a uniquely defined Poisson bracket? Of course, this question is very hard to answer, and it needs, in fact, not to be answered in this general form since most of the solutions are to be discarded on physical grounds. The situation could be compared with that of Dirac's equation with radiative reaction.[7] This equation contains time-derivatives of the third order which cause the usual Cauchy data (position and velocity at a particular time) to no longer determine a unique solution. However, the equation cannot for this reason be said to be unsatisfactory since it always seems to yield a unique physical solution.[46] However, it contains also unphysical solutions, which have to be discarded by some general and fundamental condition. Dirac imposed on his equation the asymptotic condition $\lim_{\tau\to\infty}a^\mu(\tau)=0$, where a^μ is the acceleration.

Since we intend to construct a theory in close analogy to nonrelativistic mechanics it is here natural to require that for any given Newtonian Cauchy data we must be able, in an unambiguous manner, to select a unique solution out of all possible solutions of the equation of motion of the particle in question. We need therefore a general selection rule. Dirac's asymptotic condition is only applicable to scattering processes and is therefore not general enough here as the equations of motion also contain bound-state solutions. Staruszkiewicz[47] has investigated a simple example for which he could formulate a principle of selection in two equivalent forms, one of which says that we should exclude all solutions which are unbounded for $\tau \to \infty$ or $\tau \to -\infty$, which therefore also can be considered as unphysical. However, Andersen and von Baeyer[19] found when they considered almost circular orbits that there also exist, apart from the divergent solutions, additional stable solutions. (Chern and Havas[21] seem also to have found several stable solutions for given initial data.) These extra stable solutions can be ruled out, however, if one selects those solutions which become solutions of the nonrelativistic theory as $c \to \infty$.[19,21] (This is in fact the second form of Staruszkiewicz's selection principle[47]). Thus it seems that the most general selection rule one can formulate at present is a kind of correspondence principle: The physical solutions must have a nonrelativistic limit.[19] But we would like to remark that the additional stable solutions could have a physical significance and should therefore not be ruled out. However, their inclusion would mean that we allow more degrees of freedom than the Newtonian ones by which we would depart considerably from the nonrelativistic case. On the other hand, it could also be that this selection rule is so strong that *all* solutions would be ruled out for particular initial data.[18,21]

Regarding the question whether the framework of the action principle in Sec. III can provide a selection rule, we make the following remark: Because of the fact that every infinitesimal generator $F(t)$ is not just depending on the time instant t but in general on a time interval around t, it seems as if one cannot require the action W_{21} to be stationary when the variations $\delta\vec{x}_a(t)$, $a=1,\ldots,N$, vanish at t_1 and t_2, which is possible in instantaneous action-at-a-distance theories. One exception is when $t_1 \to -\infty$ and $t_2 \to +\infty$ because in this limit the time intervals around t_1 and t_2 are effectively separated from finite time. Thus, the total action has to be stationary when the variations of the positions are zero at the infinite past and future. Feynman has proposed the use of this requirement as a selection principle. However, it seems that only the additional divergent solutions would be ruled out by this requirement.[15]

When we now turn to an explicit investigation of the Hamiltonian formalism we encounter, apart from the problem of solving the equations of motion and identifying the solution to be considered, the problem of finding the canonical variables. The actual construction will therefore always involve a great deal of guess work in this respect.

We shall here only consider the simplest type of interaction, namely, the general vector interaction given by the equations of motion (3.25). Moreover, we shall only consider a particular class of solutions which reduce to straight-line solutions in the limit $g_a \to 0$, $a=1,\ldots,N$.[14,28] This class of solutions is uniquely determined by the Newtonian Cauchy data.[14,28,48,49] They are, however, only applicable to scattering processes.[14,19,49]

By means of the matrix

$$\frac{1}{\gamma_a(t)}\left(\delta^{ij}-\frac{\dot{x}_a^i(t)\dot{x}_a^j(t)}{c^2}\right) ,$$

we may transform the Eqs. (3.25) into the following forms:

$$m_a\ddot{x}_a^i(t)=g_aR_a^i(t),\quad a=1,\ldots,N \tag{4.1}$$

where

$$R_a^i(t)=\frac{1}{\gamma_a(t)}\left[F_a^i(t)-\dot{x}_a^i(t)S_a(t)\right] , \tag{4.2}$$

$$S_a(t)=\dot{x}_a^k(t)F_a^k(t) , \tag{4.3}$$

and

$$F_a^i(t)=\sum_{b=1}^N g_b\,\dot{x}_a^k(t)\int d\xi\left(\dot{x}_b^k(t+\xi)\frac{\partial G_{ab}}{\partial x_a^i(t)}-\dot{x}_b^i(t+\xi)\frac{\partial G_{ab}}{\partial x_a^k(t)}\right)+\sum_{b=1}^N g_b\int d\xi\left(2G_{ab}{}'\xi\,\dot{x}_b^i(t+\xi)-\frac{\partial G_{ab}}{\partial x_a^i(t)}\right) . \tag{4.4}$$

The equations (3.25) themselves may be written as

$$\dot{p}_a^i(t)=g_aF_a^i(t) ,\quad a=1,\ldots,N \tag{4.5}$$

where

$$p_a^i(t)\equiv m_a\gamma_a(t)\dot{x}_a^i(t) , \tag{4.6}$$

and the Eqs. (3.26) may be written as

$$m_a\dot{\gamma}_a(t)=g_aS_a(t) . \tag{4.7}$$

We shall always consider the Eqs. (4.1) to be the basic ones. But as the solutions also satisfy (4.5) and (4.7) we shall make use of these as well whenever convenient.

Integrating (4.1) we find

$$\dot{x}_a^i(t) = \dot{x}_{a,\text{in}}^i(t) + \frac{g_a}{m_a}\int dt'\theta(t-t')R_a^i(t') \qquad (4.8)$$

and

$$x_a^i(t) = x_{a,\text{in}}^i(t) + \frac{g_a}{m_a}\int dt'\theta(t-t')(t-t')R_a^i(t') , \qquad (4.9)$$

where $x_{a,\text{in}}^i(t)$ is a straight-line solution $[\ddot{x}_{a,\text{in}}^i(t)=0]$ completely determined by the initial conditions. We impose the condition

$$\lim_{g_a\to 0} x_a^i(t) = x_{a,\text{in}}^i(t) \qquad (4.10)$$

and require that (this implies conditions on the functions G_{ab})

$$\lim_{t\to-\infty}[x_a^i(t) - x_{a,\text{in}}^i(t)] = 0 . \qquad (4.11)$$

By a change of variable, (4.8) and (4.9) may be written in the following more suitable forms:

$$x_a^i(t) = x_{a,\text{in}}^i(t) - \frac{g_a}{m_a}\int_{-\infty}^0 d\eta\,\eta R_a^i(t+\eta) , \qquad (4.12)$$

$$\dot{x}_a^i(t) = \dot{x}_{a,\text{in}}^i(t) + \frac{g_a}{m_a}\int_{-\infty}^0 d\eta\, R_a^i(t+\eta) . \qquad (4.13)$$

We shall also find it convenient to use the integrated forms of (4.5) and (4.7):

$$p_a^i(t) = p_{a,\text{in}}^i(t) + g_a\int_{-\infty}^0 d\eta\, F_a^i(t+\eta) , \qquad (4.14)$$

$$\gamma_a(t) = \gamma_{a,\text{in}}(t) + \frac{g_a}{m_a}\int_{-\infty}^0 d\eta\, S_a(t+\eta) , \qquad (4.15)$$

where $p_{a,\text{in}}^i \equiv m_a\gamma_{a,\text{in}}\dot{x}_{a,\text{in}}^i$ and $\gamma_{a,\text{in}} \equiv (1-\dot{x}_{a,\text{in}}^i\dot{x}_{a,\text{in}}^i)^{-1/2}$.

The solutions are obtained by successive approximations with respect to the coupling constants in the following fashion (cf. Yang-Feldman formalism in field theory[50]):

$$x_a^i(t) = x_{a,\text{in}}^i(t) + \sum_{n=1}^{\infty}\frac{g_a g}{m_a}\left(\frac{g^2}{m}\right)^n x_{a(n)}^i(t;\text{in}) , \quad a=1,\ldots,N \qquad (4.16)$$

where $g(g^2)^n/m^n$ stands for all possible combinations of $2n-1$ coupling constants and n masses, and where $x_{a(n)}^i$ are given expressions in terms of $x_{b,\text{in}}^i$ and $\dot{x}_{b,\text{in}}^i$, $b=1,\ldots,N$. We impose on the functions G_{ab} the condition that these iterative solutions converge for at least distant collisions [when the particles are not getting closer to each other than $\approx g^2/m$ (cf. Ref. 49)].

We make now use of the freedom to choose canonical variables which need not be the physical positions.[29,31] The reason for this is that we need explicit expressions for the Poisson brackets of the canonical variables at unequal times. The obvious choice of possible canonical variables is here $x_{a,\text{in}}^i(t)$ and $p_{a,\text{in}}^i(t)$, $i=1,2,3$, $a=1,2,\ldots,N$, because for them we have

$$[p_{a,\text{in}}^i(t), p_{b,\text{in}}^j(t+\eta)] = 0 , \qquad (4.17)$$

$$[x_{a,\text{in}}^i(t), p_{b,\text{in}}^j(t+\eta)] = \delta^{ij}\delta_{ab} , \qquad (4.18)$$

$$[x_{a,\text{in}}^i(t), x_{b,\text{in}}^j(t+\eta)] = \frac{\eta\delta_{ab}}{m_a\gamma_{a,\text{in}}}(\delta^{ij} - \dot{x}_{a,\text{in}}^i\dot{x}_{a,\text{in}}^j) . \qquad (4.19)$$

We shall first show that the generators (3.27)–(3.30) fulfill the Lie algebra (2.32) when the Poisson brackets are defined in terms of the above canonical variables. We shall do this by showing the following equalities:

$$H(t) = H_0(\text{in}) , \qquad (4.20)$$

$$P^i(t) = P_0^i(\text{in}) , \qquad (4.21)$$

$$J^{ij}(t) = J_0^{ij}(\text{in}) , \qquad (4.22)$$

$$J^{i0}(t) \equiv K^i(t) = K_0^i(\text{in}) , \qquad (4.23)$$

where

$$H_0(\text{in}) \equiv \sum_{a=1}^N m_a\gamma_{a,\text{in}} , \qquad (4.24)$$

$$P_0^i(\text{in}) \equiv \sum_{a=1}^N p_{a,\text{in}}^i , \qquad (4.25)$$

$$J^{ij}(\text{in}) \equiv \sum_{a=1}^N [p_{a,\text{in}}^j x_{a,\text{in}}^i(t) - p_{a,\text{in}}^i x_{a,\text{in}}^j(t)] , \qquad (4.26)$$

$$K_0^i(\text{in}) \equiv \sum_{a=1}^N m_a\gamma_{a,\text{in}}[x_{a,\text{in}}^i(t) - \dot{x}_{a,\text{in}}^i t] . \qquad (4.27)$$

If the relations (4.20)–(4.23) hold true, then it follows from Sec. II that the generators (3.27)–(3.30) fulfill the Lie algebra relations (2.32) of the inhomogeneous Lorentz group.

Proof of (4.20). By (4.11) and (4.7) we have [we use here and in what follows the notations $H_0(t)$, etc., when we have replaced $x_{a,\text{in}}$ by $x_a(t)$ in the expression (4.24)–(4.27)]

$$H_0(\text{in}) = H_0(t) - \int_{-\infty}^0 d\eta\,\dot{H}_0(t+\eta)$$

$$= H_0(t) - \int_{-\infty}^0 d\eta\sum_a g_a S_a(t+\eta) ,$$

which one also may obtain by a direct use of (4.15). From

$$\frac{dG_{ab}}{d\xi} - \frac{dG_{ab}}{d\eta} = 2G_{ab}'\xi - \dot{x}_a^k(t+\eta)\frac{\partial G_{ab}}{\partial x_a^k(t+\eta)} \qquad (4.28)$$

we get

$$H_0(\text{in}) = H_0(t) - \sum_{a,b=1}^{N} g_a g_b \int_{-\infty}^{0} d\eta\, \dot{x}_a^k(t+\eta) \int d\xi \left(2G_{ab}'\xi \dot{x}_b^k(t+\xi+\eta) - \frac{\partial G_{ab}}{\partial x_a^k(t+\eta)}\right)$$
$$= H_0(t) + \sum_{a,b=1}^{N} g_a g_b \int d\xi\, G_{ab} + \sum_{a,b=1}^{N} g_a g_b 2 \int_{-\infty}^{0} d\eta \int d\xi\, G_{ab}'\xi[1 - \dot{x}_a^k(t+\eta)\dot{x}_b^k(t+\eta+\xi)] ,$$

which after a change of variables in the last term $(\eta \to \eta - \xi)$ is easily shown to be exactly the expression (3.27). Equation (4.20) is thereby proved.

Proof of (4.21). By (4.5) and (4.11), or directly by (4.14), it follows that

$$P_0^i(\text{in}) = P_0^i(t) - \sum_{a=1}^{N} g_a \int_{-\infty}^{0} d\eta\, F_a^i(t+\eta)$$
$$= P_0^i(t) + \sum_{a,b=1}^{N} g_a g_b \int_{-\infty}^{0} d\eta \int d\xi \frac{\partial G_{ab}}{\partial x_a^i(t+\eta)} [1 - \dot{x}_a^k(t+\eta)\dot{x}_b^k(t+\eta+\xi)]$$
$$- \sum_{a,b=1}^{N} g_a g_b \int_{-\infty}^{0} d\eta \int d\xi \left(\frac{dG_{ab}}{d\xi} - \frac{dG_{ab}}{d\eta}\right) \dot{x}_b^i(t+\eta+\xi) , \tag{4.29}$$

where we have used (4.28). The last term may be written as

$$- \sum_{a,b=1}^{N} g_a g_b \int_{-\infty}^{0} d\eta \int d\xi \left(\frac{d}{d\xi} - \frac{d}{d\eta}\right) [G_{ab}\dot{x}_b^i(t+\eta+\xi)]$$
$$= \sum_{a,b=1}^{N} g_a g_b \int d\xi\, G_{ab}\dot{x}_b^i(t+\xi) ,$$

which together with a change of variables $(\eta \to \eta - \xi)$ in the second-to-last term in (4.29) shows that (4.29) reduces exactly to the expression (3.28). Equation (4.21) is thereby proved.

Proof of (4.22). Using (4.1), (4.5), and (4.11) we get

$$J_0^{ij}(\text{in}) = J_0^{ij}(t) + \sum_{a=1}^{N} g_a \int_{-\infty}^{0} d\eta [F_a^i(t+\eta)x_a^j(t+\eta) - F_a^j(t+\eta)x_a^i(t+\eta)] , \tag{4.30}$$

which in this case is not so easily obtained by a direct use of (4.12) and (4.14). By means of the relation

$$x_a^j(t+\eta)\dot{x}_b^i(t+\eta+\xi)\left(\frac{dG_{ab}}{d\xi} - \frac{dG_{ab}}{d\eta}\right)$$
$$= \left(\frac{d}{d\xi} - \frac{d}{d\eta}\right) (G_{ab}\dot{x}_b^i x_a^j) + G_{ab}\dot{x}_b^i x_a^j$$

and a change of variables, one easily shows that (4.30) is equal to (3.29).

Proof of (4.23). Using (4.5), (4.7), and (4.11) one obtains

$$J_0^{i0}(\text{in}) = J_0^{i0}(t) - \sum_{a=1}^{N} g_a \int_{-\infty}^{0} d\eta [m_a S_a(t+\eta)x_a^i(t+\eta) - F_a^i(t+\eta)(t+\eta)] ,$$

which after making use of the relations

$$\left(\frac{d}{d\xi} - \frac{d}{d\eta}\right)[(t+\eta)\dot{x}_b^k(t+\eta+\xi)G_{ab}]$$
$$= 2(t+\eta)\dot{x}_b^k(t+\eta+\xi)G_{ab}'\xi - (t+\eta)\dot{x}_b^k\dot{x}_a^j \frac{\partial G_{ab}}{\partial x_a^j}$$
$$- \dot{x}_b^k G_{ab} ,$$

$$\left(\frac{d}{d\xi} - \frac{d}{d\eta}\right) [x_a^i(t+\eta)G_{ab}] = 2G_{ab}'\xi x_a^i(t+\eta) - x_a^i\dot{x}_a^k \frac{\partial G_{ab}}{\partial x_a^k}$$
$$- \dot{x}_a^i G_{ab}$$

and a change of variables reduces exactly to (3.30).

Thus we have shown that the generators (3.27)–(3.30) fulfill the Lie algebra relations (2.32). Next we show that the physical positions transform in the following way [cf. (2.31)]:

$$[x_a^i(t), H(t)] = \dot{x}_a^i(t) , \tag{4.31}$$
$$[x_a^i(t), P^j(t)] = \delta^{ij} , \tag{4.32}$$
$$[x_a^i(t), J^{jk}(t)] = x_a^j(t)\delta^{ik} - x_a^k(t)\delta^{ij} , \tag{4.33}$$
$$[x_a^i(t), K^j(t)] = \dot{x}_a^i(t)x_a^j(t) - \delta^{ij}t . \tag{4.34}$$

For this purpose we shall make use of the expression (4.12). First we note that $x_{a,\text{in}}^i(t)$ transforms as $x_a^i(t)$ in (4.31)–(4.34) due to (4.20)–(4.23) and (2.31). Therefore we only need to check the consistency of (4.12) with (4.31)–(4.34) because then the right transformation properties of the first- and higher-order terms in the expansion (4.16) will follow automatically. In this consistency check below we shall make repeated use of the quantity [cf. (4.12)]

$$-\frac{g_a}{m_a}\int_{-\infty}^{0} d\eta\, \eta R_a^i(t+\eta)$$
$$= -\frac{g_a}{m_a}\sum_{b=1}^{N} g_b \int_{-\infty}^{0} d\eta\, \eta \int d\xi \frac{A_{ab}^i(t+\eta)}{\gamma_a(t+\eta)} , \tag{4.35}$$

where

$$A^i_{ab}(t+\eta)=\dot{x}^k_a(t+\eta)\left[\dot{x}^k_b(t+\eta+\xi)\frac{\partial G_{ab}}{\partial x^i_a(t+\eta)}-\dot{x}^i_b(t+\eta+\xi)\frac{\partial G_{ab}}{\partial x^k_a(t+\eta)}\right]+2G_{ab}'\xi\dot{x}^i_b(t+\eta+\xi)$$
$$-\frac{\partial G_{ab}}{\partial x^i_a(t+\eta)}-2G_{ab}'\xi\dot{x}^k_b(t+\eta+\xi)\dot{x}^k_a(t+\eta)\dot{x}^i_a(t+\eta)+\frac{\partial G_{ab}}{\partial x^k_a(t+\eta)}\dot{x}^k_a(t+\eta)\dot{x}^i_a(t+\eta)\ . \qquad (4.36)$$

(i) Consistency of (4.12) and (4.31). This is trivial since A^i_{ab}, does not contain any explicit t dependence.

(ii) Consistency of (4.12) and (4.32). Equation (4.32) implies $[\dot{x}^i_a(t), P^j]=0$, from which one realizes that $[R^i_a(t+\eta), P^j]=0$ since

$$\frac{\partial G_{ab}}{\partial x^j_a(t+\eta)}+\frac{\partial G_{ab}}{\partial x^j_b(t+\eta+\xi)}=0\ .$$

(iii) Consistency of (4.12) and (4.33). As $[\dot{x}^k_a\dot{x}^k_b, J^{ij}]=0$, etc., it follows that $[A^i_{ab}, J^{jk}]=A^j_{ab}\delta^{ik}-A^k_{ab}\delta^{ji}$.

(iv) Consistency of (4.12) and (4.34). From

$$[\dot{x}^i_a(t), K^j]=\ddot{x}^i_a(t)x^j_a(t)+\dot{x}^i_a(t)\dot{x}^j_a(t)-\delta^{ij}\ ,$$

which follows from (4.34), we get

$$\left[\frac{1}{\gamma_a(t)}, K^j\right]=\dot{x}^j_a(t)\frac{1}{\gamma_a(t)}+x^j_a\frac{d}{dt}\left(\frac{1}{\gamma_a}\right)$$

and

$$[A^i_{ab}, K^j]=\frac{d}{dt}(x^j_aA^i_{ab})+\dot{x}^i_aA^j_{ab}$$
$$+\frac{d}{d\xi}[(x^j_a-x^j_b)A^i_{ab}]$$

by a straightforward calculation.

Combining these expressions one finds

$$[R^i_a(t+\eta), K^j]=\frac{d}{d\eta}[x^j_a(t+\eta)R^i_a]+x^j_aR^i_a$$
$$+x^i_aR^j_a\ ,$$

which finally yields

$$\left[\int_{-\infty}^0 d\eta\,\eta R^i_a(t+\eta), K^j\right]=-x^j_{a,\mathrm{in}}(t)\int_{-\infty}^0 d\eta R^i_a(t+\eta)+\dot{x}^i_{a,\mathrm{in}}(t)\int_{-\infty}^0 d\eta\,\eta R^j_a(t+\eta)+\int_{-\infty}^0 d\eta R^i_a(t+\eta)\int_{-\infty}^0 d\eta'\eta' R^j_a(t+\eta')\ ,$$

where we have made use of (4.12) and (4.13) and the relations

$$x^j_{a,\mathrm{in}}(t+\eta)=x^j_{a,\mathrm{in}}(t)+\eta\dot{x}^j_{a,\mathrm{in}}(t)\ ,$$
$$\int_{-\infty}^0 d\eta f(t+\eta)\int_{-\infty}^0 d\eta' g(t+\eta+\eta')+\int_{-\infty}^0 d\eta g(t+\eta)\int_{-\infty}^0 d\eta' f(t+\eta+\eta')=\int_{-\infty}^0 d\eta f(t+\eta)\int_{-\infty}^0 d\eta' g(t+\eta')\ .$$

Thereby we have explicitly shown that we have a canonical representation in terms of the canonical variables $x^i_{a,\mathrm{in}}(t)$ and $p^i_{a,\mathrm{in}}(t)$, $i=1,2,3$, $a=1,\ldots,N$.

We could also have integrated (4.1) in the following way:

$$x^i_a(t)=x^i_{a,\mathrm{out}}(t)+\frac{g_a}{m_a}\int_0^\infty d\eta\,\eta R^i_a(t+\eta)\ , \qquad (4.37)$$

where $x^i_{a,\mathrm{out}}(t)$ is a straight-line solution completely (but indirectly) determined by the initial conditions. If we impose the conditions

$$\lim_{g_a\to 0} x^i_a(t)=x^i_{a,\mathrm{out}}(t) \qquad (4.38)$$

and

$$\lim_{t\to+\infty}[x^i_a(t)-x^i_{a,\mathrm{out}}(t)]=0\ , \qquad (4.39)$$

then one can show in exactly the same way as above that the generators (3.27)–(3.30) fulfill the Lie algebra (2.32) [the relations (4.20)–(4.23) will be replaced by $H_0(\mathrm{out})=H(t)$, etc.] and that the physical positions transform according to (4.31)–(4.34), but now with $x^i_{a,\mathrm{out}}$ and $p^i_{a,\mathrm{out}}\equiv m_a\gamma_{a,\mathrm{out}}\dot{x}^i_{a,\mathrm{out}}$ as canonical variables.

$x^i_{a,\mathrm{out}}(t)$ and $x^i_{a,\mathrm{in}}(t)$ are different in the presence of interaction. Comparing (4.12) and (4.37) one finds

$$x^i_{a,\mathrm{out}}(t)=x^i_{a,\mathrm{in}}(t)-\frac{g_a}{m_a}\int_{-\infty}^\infty d\eta\,\eta R^i_a(t+\eta)\ . \qquad (4.40)$$

Thus if $x^i_{a,\mathrm{in}}$ is known, then $x^i_{a,\mathrm{out}}$ can be calculated and vice versa. In fact $x^i_{a,\mathrm{in}}(t)$ and $x^i_{a,\mathrm{out}}(t)$ coincide with the particle trajectory at the infinite past [cf. (4.11)] and at the infinite future [cf. (4.39)], respectively. Now it is clear from the above calculations that $x^i_{a,\mathrm{in}}$, $p^i_{a,\mathrm{in}}$ and $x^i_{a,\mathrm{out}}$, $p^i_{a,\mathrm{out}}$ are canonically equivalent variables, which in terms of the bilinear covariant also can be expressed by

$$\sum_a(\delta x^i_{a,\mathrm{in}}\Delta p^i_{a,\mathrm{in}}-\delta p^i_{a,\mathrm{in}}\Delta x^i_{a,\mathrm{in}})$$
$$=\sum_a(\delta x^i_{a,\mathrm{out}}\Delta p^i_{a,\mathrm{out}}-\delta p^i_{a,\mathrm{out}}\Delta x^i_{a,\mathrm{out}})\ ,$$

where δ and Δ are two independent variations of the orbit.

The physical positions however cannot be chosen as canonical variables. Calculating the Poisson bracket between any two physical positions, using the expansions (4.16), one finds

$$[x_a^i(t), x_b^j(t)] \neq 0 \qquad (4.41)$$

for $a \neq b$ and/or $i \neq j$. Already in the first order the right-hand side of (4.41) is a quite messy expression. However, it does only depend on $x^i_{a,\text{in}}$ and $x^i_{b,\text{in}}$ which tells us that (4.41) holds true irrespective of the number of particles in our system. Thus it is the type of interaction which causes the nonzero value of (4.41). In fact (4.41) is due to the fact that the interaction is nonlocal in time. Taking the nonrelativistic limit

$$\lim_{c\to\infty} \int d\xi\, G(c^2\xi^2 - [\vec{x}_a(t) - \vec{x}_b(t+\xi)]^2) \times f(x_a(t), x_b(t+\xi), \xi) = G(-[\vec{x}_a(t) - \vec{x}_b(t)]^2) f'(x_a(t), x_b(t)) ,$$

the interaction will be local in time, (4.41) is replaced by

$$[x_a^i(t), x_b^j(t)] = 0 ,$$

and the physical positions can be chosen as canonical variables.

V. DISCUSSION OF THE RESULTS

We have shown that a wide class of particle theories formulated in terms of invariant Fokker actions can be endowed with a Lagrangian formalism. From our treatment it is obvious that one may also transform *any* parametrization-invariant Fokker action into a Lagrangian formulation in exactly the same fashion as presented here. One can even introduce N-body forces for arbitrary N. Consider, e.g., the following interaction part of an action:

$$W_I = \int\int\int dt_a dt_b dt_c f(x_a(t_a), x_b(t_b), x_c(t_c)) . \qquad (5.1)$$

Define a common parameter t by $t = \alpha t_a + \beta t_b + \gamma t_c$, where α, β, and γ are real parameters fulfilling $\alpha + \beta + \gamma = 1$. Introduce the new variables $\xi = t_a - t_c$ and $\eta = t_b - t_c$. The Lagrangian corresponding to (5.1) is now

$$L_{I\alpha,\beta}(t) = \int\int d\xi\, d\eta f(x_a(t+\xi-\rho), x_b(t+\eta-\rho), x_c(t-\rho)) ,$$

where $\rho = \alpha\xi + \beta\eta$. The present extension of the conventional action principle (2.8) is therefore more general than the finite Fokker action principle,[6,13,41] as the latter cannot be extended to actions of the type (5.1).

By the applicability of the Lagrangian formalism we expected also to have a Hamiltonian formalism. But when we turned to an explicit investigation of this formalism we encountered several new problems all due to the nonlocality in time. Owing to the fact that the generators are nonlocal in time, we have to solve the equations of motion (at least formally) in order to check the Poisson-bracket relations. However, the equations of motion are also nonlocal in time, which implies that the Newtonian Cauchy problem is not fulfilled; i.e., the specification of the position and velocity at a particular time does not yield a unique solution. We therefore need a subsidiary condition which selects only one solution out of all possible ones if the Newtonian degrees of freedom are to be retained. In the literature such a selection principle has been given which seems to be quite general (but perhaps too strong); it demands that one always chooses the solution which has a nonrelativistic limit.[19]

We have investigated a particular set of solutions, namely, the one which is obtained by the iterative procedure (4.16) which sets out from straight lines. These solutions are uniquely determined by the Newtonian initial data, but the procedure converges only for distant collisions. However, for this class of solutions we found that we indeed have a Hamiltonian formalism as expected. We also found two equivalent sets of canonical variables. A negative result for the set of solutions considered was that the physical positions cannot be chosen as canonical variables. It seems therefore as if the no-interaction theorem,[27] which is supposed to hold only for instantaneous action-at-a-distance theories, might be generalizable to noninstantaneous ones. In fact our way to construct the Hamiltonian formalism circumvents the no-interaction theorem exactly in the way proposed by Kerner and Hill[28,29] for the instantaneous case. One may therefore ask whether there is a relation between instantaneous and noninstantaneous action-at-a-distance theories, or more precisely: Do there exist equivalent equations of motion to the ones considered in the present paper (together with the selection rule) which are just differential equations of second order? We do not know, but we would like to remark that if one reduces the equations of motion to differential equations of second order by means of the solutions considered in Sec. IV,[28,48] then the solutions of these new equations will not coincide with the original ones (only the first-order terms do). Presently we have, therefore, no general reason to believe that the no-interaction theorem is applicable here. There might exist other sets of solutions (e.g., the circular ones[17-20]) than those obtained from the iterative procedure (4.16), for which the physical position can be chosen as a canonical variable.

We note in this connection that although we might have a selection principle which gives us a unique solution for any Newtonian Cauchy data, we have no uniquely defined Poisson bracket for all these solutions. The reason is that the Poisson brackets in Sec. IV are only applicable to scattering processes, where we have asymptotic straight-line solutions. Therefore we also have no hint how to find the canonical variables for the other solutions, which of course have to be investigated before one can draw any definite conclusions about the Hamiltonian formalism in general. However, we notice one general property of the Hamiltonian formalism: It is not possible to define any variable p_a^i such that the Hamiltonian is given by

$$H(t)=\sum_{a=1}^{N} p_a^i(t)\dot{x}_a^i(t)-L(t)\ , \qquad (5.2)$$

which is a heavily used relation in ordinary Hamiltonian methods (together with the fact that p_a^i and x_a^i are canonical variables there). As a consequence, the present generalized canonical formalism will always differ considerably from the corresponding nonrelativistic formalism.

VI. OUTLOOK: QUANTIZATION

Finally we would like to outline the various possible ways to quantize relativistic particle mechanics following the conventional prescriptions, and thereby in particular point to the problems.

The Heisenberg picture. We have shown that the classical theory allows for a Hamiltonian formulation and therefore a canonical quantization is also possible. Following Dirac's prescription we replace the Poisson-bracket relation (4.18) by the following canonical commutation relation:

$$p^j_{b,\text{in}}x^i_{a,\text{in}}-x^i_{a,\text{in}}p^j_{b,\text{in}}=i\hbar\delta^{ij}\delta_{ab}\ . \qquad (6.1)$$

The quantum solution $x_a^i(t)$ is now obtained by means of the expansion (4.16). (Cf. the Yang-Feldman quantization.[50] In the particle case this quantization was proposed in Ref. 51.) But because of (4.41) we arrive here at a serious ordering problem in the various dynamical quantities. However, it is possible to retain the canonical structure for appropriately ordered generators.[52] But when we impose, in a similar fashion, the canonical commutation relations (6.1) on the canonical variables $x^i_{a,\text{out}}$ and $p^i_{a,\text{out}}$, we arrive at a different quantum solution $x_a^i(t)$. This is due to the fact that here we have to choose differently ordered generators in order to retain the canonical structure. [For the Hamiltonian one gets, e.g., $H_0(\text{in})=H(t)\neq H'(t)=H_0(\text{out})$.] The origin of this nonuniqueness of the generators (and thereby of the solutions) in the quantum case is most easily seen by means of the quantum action principle.[52,53] If we let the variables in the action be operators and perform a variation, then we are here no longer allowed to assume that this variation is a c number. The reason is that c-number variations can only be used in theories where the physical positions are canonical variables. Therefore, we have here to make use of q-number variations instead. As a consequence we will also obtain an additional term in δW_{21} which contains commutators of the variations. This term can then be split in many different ways such that $\delta W_{21}=F(t_1)-F(t_2)$ still is applicable. Thus, $F(t)$ will be nonuniquely defined. As this will occur irrespective of how one symmetrizes the Lagrangian, we infer that $x^i_{a,\text{in}}$ and $x^i_{a,\text{out}}$ can never be unitarily equivalent. Hence, we conclude that one cannot in a consistent fashion perform a canonical quantization of the solutions considered in Sec. IV and probably not for any solution for which the physical position is not a canonical variable. The nonuniqueness of the generators tells us that we do not fulfill the condition of asymptotic covariance in the S-matrix theory of Fong and Sucher,[55] which implies that there does not exist any relativistically invariant S matrix. The unitary inequivalence of $x^i_{a,\text{in}}$ and $x^i_{a,\text{out}}$ implies furthermore that there does not exist any unitary S matrix. In the author's opinion no meaningful S matrix can be defined.[52]

The Feynman picture. We have shown that the classical theory allows for a Lagrangian formulation, and thus it would also be possible to perform a quantization by means of Feynman's functional method.[54] What this would mean in the present context is an open question, in particular since the relation (5.2) is not satisfied here [nor do we have $L(t)=\sum_{a=1}^{N}p^i_{a,\text{in}}\dot{x}^i_{a,\text{in}}-H(t)$]. In this connection it should also be noted that Feynman's method is not applicable to the finite Fokker action as this action does not have the necessary additivity property $W_{13}=W_{12}+W_{23}$ (required by the superposition principle).

The Schrödinger picture. Consider particles which have the same type of interaction with an overall function $G(x_a, x_b)$. If one also includes self-interaction of the same type and with the same function, then one may say that a field theory with nonlocal interaction represents the Schrödinger picture of the theory[15,56] (cf. also Ref. 57). This is perhaps not too close a correspondence but, on the other hand, one may notice that even the free Dirac or Klein-Gordon equation is not the straightforward Schrödinger picture of a free relativistic particle. Field theory with nonlocal interaction has a structure very similar to that of relativistic particle mechanics; it has, e.g., a Lagrangian and Hamiltonian formalism of the same generalized form as the one presented in this paper. But then one also encounters the same problem as above when performing any second quantization of the theory, which is therefore not possible at least within the Heisenberg picture.[52]

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APPENDIX A: THE GENERALIZED VARIATION METHOD

In this appendix the applicability of the action principle is extended to nonlocal Lagrangians of the type considered in Sec. III, i.e., Lagrangians containing integrals over time. General formulas are derived by making use of the variation method for Lagrangians with higher-order time derivatives formally generalized to infinite-order Lagrangians.

Consider the Lagrangian

$$L^n(t) \equiv L(x_a^i, Dx_a^i, \ldots, D^n x_a^i), \tag{A1}$$

where $D \equiv d/dt$ and $n < \infty$. Application of the action principle

$$\delta W_{21} = F(t_1) - F(t_2) \tag{A2}$$

to the action

$$W_{21} = \int_{t_1}^{t_2} dt\, L^n(t) \tag{A3}$$

yields the equations of motion

$$\frac{\partial L^n(t)}{\partial x_a^i(t)} - Dp_{a,1}^i = 0, \quad a = 1, \ldots, N, \tag{A4}$$

and the infinitesimal generator

$$F(t) = -\sum_{a=1}^{N} \sum_{l=0}^{n-1} p_{a,1+l}^i \delta_0 D^l x_a^i - L^n \delta t, \tag{A5}$$

where

$$p_{a,r}^i = \sum_{k=0}^{n-r} (-D)^k \frac{\partial L^n}{\partial(D^{k+r} x_a^i)}, \quad r = 1, \ldots, n, \quad a = 1, \ldots, N \tag{A6}$$

are Ostrogradsky's[58] generalized momenta. [A Hamiltonian formalism may be constructed with $p_{a,r}^i$ and $D^{r-1}x_a^i$ $(r = 1, \ldots, n,\ a = 1, \ldots, N)$ as canonical variables.[58]]

Now let $n \to \infty$. Formally one then gets the equations of motion

$$\sum_{k=0}^{\infty} (-D)^k \frac{\partial L(t)}{\partial[D^k x_a^\mu(t)]} = 0, \quad a = 1, \ldots, N, \quad \mu = (0), 1, 2, 3 \tag{A7}$$

and the infinitesimal generator

$$F(t) = -\sum_{a=1}^{N} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} (-D)^k \frac{\partial L(t)}{\partial[D^{k+l+1} x_a^\mu(t)]} \times \delta_0 D^l x_a^\mu(t) - L(t)\delta t. \tag{A8}$$

The interaction Lagrangians in Sec. III are functions and/or functionals of x_a^μ and $\dot{x}_a^\mu$. Assume that a given one has the following form:

$$L_I(t) = \int d\xi\, \mathcal{L}(x_a^\mu(t+\xi), \ldots), \tag{A9}$$

where $\mathcal{L}$ is a function of $x_a^\mu(t+\xi)$ for at least one value of a, and a function/functional of other x_a^μ and $\dot{x}_a^\mu$. Then one finds that

$$\frac{\partial L_I(t)}{\partial[D^r x_a^\mu(t)]} = \int d\xi \frac{\partial \mathcal{L}}{\partial x_a^\mu(t+\xi)} \frac{\partial x_a^\mu(t+\xi)}{\partial[D^r x_a^\mu(t)]} = \int d\xi\, f_{a\mu}(t,\xi) \frac{\xi^r}{r!}, \quad r = 0, \ldots \tag{A10}$$

where

$$f_{a\mu}(t,\xi) \equiv \frac{\partial \mathcal{L}}{\partial x_a^\mu(t+\xi)}$$

and

$$x_a^\mu(t+\xi) = \sum_{l=0}^{\infty} \frac{\xi^l}{l!} D^l x_a^\mu(t)$$

(formally). The part of the infinitesimal generator which is associated with the particular x_a^μ explicitly given in (A9) is therefore ($\delta t = 0$ here and in what follows)

$$F_a(t) = -\sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \int d\xi \frac{\xi}{(k+l+1)!} (-\xi D)^k \times f_{a\mu}(t,\xi) \delta_0[(\xi D)^l x_a^\mu(t)], \tag{A11}$$

whose time derivative is

$$DF_a(t) = \int d\xi [f_{a\mu}(t-\xi,\xi)\delta_0 x_a^\mu(t) - f_{a\mu}(t,\xi)\delta_0 x_a^\mu(t+\xi)]. \tag{A12}$$

It is a straightforward matter to show that (A11) may be written as

$$F_a(t) = -\int d\xi\, \xi \frac{\sin\frac{1}{2}i\xi D}{\frac{1}{2}i\xi D} f_{a\mu}(t - \tfrac{1}{2}\xi, \xi) \times \delta_0 x_a^\mu(t + \tfrac{1}{2}\xi), \tag{A13}$$

an expression which in turn may be transformed into a convolution

$$F_a(t) = -\int d\xi \int dt'\Delta(t-t', \tfrac{1}{2}\xi) \times f_{a\mu}(t'-\tfrac{1}{2}\xi, \xi) \times \delta_0 x_a^\mu(t'+\tfrac{1}{2}\xi)\,, \tag{A14}$$

where

$$\Delta(t,\lambda) \equiv \theta(t+\lambda) - \theta(t-\lambda)\,. \tag{A15}$$

The part of the equations of motion associated with the x_a^μ explicitly given in (A9) is, according to (A7) and (A10),

$$\int d\xi f_{a\mu}(t-\xi, \xi) \quad (+\cdots = 0)\,. \tag{A16}$$

Consider now an interaction Lagrangian of the form

$$L_I'(t) = \int d\xi\, \mathcal{L}'(\dot{x}_a^\mu(t+\xi), \ldots)\,, \tag{A17}$$

where $\mathcal{L}'$ is a function of $\dot{x}_a^\mu(t+\xi)$ for at least one value of a, and a function/functional of other x_a^μ and $\dot{x}_a^\mu$. Instead of (A10) one now finds

$$\frac{\partial L_I'(t)}{\partial[D^r x_a^\mu(t)]} = \int d\xi \frac{\partial \mathcal{L}'}{\partial \dot{x}_a^\mu(t+\xi)} \frac{\partial \dot{x}_a^\mu(t+\xi)}{\partial[D^r x_a^\mu(t)]}$$

$$= \begin{cases} \int d\xi g_{a\mu}(t,\xi) \dfrac{\xi^{r-1}}{(r-1)!}\,, & r = 1, \ldots \\ 0, & r = 0 \end{cases} \tag{A18}$$

where

$$g_{a\mu}(t,\xi) \equiv \frac{\partial \mathcal{L}'}{\partial \dot{x}_a^\mu(t+\xi)}$$

and

$$\dot{x}_a^\mu(t+\xi) = \sum_{l=0}^{\infty} \frac{\xi^l}{l!} D^{l+1} x_a^\mu(t)$$

(formally). The part of the infinitesimal generator associated with the particular $\dot{x}_a^\mu$ explicitly given in (A17) is therefore

$$F_{\dot{a}}(t) = -\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \int d\xi \frac{1}{(k+l)!}(-\xi D)^k g_{a\mu}(t,\xi) \times \delta_0[(\xi D)^l x_a^\mu(t)]\,, \tag{A19}$$

whose time derivative is

$$DF_{\dot{a}}(t) = -\int d\xi [Dg_{a\mu}(t-\xi,\xi)\delta_0 x_a^\mu(t) + g_{a\mu}(t,\xi) D\delta_0 x_a^\mu(t+\xi)]\,. \tag{A20}$$

(A19) may be rewritten as

$$F_{\dot{a}}(t) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \int d\xi \frac{\xi}{(k+l+1)!}(-\xi D)^k D g_{a\mu}(t,\xi)\delta_0[(\xi D)^l x_a^\mu(t)] - \int d\xi\, g_{a\mu}(t,\xi)\delta_0 x_a^\mu(t+\xi)\,, \tag{A21}$$

which in turn, by means of (A13) and (A14), may be transformed into

$$F_{\dot{a}}(t) = \int d\xi \int dt'\Delta(t-t', \tfrac{1}{2}\xi) D' g_{a\mu}(t'-\tfrac{1}{2}\xi, \xi)\delta_0 x_a^\mu(t'+\tfrac{1}{2}\xi) - \int d\xi\, g_{a\mu}(t,\xi)\delta_0 x_a^\mu(t+\xi)\,. \tag{A22}$$

By partial integration one also gets the following alternative expression:

$$F_{\dot{a}}(t) = -\int d\xi \int dt'\Delta(t-t', \tfrac{1}{2}\xi) g_{a\mu}(t'-\tfrac{1}{2}\xi, \xi) D'\delta_0 x_a^\mu(t'+\tfrac{1}{2}\xi) - \int d\xi\, g_{a\mu}(t-\xi,\xi)\delta_0 x_a^\mu(t)\,. \tag{A22$'$}$$

The part of the equations of motion associated with the $\dot{x}_a^\mu$ explicitly given in (A17) is, according to (A7) and (A18), here given by

$$-D\int d\xi f_{a\mu}(t-\xi,\xi) \quad (+\cdots = 0)\,. \tag{A23}$$

The Lagrangians (A9) and (A17) comprise all nonlocal Lagrangians depending on the positions and velocities of the particles. In applications one has to write the given Lagrangian like (A9) and (A17) for each position and velocity involved. The above expressions are then valid for each such separation and the final formulas are arrived at by simply adding these expressions together. The infinitesimal generator (A8) gets, e.g., the following final form:

$$F(t) = \sum_{\text{sep.}\,a} F_a(t) + \sum_{\text{sep.}\,\dot{a}} F_{\dot{a}}(t) - L(t)\delta t\,, \tag{A24}$$

where the last term comes from the end-point variation.

A corresponding variation method has also been developed for nonlocal field theories.[53]

[1]For an interesting historical account of action at a distance contra action by contact, see M. B. Hesse, *Forces and Fields* (Thomas Nelson and Sons, New York, 1961).

[2]C. F. Gauss, Werke 5, 629 (1867).

[3]A. Liénard, L'Éclairage Électrique 16, 5, 53, 106 (1898); E. Wiechert, Archives Néerland (2) 5, 549 (1900); Ann. Phys. 4, 676 (1901).

[4]K. Schwarzschild, Nachr. K. Ges. Wiss. Goett, 128 (1903); 132 (1903); 245 (1903); H. Tetrode, Z. Phys. 10, 317 (1922); G. N. Lewis, Proc. Nat. Acad. Sci 12, 22 (1926).
[5]A. D. Fokker, Z. Phys. 58, 386 (1929); Physica 9, 33 (1929); 12, 145 (1932).
[6]R. P. Feynman and J. A. Wheeler, Rev. Mod. Phys. 17, 157 (1945); 21, 425 (1949).
[7]P. A. M. Dirac, Proc. R. Soc. A167, 148 (1938).
[8]F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965).
[9]See, e.g., H. A. Lorentz, *Theory of Electrons*, second edition (1915) (Dover, New York, 1952).
[10]C. S. Shen, Phys. Rev. D 6, 3039 (1972).
[11]P. Havas, Phys. Rev. 87, 309 (1952); 91, 997 (1953).
[12]P. Ramond, Phys. Rev. D 7, 449 (1973).
[13]J. W. Dettman and A. Schild, Phys. Rev. 95, 1057 (1954).
[14]H. van Dam and E. P. Wigner, Phys. Rev. 138, B1576 (1965); 142, 838 (1966).
[15]R. P. Feynman, Phys. Rev. 74, 939 (1948).
[16]A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (MacMillan, New York, 1964).
[17]A. Schild, Phys. Rev. 131, 2762 (1963).
[18]C. M. Andersen and H. C. von Baeyer, Ann. Phys. (N.Y.) 60, 67 (1970).
[19]C. M. Andersen and H. C. von Baeyer, Phys. Rev. D 5, 802 (1972).
[20]P. Cordero and G.-C. Ghirardi, J. Math. Phys. 14, 815 (1973).
[21]D.-C. Chern and P. Havas, J. Math. Phys. 14, 470 (1973).
[22]R. P. Feynman, Science 153, 699 (1966).
[23]A. Degasperis, Phys. Rev. D 3, 273 (1971).
[24]See, e.g., J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[25]P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949); L. H. Thomas, Phys. Rev. 85, 868 (1952); B. Bakamjian and L. H. Thomas, *ibid.* 92, 1300 (1953); L. L. Foldy, *ibid.* 122, 275 (1961). [These as well as some other relevant references can be found in the following reprint collection: *The Theory of Action-at-a-Distance in Relativistic Particle Dynamics*, edited by E. H. Kerner (Gordon and Breach, New York, 1972).]
[26]A. Peres, Phys. Rev. Lett. 27, 1666 (1971); 28, 392 (1972).
[27]D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35, 350 (1963) (two particles); J. T. Cannon and T. F. Jordan, J. Math. Phys. 5, 299 (1964) (three particles); H. Leutwyler, Nuovo Cimento 37, 556 (1965) (*N* particles).
[28]E. H. Kerner, J. Math. Phys. 6, 1218 (1965).
[29]R. N. Hill and E. H. Kerner, Phys. Rev. Lett. 17, 1156 (1966); R. N. Hill, J. Math. Phys. 8, 1756 (1967).
[30]L. Bel, C. R. Acad. Sci. A 273, 405 (1971).
[31]W. Pauli, Nuovo Cimento 10, 648 (1953); J. L. Martin, Proc. R. Soc. A251, 536 (1959).
[32]D. G. Currie and T. F. Jordan, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. XA, p. 91.
[33]D. G. Currie, Phys. Rev. 142, 817 (1966).
[34]R. N. Hill, J. Math. Phys. 8, 201 (1967).
[35]E. H. Kerner, J. Math. Phys. 9, 222 (1968); F. J. Kennedy, *ibid.* 10, 1349 (1969); J. G. Wray, Phys. Rev. D 1, 2212 (1970).
[36]J. G. Wray, Nuovo Cimento 12B, 231 (1972) [missing figure in 14B, 156 (1973)].
[37]E. Noether, Nachr. Akad. Wiss. Goett. Math.-Phys. 235 (1918), or see E. L. Hill, Rev. Mod. Phys. 23, 253 (1951).
[38]A. Katz, J. Math. Phys. 10, 2215 (1969).
[39]P. Havas, in *Problems in the Foundations of Physics*, edited by M. Bunge (Springer, Berlin, 1971), p. 31.
[40]A. N. Whitehead, *The Principles of Relativity* (Cambridge Univ. Press, Cambridge, England, 1922).
[41]A. Schild, in *Evidence for Gravitational Theories, Proceedings of the International School of Physics "Enrico Fermi," Course XX*, edited by C. Møller (Academic, New York, 1963), pp. 69–115.
[42]R. Hargreaves, Trans. Camb. Philos. Soc. 22, 191 (1917); E. H. Kerner, J. Math. Phys. 3, 35 (1962); F. J. Kennedy, *ibid.* 10, 1349 (1969); H. W. Woodcock and P. Havas, Phys. Rev. D 6, 3422 (1972).
[43]R. E. Peierls, Proc. R. Soc. A214, 143 (1952).
[44]P. Gulmanelli, Nuovo Cimento 10, 1582 (1953).
[45]A. Staruszkiewicz, Acta Phys. Pol. 33, 1007 (1968).
[46]G. N. Plass, Rev. Mod. Phys. 33, 37 (1961).
[47]A. Staruszkiewicz, Ann. Phys. 23, 66 (1969).
[48]T. Taniuti, Prog. Theor. Phys. 13, 505 (1955); 15, 19 (1956).
[49]C. M. Andersen and H. C. von Baeyer, Phys. Rev. D 5, 2470 (1972).
[50]C. N. Yang and D. Feldman, Phys. Rev. 79, 972 (1950).
[51]J. Rzewuski, Nuovo Cimento 10, 90 (1953).
[52]R. Marnelius , Phys. Rev. D (to be published).
[53]R. Marnelius, Phys. Rev. D 8, 2472 (1973).
[54]R. P. Feynman, Rev. Mod. Phys. 20, 367 (1948).
[55]R. Fong and J. Sucher, J. Math. Phys. 5, 456 (1964).
[56]R. P. Feynman, Phys. Rev. 74, 1430 (1948).
[57]E. C. G. Sudarshan, Fields and Quanta 2, 175 (1972).
[58]M. Ostrogadsky, Mêm. Acad. St.-Pêt. 6, 385 (1850), or see E. T. Whittaker, *Analytical Dynamics*, fourth edition (Cambridge Univ. Press, Cambridge, England, 1937), Chap. X.