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Photon-photon scattering with synchrotron radiation

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The number and (energy as well as polarization) spectrum of photons scattered in the forward direction in the reaction center-of-mass frame are calculated when two beams of synchrotron radiation are scattered on each other. The synchrotron radiation is produced by electrons or positrons passing through a locally intense magnetic field which causes a "break" or "wave" in their orbit. Its properties depend on the following parameters: energy and local radius of orbit curvature of the emitting particles, intensity of the emitting current, and angle of emission of the synchrotron radiation. The dependence of the calculated numbers on these parameters is exhibited.

I. BACKGROUND

Quantum electrodynamics predicts the (elastic as well as inelastic) scattering of two photons on each other. This was first noted by Halpern.[1] Shortly thereafter, Heisenberg and Euler[2,3] performed careful calculations to evaluate this effect, concentrating their attention on low-energy scattering. Akhiezer[4] studied photon-photon scattering at high energies. Several years later, Karplus and Neuman[5] gave a general discussion of all the photon-photon scattering amplitudes in quantum electrodynamics up to fourth order, and evaluated them for certain selected values of the scattering parameters. The subject was well reviewed and simplified and the discussion was made more complete by Jauch and Rohrlich.[6] Subsequently Sannikov,[7] De Tollis,[8] and Violini[8] applied the techniques of dispersion relations and succeeded in considerably simplifying the treatment of this nevertheless still complicated theoretical problem.

The calculated photon-photon cross sections are very small, and are beyond the range of traditional experimental techniques. For photons in the visible spectrum, the elastic cross section is about 10^{-65} cm^2. Although the first calculations were performed several decades ago, elastic scattering of real photons on each other in vacuum was never observed. During the last few years, experimental technology started on a new "spiral" of development which is still in progress. This progress is tied to the availability of new accelerators, storage rings, and the accompanying detection devices. The question arises as to whether the new technology can be used to study photon-photon scattering either at low or at high energies. In earlier works by one of the authors[9] various possibilities were surveyed, the obtainable counting numbers were estimated, and it was concluded that several new approaches may become feasible in the foreseeable future. It was found that one of the most promising techniques would consist of using storage rings with a locally intense magnetic field which would cause a sharp "break" or "wave" in the circulating particle orbit to produce two beams of synchrotron radiation, and then scattering two such beams of radiation on each other.

At the present time, several new electron storage rings are being planned or are under construction. The time seems appropriate to make a more detailed study of the possibility of observing photon-photon scattering in the collision of two beams of synchrotron radiation.

II. PURPOSE OF THE PRESENT PAPER

Our purpose is to calculate the properties of elastically scattered photons which are produced in the collision of two beams of synchrotron radiation.

The two beams of synchrotron radiation are produced by electron or positron beams circulating in one or more rings (see Fig. 1). One can use two beams in one ring, or two rings with one beam in each. Either or both beams can be electron or positron beams. In the following, unless explicitly stated otherwise, "electron" means either an electron or a positron. The energy spectrum and polarization of the radiation can be varied by changing the energy of the circulating particles, by varying the angle of emission of the radiation used, and by adjusting the local radius of curvature of the electron orbits at the points where the

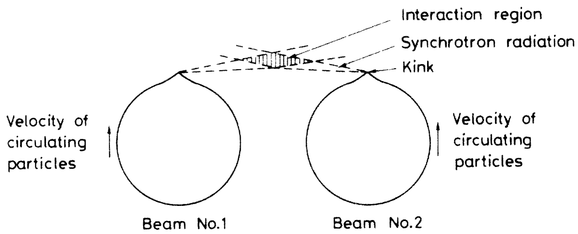

FIG. 1. Two circulating (electron or positron) beams produce intense synchrotron radiation at the points where the local radius of curvature of the beam is small. The two beams may circulate in one ring, or (as drawn in the figure) in two rings. Two beams of intense synchrotron radiation are directed towards the interaction region, where photon-photon scattering takes place.

synchrotron radiation is emitted. The influence of these parameters on the properties of the scattered photons is studied in this paper.

In order to produce photon-photon scattering experiments one wants to produce two intense photon beams to scatter on each other. The intensity of the emitted synchrotron radiation can be increased by decreasing the local radius of curvature of the circulating electron beam at those points where the radiation is emitted. One may be tempted to conclude from this that it is not possible to perform photon-photon scattering in this way because one could argue as follows: To obtain two intense synchrotron radiation beams, one needs storage rings with a small radius. But such rings are economically unfeasible, because in them a circulating electron would lose an unacceptable amount of energy during each revolution around the ring, simply by radiating it out. This argument is wrong, however, because one does not need rings with a small radius. What one does need is only one section in each of the two circulating electron beam orbits, where the *local* radius of curvature is small. Along the rest of the orbit, the radius of curvature may be as large as usual. This can be achieved in practice by introducing a short "break" or "wave" into the otherwise smoothly circulating electron beam (see Fig. 1). This proposed modification will make this short orbit section an intense source of synchrotron radiation, but will leave the rest of the ring unchanged. Such a modification is technically feasible; it requires the use of some compensating magnets. It permits the ring to be used for a variety of experiments with synchrotron radiation, including photon-photon scattering, and at the same time all the usual experiments with storage rings can also be performed. (Nowadays, a typical storage ring has a radius of the order of 100 m. Introducing a break with a length of 1 cm would increase radiation losses over only a fraction 10^{-4} of the total orbit. In this short section, the *local* radius of curvature, R, can be much less than the average radius of the ring, and a particularly interesting case is when R is in the region between 10 and 300 cm.)

In the type of experiment outlined in Fig. 1, the average reaction center-of-mass frame is the laboratory rest frame, and the calculated differential cross section as a function of angle does not have any sharp peaks in this frame. One can measure the spectrum of photons leaving the interaction region along any particular line. The elimination of the background is easiest if one measures photons leaving with a momentum perpendicular to the plane (x, y) in which both electron beams cirulate. Detectors can be placed far away from this plane, and the relatively high energy photons detected without strong background interference. At least from this point of view, the more interesting quantities we could have calculated are the polarization and energy spectrum and the total number of photons leaving the interaction region with a momentum perpendicular to the (x, y) plane. Instead, we chose to calculate at first the properties of those photons which are scattered in the forward direction in the reaction center-of-mass frame. The calculation of these is somewhat simpler and it seemed reasonable to perform this calculation first. The results are exact for forward scattering. They are approximately valid also near the forward direction, because we know from the second of Refs. 8 that the differential scattering cross section for photon-photon scattering at the forward direction (i.e., $\theta'=0$) has zero derivative as a function of the reaction center-of-mass scattering angle θ'. For the range of parameters to be discussed below, the elastic photon-photon differential cross section for $|\theta'| \leq 10°$ should differ from the corresponding value at $\theta'=0°$ by not more than about 4%. In these experiments the reaction center-of-mass energy and angle of each scattered photon can be determined, and, in fact, the average reaction center-of-mass frame is just the laboratory frame. Consequently, when designing a realistic experiment, the results reported in this paper should be helpful. They are summarized in the figures. At a later date we plan to study the spectrum and the number of photons scattered perpendicularly to the (x, y) plane.

In the Appendix we list what we believe are misprints or arithmetic errors in the published literature concerning photon-photon scattering.

III. CALCULATIONS

At each point B where synchrotron radiation is emitted, we define a local Cartesian coordinate frame as explained in Fig. 2. The direction of any vector $\vec{k}$ originating at B can be characterized in this coordinate system by two angles ϕ and ψ.

With each photon of momentum $\vec{k}$, emitted at B, we associate a Cartesian coordinate system whose axes are $\hat{\epsilon}_{\parallel}$, $\hat{\epsilon}_{\perp}$, and $\hat{k}$ (see Fig. 2). The component of the electric vector $\vec{E}$ parallel to $\hat{\epsilon}_{\parallel}$ is $E_{\parallel}$, and $E_{\perp}$ is defined similarly. Circular polarization states are defined by the components.

$$E_{\pm} \equiv \frac{1}{\sqrt{2}}(E_{\perp} \mp iE_{\parallel}) \ . \tag{1}$$

When the complete orbit of the circulating electrons is a perfect circle and radiation emitted at every point along the orbit is permitted to reach a certain point A, then, due to the circular symmetry, the observed synchrotron radiation will depend only on the distance (A, C) and on the z coordinate of the position vector of A. It will not depend on the azimuthal angle of A, which determines the position of the projection of A onto the (x, y) plane. When the orbit is not a perfect circle, or if radiation from some orbit points is not permitted to reach A, then the observed synchrotron radiation will depend on the azimuthal angle. For cases of practical interest, the synchrotron radiation emitted at any point B is strongly peaked along y, because the synchrotron radiation is emitted within a cone of half angle $\Delta\psi \simeq 1/\gamma$ around the axis of the circulating electron beam. Therefore, in cases of practical interest, the value of the radiation at a point A depends only on a very short section of the electron orbit. For example,

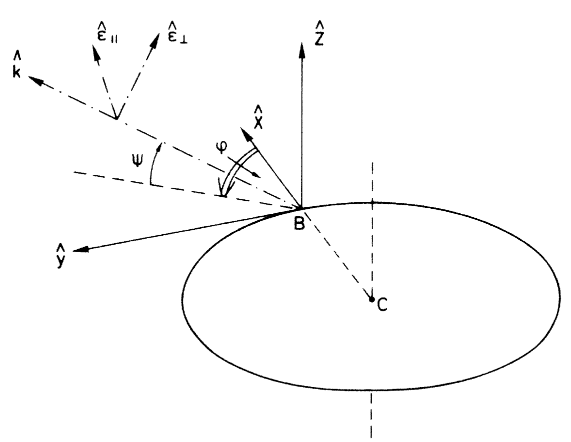

FIG. 2. The figure shows the axes $\hat{x}$, $\hat{y}$, and $\hat{z}$ of a Cartesian coordinate frame located at the point B of emission of the synchrotron radiation. The x axis is parallel to the position vector of B measured from the point C which is the center of the circle tangent to the particle orbit at B. When the circulating particle is an electron, then $\hat{y}$ is parallel to the velocity of the circulating particle at B. If the particle is a positron, $\hat{y}$ is antiparallel to its velocity at B; $\hat{z} \equiv \hat{x} \times \hat{y}$. The unit vector $\hat{\epsilon}_{\parallel}$ is parallel to $\hat{x}$, and $\hat{\epsilon}_{\perp} \equiv \hat{k} \times \hat{\epsilon}_{\parallel}$.

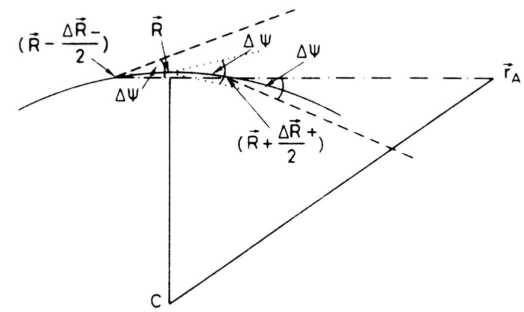

FIG. 3. Synchrotron radiation emitted at any point is concentrated within the angular interval $\Delta\psi$. Only points between $[\vec{R} - \Delta\vec{R}_-/2]$ and $[\vec{R} + \Delta\vec{R}_+/2]$ contribute significantly to the radiation at A. If $R \gg \Delta R_+$, ΔR_-, then $|\Delta\vec{R}_+| \approx |\Delta\vec{R}_-|$

if $\gamma = 11\,742.1$, then $\Delta\psi \simeq 10^{-4}$ rad, and if $R = 10^2$ cm, then the length of the beam orbit which contributes to point A_1 is approximately $10^{-4} \times 10^2$ cm $= 10^{-2}$ cm. Thus, as far as the intensity of the synchrotron radiation at A is concerned, it suffices if the break or wave in the electron orbit has a length of only about 10^{-2} cm. In practice, we are interested in the radiation intensity not only at point A, but also within T, a finite neighborhood around A. Nevertheless, for most cases of practical interest, the radiation intensity in T will still depend only on a short section of the electron orbit, with a length of the order of 1 cm or less. As long as the break or kink has this length or more, the distribution of synchrotron radiation within T is as if the radiation had been emitted from a completely circular electron orbit with radius R. In particular, within T the radiation will then be independent of the azimuthal angle ϕ.

We will use the following conventions and notation:

"local": value calculated at point B,
"electron": either electron or positron, unless explicitly stated otherwise,
e: electron charge,
c: velocity of light in vacuum,
m: rest mass of the electron,
v: velocity of circulating electrons,
$\mathcal{E}$: local energy of the circulating electron,
$\gamma \equiv \mathcal{E}/mc^2$,
R: local radius of curvature of circulating electrons,
ω: (2π/period) of emitted synchrotron radiation,
$\vec{k}$: photon wave vector (i.e., momentum/$\hbar$),
$\omega_c \equiv \frac{3}{2}(c/R)\gamma^3$,
$\xi \equiv \frac{1}{2}(\omega/\omega_c)(1+\gamma^2\psi^2)^{3/2}$,
$P_{\parallel}$, $P_{\perp}$: power radiated by one circulating electron in the form of synchrotron radiation polarized, respectively, along $\hat{\epsilon}_{\parallel}$ and $\hat{\epsilon}_{\perp}$,
$\rho_{\parallel\nu}(\vec{r}_{A\nu}, \omega)$, $\rho_{\perp\nu}(\vec{r}_{A\nu}, \omega)$: density at point A of photons of frequency ω, emitted by

beam No. ν, polarized, respectively, along $\hat{\epsilon}_{\parallel}$ and $\hat{\epsilon}_{\perp}$; the position vector $\vec{r}_{A\nu}$ of point A is measured from point C,

$\rho_{+\nu}(\vec{r}_{A\nu}, \omega)$, $\rho_{-\nu}(\vec{r}_{A\nu}, \omega)$: density at point A of photons of frequency ω, emitted by beam No. ν, with polarization E_+ and E_-, respectively,

n: number of electrons in beam per unit time, passing through a surface perpendicular to the beam,

subscript ν: ($\nu = 1, 2$) means that the subscripted quantity refers to beam No. ν, or to photons emitted by that beam,

prime: a prime indicates that the quantity is evaluated in the reaction center-of-mass frame (see below).

(2)

The intensity of the synchrotron radiation emitted at B has been calculated and evaluated for various values of the parameters.[10-12] The result is

$$\frac{\partial}{\partial\psi}\frac{\partial}{\partial\omega}P_{\parallel}=\frac{3}{4\pi^2}\frac{e^2}{R}\left(\frac{\omega}{\omega_c}\right)^2\gamma^2(1+\gamma^2\psi^2)^2K_{2/3}{}^2(\xi)\ , \tag{3a}$$

$$\frac{\partial}{\partial\psi}\frac{\partial}{\partial\omega}P_{\perp}=\frac{3}{4\pi^2}\frac{e^2}{R}\left(\frac{\omega}{\omega_c}\right)^2\gamma^2(1+\gamma^2\psi^2)^2 \times\frac{\gamma^2\psi^2}{1+\gamma^2\psi^2}K_{1/3}{}^2(\xi)\ , \tag{3b}$$

$$\frac{E_{\parallel}}{E_{\perp}}=i\,\frac{(1+\gamma^2\psi^2)^{1/2}}{\gamma\psi}\,\frac{K_{2/3}(\xi)}{K_{1/3}(\xi)}\equiv i\bar{Q}\ . \tag{4}$$

Both $P_{\parallel}$ and $P_{\perp}$ are found to go to zero rapidly for $\psi \gtrsim 1/\gamma$.

If the total energy carried by all the photon is given, then the number of photons emitted per unit length of beam No. ν is inversely proportional to v_ν. In all cases of interest $v_1 = v_2 \approx c$, and we will assume this. The emitted photons also travel with velocity c, so that

$$\rho_{\parallel\nu}(\vec{r}_{A\nu},\omega_\nu)=\frac{1}{c^2}\,n_\nu\,\frac{1}{\hbar\omega_\nu}\times\int d\psi_\nu\,\frac{\partial}{\partial\psi_\nu}\,\frac{\partial}{\partial\omega_\nu}\,P_{\parallel\nu}(\omega_\nu,\psi_\nu)\,G_\nu(\psi_\nu,\vec{r}_{A\nu})\ ,$$

$$\rho_{\perp\nu}(\vec{r}_{A\nu},\omega_\nu)=\frac{1}{c^2}\,n_\nu\,\frac{1}{\hbar\omega_\nu}\times\int d\psi_\nu\,\frac{\partial}{\partial\psi_\nu}\,\frac{\partial}{\partial\omega_\nu}\,P_{\perp\nu}(\omega_\nu,\psi_\nu)\,G_\nu(\psi_\nu,\vec{r}_{A\nu})\ , \tag{5}$$

where $\int d\psi_\nu$ is an integration over all those values of ψ_ν which contribute (from various points in the beam) photons with frequency ω_ν to point A, and $G_\nu(\psi_\nu, \vec{r}_A)$ is a geometrical factor. Incidentally, the expression preceding $G_\nu(\psi_\nu, \vec{r}_A)$ on the right-hand side of Eq. (5), when multiplied by an interval $\Delta\omega_\nu$, gives the space density of emitted photons with circular frequency in the interval $(\omega_\nu - \frac{1}{2}\Delta\omega_\nu, \omega_\nu + \frac{1}{2}\Delta\omega_\nu)$ immediately to the side of an electron beam of unit height and zero width.

For example, let us make the assumption that each electron beam is such that when B_ν is in the interval $(\vec{R}_\nu - \frac{1}{2}\Delta\vec{R}_{-\nu}, \vec{R}_\nu + \frac{1}{2}\Delta\vec{R}_{+\nu})$ along the electron orbit, then the height and the width of the beam are small compared with $|\vec{R}_\nu|$ as well as $|\vec{r}_{A\nu}|$. Furthermore, let us assume also that the point A is located so that only those photons can reach it which are emitted within a very narrow ψ_ν range, $d\psi_\nu$. If the integrand varies only negligibly over $d\psi_\nu$, then it can be taken outside the integral sign, giving

$$\rho_{\parallel\nu}(\vec{r}_{A\nu},\omega_\nu)=\frac{1}{c^2}\,n_\nu\,\frac{1}{\hbar\omega_\nu}\,\frac{\partial}{\partial\psi_\nu}\,\frac{\partial}{\partial\omega_\nu}\,P_{\parallel\nu}(\omega_\nu,\psi_\nu)\,d\psi_\nu\,G_\nu(\psi_\nu,\vec{r}_A)\ ,$$

$$\rho_{\perp\nu}(\vec{r}_{A\nu},\omega_\nu)=\frac{1}{c^2}\,n_\nu\,\frac{1}{\hbar\omega_\nu}\,\frac{\partial}{\partial\psi_\nu}\,\frac{\partial}{\partial\omega_\nu}\,P_{\perp\nu}(\omega_\nu,\psi_\nu)\,d\psi_\nu\,G_\nu(\psi_\nu,\vec{r}_A)\ . \tag{6}$$

If it is also true that $|\Delta\vec{R}_{-\nu}|$ and $|\Delta\vec{R}_{+\nu}|$ are both small compared with $|\vec{r}_{A\nu} - \vec{R}_\nu|$ and also $|\vec{r}_{A\nu}|$, then all points B_ν which significantly contribute photons to point A are, to a good approximation, $|\vec{r}_A - \vec{R}_\nu|$ distance away from point A. In this case, the geometrical factor $G_\nu(\psi_\nu, \vec{r}_A)$ is easy to evaluate. For example, if $(\vec{r}_{A\nu} - \vec{R}_\nu)$ forms an angle ψ_ν with the plane of that circle (assumed to be horizontal) which is tangent to the electron beam at B_ν and which best approximates the beam orbit there, then $G_\nu(\psi_\nu, \vec{r}_{A\nu})$ is simply the factor $f(\psi_\nu)\,d\psi_\nu$ giving the probability that a photon will be emitted into the angular range $(\psi_\nu - \frac{1}{2}d\psi_\nu, \psi_\nu + \frac{1}{2}d\psi_\nu)$ multiplied by the factor taking into account the fact that near A the photons emitted at B_ν are distributed over a section of a circle whose radius is $|\vec{r}_{A\nu}|$ instead of $|\vec{R}_\nu|$, and that they are spread within a vertical distance above and below A which is not the unit length (denoted by u), but $\frac{1}{2}d\psi_\nu\,|\vec{r}_{A\nu}|$, so that the total vertical spread is twice this, and we obtain, to a good approximation,

$$G_\nu(\psi_\nu,\vec{r}_{A\nu})=f(\psi_\nu)\,\frac{u}{|\vec{r}_{A\nu}-\vec{R}_\nu|}\,\frac{|\vec{R}_\nu|}{|\vec{r}_{A\nu}|}\ . \tag{7}$$

If the geometry of the experiment is such that the above-stated simplifying assumptions do not hold to a good enough approximation, then, in general, $G_\nu(\psi_\nu, \vec{r}_{A\nu})$ will not satisfy Eq. (7). However, for any experimental arrangement $G_\nu(\psi_\nu, \vec{r}_{A\nu})$ can be obtained from straightforward geometrical considerations.

We will describe elastic photon-photon scattering in the reaction center-of-mass frame. In this frame, we denote the wave vectors of the two incoming photons by $\vec{k}_1'$ and $\vec{k}_2'$, and those of the outgoing photons by $\vec{k}_3'$ and $\vec{k}_4'$. We define the unit vector $\hat{\epsilon}_\perp' \equiv \hat{k}_3' \times \hat{k}_1'$, and the unit vectors $\hat{\epsilon}_{\parallel\mu}' \equiv \hat{k}_\mu' \times \hat{\epsilon}_\perp'$. The scattering angle θ' is defined as the angle between $\vec{k}_1'$ and $\vec{k}_3'$, and takes $\vec{k}_3'$ into $\vec{k}_1'$, through a right-handed rotation around $\hat{\epsilon}_\perp'$.

We describe the polarization state of the photon, whose wave vector is $\vec{k}_\mu$, in the right-handed Cartesian frame whose three axes are defined by $\hat{\epsilon}_{\perp\mu}'$, $\hat{\epsilon}_{\parallel\mu}'$, and $\hat{k}_\mu'$. For example, the electric field $\vec{E}_1$ emitted by the first beam has two components, $E_{\perp 1}'$ parallel to $\hat{\epsilon}_\perp'$, and $E_{\parallel 1}'$ parallel to $\hat{\epsilon}_{\parallel 1}'$. These describe photons polarized linearly along $\hat{\epsilon}_\perp'$ and $\hat{\epsilon}_{\parallel 1}'$, respectively. Circular polarization states are defined by components

$$E_{\pm\mu}' \equiv \frac{1}{\sqrt{2}}(E_{\perp\mu}' \mp iE_{\parallel\mu}') \ . \tag{8}$$

In the following we will represent all photon polarization states as a linear superposition of circular polarization states. We denote by λ_μ the circular polarization state of the photon whose wave vector is $\vec{k}_\mu'$, where $\lambda_\mu = +$ or $-$ for $\mu = 1, 2, 3, 4$.

The differential (in θ') elastic cross section of photons with frequency ω' in the reaction center-of-mass frame is[5]

$$\frac{d}{d\theta'}\sigma_{\lambda_1\lambda_2\lambda_3\lambda_4}(\theta', \omega') = \frac{\alpha^4\hbar^2}{4\pi^2 m^2 c^2}\left|\frac{1}{\omega'}M_{\lambda_1\lambda_2\lambda_3\lambda_4}(\theta', \omega')\right|^2 . \tag{9}$$

The functions M have been calculated by Karplus and Neuman in the special cases $\theta' = 0$ and $\theta' = \pi/2$, in terms of three transcendental functions, B, T, and I, defined by them. De Tollis[8] succeeded in expressing M as a function of B, T, and I for all values of θ'. The definitions of B, T, and I in terms of the kinematical variables of the scattering, as well as the expression of M in terms of B, T, and I are lengthy and will not be reproduced here. They can be found in Refs. 5 and 8 (see Appendix for certain misprints).

Suppose that two beams of synchrotron radiation collide and scatter in an interaction region T. The number of events per unit time, in which two photons with polarizations λ_3 and λ_4 are produced, is

$$N_{\lambda_3\lambda_4} = \frac{n_1 n_2 \alpha^4}{4\pi^2 c^5 m^2}\int_{-\pi/2}^{+\pi/2} d\theta' \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d\psi_1 \int d\psi_2 \left| \sum_{\lambda_1,\lambda_2} \frac{1}{\omega'^2} M_{\lambda_1\lambda_2\lambda_3\lambda_4}(\theta', \omega') \times \left[\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\psi_1} P_{\lambda_1}(\omega_1, \psi_1)\right]^{1/2} \times \left[\frac{\partial}{\partial\omega_2}\frac{\partial}{\partial\psi_2} P_{\lambda_2}(\omega_2, \psi_2)\right]^{1/2} \right|^2 \times \int d\vec{r}_A\, G_1(\psi_1, \vec{r}_{A\nu})\, G_2(\psi_2, \vec{r}_{A\nu}) \ , \tag{10}$$

where $\int d\vec{r}_A$ is taken over the whole interaction volume, $\omega' = (\omega_1\omega_2)^{1/2}$, and using Eqs. (4) and (8) we define (ignoring an uninteresting over-all phase) P_λ to be proportional to E_λ^2:

$$\frac{\partial}{\partial\psi_\nu}\frac{\partial}{\partial\omega_\nu}P_{\lambda_\nu} = \frac{\partial}{\partial\psi_\nu}\frac{\partial}{\partial\omega_\nu}\frac{P_\parallel + P_\perp}{2(1+\overline{Q}_\nu^{\,2})}(1+\lambda_\nu\overline{Q}_\nu)^2 \qquad (\lambda_\nu = \pm) \ . \tag{11}$$

The last integral on the right-hand side of Eq. (10) is a purely geometrical factor which can be evaluated directly for any experimental arrangement. In this paper we estimate it only for the case when the conditions stated before Eq. (6), and also before Eq. (7), hold, and if, furthermore, all dimensions of the interaction region are small compared with $|\vec{r}_{A\nu}|$ and $|\vec{r}_{A\nu} - \vec{R}_\nu|$. Then, to a good approximation

$$G \equiv \int d\vec{r}_A\, G_1(\psi_1, \vec{r}_{A\nu})\, G_2(\psi_2, \vec{r}_{A\nu}) = f(\psi_1)f(\psi_2)\frac{u^2}{|\vec{r}_{A\nu} - \vec{R}_1|\,|\vec{r}_{A\nu} - \vec{R}_2|}\,\frac{R_1 R_2}{|\vec{r}_{A_1}|\,|\vec{r}_{A_2}|}\,V \ , \tag{12}$$

where V is the volume of the interaction region. Since the $G_\nu(\psi_\nu, \vec{r}_{A\nu})$ are dimensionless, the G has the dimension of a three-dimensional volume element. If, in addition,

$$R_1 = R_2 \equiv R \;,$$
$$|\vec{r}_{A1} - \vec{R}_1| = |\vec{r}_{A2} - \vec{R}_2| \equiv |\vec{r}_A - \vec{R}| \;,$$
$$|d\psi_1| = |d\psi_2| \equiv |d\psi| \;, \tag{13}$$
$$f(\psi_1) = f(-\psi_1) \;,$$

and if both beams are electron beams or both beams are positron beams, then $\psi_1 = -\psi_2$, but if one of the beams contains electrons and the other positrons, then $\psi_1 = +\psi_2$. Equation (12) now reduces to

$$G = \delta(\psi_1 - (\pm 1)\psi_2) f^2(\psi_1) \left(\frac{uR}{|\vec{r}_A - \vec{R}|\,|\vec{r}_A|} \right)^2 V$$
$$= \delta(\psi_1 - (\pm 1)\psi_2) f^2(\psi_1) g(\psi_1)\, d\psi_1 \;, \tag{14}$$

where $\delta(x)$ is the usual δ function of x, and the + sign hold if one beam is an electron beam and the other is a positron beam, otherwise the − sign holds. The g can be rewritten as

$$g(\psi_1) = \left(\frac{uR}{|\vec{r}_A - \vec{R}|\,|\vec{r}_A|} \right)^2 |\vec{r}_A - \vec{R}| S(\psi_1) \;, \tag{15}$$

where S is the cross section of the interaction region parallel to the common plane of the two circulating beams at the value of ψ_1. In most cases of practical interest $S(\psi_1)$ will be independent of ψ_1. For example, consider the case when both beams have a vertical and horizontal emittence of 10^{-6} cm rad. At points B_1 and B_2, focus the beam to a height and width of 10^{-1} cm. Then 10 cm away from these points, the beam height will be $(10^{-1} + 10^{-4})$ cm. Let $\gamma = 1.2 \times 10^4$, so that $\Delta\psi \simeq 1/\gamma \simeq 8 \times 10^{-5}$. Choose $|\vec{r}_A - \vec{R}| = 10^4$ cm, so that at the interaction region the synchrotron radiation will be appreciable within a distance $[\frac{1}{2} 10^{-1} + (8 \times 10^{-5} + 10^{-5}) \times 10^4]$ cm $= 0.95$ cm from the common plane of the two beams. Let the interaction region extend 1 cm above and below this plane, so that $d\psi_v \simeq 10^{-4}$ rad, and let its projection onto this plane be a circle with radius 2.4×10^2 cm, independent of ψ_1. Choose $R = 10^2$ cm. Then $g \simeq 10^{-3}$ cm^3.

In practice, one will want to increase the counting rate by increasing the interaction region. It may then happen that ψ_1 and ψ_2 can no longer be considered constant over the whole interaction region. In this case, if the interaction is such that all points A in this region satisfy Eq. (13) to a good approximation, then from Eq. (14) we find

$$N_{\lambda_3\lambda_4} = \frac{n_1 n_2 \alpha^4}{4\pi^2 m^2 c^5}\, 2\pi \int_{-\pi/2}^{\pi/2} d\theta' \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int_{-\pi/2}^{\pi/2} d\psi_1 \left| \sum_{\lambda_1,\lambda_2} \frac{1}{\omega'^2} M_{\lambda_1\lambda_2\lambda_3\lambda_4}(\theta', \omega') \left[\frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\psi_1} P_{\lambda_1}(\omega_1, \psi_1) \right]^{1/2} \right.$$
$$\left. \times \left[\frac{\partial}{\partial\omega_2} \frac{\partial}{\partial(\pm\psi_1)} P_{\lambda_2}(\omega_2, \psi_2 = \pm\psi_1) \right]^{1/2} \right|^2 g(\psi_1). \tag{16}$$

In this type of experiment, the laboratory momenta, the reaction center-of-mass momenta, and the approximate point of emission of the scattered photons can be determined. However, if the dimensions of the interaction region are sufficiently small compared with $|\vec{r}_A - \vec{R}|$, then all photons which are emitted at one common point and reach the interaction region can be considered as having essentially parallel momenta (narrow-beam approximation). The number of photon pairs[13] scattered in the forward direction per unit time for a certain set of final polarizations is obtained from $N_{\lambda_3\lambda_4}$ by dividing by 2π, omitting $\int d\theta'$, and evaluating M at $\theta' = 0$:

$$\frac{\partial}{\partial\Omega'} N_{\lambda_3\lambda_4}(\theta' = 0) \equiv \frac{\partial}{\partial\Omega'} \bar{\sigma}_{\lambda_3\lambda_4}(\theta' = 0) g \;, \tag{17a}$$

where the integrated (over ω_1 and ω_2) differential cross section is defined as

$$\frac{\partial}{\partial\Omega'} \bar{\sigma}_{\lambda_3\lambda_4}(\theta' = 0) \equiv \frac{n_1 n_2 \alpha^4}{4\pi^2 m^2 c^5} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int_{-\pi/2}^{\pi/2} d\psi_1 \left| \sum_{\lambda_1,\lambda_2} \frac{1}{\omega'^2} M_{\lambda_1\lambda_2\lambda_3\lambda_4}(\theta' = 0, \omega') \left[\frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\psi_1} P_{\lambda_1}(\omega_1, \psi_1) \right]^{1/2} \right.$$
$$\left. \times \left[\frac{\partial}{\partial\omega_2} \frac{\partial}{\partial\psi_2} P_{\lambda_2}(\omega_2, \psi_2 = (\pm 1)\psi_1) \right]^{1/2} \right|^2 . \tag{17b}$$

The number of photons with a certain frequency and given polarization state scattered in a direction parallel to the momenta in one of the incoming photon beams, say the first one,[14] is

$$\frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\Omega'} N_{\lambda_3}(\theta' = 0, \omega_1) = \sum_{\lambda_4} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\Omega'} \bar{\sigma}_{\lambda_3\lambda_4}(\theta' = 0, \omega_1) g \;, \tag{18a}$$

where

$$\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\Omega'}\bar{\sigma}_{\lambda_3\lambda_4}(\theta'=0,\omega_1)\equiv\frac{n_1 n_2 \alpha^4}{4\pi^2 m^2 c^5}\int_0^\infty d\omega_2\int_{-\pi/2}^{\pi/2}d\psi_1\left|\sum_{\lambda_1,\lambda_2}\frac{1}{\omega'^2}M_{\lambda_1\lambda_2\lambda_3\lambda_4}(\theta'=0,\omega')\left[\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\psi_1}P_{\lambda_1}(\omega_1,\psi_1)\right]^{1/2}\right.$$
$$\left.\times\left[\frac{\partial}{\partial\omega_2}\frac{\partial}{\partial\psi_2}P_{\lambda_2}(\omega_2,\psi_2=(\pm 1)\psi_1)\right]^{1/2}\right|^2. \quad (18b)$$

The scattering physics is contained in

$$\frac{\partial}{\partial\Omega'}\bar{\sigma}_{\lambda_3\lambda_4}(\theta'=0)$$

and in

$$\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\Omega'}\bar{\sigma}_{\lambda_3\lambda_4}(\theta'=0,\omega_1).$$

These quantities have been computed for certain values of the parameters, and are shown in Figs. 4, 5, and 6. They are normalized so that $n_1 n_2 = (6.24)^2\times 10^{36}$, corresponding to two beams, each with an intensity of 1 A.

Figures 4 and 5 are to be used as follows. It is assumed that both circulating electron beams have the same energy, i.e., they have one common γ value, and that the local radius of curvature near B_v in both beams is the same, i.e., $|\vec{R}_1| = |\vec{R}_2| \equiv R$. Only if these (usually valid) assumptions are valid can the figures be used directly to design a photon-photon scattering experiment. To calculate the number of photon pairs scattered near the forward direction per second and per steradian, if the helicities of the two scattered photons are λ_3 and λ_4, then, as a first step, one has to look up in the figures the value of $(\partial/\partial\Omega')\bar{\sigma}_{\lambda_3\lambda_4}(\theta'=0)$ for the appropriate values of γ, R, λ_3, and λ_4. For example, if $\gamma = 11\,742.1$, $R = 10^2$ cm, $\lambda_3 = \lambda_4 = +1$, and when one of the circulating beams is an electron beam, the other a positron beam, then $(\partial/\partial\Omega')\bar{\sigma}_{++}(\theta'=0) = 1.8\times 10^{-2}\ \mathrm{cm^{-3}\,sec^{-1}\,sr^{-1}}$. The second step is [using Eq. (16) and making the generally valid assumption that $g(\psi_1)$ is independent of ψ_1] to multiply this number by g. For example, if the experimental setup is such [as it is to a good approximation in the case described after

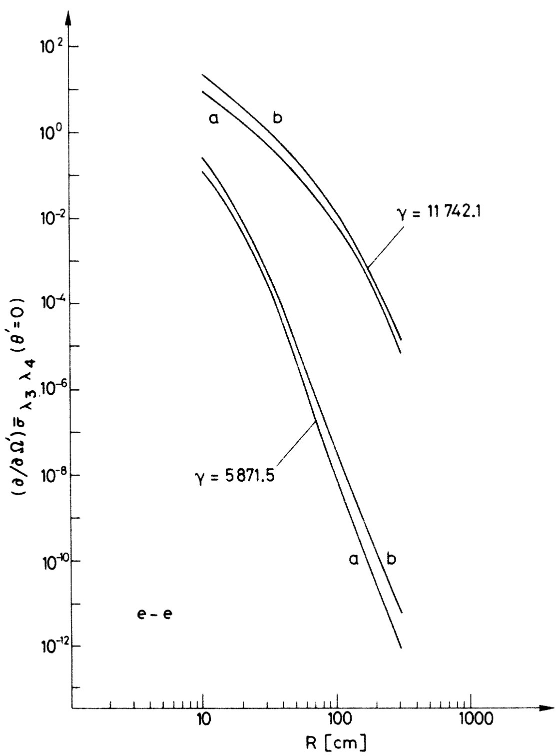

FIG. 4. $(\partial/\partial\Omega')\bar{\sigma}_{\lambda_3\lambda_4}(\theta'=0)$ in units of $\mathrm{cm^{-3}\,sec^{-1}\,sr^{-1}}$ as a function of R, for $\gamma = 5.8715\times 10^3$ and 1.17421×10^4, when both circulating beams are 1 A electron beams [i.e., $\psi_2 = -\psi_1$ in the argument of P_{λ_2} in Eq. (16)]. (a) $\lambda_3 = +$, $\lambda_4 = +$, or $\lambda_3 = -$, $\lambda_4 = -$; (b) $\lambda_3 = +$, $\lambda_4 = -$, or $\lambda_3 = -$, $\lambda_4 = +$.

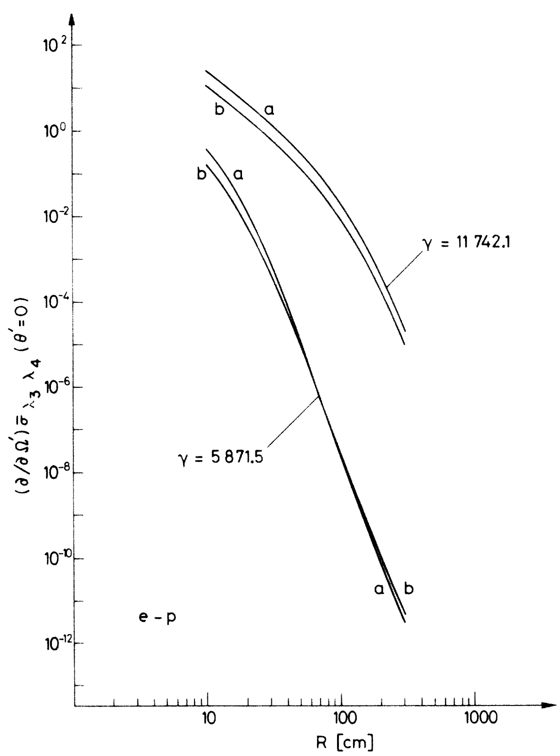

FIG. 5. $(\partial/\partial\Omega')\bar{\sigma}_{\lambda_3\lambda_4}(\theta'=0)$ in units of $\mathrm{cm^{-3}\,sec^{-1}\,sr^{-1}}$ as a function of R, for $\gamma = 5.8715\times 10^3$ and 1.17421×10^4, when one of the two circulating beams is a 1 A electron beam, and the other is a 1 A positron beam [i.e., $\psi_2 = +\psi_1$ in the argument of P_{λ_2} in Eq. (16)]. (a) $\lambda_3 = +$, $\lambda_4 = +$, or $\lambda_3 = -$, $\lambda_4 = -$; (b) $\lambda_3 = +$, $\lambda_4 = -$, or $\lambda_3 = -$, $\lambda_4 = +$.

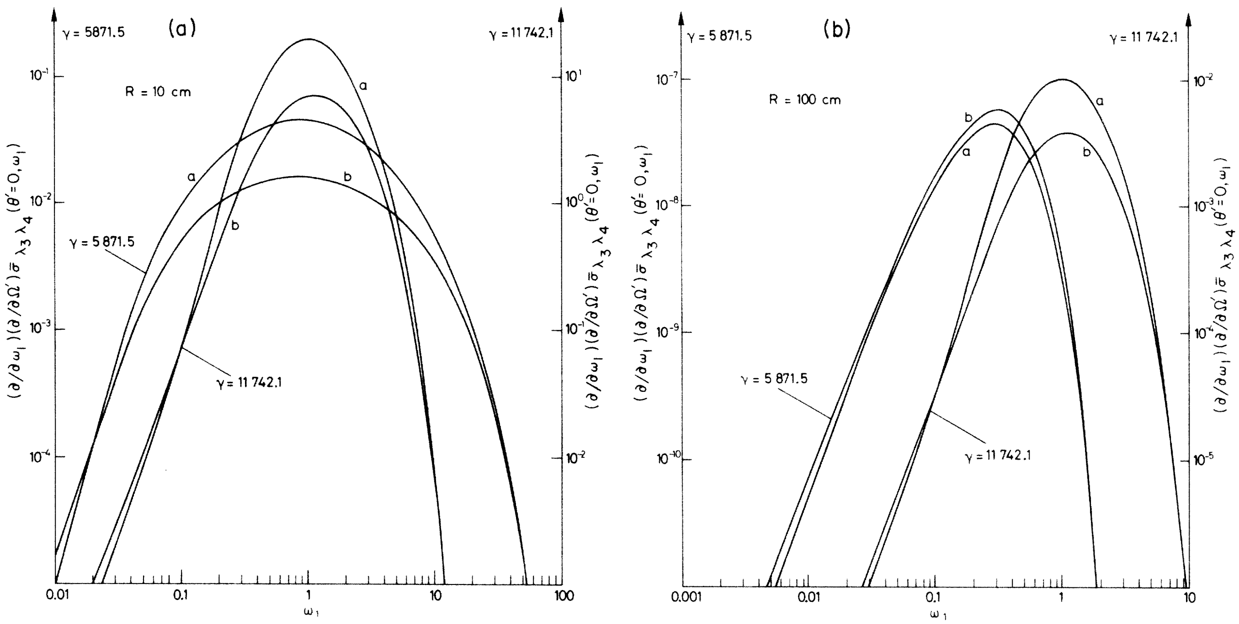

FIG. 6. $(\partial/\partial\omega_1)(\partial/\partial\Omega')\bar{\sigma}_{\lambda_3\lambda_4}(\theta'=0,\omega_1)$ in units of $\mathrm{cm}^{-3}\mathrm{sec}^{-1}\mathrm{sr}^{-1}$ $(\hbar^{-1}\times$ electron rest energy$)^{-1}$ as a function of ω_1, for $R=10$ cm and 10^2 cm, and for $\gamma=5.8715\times10^3$ and 1.17421×10^4, when one of the two circulating beams is a 1 A electron beam, and the other is a 1 A positron beam [i.e., $\psi_2=+\psi_1$ in the argument of P_{λ_2} in Eq. (16)]. (Note the different scales.) (a) $\lambda_3=+$, $\lambda_4=+$, or $\lambda_3=-$, $\lambda_4=-$; (b) $\lambda_3=+$, $\lambda_4=-$, or $\lambda_3=-$, $\lambda_4=+$.

Eq. (16)] that $g=10^{-3}$ cm^3, then we obtain 1.8 $\times10^{-5}$ sec^{-1} sr^{-1}, which is the number of forward-scattered photon pairs within the whole interaction region, if both photons have helicity $+1$, and if the intensity of both circulating beams is 1 A. The third step consists of multiplying this by the product of the intensities of both circulating beams in amperes. For example, if both beams have 20 A circulating, then we have to multiply by 400, and the final result is 7.2×10^{-3} sec^{-1} sr^{-1} = 2.6 $\times10$ h^{-1} sr^{-1}. When both photons have helicity -1, then we find similarly from Fig. 5 that the result is the same as above, while if one photon has helicity $+1$ and the other -1, then the result is 1.7 $\times10$ h^{-1} sr^{-1}. The total number of forward-scattered photon pairs produced in the whole interaction region in such an experiment is the sum over all helicity combinations, i.e., 8.6×10 h^{-1} sr^{-1}. (The results given in this paper do not enable us to calculate the number of photon pairs scattered in other than the forward direction. Nevertheless, we can roughly estimate this number using the fact that the photon-photon elastic cross section is not a strongly varying function of θ', and neglecting this variation. Then the total number of elastically scattered photons produced in the whole interaction region in this experiment is roughly 4π times more, i.e., 10^3 per hour.) The number of the scattered photon beams can be increased in general by increasing the circulating beam intensity, or γ, the size of the interaction region, or by decreasing $|\vec{R}|$,[15] or by a combination of these.

The energy distribution of the forward-scattered photons can be obtained from Fig. 6; it gives the laboratory energy of the two outgoing scattered photons. The normalization of $(\partial/\partial\omega_1)(\partial/\partial\Omega')\bar{\sigma}_{\lambda_3\lambda_4}(\theta'=0,\omega_1)$ is fixed by the fact that its integral over the whole range of ω_1 gives $(\partial/\partial\Omega')\bar{\sigma}_{\lambda_3\lambda_4}(\theta'=0)$.

IV. COMPUTATION

The integrals in Eqs. (17a) and (18a) have been evaluated by numerical integration on a CDC 7600 computer at CERN. Because of the fact that for $\theta'=0$

$$M_{++++}=M_{----},$$
$$M_{++--}=M_{--++},$$
$$M_{+-+-}=M_{-+-+},$$

and all other M are zero, the sum over λ_1 and λ_2 simplifies considerably. In fact, it reduces to two terms in the cases $\lambda_3=+$, $\lambda_4=+$; $\lambda_3=-$, $\lambda_4=-$, and to one term for $\lambda_3=+$, $\lambda_4=-$; $\lambda_3=-$, $\lambda_4=+$.

Further, it is found that for both the electron-electron and electron-positron cases

$$\frac{\partial}{\partial\Omega'}\bar{\sigma}_{++}(\theta'=0)=\frac{\partial}{\partial\Omega'}\bar{\sigma}_{--}(\theta'=0)$$

and

$$\frac{\partial}{\partial\Omega'}\bar{\sigma}_{+-}(\theta'=0)=\frac{\partial}{\partial\Omega'}\bar{\sigma}_{-+}(\theta'=0).$$

These equalities are also true for the quantity defined in Eq. (18a). After some experimentation in order to determine the behavior of the integrand, the following computing procedure was found adequate. First, the integration was performed over ω_2. Instead of the upper limit ∞, the straight line $\omega_2 = \omega^* - \omega_1$ was taken, where

$$\omega^* = 2\omega_c\,\xi^*(1+\gamma^2\psi^2)^{-3/2}$$

is defined in such a way that the argument $\xi = \xi^*$ in the modified Bessel function is large enough to make the contribution of the remaining integrand to the integral negligible. It was found that $\xi^* = 15$ is a safe value, which could be made smaller in most cases. Since $|M_{++++}|^2$ and $|M_{++--}|^2$ are not differentiable at $\omega' = (\omega_1\omega_2)^{1/2} = 1$, care was taken to integrate separately on both sides of the hyperbola $\omega_2 = 1/\omega_1$. For $\lambda_3 = +$, $\lambda_4 = -$, this separation is not necessary.

The next step, the integration over ψ, was performed with a trapezoidal rule, using unequally spaced points in the range $|\psi| \leq 5\times10^{-4}$, instead of $|\psi| < \pi/2$. This reduced range was found to be sufficient. The results obtained correspond to Eq. (18a).

In order to obtain the values of Eq. (17a), another trapezoidal rule integration was applied, taking in most cases $0.01 \leq \omega_1 \lesssim 30$, except for $R \geq 100$ and $\gamma = 5870.5$, where a range $0.001 \leq \omega_1 \lesssim 3$ is adequate. These results have been tested in some cases by integrating over both ω_1 and ω_2 with an adaptive Gaussian quadrature rule.

For the computation of $K_{1/3}(x)$, $K_{2/3}(x)$, the Chebyshev approximations given by Luke[16] were used.

ACKNOWLEDGMENTS

We wish to thank R. Haensel, R. Karplus, and F. Rohrlich for helpful correspondence concerning this problem. A detailed correspondence with B. De Tollis was particularly valuable in aiding us to locate and confirm misprints in the published literature. One of us (P.L.C.) wishes to acknowledge the hospitality of the Theory Division at CERN, where part of this work was done.

APPENDIX

The calculation of photon-photon scattering to lowest order is quite cumbersome, and it is inevitable that some misprints should occur in these calculations. (It would be a miracle if our own calculations should be an exception.) Several of these have been corrected by the authors themselves in subsequent publications; others were either found by us, or have already been corrected in the literature. Since tracing the source of a discrepancy is often quite time-consuming, it seemed worthwhile to list here those instances which at this time we believe to be misprints or in error. B. De Tollis, R. Karplus, and F. Rohrlich have been most cooperative and helpful in discussing these.

Reference numbers refer to the numbering employed in the present paper.

(i) R. Karplus and M. Neuman, Phys. Rev. 83, 776 (1951):

Five lines before Eq. (3): For $hc\kappa\omega$ read $\hbar c\kappa\omega$ [compare Phys. Rev. 80, 380 (1950), Eq. (4)].

Two lines before Eq. (9): For $hc\kappa\omega$ read $\hbar c\kappa\omega$.

Eq. (20″): For $[v(a+b(u))^2]$ read $|v[a+b(u)]^2|$, for $[u(a+b(v))^2]$ read $|u[a+b(v)]^2|$.

Eq. (23): For $M_{++++}(\theta,\omega)$ read $M_{++--}(\theta,\omega)$, for $M_{++--}(\theta,\omega)$ read $M_{++++}(\theta,\omega)$, add $M_{+++-}(\theta,\omega) = \frac{2}{315}\omega^6\sin^2\theta$.

Eq. (26): For $M_{++++}(\pi/2,\omega)$ read $M_{++--}(\pi/2,\omega)$, for $M_{++--}(\pi/2,\omega)$ read $M_{++++}(\pi/2,\omega)$.

Eq. (30): First line: For $-\pi i(\log 4\omega^2 - 2)$ read $-\pi i$; third line: for $-\pi i$ read $-\pi i(\log 4\omega^2-2)$.

Eq. (31) should read

$$\sigma_p = \pi\frac{\alpha^2}{\kappa^2\omega^2}\left[-\left(1+\frac{1}{\omega^2}\right)\left(1-\frac{1}{\omega^2}\right)^{1/2} + \left(2+\frac{2}{\omega^2}-\frac{1}{\omega^4}\right)\cosh^{-1}\omega\right].$$

(Pointed out by B. De Tollis. See also Ref. 6, p. 301, footnote.)

Table II: second row, last column: For $-1+2/\omega^2$ read $-1+2/\omega^4$, for M_{++++} read M_{++--}, for M_{++--} read M_{++++}.

Table III: second row, second column: For $-2/\omega^2$ read $-1/\omega^2$.

Fig. 2: This figure should be replaced by Fig. 7 of the present paper.

Table IV: There are several inaccurate numerical values in this table.

(ii) B. De Tollis, Nuovo Cimento, 32, 757 (1964) (the following misprints are pointed out by B. De Tollis in the second of Refs. 8, footnote 1):

p. 759, line 10: For "... interchanges $2 \rightarrow 3$ and $3 \rightarrow 4$..." read "... interchanges $2 \rightarrow 4$ and $3 \rightarrow 4$...".

Eq. (II.4): Lower limit of integration is 1 instead of 0. Further, Eq. (II.3): For $\{s(a+b(r))^2\}$ read $|s[a+b(r)]^2|$, for $\{r(a+b(s))^2\}$ read $|r[a+b(s)]^2|$.

(iii) B. De Tollis, Nuovo Cimento, 35, 1182 (1965) (pointed out by B. De Tollis and G. Violini

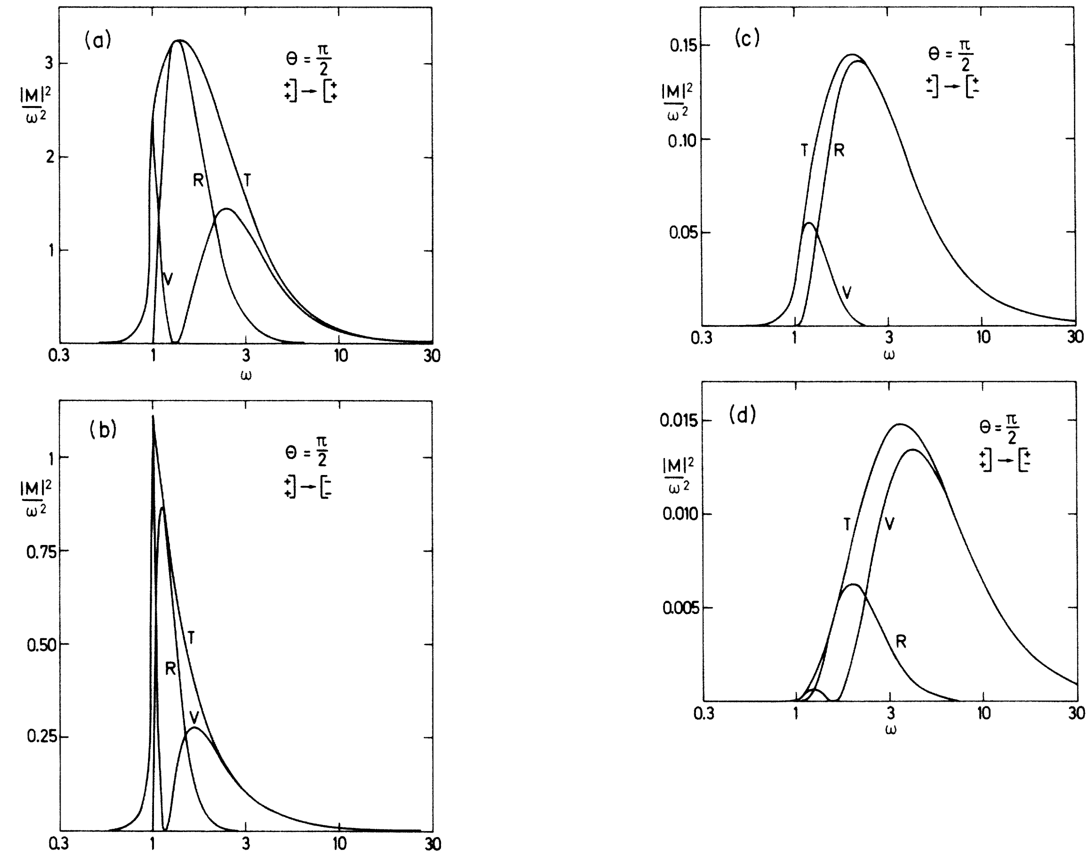

FIG. 7. The figure shows $V=[\mathrm{Re}(M)/\omega]^2$, $R=[\mathrm{Im}(M)/\omega]^2$, and $T=|M|^2/\omega^2$ evaluated at $\theta'=\pi/2$. (Note the different scales.) (a) For $\lambda_1=+$, $\lambda_2=+$, $\lambda_3=+$, $\lambda_4=+$, (b) for $\lambda_1=+$, $\lambda_2=+$, $\lambda_3=-$, $\lambda_4=-$, (c) for $\lambda_1=+$, $\lambda_2=-$, $\lambda_3=+$, $\lambda_4=-$, (d) for $\lambda_1=+$, $\lambda_2=+$, $\lambda_3=+$, $\lambda_4=-$. For the other λ values, the V, R, and T can be obtained from the above ones with the help of the symmetry relations $M_{++++}=M_{----}$, $M_{++--}=M_{--++}$, $M_{+-+-}=M_{-+-+}=M_{+--+}=M_{-++-}$, and all the remaining M's are equal to each other and to M_{+++-}. The ω is measured in units of mc^2.

in the third of Refs. 8, footnote 1):
Eq. (10), in the last term of the fourth equation:
For $M^{(1)}_{1122}(s,t)$ read $M^{(1)}_{1122}(t,s)$.

(iv) J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955) (pointed out by B. De Tollis in the second of Refs. 8, p. 1185, footnote):
p. 294: The footnote is in error, and the weight factors in Ref. 5 are correct.

(v) J. Schwinger, Phys. Rev. 75, 1912 (1949):
Eq. (II.37): For $\pi^2/4$ read $\pi^2/8$.

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[12]R. Haensel, DESY Report No. A 2.101, 2. verbesserte Auflage, 1965 (unpublished).
[13]The number of photons scattered in the forward direc-

tion is twice the number of such scattered pairs.

[14]The number of photons with the same frequency scattered in a direction parallel to the second incoming beam is the same.

[15]We remind the reader that $|\vec{R}|$ is the *local* radius of curvature of the circulating beam at the point where the photons are emitted, and *not* the average radius of the circulating beam. The latter will, in general, be much larger than $|\vec{R}|$ to reduce unnecessary radiation losses.

[16]Y. L. Luke, *The Special Functions and Their Approximations* (Academic, New York, 1969), Vol. 2, p. 361.

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Inclusion of resonances in the multiperipheral model*

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A simple generalization of the multiperipheral model (MPM) and the Mueller-Regge model (MRM) is given which has improved phenomenological capabilities by explicitly incorporating resonance phenomena, and still is simple enough to be an important theoretical laboratory. The model is discussed both with and without charge. In addition, the one-channel, two-channel, three-channel, and N-channel cases are explicitly treated. Particular attention is paid to the constraints of charge conservation and positivity in the MRM. The recently proven equivalence between the MRM and MPM is extended to this model, and is used extensively.

I. INTRODUCTION

For many years the Chew-Pignotti multiperipheral model[1] (CPM) has been an important testing ground for various theoretical ideas. This is in spite of the fact that it is fundamentally at odds with the data.[2] Recently, it has been proved that the N-channel multiperipheral model is totally equivalent to the N-channel Mueller-Regge model, and that one can derive either model from the other quite simply.[3]

We want to point out a simple generalization of the CPM and the equivalent Mueller-Regge model which (1) provides a generalization of CPM which has improved phenomenological capabilities, (2) still is simple enough to be an important theoretical laboratory, and (3) adds additional phenomena that are expected on physical grounds.

This model has been known in gas dynamics for years as the δ-function interaction, but in high-energy physics might justifiably be called a resonance model.[4] In the Veneziano model[5] resonances are dual to an infinite set of Regge trajectories. In this model the δ-function interaction, while not identical to either real resonance or Regge exchanges, gives effects similar to low-mass resonances or to low-lying Regge trajectories.

From the CPM point of view what this means is that one selects a subset of the infinite set of trajectories and daughters to be treated in the normal way and approximates the remaining trajectories by resonances (which we take here to be δ functions).

In Sec. II we discuss the simple scalar model. Section II contains most of the physics implications, without the complications of the more detailed models which are discussed in Secs. III and IV. In Sec. III we generalize the problem to N channels with k channels containing normal Regge poles and $N-k$ channels containing δ functions. In Sec. IV, a specific three-channel example with charge is worked out in great detail. Whereas in Sec. II we start the discussion from the multiperipheral model, deriving the equivalent Mueller model in the end, in Sec. IV we start by contrast with the Mueller model, and in the end find the equivalent multiperipheral model. This is instructive to the demonstration of the complete equivalence of the Mueller and multiperipheral models.