(at $s \approx 2500$ GeV^2). One model which attempts to explain (and in fact predicted) this behavior is the eikonal model of Cheng and Wu. This model is a perturbative calculation in massive QED, where the tower graphs are taken as a Born term in an eikonal expansion for the scattering amplitude. The result they find for total cross sections is $\sigma_{tot} \propto (\ln s)^2$, saturating the Froissart bound. This saturation depends critically on the fact that the tower graphs themselves (the Born term) violate the Froissart bound by a power. A model of this type will have difficulty coexisting with the conditions leading to the bounds here discussed. It may be that the CERN results coupled with the Cheng-Wu model offer experimental evidence that the virtual Compton amplitude violates the Froissart bound at energies of $s \approx 2500$ GeV^2 for spacelike photons.

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Possibility that massless Yang-Mills fields generate massive vector particles*

Jan Smit
Department of Physics, University of California, Los Angeles, California 90024
(Received 15 July 1974)

It is examined whether the massless Yang-Mills field theories can describe massive vector particles which have a common mass. The nonzero mass is supposed to be accompanied by poles in vertex functions. It is demonstrated that such poles can be consistent with the Ward identities. Homogeneous linear integral equations describing approximately the residues of these poles are shown to have asymptotic nontrivial solutions.

I. INTRODUCTION

It was conjectured[1] some time ago that theories with a local gauge group may describe massive vector particles, where the mass is of purely dynamical origin. One way to achieve this is by means of the so-called "Higgs mechanism" leading to the classes of spontaneously broken gauge theories[2] which have received much interest in recent times. Because these models can be treated by perturbation theory, they are practical, at least for the description of weak interactions. Yet, because of the often large number of scalar fields that have to be introduced, there can be many free parameters resulting from the multitude of possible couplings of the scalar fields among themselves. It is therefore useful to investigate the possibility that vector particles may acquire a mass without the introduction of scalar mesons. This has been done,[3-5] essentially by exploiting the analogy with the Meissner effect in the theory of superconductivity, with the qualitative conclusion that this spontaneous mass generation is indeed possible.

The present investigation deals with the possibility that the pure Yang-Mills field, without other couplings, may generate massive vector particles, specifically, without breaking the global symmetry so that the particles have equal mass.

It is known[1,6] that the vector particles can only be massive if their proper self-energy function has a pole at $p^2 = 0$. In Sec. II a simple mechanism that may lead to the formation of such a pole is proposed, and it is reviewed how this pole decouples from the physical S matrix. In this section is also reported the result of a one-loop calculation in self-consistent perturbation theory (a very crude form of the Hartree-Fock approximation). In Sec. III the renormalized Schwinger-Dyson equations are approximated with an eye on the mechanism proposed in Sec. II. The resulting equations for the cubic vertex functions are of the Bethe-Salpeter form in the ladder-chain approximation. Section IV deals with the consistency with the Ward identities. In Sec. V homogeneous integral equations describing the residues of the poles in the vertex functions are written down explicitly, and it is argued that there are nontrivial solutions to these equations (for any nonzero value of the coupling constant). This is tentatively interpreted in Sec. VI as evidence that the vertex functions have indeed poles which "cause" the pole in the self-energy function assumed already. An equation for the mass emerges that is, under a certain assumption, in form consistent with the one-loop calculation of Sec. II. The mass is proportional to the cutoff and depends on the coupling constant in the nonanalytic way that is familiar from the theory of superconductivity. For small coupling constants the cutoff can be very large. In Sec. VII comments on the significance of these results are given. Section VIII contains some conclusions. There are two appendixes: Appendix A contains formulas in relation to the integral kernels, and in Appendix B a mathematical example is exposed that serves to illustrate some comments made in Sec. VII.

II. PRELIMINARIES

It is expected that gauge particles remain massless to any finite order in perturbation theory,[6] for it follows from gauge invariance that the self-energy function is transverse:

$$\Sigma_{\mu\nu}(p) = (g_{\mu\nu}p^2 - p_\mu p_\nu)\Sigma(p^2)\,. \tag{2.1}$$

Accordingly, the propagator has the form (ξ specifies the gauge)

$$G^{\mu\nu}_{mn}(p) = \delta_{mn}\left[\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\frac{1}{p^2 + p^2\Sigma(p^2)} + \frac{1}{\xi}\frac{p^\mu p^\nu}{(p^2)^2}\right], \tag{2.2}$$

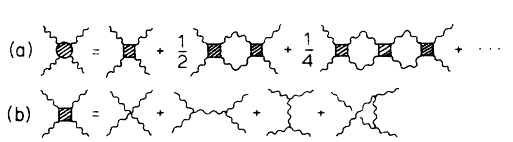

FIG. 1. Basic mechanism that may lead to the formation of a zero-mass bound state.

which shows that the position of the mass is given by the solution of the equation

$$m^2 + m^2\Sigma(-m^2) = 0\,. \tag{2.3}$$

Hence $m^2 = 0$, unless $\Sigma(p^2)$ has a pole at $p^2 = 0$. From Eq. (2.1) it follows then that the $p_\mu p_\nu$ part of $\Sigma_{\mu\nu}$ also has a pole. Such a pole can presumably not occur in any finite order of perturbation theory. It may be interpreted as the formation of a zero-mass bound state.

A simple mechanism that could generate such a bound state out of two vector particles is depicted in Fig. 1. In Fig. 1(a) the two (virtual) particles are bound through an arbitrary number of elementary scattering processes. The elementary scattering process is exhibited in Fig. 1(b). The set of graphs of Fig. 1(a) leads to a contribution to the propagator as shown in Fig. 2. Of course, in most gauges there will also be essential contributions from the ghost particles (Feynman-DeWitt-Faddeev-Popov[2,7]). These contributions will be taken into account later. For the moment we may think for convenience of a special gauge in which the ghost particles decouple (the so-called axial gauge[8]). Figures 1(a) and 2 suggest that a partial summation of the perturbation series is being made. It would, however, be a poor approximation to start with the bare massless propagator. Rather, the lines are assumed to correspond to a simple massive propagator, consistent with gauge invariance,

$$G^{\mu\nu}_{mn}(p)^{(0)} = \left[\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\frac{1}{m^2 + p^2} + \frac{1}{\xi}\frac{p^\mu p^\nu}{(p^2)^2}\right]\delta_{mn}\,, \tag{2.4}$$

corresponding to the approximation

$$\Sigma(p^2) \to \frac{m^2}{p^2}\;. \tag{2.5}$$

[Equation (2.4) is the covariant version to be used later; it does not correspond to the axial-vector gauge.] The expression (2.4) can be used as a starting point for self-consistent perturbation theory as follows. The propagator is the solution of the equation

$$[\delta_{kl}(g_{\kappa\lambda}p^2 - p_\kappa p_\lambda + \xi p_\kappa p_\lambda) + \Sigma^{kl}_{\kappa\lambda}(p)]\,G^{\lambda\mu}_{lm}(p) = +g_\kappa{}^\mu\delta_{km}\,, \tag{2.6}$$

or

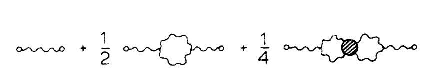

FIG. 2. How the basic mechanism contributes to the propagator.

$$\left[\delta_{kl}\left(g_{\kappa\lambda}-\frac{p_\kappa p_\lambda}{p^2}\right)(m^2+p^2)+\xi\delta_{kl}p_\kappa p_\lambda+\Sigma^{kl}_{\kappa\lambda}(p)+C^{kl}_{\kappa\lambda}(p)\right]G^{\lambda\mu}_{lm}(p)=+g_\kappa{}^\mu\delta_{km}, \tag{2.7}$$

where

$$C^{kl}_{\kappa\lambda}(p)=-\delta_{kl}(g_{\kappa\lambda}p^2-p_\kappa p_\lambda)\frac{m^2}{p^2} \tag{2.8}$$

is a nonlocal counterterm. The lowest-order approximation is obtained by setting $\Sigma+C=0$, which corresponds to (2.5) and leads to the solution (2.4). The perturbative solution for G then takes the form

$$G=G_0+G_0(\Sigma+C)G_0+G_0(\Sigma+C)G_0(\Sigma+C)G_0+\cdots, \tag{2.9}$$

and Fig. 2 must be supplemented by the contribution of the counterterms C. The word "counterterms" should not lead to the impression that C is infinite: In Eq. (2.8) m is the actual, physical mass of the vector meson.

The series in Fig. 2 can be rewritten by focusing attention on the proper part Σ. The contribution to the proper self-energy function Σ implied by the series is denoted by Σ^s. This leads to Fig. 3. There Σ^s is written in terms of the vertex function Γ^s, which satisfies the integral equation of Fig. 3(c). If the iterative solution of this equation is inserted in Fig. 3(b), and this result in turn in Fig. 3(a), then one obtains Fig. 2 again.

It is assumed that the vertex function Γ^s contains the zero-mass bound-state pole, and that this pole is carried over to Σ^s via the equation in Fig. 3(b). The vertex function Γ^s defined by Fig. 3(c) does not satisfy the requirements of Bose symmetry: All three external legs are not equivalent. The complete vertex function Γ should be symmetric under interchange of any two external legs. The idea is that there exists a consistent approximation Γ_{appr} to the exact vertex function Γ. If the momentum squared of one of the external legs of Γ_{appr} approaches zero, then Γ_{appr} becomes equal to a Γ^s. Thus the Γ^s functions contain only the singular part of Γ_{appr}. Similar words apply to Σ^s.

The pole is assumed to correspond to a scalar particle that couples to one and two vector particles, as illustrated in Fig. 4. The residue of the pole in Σ is called κ^2. Figure 4(a) stands for

$$-i\Sigma_{\mu\nu}(p)\big|_{\text{pole}}=(-\kappa p_\mu)\frac{-i}{p^2}(\kappa p_\nu)$$
$$=-i(-p_\mu p_\nu)\frac{\kappa^2}{p^2} \tag{2.10}$$

(again, this is only correct in a covariant gauge). In Fig. 4(b), Γ' is the regular part of Γ with respect to the external leg under consideration [Γ' has still the poles of the other two legs; compare Eq. (4.9)]. Comparison of the pole parts in Figs. 3(b) and 4(b) leads to an equation for κ (cf. Fig. 12).

The bound-state particle is not physical; its pole decouples from the S matrix. This phenomenon is the same as mentioned in Refs. 3, 4, and 5. For completeness the arguments are repeated here, for a covariant gauge. Consider elastic scattering of two particles. The (connected) S matrix is given by Fig. 1(b), if one interprets the internal lines and vertices to stand for the exact propagators and irreducible vertices. Since the external lines are on the mass shell and contracted with physical polarization vectors, the $p_\mu p_\nu$ parts of the propagators do not contribute. This expresses current conservation. For a pole in the s channel the first two diagrams of Fig. 1(b) are relevant. The pole parts are shown in Fig. 5, where (1) represents the pole in the quartic vertex, and (2), (3), and (4) come from the two cubic vertices. The vertex Γ' is the nonpole part of Γ as it is defined in Fig. 4(b). From the expressions in Figs. 4(b), 4(c), and current conservation, it follows that each of the diagrams (3) and (4) is equal to minus diagram (2). This leads to the equality b. The pole parts in b cancel: As $p^2\to 0$,

$$\frac{-i}{p^2}-\frac{-i}{p^2}(\kappa p_\nu)\frac{-ig^{\mu\nu}}{p^2+p^2\Sigma(p^2)}(-\kappa p_\mu)\frac{-i}{p^2}$$
$$=\frac{-i}{p^2}\left[1-\frac{\kappa^2}{\kappa^2}+O(p^2)\right]. \tag{2.11}$$

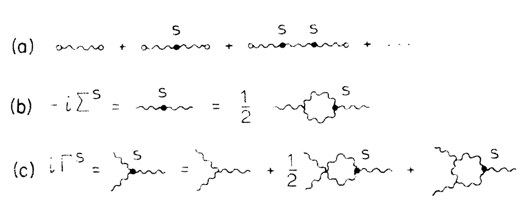

FIG. 3. (a) Rearrangement of the series in Fig. 2, (b) Σ^s in terms of Γ^s; (c) integral equation for Γ^s.

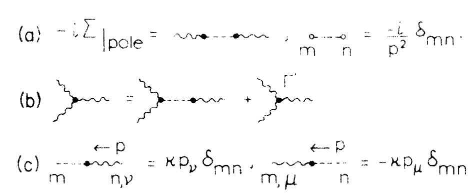

FIG. 4. The pole parts of the self-energy and the cubic vertex functions.

The generalization to many particle processes is straightforward. To substantiate these arguments by a proof, one must show that the zero-mass pole really behaves as a particle. In the argument above it was assumed that the residue of the pole in the quartic vertex factorizes in a product of two residues of the pole connected with the cubic vertex. This may be shown to hold within the approximation of the "basic mechanism." In general, the pole of the vertex functions must be also shown to reside in the correct invariant functions. For example, for the cubic vertex, the Lorentz tensors that multiply the invariant functions that have the pole must all contain the factor p_ν, as implied by Figs. 4(b) and 4(c).

Actually, the assumption that the pole corresponds to a two-particle bound state means that the coupling to multiparticle vertex functions has to be negligible. This is an assumption of simplicity. It may be justifiable for a small coupling constant, $g \to 0$, if that does not obliterate the bound state.

The following sections are dedicated to the possibility of the existence of the pole in the vertex functions and in Σ. The remainder of this section explores possible hints that self-consistent perturbation theory may give.

Let us write

$$\Sigma_{\mu\nu}(p) = g_{\mu\nu}\Sigma_1(p^2) - p_\mu p_\nu \Sigma(p^2) \,. \tag{2.12}$$

Equation (2.3) can be written

$$m^2 = \Sigma_1(-m^2) \,. \tag{2.13}$$

A correct approximation should respect the relation $\Sigma_1 = p^2\Sigma$, but since Σ_1 does not need the pole at $p^2 = 0$, it is tempting to compute the one-loop approximation to $\Sigma_1(-m^2)$. Of course in this approximation $\Sigma_1 \neq p^2\Sigma$, if $m^2 \neq 0$. The relevant diagrams are shown in Fig. 6(a), provided that the full vertices are replaced by the elementary ones. The rules for the diagrams are given in Fig. 7 and Eqs. (5.1)–(5.3). A covariant gauge is chosen, characterized by the parameter ξ in (2.4). The ghost particle is taken to be massless. A justification for this is given in Sec. IV [for convenience a Hermitian ghost field is used, which is related to the complex field by Eq. (3.1)]. The diagrams are conveniently evaluated with the n-dimensional

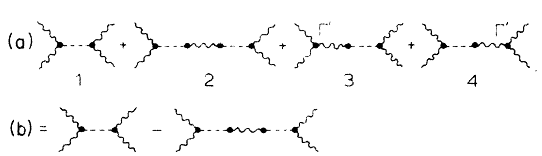

FIG. 5. Decoupling of the zero-mass pole from the S matrix.

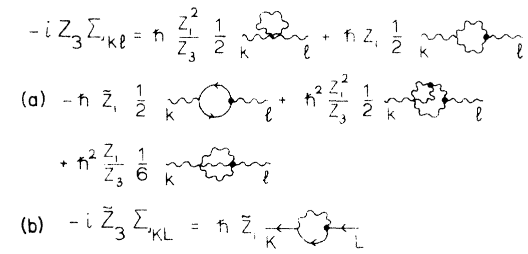

FIG. 6. (a) Equation for the vector function; and (b) equation for the ghost self-energy function.

regularization method.[9] Suppose n, the dimension of space time, is real and positive. For $n < 2$ the individual diagrams are convergent, provided that in the longitudinal part of the vector-meson propagator $1/p^2$ is replaced by $1/(p^2 + \mu^2)$, in order to suppress an infrared divergence. The individual diagrams have a pole at $n = 2$, but the poles cancel in the sum. So n may be pushed to $n > 2$. Then the limit $\mu \to 0$ can be taken. The next poles appear as n approaches 4; then

$$\Sigma_1(-m^2) = m^2\lambda \frac{2}{4-n}\left(\tfrac{75}{12} + \tfrac{1}{6}\xi^{-1}\right) + \cdots \,, \tag{2.14}$$

where

$$\lambda = \frac{cg^2}{32\pi^2}\hbar \,, \tag{2.15}$$

and where the constant c depends on the group according to Eq. (5.4). The dots indicate terms that remian finite as $n \to 4$; they contain the terms proportional to ξ^{-2}. Equation (2.14) can be translated unambiguously into the cutoff language. The only reason for using the n-dimensional regularization scheme is that it simplifies the calculations considerably. For one-loop calculations there exist schemes[10,11] that employ large masses

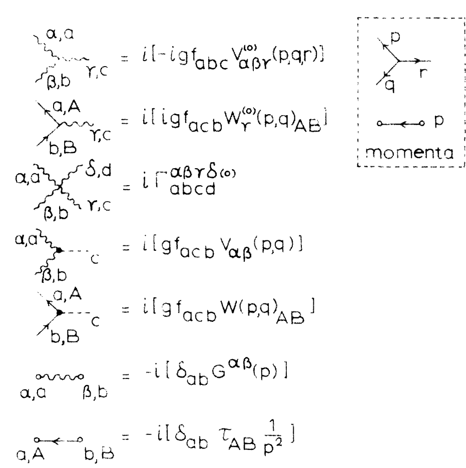

FIG. 7. Vertices and propagators used.

Λ, which would give the result

$$\Sigma_1(-m^2) = m^2(\tfrac{75}{12} + \tfrac{1}{6}\xi^{-1})\lambda \ln\frac{\Lambda^2}{m^2} + \cdots \,. \tag{2.16}$$

In Λ language, the cancellation of the poles at $n=2$ means that the quadratic dependence on Λ cancels. This is to be expected, since the quadratic divergences are mass-independent and they cancel in the massless Yang-Mills theory. One expects this to happen to any order in perturbation theory, since the massless Yang-Mills theory has no counterterm available to absorb a p^2-independent but cutoff-dependent term in Σ_1. So in general $\Sigma_1(-m^2)$ is expected to be proportional to m^2, and to have the form of a power series in $\ln\Lambda^2$. The number in front of $\lambda\ln(\Lambda^2/m^2)$ is positive, since ξ is assumed to take on only positive values [an argument for $\xi > 0$ comes from the path-integral derivation of the Feynman rules; see, for instance, the derivation in Ref. 12]. Of course the mass should not depend on ξ. The ξ dependence in Eq. (2.16) should cancel against higher order contributions. Summing the perturbation series would presumably give an equation for the mass of the form

$$\begin{aligned} m^2 &= m^2 F(z, \lambda, \xi^{-1}), \\ F(z, \lambda, \xi^{-1}) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} f_{nm}(\xi)\lambda^{n-m}z^m , \end{aligned} \tag{2.17}$$

where $z = \lambda\ln(\Lambda^2/m^2)$ and where λ is supposed to be the renormalized coupling constant. In the form (2.17) for F the reasonable assumption has been made that to order λ^n the maximum power of $\ln(\Lambda^2/m^2)$ is also n. If $m^2 \neq 0$, then m^2 is determined by the equation

$$1 = F(z, \lambda, \xi^{-1}) \,. \tag{2.18}$$

Suppose solutions exist for arbitrarily small λ. Then in the limit $\lambda \to 0$, $\Lambda \to \infty$, z fixed, Eq. (2.18) becomes

$$1 = F(z, 0, \xi^{-1}) \,. \tag{2.19}$$

At some point $z = z_0$ the miraculous cancellation in the ξ dependence should occur:

$$1 = F(z_0, 0, \xi^{-1}) = F(z_0, 0, \xi^{-1} + \eta) = F(z_0, 0, 0)\,; \tag{2.20}$$

then (2.18) becomes the formula for the mass in terms of the cutoff and λ:

$$m^2 = \Lambda^2 e^{-z_0/\lambda} \,. \tag{2.21}$$

The approximation (2.16) gives for z_0 the value $\frac{12}{75}$. For $\lambda \neq 0$ the cancellation of the ξ dependence in (2.18) would be an even greater miracle. It would define a set of points in the z-λ plane, a line, or maybe a single point.

The dependence in (2.21) on the coupling constant is familiar from the theory of superconductivity and from the Nambu model.[13] In these models the Goldstone phenomenon plays a role. It is not clear that this phenomenon has something to do with the model discussed here, where all vector mesons are assumed to have the same mass.

If Eq. (2.21) is indeed correct, then, for small coupling constants, the cutoff is innocent. For instance, with $4\pi/(g^2h) = 137$, $z_0 = \frac{1}{6}$ and for the group SU(2) [$c=2$ in Eq. (2.15)], $z_0/\lambda \approx 172$, and Eq. (2.21) gives $\Lambda^2/m^2 \approx \exp(172) \approx 10^{75}$, which for practical purposes seems to be close enough to infinity.

Of course, the reasoning above is pure speculation, and it might well be that the only ξ-independent solution of Eq. (2.17) is $m^2 = 0$.

III. THE SCHWINGER-DYSON EQUATIONS

For practical calculations a covariant gauge is almost indispensable. This means that the contribution of the ghost particles must be taken into account. This will be done by looking at the hierarchy of renormalized Schwinger-Dyson equations. Then these equations will be approximated so as to obtain back the simple mechanism of Sec. II, amended by the inclusion of the ghost particle as well as by renormalization effects. Hermitian as well as non-Hermitian ghost fields will be used. The two are related by

$$\psi_a(x) = \frac{1}{\sqrt{2}}[\chi_a^1(x) - i\chi_a^2(x)] \,. \tag{3.1}$$

Here a is a group index and $\chi_a^A(x)$, $A = 1, 2$, is the Hermitian ghost field. By the Schwinger-Dyson equations is meant an infinite set of coupled integral equations among the one-particle irreducible vertices. The equation for the vector particle self-energy is given in Fig. 6(a). The wavy line represents the exact vector-meson propagator, the line with the arrow the exact ghost propagator. The dots represent the exact irreducible vertices. All quantities in the diagrams are renormalized. The constant g in the elementary vertices is the renormalized coupling constant. The notation of Ref. 2 is used for the renormalization constants Z. The object $Z_3\Sigma$ is in general not a renormalized quantity, except at $p=0$. By Eqs. (2.1), (2.6), and (2.10),

$$Z_3\kappa^2 g^{\mu\nu} = \lim_{p\to 0} Z_3 p^2\Sigma(p^2)g^{\mu\nu} = Z_3 G^{\mu\nu}(0)^{-1} \,. \tag{3.2}$$

Here Z_3G^{-1} is the renormalized inverse propagator, and (3.2) shows that $Z_3\Sigma$ is a useful quantity if one is interested in the residue of the pole at

$p^2=0$. In particular $Z_3\kappa^2$ is a renormalized quantity which deserves a special notation:

$$\kappa_R{}^2 = Z_3\kappa^2 . \tag{3.3}$$

Note, however, that in Eq. (2.3) for the mass Σ is an unrenormalized object. The propagator of the ghost field is related to the ghost self-energy by an equation similar to (2.6) (τ is the second Pauli matrix):

$$[\,\delta_{ab}\,\tau_{AB}\,p^2 + \Sigma_{ab}^{AB}(p^2)]\,G_{bc}^{BC} = \delta_{ac}\,\delta_{AC} . \tag{3.4}$$

The integral equation relating Σ_{AB} to the other vertices is illustrated in Fig. 6(b). Note that $\tilde{Z}_3\Sigma_{KL}$ is not a renormalized quantity. For the derivation of these equations, external-source techniques have been used as described in Refs. 2, 14, and 15, for instance. The propagators and exact vertices in Fig. 6 may be considered to be a functional of a field $\phi_m^\mu(x)$, which is the vacuum expectation value of the vector field in the presence of an external source. A similar ghost-type field has been set equal to zero. Functional differentiation of the equations in Fig. 6 with respect to $\phi_m^\mu(x)$ yields the equations for the cubic vertices, quartic vertices, and so on. The equations for the cubic vertices are represented by Fig. 8. The h dependence is explicit, provided one does not count the factors h hidden in g, m, etc., which are absent in classical field theory. With each loop in the equations goes an explicit factor $\hbar$. The $O(\hbar^2)$ terms in Fig. 8 are equal to the once-differentiated version of the $O(\hbar^2)$ terms in Fig. 6.

The infinite set of integral equations must now be approximated into a manageable closed set that is as simple as possible. The purpose is not to obtain a consistent quantitative approximation in some sense. Rather, the spirit is to obtain linear integral equations of the type introduced in the previous section (Fig. 3). First of all, the $O(\hbar^2)$ terms in the integral equations are neglected. This might be justifiable if $\lambda\ln(\Lambda^2/m^2)$ is small compared to one. Suppose now that we are looking for the bound-state pole in the leg labeled "m" in Fig. 8. With the assumption that the bound state couples negligibly to three particles, comparison of the pole terms leads to the homogeneous set of equations of Fig. 9. In these diagrams also a nonlinearity has been removed by replacing the full cubic vertices attached to the legs "l" and "L" by the elementary ones. This replacement would be disastrous if the poles that are supposed to be present in the full vertex functions would play an essential role in the loop integral. There is, however, one gauge, the Landau gauge ($\xi\to\infty$), in which this pole does not contribute [compare Fig. 4(c)]. The homogeneous equations of Fig. 9 are of a Bethe-Salpeter (BS) type in the lowest-order approximation. The Z factors have been left out, since they differ from one only by terms of order h. A Bethe-Salpeter–type approach would not produce them in the homogeneous equations since the BS kernels can be expressed in a form where the explicit dependence on the elementary vertices has disappeared.

To have a pole-generating mechanism, an inhomogeneous term must be added to the homogeneous equations. The diagrams in Fig. 8 that do not contain the pole in the leg "m" act as inhomogeneous terms to the homogeneous equations of Fig. 9. The simplest terms are the elementary vertices, but their form may not be general enough to absorb renormalization effects. The inhomogeneous terms proposed are

$$Z_1\hat{\Gamma}^{kml}_{\kappa\mu\lambda}(p,q)^{(0)} = Z_1(-igf_{kml})\{V^{(0)}_{\kappa\mu\lambda}(p,r,q) + z(r_\kappa g_{\mu\lambda} - r_\lambda g_{\mu\kappa}) + z'[(p-q)_\kappa g_{\mu\lambda} + (p-q)_\lambda g_{\mu\kappa}]\}, \tag{3.5a}$$

$$\tilde{Z}_1\hat{\Gamma}^{KL}_{klm\mu}(p,q)^{(0)} = \tilde{Z}_1(igf_{kml})\{W^{(0)}_\mu(p,q)_{KL} + z''[(p+q)_\mu\delta_{KL} - (p-q)_\mu\tau_{KL}]\}. \tag{3.5b}$$

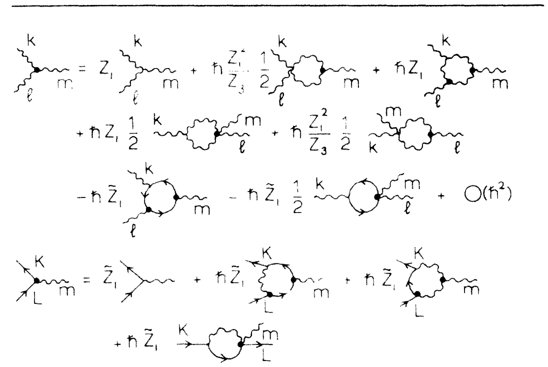

FIG. 8. The Schwinger-Dyson equations for the cubic vertices.

Here $V^{(0)}$ and $W^{(0)}$ are the elementary vertices [Eqs. (5.1)–(5.3)]. Figure 10 is the improved version of Fig. 3(c). The z, z', z'' terms introduced above represent effects from the loop diagrams

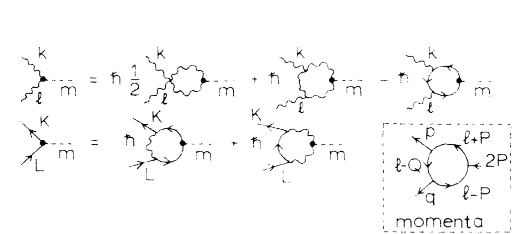

FIG. 9. The homogeneous integral equations for the residues.

left out; they are of order h. In contrast to the elementary vertex $\Gamma^{klm(0)}_{\kappa\lambda\mu}$, the function $\hat{\Gamma}^{kml(0)}_{\kappa\mu\lambda}$ is no longer completely symmetric in all three external legs; it is only symmetric under $(p, \kappa, k) \leftrightarrow (q, \lambda, l)$. The $\hat{\Gamma}$ vertices are of a rather general form: linear and homogeneous in the momenta, consistent with the symmetries of homogeneous equations of Fig. 9. The constants Z_1, $\tilde{Z}_1$, z, z', z'' must be chosen such that the solution of the inhomogeneous equations is cutoff-independent.

The inhomogeneous equations proposed in Fig. 10 are only mentioned for later discussion. These equations are very difficult to handle and they will not be considered in any more detail. Only the homogeneous equations will be studied further. First, however, it is necessary to check that the presence of a pole in the cubic vertices does not violate gauge invariance, as this is expressed by the Ward identities.

IV. CONSISTENCY WITH THE WARD IDENTITIES

Recently[16] Ward identities for the generating functional of irreducible vertices have been derived and used[17] to simplify the renormalization procedure of spontaneously broken gauge theories. In this section it is shown that the presence of a pole in the cubic vertices can be consistent with these identities. All quantities are unrenormalized in this section.

The Ward identity for the vector self-energy function simply states that it is transverse [Eq. (2.1)], which has been taken into account throughout. At this point it is perhaps useful to stress that the longitudinal part of the vector-meson propagator is unchanged by the interaction. This follows from the form (2.1) for $\Sigma_{\mu\nu}$ together with Eq. (2.6).

The Ward identity for the cubic vector vertex can be written $(p+q+r=0)$

$$-ip^{\kappa}L_{ka}(p^2)\Gamma^{klm}_{\kappa\lambda\mu}(p,q,r) = L^{\kappa}_{kam\mu}(q,p,r)(g_{\kappa\lambda}q^2 - q_{\kappa}q_{\lambda})[\delta_{kl} + \Sigma_{kl}(q^2)] + (\lambda, l, q) \leftrightarrow (\mu, m, r). \tag{4.1}$$

The function $L_{ka}(p^2)$ is closely related to the inverse complex ghost propagator:

$$G_{ab}(p)^{-1} = p^2 L_{ab}(p^2) \equiv p^2[\delta_{ab} + M_{ab}(p^2)], \tag{4.2}$$

where M_{ab} is of order h. The function $L^{\kappa}_{kam\mu}$ can be written as

$$L^{\kappa}_{kam\mu}(q,p,r) = -g^{\kappa}{}_{\mu} g f_{kam} + M^{\kappa}_{kam\mu}(q,p,r), \tag{4.3}$$

where $M^{\kappa}_{kam\mu}$ is defined in Fig. 11. If the M terms were zero in (4.2) and (4.3), then Eq. (4.1) would have a form familiar from quantum electrodynamics in the Landau gauge: The divergence of the vertex function is the difference between two inverse propagators that are multiplied by the charge matrix.

Now let $p \to 0$ in Eq. (4.1). The function $L_{ka}(p^2)$ presumably approaches a constant. For instance, a pole in the $L_{ka}(p^2)$ at $p^2=0$ would, by (4.2), imply that the ghost particle is massive. But the ghost particle is there to compensate for unphysical features of the longitudinal part of the vector-meson propagator. Since the latter has a singularity only at $p^2=0$ [Eq. (2.2)], the ghost should remain massless (in spontaneously broken gauge theories involving scalar mesons the ghost may become massive, while simultaneously the longitudinal part of the vector-meson propagator is singular at the ghost mass). So, by Eq. (4.2), $L_{ab}(0) = \tilde{Z}_3^{-1}\delta_{ab}$, where $\tilde{Z}_3$ is the wave-function renormalization constant of the ghost. Consider next $L^{\kappa}_{kam\mu}(q,p,r)$ as $p \to 0$. From Eq. (4.3) and Lorentz invariance,

$$L^{\kappa}_{kam\mu}(q,0,-q) = -g^{\kappa}{}_{\mu} g f_{kam} - g t_{kam} g^{\kappa}{}_{\mu} N(q^2) + q^{\kappa}q_{\mu} \text{ term}. \tag{4.4}$$

The $q^{\kappa}q_{\mu}$ term does not contribute in (4.1). From the requirement that the global symmetry shall not be broken follows that t_{kam} is an invariant tensor in group space. If

$$t_{kam} = f_{kam}, \tag{4.5}$$

then Eq. (4.1) becomes as $p \to 0$, in matrix notation for the group indices,

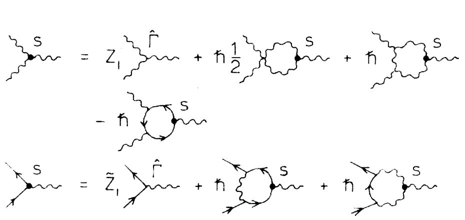

FIG. 10. The inhomogeneous integral equations replacing Fig. 3(c).

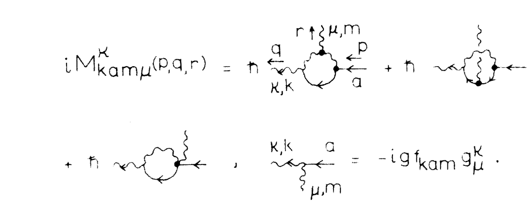

FIG. 11. The function $M^{\kappa}_{kam\mu}$ occurring in the Ward identity (4.1) (here the ghost propagator corresponds to the complex ghost field).

$$L_{ka}(0)\lim_{p\to 0} p^\kappa \Gamma^{klm}_{\kappa\lambda\mu}(p,q,r) = (g_{\lambda\mu}q^2 - q_\lambda q_\mu)[1+N(q^2)]\{-[1+\Sigma^T(q^2)]_g F_a + gF_a[1+\Sigma(q^2)]\}_{lm}$$
$$\propto [F_a, 1+\Sigma(q^2)]_{lm}, \tag{4.6}$$

where $(F_a)_{bc} = if_{bac}$, and where the symmetry of Σ in the group indices has been used. But the matrix Σ is just a multiple of the identity because of the global symmetry, and

$$\lim_{p\to 0} p^\kappa \Gamma^{klm}_{\kappa\lambda\mu}(p,q,r) = 0\,. \tag{4.7}$$

Equation (4.5) is true for the group SU(2), since there is no other invariant tensor with three indices than ϵ_{abc}. Low orders of perturbation theory indicate that (4.5) is probably true for general SU(n). In the Landau gauge, however, the t_{kam} term is even zero to all orders of perturbation theory (this observation was pointed out to the author by F. Feinberg); also, the possibly nonperturbative pole parts of the vertex functions do not contribute in this gauge, since they are purely longitudinal. We shall assume Eq. (4.7) to be valid as a consequence of global symmetry.

Conversely, if one is looking for a solution that violates the global symmetry, then the commutator in (4.6) is nonzero [there may also be additional terms, since now Eq. (4.5) may be violated]. Hence in case of broken global symmetry

$$\lim_{p\to 0} p^\kappa \Gamma^{klm}_{\kappa\lambda\mu}(p,q,r) \neq 0\,, \tag{4.8}$$

showing that the vertex must have a pole at $p^2=0$, which is a hint at the Goldstone phenomenon. The pole in the vertex then causes, through the Schwinger-Dyson equations, a pole in the vector self-energy, and it seems quite natural that in case of broken symmetry the massless Yang-Mills theory leads to massive vector particles.[18] The situation is very similar to that of Refs. 3–5.

However, the present paper deals with the possibility of unbroken global symmetry. Then it seems at first sight that the behavior (4.7) is not possible, since the vertex is supposed to have a pole at $p^2=0$. To see that no contradiction needs to arise, consider the following form for the vertex function:

$$\Gamma^{klm}_{\kappa\lambda\mu}(p,q,r) = igf_{kml}V_{\kappa\lambda}(p,q)\frac{1}{r^2}\kappa r_\mu$$
$$+\text{two permutations}$$
$$+\Gamma''^{klm}_{\kappa\lambda\mu}(p,q,r), \tag{4.9}$$

where Γ'' does not contain the poles. One may write

$$V_{\kappa\lambda}(p,q) = g_{\kappa\lambda}R_1' + Q_\kappa Q_\lambda R_2' + (P_\kappa Q_\lambda + Q_\kappa P_\lambda)R_3$$
$$+P_\kappa P_\lambda R_4' + (P_\kappa Q_\lambda - Q_\kappa P_\lambda)R_5', \tag{4.10}$$

where

$$P=\tfrac{1}{2}(p+q),\quad Q=\tfrac{1}{2}(p-q)\,. \tag{4.11}$$

The R's are functions of the invariants Q^2 and $P\cdot Q$ only, since $V_{\kappa\lambda}(p,q)$ is a factor of the residue of the pole in $r^2=4$, $P^2=0$. Bose symmetry requires

$$V_{\kappa\lambda}(p,q) = -V_{\lambda\kappa}(q,p)\,, \tag{4.12}$$

or

$$R_i'(Q^2, P\cdot Q) = -R_i'(Q^2, -P\cdot Q)\,,\quad i=1,2,4,5\,,$$
$$R_3(Q^2, P\cdot Q) = R_3(Q^2, -P\cdot Q)\,. \tag{4.13}$$

It follows that the R' must vanish as $P\cdot Q\to 0$, and it is convenient to write

$$R_i'(Q^2, P\cdot Q) = P\cdot Q R_i(Q^2, P\cdot Q)\,,\quad i=1,2,4,5\,. \tag{4.14}$$

All functions R_i are now indifferent under a sign change of $P\cdot Q$, and they will become nontrivial functions of Q^2, if $P\cdot Q=0$. Then, as $p\to 0$,

$$V_{\kappa\lambda}(p,q) \sim g_{\kappa\lambda}P\cdot QR_1(Q^2) + Q_\kappa Q_\lambda P\cdot QR_2(Q^2)$$
$$+(P_\kappa Q_\lambda + Q_\kappa P_\lambda)R_3(Q^2)\,. \tag{4.15}$$

Now observe that the requirement (4.7) is satisfied by the ansatz (4.10)–(4.15). The vertex function has poles in p^2, q^2, and r^2, but not in p, q, and r. It is the very global symmetry that saves the situation, since that allows for the possibility that the pole parts of the vertex function are proportional to the totally antisymmetric tensor f_{klm}. Note furthermore that the consequence of (4.1),

$$p^\kappa q^\lambda q^\mu \Gamma^{klm}_{\kappa\lambda\mu}(p,q,r) \equiv 0\,, \tag{4.16}$$

implies only a relation between the pole parts of Γ and the regular part Γ''.

Consider next the ghost-ghost-vector vertex. The Ward identity for this vertex is more complicated than that for the cubic vector vertex. It is unlikely that in general

$$\lim_{r\to 0} r^\mu \Gamma^{AB}_{abm\mu}(p,q,r) = 0\,, \tag{4.17}$$

if this vertex has a pole at $r^2=0$. Let us examine (4.17) for an ansatz for the pole part of Γ. The lowest-order form is

$$\Gamma^{AB}_{abm\mu}(p,q,r) = igf_{amb}(P_\mu + Q_\mu\tau)_{AB} \tag{4.18}$$

(the Hermitian ghost is now under consideration; τ is the second Pauli matrix). The ansatz for the exact vertex is

$$\Gamma^{AB}_{abm\mu}(p,q,r)=igf_{amb}W(p,q)_{AB}\frac{1}{r^2}mr_\mu + \Gamma'^{AB}_{abm\mu}(p,q,r), \qquad (4.19a)$$

$$W(p,q)=R_6(Q^2,P\cdot Q)+\tau R_7'(Q^2,P\cdot Q). \qquad (4.19b)$$

Again Γ' is the regular part. The antisymmetry in the fermion indices implies that

$$W(p,q)_{AB}=W(q,p)_{BA}, \qquad (4.20)$$

or

$$R_6(x,y)=R_6(x,-y), \quad R_7'(x,y)=-R_7(x,-y), \qquad (4.21)$$

so that R_7' must vanish at $P\cdot Q=0$. Therefore write

$$W(p,q)=R_6(Q^2,P\cdot Q)+\tau P\cdot Q R_7(Q^2,P\cdot Q), \qquad (4.22)$$

where R_7 need not be zero at $P\cdot Q=0$. Then, as $P\to 0$,

$$W(p,q)\sim R_6(Q^2)+\tau P\cdot Q R_7(Q^2), \qquad (4.23)$$

which shows that, since $R_6(Q^2)$ need not be zero, the global symmetry does not necessarily imply the behavior (4.17). It is found in the next section that, in studying the homogeneous equations of Fig. 9 in the Landau gauge, the function $R_6(Q^2)$ decouples from the other invariant functions. The question is open whether R_6 has to be zero or nonzero.

The approximate equations of Figs. 9 and 10 are not consistent with gauge invariance. Too many diagrams have been left out. The inhomogeneous equations are probably useless for practical calculations. The assumption is that these equations can give a correct answer to the question if the zero-mass bound state exists or not. The situation may be compared with that in quantum electrodynamics. It is known[19] that the Bethe-Salpeter equation in the ladder approximation gives correct gauge-invariant results only in the nonrelativistic limit. Higher order corrections have to be added to the Bethe-Salpeter kernel to improve the gauge invariance as well as the numerical results in the relativistic domain. The qualitative question regarding the existence or nonexistence of bound states is, however, correctly answered by the Bethe-Salpeter equation in the ladder approximation.

V. THE HOMOGENEOUS INTEGRAL EQUATIONS

In this section the homogeneous integral equations of Fig. 9 will be written down and a preliminary discussion of them is given. For clarity, the propagators and vertices used are given in Fig. 7, where

$$V^{(0)}_{\alpha\beta\gamma}(p,q,r)=(q-p)_\gamma g_{\alpha\beta}+(r-q)_\alpha g_{\beta\gamma} + (p-r)_\beta g_{\alpha\gamma}, \qquad (5.1)$$

$$\Gamma^{abcd(0)}_{\alpha\beta\gamma\delta}=-g^2[f_{abs}f_{cds}(g_{\alpha\gamma}g_{\beta\delta}-g_{\beta\gamma}g_{\alpha\delta}) + f_{acs}f_{bds}(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\delta}g_{\beta\gamma}) + f_{ads}f_{bcs}(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\gamma}g_{\beta\delta})], \quad (5.2)$$

$$W^{(0)}_\gamma(p,q)_{AB}=\tfrac{1}{2}(p+q)_\gamma\delta_{AB}+\tfrac{1}{2}(p-q)\tau_{AB}, \qquad (5.3)$$

and where $G^{\alpha\beta}(p)$ is given in Eq. (2.4). This propagator is supposed to be an approximation to the renormalized exact one (the parameter ξ is then actually $\xi_R=Z_3{}^{-1}\xi$). The ghost propagator is taken to be the elementary massless one. In general, Lorentz invariance, invariance under the global Lie group, and Fermi-Dirac symmetry require the ghost propagator to be of the form $\delta_{ab}\tau_{AB}\times$(invariant function of p^2). This invariant function is approximated by $1/p^2$, since, as argued in the previous section, the ghost remains massless. The symmetry properties of the functions $V_{\kappa\lambda}(p,q)$ and $W(p,q)$ have been given in the previous section, Eqs. (4.12) and (4.20). These forms for $V_{\kappa\lambda}$ and W are consistent with charge conjugation invariance.

The contribution of the diagram involving the elementary quartic vertex turns out to be zero. With the help of the relations

$$f_{pmq}f_{qnp}=-c\delta_{mn}, \qquad (5.4)$$

$$\Gamma^{klab(0)}_{\kappa\lambda\alpha\beta}f_{bma}=\tfrac{3}{2}cg^2f_{kml}(g_{\kappa\mu}g_{\lambda\nu}-g_{\kappa\nu}g_{\lambda\mu}), \qquad (5.5)$$

its contribution to $V_{\kappa\lambda}(p,q)$ can be written in the form

$$\tfrac{3}{4}cg^2\hbar i\int\frac{d^4l}{(2\pi)^4}\{[G(l+P)V(l+P,-l+P)G(-l+P)]_{\kappa\lambda} - [\kappa\leftrightarrow\lambda]\}, \qquad (5.6)$$

in obvious matrix notation; P and Q are defined in Eq. (4.11). The integrand here is independent of Q. Since there is no antisymmetric tensor that can be formed with only one vector P available, the expression (5.6) equals zero. The fact that this diagram vanishes eliminates a possible constant term (as a function of Q) from the integral equations, and this makes it possible later to obtain simple power-type asymptotic solutions for $Q^2\to\infty$.

The integral equations are coupled equations involving seven functions $R_i(Q^2,P\cdot Q)$ of two variables. Choosing the Landau gauge ($\xi=\infty$) simplifies things greatly. The functions R_2-R_5 become irrelevant since they describe the longitudinal part of $V_{\kappa\lambda}(p,q)$. The function R_6 satisfies its

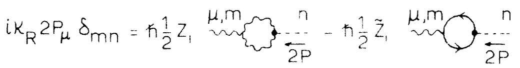

FIG. 12. The approximate equation for κ_R.

own integral equation and R_1 and R_7 satisfy two coupled equations. It is sufficient, for the immediate point of interest [namely, the equation for the residue of the pole in Σ (Fig. 12)] to study these equations in the limit $P \to 0$. This eliminates the variable $P\cdot Q$ in the functions R_i. After a Wick rotation, which is presumably valid for $P^2>0$, the metric is Euclidean, and the angular integrations can be performed in the limit $P\to 0$. The result is the following set of integral equations:

$$R_1(Q^2)=\lambda\int_0^\infty \frac{dl^2}{l^2}\left[\left(\frac{l^2}{m^2+l^2}\right)^2 K_{11}(Q^2,l^2;m^2)R_1(l^2) + K_{17}(Q^2,l^2)R_7(l^2)\right] , \tag{5.7a}$$

$$R_7(Q^2)=\lambda\int_0^\infty \frac{dl^2}{l^2}\left[\left(\frac{l^2}{m^2+l^2}\right)^2 K_{71}(Q^2,l^2)R_1(l^2) + K_{77}(Q^2,l^2;m^2)R_7(l^2)\right] , \tag{5.7b}$$

$$R_6(Q^2)=\lambda\int_0^\infty \frac{dl^2}{l^2} K_6(Q^2,l^2;m^2)R_6(l^2) , \tag{5.8}$$

where λ is defined in Eq. (2.15). Some details are given in Appendix A. The kernels K_{17} and K_{71} [Eqs. (A5) and (A6)] are relatively simple, since they correspond to the exchange of a massless particle. The kernels K_{11}, K_{77}, and K_{66} [Eqs. (A7), (A9), and (A11)] are messy; they correspond to the exchange of a massive particle.

The integral equations (5.7) and (5.8) cannot be solved exactly. The existence of solutions is determined by the domain of eigenvalue λ^{-1}. If this domain does not include the positive real axis, then no nontrivial solution exists. Let us look at the equations for large Q^2. This involves the behavior of the functions K_0, I_0, defined in (A1) and (A3), that appear linearly in the kernels K_{11}, K_{77}. An expansion in $1/Q^2$ makes no sense under the integral, since l^2 may always be much larger than Q^2. Therefore it is useful first to make a transformation of variables $l^2 \to t = l^2/Q^2$. Then the explicit Q dependence in the kernels is only through m^2/Q^2, and $Q^2\to\infty$ means setting $m=0$ in the kernels. The $m=0$ version of K_{11}, K_{77}, and K_{66} is shown in (A8), (A10), and (A12). Consider first the equation for R_6. For $Q^2 \gg m^2$, (5.8) becomes, in the t language,

$$\lambda^{-1}R_6(Q^2)\sim\int_0^1 \frac{dt}{t}\,\frac{3}{4}\,tR_6(tQ^2)+\int_1^\infty \frac{dt}{t}\,\frac{3}{4}\,\frac{1}{t}R_6(tQ^2) . \tag{5.9}$$

Only the neighborhood of $t=0$ includes nonasymptotic values of R_6. Away from $t=0$ R_6 may be replaced by its asymptotic form R_6^{as}. Suppose $R_6(Q^2)\approx R_6^{as}(Q^2)$ for $Q^2>M^2$, where M^2 is some constant $\gg m^2$. In t language, the region $0\le Q^2 \le M^2$ corresponds to $0\le t\le M^2/Q^2$. If the function R_6 is bounded in this region, then the contribution of this region to the integral vanishes like $1/Q^2$, as $Q^2\to\infty$. Assuming that this is the case and that R_6^{as} does not vanish as fast as $1/Q^2$, (5.9) is replaced by

$$\lambda^{-1}R_6^{as}(Q^2)=\int_0^1 \frac{dt}{t}\,\frac{3}{4}\,tR_6^{as}(tQ^2)+\int_1^\infty \frac{dt}{t}\,\frac{3}{4}\,\frac{1}{t}R_6^{as}(tQ^2) . \tag{5.10}$$

The solutions of this equation are simple powers. The ansatz $R_6^{as}(Q^2)=m(Q^2/m^2)^\gamma$ reproduces itself upon integration, provided that

$$\lambda^{-1}=\frac{3}{2}\,\frac{1}{1-\gamma^2} , \quad -1<\mathrm{Re}\gamma<1 . \tag{5.11}$$

The inequalities in (5.11) are convergence conditions for the integrals in (5.10). Since γ may be chosen real, or purely imaginary, the domain of eigenvalues of (5.10) includes the entire positive real axis in the λ plane. For $\gamma=0$ there are still two solutions: a constant and $\ln Q^2$. Note that if λ turned out to be negative, then the only solution of (5.10) would be $R_6^{as}=0$.

Consider next what happens as $Q^2\to 0$. Since K_{66} is symmetric, its behavior as $Q^2\to 0$ is obtained from (A14) by interchanging l and Q. Thus, for $Q^2\to 0$

$$R_6(Q^2)=\tfrac{1}{4}\lambda\int_0^\infty \frac{dl^2}{l^2}\,\frac{m^2}{m^2+l^2}\,R_6(l^2)+\lambda\frac{Q^2}{2m^2}\int_0^\infty \frac{dl^2}{l^2}\left[1+\frac{l^2}{m^2+l^2}-\tfrac{1}{2}\left(\frac{l^2}{m^2+l^2}\right)^2\right]\frac{m^2}{m^2+l^2}R_6(l^2)+\cdots . \tag{5.12}$$

In order that the integral in Eq. (5.8) converges at the lower limit, the function R_6 must vanish at $Q^2=0$ [cf. (A14)]. The existence of a nontrivial solution then would imply that the first integral in (5.12) converges but vanishes, while the second integral in (5.12) also converges, so that

$$R_6(Q^2)\sim \text{const}\times Q^2, \quad Q^2\to 0 . \tag{5.13}$$

However, the fact that the behavior at the origin, necessary for convergence, is not an automatic consequence of the integral equation might indicate that no nontrivial solution exists. Summarizing: Equation (5.8) might have solutions for any $\lambda > 0$ whose behavior at the origin is given by (5.13) and whose asymptotic behavior is given by

$$\frac{1}{m} R_6(Q^2) \sim b_+ \left(\frac{Q^2}{m^2}\right)^{\gamma} + b_- \left(\frac{Q^2}{m^2}\right)^{-\gamma} ,$$

$$\gamma = (1 - \tfrac{2}{3}\lambda)^{1/2} . \qquad (5.14)$$

One expects that in a more complete asymptotic expansion, each power in (5.14) is accompanied by a descending series in Q^2, possibly involving logarithms.

The coupled equations for R_1 and R_7 are more interesting. The asymptotic behavior of possible solutions is obtained from (5.7) and Eqs. (A8) and (A10) in the same way as for R_6:

$$\lambda^{-1} R_1^{\text{as}}(Q^2) = \int_0^1 \frac{dt}{t}\left[\left(t + \frac{25t^2}{12} - \frac{t^3}{4}\right) R_1^{\text{as}}(tQ^2) + \left(-\frac{t^2}{6} + \frac{t^3}{12}\right) R_7^{\text{as}}(tQ^2)\right]$$
$$+ \int_1^\infty \frac{dt}{t}\left[\left(1 + \frac{25}{12t} - \frac{1}{4t^2}\right) R_1^{\text{as}}(tQ^2) + \left(-\frac{1}{6} + \frac{1}{12t}\right) R_7^{\text{as}}(tQ^2)\right] , \qquad (5.15\text{a})$$

$$\lambda^{-1} R_7^{\text{as}}(Q^2) = \int_0^1 \frac{dt}{t}\left[\left(-\frac{t}{4} + \frac{t^2}{8}\right) R_1^{\text{as}}(tQ^2) + \frac{t^2}{2} R_7^{\text{as}}(tQ^2)\right]$$
$$+ \int_1^\infty \frac{dt}{t}\left[\left(-\frac{1}{4t} + \frac{1}{8t^2}\right) R_1^{\text{as}}(tQ^2) + \frac{1}{2t} R_7^{\text{as}}(tQ^2)\right] . \qquad (5.15\text{b})$$

The ansatz $R_i^{\text{as}}(Q^2) = m^{-1} r_i (Q^2/m^2)^{-1/2+\delta}$, $i = 1, 7$, leads to the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r_1 \\ r_7 \end{pmatrix} = \lambda^{-1}\begin{pmatrix} r_1 \\ r_7 \end{pmatrix} ,$$

$$a = (\tfrac{1}{4} - \delta^2)^{-1} + \tfrac{25}{12}(\tfrac{9}{4} - \delta^2)^{-1} - \tfrac{1}{4}(\tfrac{25}{4} - \delta^2)^{-1} ,$$

$$b = -[(\tfrac{1}{2} - \delta)(\tfrac{9}{4} - \delta^2)(\tfrac{5}{2} + \delta)]^{-1} ,$$

$$c = -\tfrac{3}{2}[(\tfrac{1}{2} + \delta)(\tfrac{9}{4} - \delta^2)(\tfrac{5}{2} - \delta)]^{-1} ,$$

$$d = \tfrac{1}{2}(\tfrac{9}{4} - \delta^2)^{-1} . \qquad (5.16)$$

The condition for convergence of the integrals in (5.15) is given by

$$-\tfrac{1}{2} < \text{Re}\,\delta < \tfrac{1}{2} . \qquad (5.17)$$

The eigenvalue λ^{-1} is related to the exponent δ by the equation

$$\lambda^{-1} = \frac{1}{2}\;\frac{P(\delta^2) \pm \{[P(\delta^2)]^2 - 4(\tfrac{1}{4} - \delta^2)(\tfrac{25}{4} - \delta^2) Q(\delta^2)\}^{1/2}}{(\tfrac{1}{4} - \delta^2)(\tfrac{9}{4} - \delta^2)(\tfrac{25}{4} - \delta^2)}$$

$$P(\delta^2) = \tfrac{881}{192} - \tfrac{592}{24}\delta^2 + \tfrac{10}{3}\delta^4,$$

$$Q(\delta^2) = \tfrac{3161}{192} - \tfrac{257}{24}\delta^2 + \tfrac{17}{12}\delta^4 . \qquad (5.18)$$

Of special interest are small values of λ. For small λ there are in the range (5.17) two possibilities for δ: $\delta = \pm(\tfrac{1}{2} - \epsilon)$, ϵ small and positive. Only the plus sign in (5.18) turns out to be consistent with a small λ, and expressing ϵ as a function of λ gives $\epsilon = \lambda + O(\lambda^2)$. The two eigenvectors are given by

$$\begin{pmatrix} r_1 \\ r_7 \end{pmatrix}\bigg|_{\delta = -1/2+\lambda} = \begin{pmatrix} 1 \\ -\tfrac{1}{4} \end{pmatrix} , \quad \begin{pmatrix} r_1 \\ r_7 \end{pmatrix}\bigg|_{\delta = 1/2-\lambda} = \begin{pmatrix} 1 \\ O(\lambda) \end{pmatrix} . \qquad (5.19)$$

The eigenvectors corresponding to the asymptotically fastest decreasing solution $[\sim (Q^2)^{-1+\lambda}]$ makes full use of the benefits of the ghost: Both r_1 and r_7 are comparable in magnitude. In contrast, for the slower decreasing solution $[\sim (Q^2)^{-\lambda}]$ the component r_7 is negligible compared to r_1, as $\lambda \to 0$.

The behavior for small Q^2 follows from (A13) and (5.7), since K_{11} and K_{77} are of the form $Q^{-2} \times$(symmetric function of l^2 and Q^2). By inspection one finds that

$$R_1(Q^2) \to A, \quad R_7(Q^2) \to BQ^2, \quad Q^2 \to 0 , \qquad (5.20)$$

where A and B are constants. In contrast to the situation with R_6, here the low-Q^2 behavior suggested by the integral equations is consistent with the convergence conditions at $l^2 = 0$ that follow from (A13). Summarizing: It is very likely that Eq. (5.7) has solutions for any $\lambda > 0$, whose low momentum behavior is given by (5.20) and whose asymptotic behavior is, for $\lambda \to 0$, given by

$$m R_1(Q^2) \sim a_- \left(\frac{Q^2}{m^2}\right)^{-1+\lambda} + a_+ \left(\frac{Q^2}{m^2}\right)^{-\lambda} , \qquad (5.21\text{a})$$

$$m R_7(Q^2) \sim -\tfrac{1}{4} a_- \left(\frac{Q^2}{m^2}\right)^{-1+\lambda} + O(\lambda) . \qquad (5.21\text{b})$$

In conclusion, it can be safely assumed that the homogeneous integral equations of Fig. 9 have nontrivial solutions for any positive λ.

VI. THE EQUATION FOR THE MASS

Appealing to the renormalizability of the Green's functions of the theory (not merely of the S matrix), one may suppose a relation between κ_R^2 defined in Eq. (3.3) and m^2 of the form

$$m^2 = \kappa_R^{\,2} f(g^2) \; , \qquad (6.1)$$

where f is some function of the renormalized coupling constant g, independent of the cutoff. It will be assumed that $f(g^2)$ approaches some real number as $g \to 0$, provided of course that massive solutions exist in that limit. With some unconventional way of fixing the renormalization constants one could presumably arrange for $f = 1$.

Assuming that the coupling of the bound state to more than two particles can be neglected, or alternatively, ignoring $O(\hbar^2)$ terms, comparison of the pole terms in Fig. 6(a) leads to the equation for κ_R given in Fig. 12. The renormalization constants may again be replaced by one, for small $g^2\hbar$. Evaluation of the diagrams in Fig. 12 in the Landau gauge and for $P \to 0$ leads to the relation

$$m = f^{1/2}\lambda \int \frac{dl^2}{l^2}\left[\left(\frac{l^2}{m^2 + l^2}\right)^2 \tfrac{3}{4} l^2 R_1(l^2) + \tfrac{1}{4} l^2 R_7(l^2) - R_6(l^2)\right] \; , \qquad (6.2)$$

where the integrals have again been converted to the Euclidean form and where Eq. (6.1) has been used. Since the functions $R_i(l^2)$ have to be real by time-reversal invariance, κ_R is real and hence f is positive. In Eq. (6.2) it is assumed that κ_R is positive. If κ_R is negative, then $f^{1/2}$ should be replaced by $-f^{1/2}$.

From the expressions for the asymptotic behavior of the solutions found in the preceding section, (5.14) and (5.21), it follows that the integral in (6.2) needs a regularization at high momenta. The integral will be cut off in a simpleminded way at $l^2 = \Lambda^2$, where Λ^2 is very much larger than m^2. Since the asymptotic region dominates, the functions $R_i(l^2)$ may be approximated by their asymptotic form. If the asymptotic forms (5.14) and (5.21) are inserted into Eq. (6.2), then a strong cutoff dependence emerges. The a_+, b_+ terms lead to integrals that vary as $(\Lambda^2)^{1-\lambda}$ and $(\Lambda^2)^{1-1/3\lambda}$, respectively, with the cutoff. Such a strong cutoff dependence is not expected for small coupling constants. If the homogeneous equations of Fig. 9 are to make any sense with regard to the bound-state problem, then there should be solutions for which the a_+ and b_+ terms are absent. This probably means that there is a degeneracy: a_+, b_+ and a_-, b_- type solutions are separately possible. The integral equations (5.7) and (5.8) are sufficiently singular and complicated, especially because of the presence of the massless ghost field, to allow for this possibility. The next section contains more discussion on this point.

Assuming that solutions exist without the a_+, b_+ terms in the asymptotic behavior, insertion of the forms (5.14) and (5.21) into Eq. (6.2), while replacing the lower limit by some mass M^2 ($m^2 \ll M^2 \lll \Lambda^2$), leads to

$$m = m\lambda\alpha\,\frac{(\Lambda^2/m^2)^\lambda - (M^2/m^2)^\lambda}{\lambda} \approx m\lambda\alpha\,\frac{(\Lambda^2/m^2)^\lambda - 1}{\lambda} \; , \qquad (6.3)$$

for small λ. Here $\alpha = a_-(\frac{11}{16})$, and the contribution of R_6 has been left out since it is cutoff-independent in the limit $\Lambda \to \infty$. The positive constant α is of course undetermined by the homogeneous equations. In Eq. (6.3) only the first term in an epansion in λ should be kept, since this was the guiding idea in making all approximations. Then Eq. (6.3) may be compared with the $\xi^{-1} = 0$ version of (2.16).

VII. COMMENTS

The author does not want to exclude the possibility that the results obtained so far make sense, but the situation at this point is quite unsatisfactory, for reasons to be discussed below.

The fact that the homogeneous $P^2 = 0$ integral equations have solutions does not necessarily imply that the solutions of the inhomogeneous equations (Fig. 10) have a pole at $P^2 = 0$. This would be the case if the integral operators would obey the rules of ordinary matrix theory, that is, if the kernels would correspond to compact (completely continuous), Fredholm– or Hilbert-Schmidt–type operators. They are obviously not, since the spectrum is continuous. It is very likely that the $P^2 \neq 0$ homogeneous equations corresponding to Fig. 10 also have a continuous spectrum, so that for any $\lambda > 0$ the $P^2 \neq 0$ homogeneous equations have solutions. Under these circumstances there is no reason why the solutions of the inhomogeneous equations should have a pole. Apparently, the pole has to be inserted by hand. One simply takes a particular solution of the inhomogeneous equation and adds to it a solution of the homogeneous equation, divided by P^2. One cannot divide by $(P^2)^{10}$, since this would violate the Ward identities. This does not look very convincing. The whole idea of a pole-generating mechanism, such as suggested in Sec. II, becomes irrelevant. However, there seems to be no fundamental objection against this possibility as long as the Schwinger-Dyson equations and the Ward identities are satisfied. These remarks apply also to the models of Refs. 4 and 5 (but not to that of Ref. 3). One may also compare with Ref. 20.

A related question concerns the presence of the cutoff. Since a cutoff is needed in the equation for the mass, one cannot exclude its presence from the homogeneous equations, although it is not needed there. Now it could well be that, in a certain sense, the integral equations of the type encountered here are very unstable against the introduction of a cutoff. It may be that there is a minimum value of the coupling constant: For $\lambda < \lambda_{\min}$ the homogeneous equations with cutoff have no solution and the solutions of the inhomogeneous equations do not have poles; for $\lambda > \lambda_{\min}$ the homogeneous equations with cutoff have solutions which are arbitrarily close to the solutions of the equations with no cutoff, the spectrum is dense $[\Delta\lambda = O((\ln\Lambda^2)^{-1})]$, and the inhomogeneous equations generate poles. Appendix B given a simple example of these phenomena. Typically $\lambda_{\min}$ is so large (of the order 1) that the equations cannot be trusted any more. This type of behavior can be expected in theories with dimensionless coupling constants. The $\lambda_{\min}$ effect does not seem to depend on the type of regularization used. The sharp cutoff, the Pauli-Villars suppression factor and the analytical regularization procedure (closely related to the dimensional continuation method) all yield the same $\lambda_{\min}$.

Consider Eq. (5.8) for R_6 and suppose the upper limit of the integral is replaced by Λ^2, where Λ^2 is very large. Both Q^2 and l^2 are restricted to be smaller than Λ^2. The solutions of the cutoff equations will be arbitrarily close to the solutions of the equations without cutoff, if one may choose Λ arbitrarily large. In particular the asymptotic behavior (5.14) will hold, for $m^2 \ll Q^2 \leq \Lambda^2$. Equation (5.10) determines again the asymptotic behavior, with the upper limit replaced by Λ^2/Q^2. It can be rewritten in the form

$$R_6^{\mathrm{as}}(Q^2) = \lambda \int_0^{\Lambda^2} \frac{dl^2}{l^2} K_{66}(Q^2, l^2; m=0) R_6^{\mathrm{as}}(l^2) . \tag{7.1}$$

Now the $m=0$ kernels may be considered as some sort of Green's functions for differential operators of the Euler type. They are polynomials in l^2/Q^2 and Q^2/l^2, multiplied by step functions, and they satisfy equations of the form

$$\prod_k (n_k - D) K(Q^2, l^2; 0) = \sum_k a_k D^k\, l^2\, \delta(Q^2 - l^2) , \tag{7.2}$$

where $D = Q^2\, \partial/\partial Q^2$, the n_k are integers, and the a_k are numerical constants. Thus the asymptotic forms R_i^{as} are solutions of differential equations, which are unaffected by the presence of the cutoff. The integral equation (7.1) puts conditions on those solutions. The general solution to the differential equation corresponding to (7.1) is given by (5.14). If $\Lambda \neq \infty$, then insertion of (5.14) into (7.1) leads to the condition

$$b_+ \frac{C^{\gamma}}{1-\gamma} + b_- \frac{C^{-\gamma}}{1+\gamma} = 0 , \quad C = \frac{\Lambda^2}{m^2} . \tag{7.3}$$

Now, if for $\Lambda = \infty$ there is degeneracy, so that b_+ and b_- are unrelated, then with $\Lambda \neq \infty$ this degeneracy is removed by the condition (7.3). For small λ, $\gamma \approx 1 - \lambda/3$, and Eq. (7.3) becomes

$$b_+ \approx \tfrac{1}{6}\lambda C^{-2+2\lambda/3} b_- . \tag{7.4}$$

It follows that the b_- term dominates for $m^2 \ll Q^2 \ll \Lambda^2$, while for $Q^2 \approx \Lambda^2$ the b_+ term is still smaller than the b_- term by a factor λ. These arguments are not rigorous, and one must be careful especially for small λ (γ near 1) since $O(Q^{-2})$ terms coming from the neighborhood of $t=0$ in (5.9) have been neglected, where $Q^2 \leq \Lambda^2$. So small coupling constants need large cutoffs.

If, for $\Lambda = \infty$, there is no degeneracy, then one expects that both b_+ and b_- in (7.3) are certain nonzero single valued functions of γ, and Eq. (7.3) has no solution for real γ (that is, for $\lambda < \frac{3}{2}$): For large enough C, the b_+ term always dominates. For $\lambda > \frac{3}{2}$ there are real oscillating solutions, behaving like $\cos[|\gamma| \ln(Q^2/m^2) + \phi]$, and (7.3) may be satisfied.

A similar discussion can be given for the equations for R_1 and R_7, expect that there are more (four) equations of the type (7.3) that have to be satisfied. For small λ, one must now also invoke the power solutions with δ in (5.16) near $\frac{3}{2}$ and $\frac{5}{2}$, which may be present in case of a cutoff. The results are similar to (7.4): The lowest power (δ near $-\frac{1}{2}$) dominates and the higher powers are suppressed by Λ-dependent factors that vanish as $\Lambda \to \infty$ as well as by a factor λ. Again this is only possible if there is degeneracy for the $\Lambda = \infty$ case. If there is no degeneracy, then δ has to be purely imaginary. From Eq. (5.18) it can be seen that this is possible for $\lambda \gtrsim 0.2$. In the previous section it was already assumed that there is degeneracy, so that the results obtained there are stable against the introduction of the cutoff.

It appears that if there is degeneracy for $\Lambda = \infty$, then there is still a continuous spectrum for finite Λ. This is compatible with the fact that the integral operators are not of the Hilbert-Schmidt type even in case of the cutoff. The trace of $K^T K$ diverges at ths origin; due to the masslessness of the ghost particle. Now it is not only the $P^2 = 0$ homogeneous equation that is relevant for the question if there is a bound state at $P^2 = 0$. More important is the behavior of the kernels in the neighborhood of $P^2 = 0$. For nonzero P^2 there is an effective mass in the ghost particle propagators, and the integral operator (having discrete indices acting on the various invariant functions as well as continuous ones) is of the Hilbert-

Schmidt type, if Λ is finite. A theorem than tells us that the spectrum is discrete. As $P^2 \to 0$ it might go over smoothly into a continuous one, but the example of Appendix B shows that this is not necessarily the case. Again λ may have to exceed a minimum value in order to achieve a continuous transition. The example in the appendix may be misleading in that the mass of the exchanged particle is zero. Also, the coupled integral equations involving massive vector particles coupled among themselves and to massless scalar fermions are much richer in structure. Last but not least, the simpleminded approximations made in arriving at these equations may be reason for the possible sensitivity to the presence or absence of the cutoff.

The occurrence of a minimum (or maximum) λ has been discovered earlier by Goldstein[21] within the context of the Bethe-Salpeter equation in the ladder approximation for the scattering of two spinor particles. The so-called Goldstein equations has the same qualitative properties as the equation discussed in Appendix B. In the cutoff case, Goldstein only accepted the value $\lambda = \frac{1}{4}$, but it turned out later[22] that exactly at this value of λ there is no solution. This is what is also found in Appendix B, but the reasons for accepting or rejecting solutions adopted there are different from the ones used in the references quoted. Subsequently Mandelstam[23] formulated criteria that Bethe-Salpeter amplitudes must satisfy in order that they have a physical interpretation. All the solutions found by Goldstone are to be rejected according to these criteria. It is not clear that these criteria are applicable to the present case, since the zero-mass bound state particle decouples from the physical state vector space.

Finally, the author is of the opinion that most of the doubts raised here, implicitly, with regard to the effect that the cutoff may have, do also apply to the models of Refs. 4, 5, and 18. Although a cutoff is not needed in these models, its presence should not do harm either.

VIII. FINAL REMARKS

There is evidence that symmetry-preserving solutions may exist for the pure Yang-Mills theory. The salient points are that the Ward identities admit a pole in the cubic vector vertex functions; the approximate homogeneous integral equations for the residue of the pole in the vertex function have asymptotic solutions.

It has been conjectured[1] that there is a minimum value for the coupling constant, in order that a vector-meson mass may be generated dynamically. A possible hint in this direction is found in the mathematical example of Appendix B, although in this paper this hint is assumed not to be relevant. More work has to be done in order that a convincing picture may result.

The nonperturbative investigations are interesting, but one seems to be far from any practical calculation. The quasiparticle approach[24] could be more successful: The bound state one is looking for is introduced as an elementary field, but one requires its wave-function renormalization constant to vanish, order by order in perturbation theory.

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APPENDIX A: THE INTEGRAL KERNELS

The angular integrations are $2\pi^2$ times the angular average. The basic angular average is

$$K_0(Q^2, l^2; m^2) = \left\langle \frac{1}{m^2 + (l-Q)^2} \right\rangle$$

$$= \frac{2}{\pi} \int_0^\pi d\theta \frac{\sin^2\theta}{m^2 + l^2 + Q^2 - 2lQ\cos\theta}$$

$$= \frac{1}{2l^2Q^2}\{m^2 + l^2 + Q^2 - [(m^2 + l^2 + Q^2)^2 - 4l^2Q^2]^{1/2}\} \tag{A1}$$

$$= Q^{-2}\,\theta(Q-l) + l^{-2}\,\theta(l-Q), \quad m = 0\,. \tag{A2}$$

A related function is

$$I_0(Q^2, l^2; m^2) = \left\langle \frac{1}{(l-Q)^2[m^2+(l-Q)^2]} \right\rangle$$

$$= m^{-2}[K_0(Q^2, l^2; 0) - K_0(Q^2, l^2; m^2)] \tag{A3}$$

$$= [-Q^{-2}\,\theta(Q-l) + l^{-2}\,\theta(l-Q)](l^2-Q^2)^{-1}, \quad m=0\,. \tag{A4}$$

The complete kernels are continuous at $l=Q$, even for $m=0$, and it is appropriate to set $\theta(0)=\frac{1}{2}$. For the kernels one finds

$$K_{17}(a,b) = \theta(a-b)\left(-\frac{1}{6}\frac{b^2}{a^2} + \frac{1}{12}\frac{b^3}{a^3}\right) + \theta(b-a)\left(-\frac{1}{6} + \frac{1}{12}\frac{a}{b}\right), \tag{A5}$$

$$K_{71}(a,b) = \theta(a-b)\left(-\frac{1}{4}\frac{b}{a} + \frac{1}{8}\frac{b^2}{a^2}\right) + \theta(b-a)\left(-\frac{1}{4}\frac{a}{b} + \frac{1}{8}\frac{a^2}{b^2}\right), \tag{A6}$$

$$\begin{aligned} K_{11}(a,b;m^2) = a^{-1}\frac{1}{24ab}\big(&8(a^3+b^3) - 22(a^2b+b^2a) + m^2[18(a^2+b^2)+31ab] + m^4 8(a+b) - m^6 \\ &+\{-7(a^4+b^4) + 32(a^3b+b^3a) + 46a^2b^2 + m^2[-29(a^3+b^3) - 21(a^2b+b^2a)] \\ &+ m^4[-15(a^2+b^2) - 26ab] + m^6\,3(a+b) + m^8\}K_0(a,b;m^2) \\ &+[-a^4+b^4-10(a^3b-b^3a)](a-b)\,I_0(a,b;m^2)\big), \end{aligned} \tag{A7}$$

$$K_{11}(a,b;0) = \theta(a-b)\left(\frac{b}{a} + \frac{25}{12}\frac{b^2}{a^2} - \frac{1}{4}\frac{b^3}{a^3}\right) + \theta(b-a)\left(1 + \frac{25}{12}\frac{a}{b} - \frac{1}{4}\frac{a^2}{b^2}\right), \tag{A8}$$

$$\begin{aligned} K_{77}(a,b;m^2) = a^{-1}\tfrac{1}{8}\{&-2(a+b) - m^2 + [3(a^2+b^2) + 2ab + m^2\,3(a+b) + m^4]K_0(a,b;m^2) \\ &-(a^2-b^2)(a-b)\,I_0(a,b;m^2)\}, \end{aligned} \tag{A9}$$

$$K_{77}(a,b;0) = \theta(a-b)\,\frac{1}{2}\frac{b^2}{a^2} + \theta(b-a)\,\frac{1}{2}\frac{a}{b}, \tag{A10}$$

$$K_{66}(a,b;m^2) = -\tfrac{1}{4} + \tfrac{1}{2}(a+b+m^2)K_0(a,b;m^2) - \tfrac{1}{4}(a-b)^2 I_0(a,b;m^2), \tag{A11}$$

$$K_{66}(a,b;0) = \theta(a-b)\,\frac{3}{4}\frac{b}{a} + \theta(b-a)\,\frac{3}{4}\frac{a}{b}, \tag{A12}$$

where $a=Q^2$, $b=l^2$. For $l\to 0$, Q fixed

$$K_{11} \sim \frac{Q^2}{m^2+Q^2}\,\frac{l^2}{m^2+Q^2}, \qquad K_{77} \sim \frac{1}{4}\left(\frac{m^2+Q^2}{Q^2}+1\right)\frac{l^4}{(m^2+Q^2)^2}, \tag{A13}$$

$$K_{66} \sim \frac{1}{4}\,\frac{m^2}{m^2+Q^2} + \frac{1}{2}\left[1 + \frac{Q^2}{m^2+Q^2} - \frac{1}{2}\,\frac{Q^4}{(m^2+Q^2)^2}\right]\frac{l^2}{m^2+Q^2}\,. \tag{A14}$$

The expressions in (A13) and (A14) can be used to obtain the behavior for $Q\to 0$, uniform in l. For $l\to\infty$, Q fixed

$$\begin{aligned} &K_{11} \sim 1, \\ &K_{77} \sim \frac{1}{2}\,\frac{Q^2}{l^2}, \\ &K_{66} \sim \left(\frac{1}{4} + \frac{1}{2}\,\frac{Q^2}{m^2+Q^2}\right)\frac{m^2+Q^2}{l^2} \end{aligned} \tag{A15}$$

APPENDIX B: A MATHEMATICAL EXAMPLE

Suppose an invariant function occurring in a vertex function is a solution of the equation

$$\begin{aligned} F(Q^2, (P\cdot Q)^2; P^2) &= Z_1 + \tfrac{1}{2}cg^2\hbar\int\frac{d^4l}{(2\pi)^4}\,\frac{l^2}{\mu^2+(l-Q)^2} \\ &\quad\times\frac{F(l^2,(l\cdot P)^2;P^2)}{[\nu^2+(l+P)^2][\nu^2+(l-P)^2]}, \end{aligned} \tag{B1}$$

where the metric is Euclidean. The constant Z_1 is a vertex renormalization constant. We shall feel free to give it any value that makes the solution of (B1) cutoff-independent. The interest is in a possible pole of F at $P^2=0$. To solve (B1), an expansion in spherical harmonics appropriate to four dimensions may be tried:

$$F(Q^2, (P\cdot Q)^2; P^2) = [\nu^2 + (P+Q)^2][\nu^2 + (P-Q)^2] \times \sum_{n\text{ even}} G_n(Q^2; P^2) U_n\left(\frac{P\cdot Q}{PQ}\right), \tag{B2}$$

$$[\mu^2 + (l-Q)^2]^{-1} = \sum_{n=0}^{\infty} K_n(Q^2; l^2; \mu^2) U_n\left(\frac{l\cdot Q}{lQ}\right), \tag{B3}$$

where the U_n are Tchebyscheff polynomials of the second kind. Performing the angular integrations one finds the set of equations

$$(\nu^2 + P^2 + Q^2)^2 G_0 - P^2Q^2(G_0 + G_2) = Z_1 + \lambda \int_0^\infty dl^2 l^4 K_0 G_0, \tag{B4a}$$

$$(\nu^2 + P^2 + Q^2)^2 G_n - P^2Q^2(G_{n-2} + 2G_n + G_{n+2}) = \lambda \int_0^\infty dl^2 l^4 K_n G_n. \tag{B4b}$$

In Eq. (B4b), $n = 2, 4, \ldots$. One may assume solutions such that only G_0 has the pole at $P^2 = 0$. Thus one is led to consider Eq. (B4a) with $G_2 = 0$. Remarkable enough it is consistent with (B4) that $G_2 \equiv 0$, but the resulting series may not converge. To regain the similarity with the equations of the main text introduce

$$F_0(Q^2; P^2) = [(\nu^2 + P^2 + Q^2)^2 - P^2Q^2] G_0(Q^2; P^2). \tag{B5}$$

Then (B4a), with $G_2 = 0$, becomes an equation for F_0, which is expected to make sense near $P^2 = 0$. For $\mu = 0$, this equation is explicitly solvable in terms of hypergeometric functions, since it is then equivalent to a second-order differential equation. To simplify the example to the bone, the replacement

$$(\nu^2 + P^2 + Q^2)^2 - P^2Q^2 \to (\nu^2 + P^2 + Q^2)^2 \tag{B6}$$

will be made. The resulting equations are more transparent, while the qualitative features that this appendix wants to emphasize are retained. Introducing dimensionless variables $Q^2 = m_1^2 x$, $l^2 = m_1^2 y$, $s = (\nu^2 + P^2)/m_1^2$, $F_0(Q^2; P^2) = \phi(x, s)$, and using Eq. (A2), the proposed equation for F_0 takes the form

$$\phi(x,s) = Z_1 + \lambda \int_0^\infty \frac{dy}{y}\left(\frac{y}{s+y}\right)^2 \left[\theta(x-y)\frac{y}{x} + \theta(y-x)\right] \times \phi(y,s). \tag{B7}$$

If $\nu \neq 0$, then it is natural to take $m_1 = \nu$; for $\nu = 0$, m_1 is some reference mass. So for $\nu \neq 0$, $s \geq 1$; for $\nu = 0$, $s \geq 0$. Consider first the homogeneous equation obtained by setting $Z_1 = 0$. Differentiation of (B7) shows that the solutions of (B7) are to be found among the solutions of the differential equation

$$x(x\phi)'' + \lambda w \phi = 0, \tag{B8}$$

where $w(x) = (x/(s+x))^2$. The solutions of (B8) are for this $w(x)$ simple powers $x\phi = (s+x)^\alpha$, $(s+x)^{1-\alpha}$, with $\lambda = \alpha(1-\alpha)$. The real solutions of (B7) with $Z_1 = 0$ are

$$s > 0:\ \phi = a x^{-1}(s+x)^{1/2}\left[\left(1+\frac{x}{s}\right)^{\beta/2} - \left(1+\frac{x}{s}\right)^{-\beta/2}\right], \quad 0 < \lambda < \tfrac{1}{4} \tag{B9a}$$

$$= b x^{-1}(s+x)^{1/2} \ln\left(1+\frac{x}{s}\right), \quad \lambda = \tfrac{1}{4} \tag{B9b}$$

$$= c x^{-1}(s+x)^{1/2} \sin\left[\tfrac{1}{2}|\beta| \ln\left(1+\frac{x}{s}\right)\right], \quad \lambda > \tfrac{1}{4} \tag{B9c}$$

$$s = 0:\ \phi = x^{-1/2}(a_+ x^{\beta/2} + a_- x^{-\beta/2}), \quad 0 < \lambda < \tfrac{1}{4} \tag{B10a}$$

$$= x^{-1/2}(b_1 + b_2 \ln x), \quad \lambda = \tfrac{1}{4} \tag{B10b}$$

$$= x^{-1/2}\{c_1 \cos[\tfrac{1}{2}|\beta|\ln x] + c_2 \sin[\tfrac{1}{2}|\beta|\ln x]\}, \quad \lambda > \tfrac{1}{4} \tag{B10c}$$

where $\beta = (1-4\lambda)^{1/2}$ and where the a's, b's, and c's are real. For $s > 0$ there is only one solution; for $s = 0$ there are two independent solutions. In both cases λ may be any positive real number. The inhomogeneous equation (B7) ($Z_1 \neq 0$) has no solution for nonzero λ. The differential equation is unchanged, and there is no superposition of its two independent solutions that satisfies (B7). The trouble comes from the infinite range of the integral: (B7) suggests that $\phi \to Z_1$ as $x \to \infty$, but this is incompatible with the convergence condition at $y = \infty$.

Returning to the homogeneous equation, suppose the upper limit in (B7) is replaced by a large number C, and that $x, y \leq C$. Since the differential equation is unchanged as well as the boundary condition at the origin, the possible solutions are again to be found among the functions listed in (B9) and (B10). Direct substitution of these functions in (B7) ($Z_1=0$) leads to conditions [such as (B12) below] that restrict λ again to be positive. For positive λ, $\phi\to 0$ as $x\to\infty$, and the new condition may be phrased as

$$\int_C^\infty \frac{dy}{y}\left(\frac{y}{s+y}\right)^2 \phi(y)=0\,. \tag{B11}$$

Equation (B11) is a nonlocal condition that states that those solutions of the noncutoff equation that make the integral vanish are also solutions of the cutoff equation. There are no more solutions as can be checked by direct substitution. For $s=0$, Eq. (B11) can always be satisfied by a suitable choice of the constants $a_\pm$, $b_{1,2}$, $c_{1,2}$. The spectrum is again given by $0<\lambda<\infty$. The integral operator is not of the Hilbert-Schmidt type, since trace (K^TK) diverges at the origin. For $s>0$, the integral operator is of the Hilbert-Schmidt type and one expects a discrete spectrum. In fact (B11) determines the eigenvalues λ^{-1}. Consider first the case $\lambda\neq\frac{1}{4}$. Inserting (B9a), which includes essentially also (B9c) into (B11), leads to the condition

$$(1+\beta)\left(1+\frac{C}{s}\right)^{\beta/2}-(1-\beta)\left(1+\frac{C}{s}\right)^{-\beta/2}=0\,. \tag{B12}$$

This condition depends only on C/s, which is due to the simplification (B6). For large enough C, Eq. (B12) has no solution if β is real. For $\lambda>\frac{1}{4}$ β is purely imaginary, and (B12) may be rewritten in the form

$$\tan\left[\tfrac{1}{2}|\beta|\ln\left(1+\frac{C}{s}\right)\right]=-|\beta|\,, \tag{B13}$$

which shows that there is a dense set of eigenvalues for $\lambda>\frac{1}{4}$, with interspacing, in terms of β, of the order $2\pi/\ln C$. The point $\beta=0$, $\lambda=\frac{1}{4}$ does not belong to the spectrum as can be seen by inserting (B9b) into (B11). Suppose $s=1$ and we let C increase. Then, if we follow a particular $|\beta|$, this $|\beta|$ decreases. This phenomenon is familiar from the quantization procedure of a field enclosed in a box. There one has $k_n=2\pi n/L$, and as $L\to\infty$, $k_n\to 0$. Of course one lets n vary with L, such that k_n approaches a limit as $L\to\infty$. So there is no reason to reject solutions of (B13) because a particular β is strongly cutoff dependent. As $C\to\infty$, the spectrum becomes continuous again, but there is a gap $0<\lambda\leq\frac{1}{4}$. To see what happens if the large momenta are suppressed, rather than left out, consider a Pauli-Villars-type factor $C/(C+x)$ inserted into the integral in (B7). Then $w(x)$ in (B8) becomes

$$w(x)=\left(\frac{x}{s+x}\right)^2\frac{C}{C+x}\,, \tag{B14}$$

while ϕ has to satisfy the boundary conditions, for $s>0$,

$$\phi\sim\text{const}\,,\ x\to 0\,;\quad \phi\sim\text{const}\,x^{-1}\,,\ x\to\infty\,. \tag{B15}$$

The solution that satisfies the boundary condition at the origin is

$$\phi=x^{-1}(s+x)^{1/2}\left[\left(1+\frac{x}{s}\right)^{\beta/2}\frac{F(\frac{1}{2}+\frac{1}{2}\beta,-\frac{1}{2}+\frac{1}{2}\beta;-(s+x)/C)}{F(\frac{1}{2}+\frac{1}{2}\beta,-\frac{1}{2}+\frac{1}{2}\beta;-s/C)}-(\beta\to-\beta)\right]\,, \tag{B16}$$

where F is the standard hypergeometric function. For large C, the boundary condition at infinity requires that

$$(2C)^\beta=\frac{\Gamma(1-\frac{1}{2}\beta)\,\Gamma(\frac{3}{2}+\frac{1}{2}\beta)}{\Gamma(\frac{3}{2}-\frac{1}{2}\beta)\,\Gamma(1+\frac{1}{2}\beta)}\,. \tag{B17}$$

Again, this equation has no solution for $0<\beta<1$ ($0<\lambda<\frac{1}{4}$). Finally, suppose we insert a factor $(s+x)^{-\epsilon}$ under the integral in (B7). Small positive ϵ corresponds to a large cutoff. In the differential equation now $w(x)=x^2/(s+x)^{2+\epsilon}$, while the boundary conditions are still given by (B15). The solution that satisfies the boundary condition at infinity is

$$\phi=x^{-1}(s+x)^{1/2}J_{\epsilon^{-1}}(2\lambda^{1/2}\epsilon^{-1}(s+x)^{-1/2\epsilon})\,, \tag{B18}$$

where J is the standard Bessel function. The boundary condition at the origin now requires that the Bessel function vanishes at $x=0$. For small ϵ, the first nontrivial zero of $J_{\epsilon^{-1}}$ occurs at

$$2\lambda^{1/2}\epsilon^{-1}s^{-\epsilon/2}=\epsilon^{-1}+(1.85575\ldots)\epsilon^{-1/3}+O(\epsilon^{1/3})\,, \tag{B19}$$

so that the minimum value of λ is again close to $\frac{1}{4}$.

Now let us look again at the inhomogeneous equation (B7). With the crude cutoff, it is satisfied by the functions in (B9) and (B10), provided that

$$Z_1-\lambda\int_C^\infty\frac{dy}{y}\left(\frac{y}{s+y}\right)\phi(y)=0\,. \tag{B20}$$

For the other two regularization methods the boundary condition at infinity is now $\phi\to Z_1$, $x\to\infty$. From now on the first method will be used. The combination of (B9a) and (B20) gives an equation for the factor $a=a(s)$ that determines the scale of ϕ. For large C this equation can be written

$$a(s)^{-1}=\tfrac{1}{2}Z_1{}^{-1}C^{-1/2}[(1+\beta)s^{-\beta/2}C^{\beta/2} - (1-\beta)s^{\beta/2}C^{-\beta/2}]. \qquad \text{(B21)}$$

It can be seen that for $0<\beta<1$ there is no pole in $a(s)$, either at $s=1$, or at $s=0$. We can make $a(s)$ cutoff-independent by choosing $Z_1 = A\frac{1}{2}(1+\beta)C^{-1/2+\beta/2}$, where A is some constant. Then $a(s)=As^{\beta/2}$, and

$$\phi(x,s)=Ax^{-1}(s+x)^{1/2}[(s+x)^{\beta/2}-s^{\beta}(s+x)^{-\beta/2}]. \qquad \text{(B22)}$$

For $\beta=i|\beta|$, the form (B9c) must be used, which means $a^{-1}\to 2ic^{-1}$ in (B21). For $\nu=0$, $s\geq 0$, and (B21) shows that there is still no pole possible at $s=0$. One finds the same result if the simplification (B6) is not made. For $\nu\neq 0$, one may choose C and β such that (B13) is satisfied for $s=1$. Then (B21) shows that there is a pole at $s=1$. With a suitable choice for Z_1, $Z_1=\pm\frac{1}{2}A(1+|\beta|^2)^{1/2}C^{-1/2}$, where the sign is the sign of $\cos[\frac{1}{2}|\beta|\ln C]$, the solution can be written as

$$\phi(x,s)=Ax^{-1}(s+x)^{1/2}\frac{\sin\left[\frac{1}{2}|\beta|\ln\left(1+\frac{x}{s}\right)\right]}{\sin[\frac{1}{2}|\beta|\ln s]}. \qquad \text{(B23)}$$

Equation (B23) shows clearly the pole at $P^2=\nu^2(s-1)=0$, as well as at other values of P^2. The spacelike poles at large positive P^2 are not due to the replacement (B6). Presumably they are not present in the solutions of Eq. (B1). The pole at $P^2=0$ has of course been inserted by hand, by satisfying (B13) at $s=1$. It could have been done more directly by taking a solution of (B7), without cutoff and with $Z_1=0$, and by dividing by P^2. The point is that with cutoff one cannot get the pole if $\lambda\leq\frac{1}{4}$, and for $\lambda>\frac{1}{4}$, there is still no pole possible in the $\nu=0$ case.

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