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Effective action for composite operators*

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An effective action and potential for composite operators is obtained. The formalism is used to analyze, by variational techniques, dynamical symmetry breaking and coherent solutions to field theory. A Rayleigh-Ritz procedure is introduced which replaces arbitrary variations with parametric variations. Previously unsolved nonlinear equations become, in the Rayleigh-Ritz approximation, solvable algebraic equations.

I. INTRODUCTION

Field-theoretic descriptions of natural processes suffer from a serious shortcoming: The only available method for solving dynamical equations is the perturbative expansion. Yet it is clear that there exist phenomena, apparently physically important, which cannot be easily seen in the perturbation series. Examples are spontaneous symmetry violation, bound states, entrapment of various experimentally unobserved excitations, etc. What is needed is an approximation scheme that preserves some of the nonlinear features of field theory, which presumably lead to these cooperative and coherent effects.

Recently in the course of various investigations of spontaneous symmetry violation at zero and finite temperature, it became possible to sum large classes of ordinary perturbation-series diagrams which contribute to the effective action $\Gamma(\phi)$ (the generating functional of single-particle irreducible n-point functions), and which preserve a much richer nonlinear structure than the familiar classical (tree) approximation.[1-3] In the present paper we continue that development. We study a generalization of the effective action, $\Gamma(\phi, G)$, which depends not only on $\phi(x)$—a possible expectation value of the quantum field $\Phi(x)$—but also on $G(x,y)$—a possible expectation value of $T\Phi(x)\Phi(y)$. Physical solutions require

$$\frac{\delta\Gamma(\phi,G)}{\delta\phi(x)}=0, \tag{1.1a}$$

$$\frac{\delta\Gamma(\phi,G)}{\delta G(x,y)}=0. \tag{1.1b}$$

Hence the formalism is especially appropriate for the study of dynamical symmetry violation, which is characterized by the fact that even though (1.1a) has only the symmetric solution $\phi(x)=0$, symmetry-breaking solutions exist for (1.1b).

In Sec. II we define $\Gamma(\phi, G)$ and derive a series expansion for it which is analogous to the WKB loop expansion previously obtained for $\Gamma(\phi)$.[4] In Sec. III we show that the bubble sum for $\Gamma(\phi)$, which has been recently performed[1-3] and which is dominant in an O(N)-invariant spinless theory for large N, is trivially obtained in the present formalism. It corresponds to a single graph—that of the Hartree-Fock approximation. Section IV is devoted to spontaneous symmetry violation by bound states. We show that the Hartree-Fock approximation to $\Gamma(\phi, G)$ leads to a gap equation for G, which upon linearization gives the ladder Bethe-Salpeter model that recently has been offered as an example of dynamical symmetry violation.[5] We also demonstrate how the nonlinear aspects can be analyzed. Rather than considering the arbitrary variation (1.1), which leads to an intractable nonlinear integral equation, we perform a Rayleigh-Ritz variation which gives a nonlinear algebraic equation. In Sec. V we adopt our formalism to the study of time-independent but position-dependent solutions to (1.1). We show that $\Gamma(\phi, G)$ in that case corresponds to the stationary expectation of the Hamiltonian in a normalized state $|\psi\rangle$ for which

$$\begin{aligned}&\langle\psi|\Phi(x)|\psi\rangle=\phi(\vec{x}),\\&\langle\psi|\Phi(x)\Phi(y)|\psi\rangle|_{x_0=y_0}=\phi(\vec{x})\phi(\vec{y})+\hbar G(\vec{x},\vec{y}).\end{aligned}\tag{1.2}$$

Now, in the Hartree-Fock approximation, (1.1) becomes equivalent to the variational equations derived by Kuti,[6] in his interesting development of a functional Schrödinger picture for field theory.

Studies of the sort here presented were initiated years ago by Lee and Yang,[7] and others.[8] These authors concerned themselves with nonrelativistic statistical mechanics. With the exception of a few isolated works,[9] little has been done to extend and apply these techniques to relativistic field theories. We hope that our use of functional methods to replace combinatorial analysis makes the general proofs more transparent, and that our sample applications demonstrate the utility of these ideas for practical calculations.

II. THE LOOP EXPANSION FOR $\Gamma(\phi,G)$

A. Definitions

We define $Z(J,K)$, the generating functional for Green's functions of nonlocal, composite fields (the derivation will be carried out for Bose fields; the generalization to fermions is trivial, and will be indicated in Sec. IV):

$$\begin{aligned} Z(J,K) &= e^{(i/\hbar)W(J,K)} \\ &= \int d\Phi \exp\left\{\frac{i}{\hbar}\left[I(\Phi) + \int d^4x\, \Phi(x)J(x) + \tfrac{1}{2}\int d^4x\, d^4y\, \Phi(x)K(x,y)\Phi(y)\right]\right\}. \end{aligned} \tag{2.1}$$

The Φ integration is functional. $I(\Phi)$ is the classical effective action

$$I(\Phi) = \int d^4x\, \mathcal{L}(x). \tag{2.2a}$$

$\mathcal{L}$ is the effective Lagrangian, containing gauge and ghost terms if a gauge theory is discussed. The field $\Phi(x)$ can possess components; the specifying index is suppressed. The classical action (2.2a) may also be written as

$$I(\Phi) = \int d^4x\, d^4y\, \Phi(x) iD^{-1}(x-y)\Phi(y) + I_{\text{int}}(\Phi),$$

$$I_{\text{int}}(\Phi) = \int d^4x\, \mathcal{L}_{\text{int}}(x), \tag{2.2b}$$

where $D(x-y)$ is the free propagator

$$iD^{-1}(x-y) = -(\Box + m^2)\delta^4(x-y) \tag{2.3}$$

and the interaction Lagrangian $\mathcal{L}_{\text{int}}$ is at least cubic in the fields.

$\Gamma(\phi, G)$ is a double Legendre transform of $W(J,K)$. We define

$$\frac{\delta W(J,K)}{\delta J(x)} = \phi(x), \tag{2.4a}$$

$$\frac{\delta W(J,K)}{\delta K(x,y)} = \tfrac{1}{2}[\phi(x)\phi(y) + \hbar G(x,y)]. \tag{2.4b}$$

Eliminate J and K in favor of ϕ and G and set

$$\begin{aligned} \Gamma(\phi, G) = W(J,K) &- \int d^4x\, \phi(x)J(x) \\ &- \tfrac{1}{2}\int d^4x\, d^4y\, \phi(x)K(x,y)\phi(y) \\ &- \tfrac{1}{2}\hbar \int d^4x\, d^4y\, G(x,y)K(y,x). \end{aligned} \tag{2.5}$$

Evidently it is also true that

$$\frac{\delta\Gamma(\phi, G)}{\delta\phi(x)} = -J(x) - \int d^4y\, K(x,y)\phi(y), \tag{2.6a}$$

$$\frac{\delta\Gamma(\phi, G)}{\delta G(x,y)} = -\tfrac{1}{2}\hbar K(x,y). \tag{2.6b}$$

Since physical processes correspond to vanishing sources J and K, Eqs. (2.6) provide a derivation of the stationarity requirement (1.1).

Let us observe that the conventional effective action is merely $\Gamma(\phi, G)$ at $K=0$, or equivalently from (2.6b), it is $\Gamma(\phi, G)$ at that value of $G(x,y)$ for which (2.6b) vanishes:

$$\begin{aligned} &\Gamma(\phi) = \Gamma(\phi, G_0), \\ &\frac{\delta\Gamma(\phi, G_0)}{\delta G_0(x,y)} = 0. \end{aligned} \tag{2.7}$$

Furthermore it is known that $\Gamma(\phi, G)$ is the generating functional in ϕ for *two*-particle irreducible Green's functions expressed in terms of the propagator G.[10] For example the diagrammatic expansion of

$$\left.\frac{\delta\Gamma(\phi, G)}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=0}$$

is the Feynman-Dyson series for the inverse 2-point function of the theory, with *two*-particle reducible graphs deleted, and with lines representing $\hbar G(x,y)$. (The reader may convince himself of this by working out all the differentiations.[11])

We now describe the series expansion for $\Gamma(\phi, G)$. We introduce the functional operator $\mathfrak{D}^{-1}(\phi)$ by the definition

$$\begin{aligned} i\mathfrak{D}^{-1}\{\phi; x,y\} &= \frac{\delta^2 I(\phi)}{\delta\phi(x)\delta\phi(y)} \\ &= iD^{-1}(x-y) + \frac{\delta^2 I_{\text{int}}(\phi)}{\delta\phi(x)\delta\phi(y)}. \end{aligned} \tag{2.8}$$

The required series is

$$\Gamma(\phi, G) = I(\phi) + \tfrac{1}{2}i\hbar\, \text{Tr Ln}G^{-1} + \tfrac{1}{2}i\hbar\, \text{Tr}\, \mathfrak{D}^{-1}(\phi)G + \Gamma_2(\phi, G) + \text{const}. \tag{2.9a}$$

The trace, the logarithm, and the product $\mathfrak{D}^{-1}G$ in the second and third terms are taken in the functional sense. The constant, independent of ϕ and G, is evaluated so that (2.7) is satisfied:

$$\Gamma(\phi, G) = I(\phi) + \tfrac{1}{2}i\hbar\, \text{Tr Ln}DG^{-1} + \tfrac{1}{2}i\hbar\, \text{Tr}\mathfrak{D}^{-1}(\phi)G + \Gamma_2(\phi, G) - \tfrac{1}{2}i\hbar\, \text{Tr}1. \tag{2.9b}$$

$\Gamma_2(\phi, G)$ is computed as follows. In the classical action $I(\Phi)$ shift the field Φ by $\phi(x)$. The new action $I(\Phi+\phi)$ possesses terms cubic and higher in Φ; these define an "interaction" part $I_{\text{int}}(\phi;\Phi)$ where the vertices depend on $\phi(x)$. $\Gamma_2(\phi, G)$ is given by all the two-particle irreducible vacuum graphs in a "theory" with vertices determined by $I_{\rm int}(\phi;\Phi)$ and propagators set equal to $G(x,y)$. Another way to say it is that only those vacuum graphs are kept which, upon opening one line, yield proper self-energy graphs. Since the vertices depend on $\phi(x)$, and $G(x,y)$ is not merely a function of $x-y$, this is not a translationally invariant theory. Nevertheless, $\Gamma_2(\phi, G)$ is easily determined by the usual Feynman-Dyson-Wick expansion. $\Gamma_2(\phi, G)$ is of order $\hbar^2$, and the number of loops corresponds to powers of $\hbar$.

In subsequent sections, formula (2.9) will be evaluated in illustrative examples and applied in various contexts. The remainder of Sec. II is devoted to the derivation.

B. Derivation of (2.9)

The derivation of (2.9) is facilitated by two observations. First, note that inasmuch as $\Gamma(\phi, G)$ is the generating functional (in ϕ) for two-particle irreducible n-point functions for the theory governed by the action $I(\Phi)$ (and lines set equal to $\hbar G$), it follows that $\Gamma(0, G)$ is the sum of all two-particle irreducible vacuum graphs of the same theory. According to (2.5) and (2.6) this is also given by

$$\Gamma(0,G)=\operatorname{Tr}G\frac{\partial\Gamma(0,G)}{\partial G}-i\hbar\ln\int d\Phi\exp\left\{\frac{i}{\hbar}\left[\tfrac{1}{2}\Phi iD^{-1}\Phi+I_{\rm int}(\Phi)+\Phi J^0-\frac{1}{\hbar}\Phi\frac{\partial\Gamma(0,G)}{\partial G}\Phi\right]\right\}$$
$$+i\hbar\ln\int d\Phi\exp\left\{\frac{i}{\hbar}\left[\tfrac{1}{2}\Phi iD^{-1}\Phi\right]\right\}$$
$$=\text{two-particle irreducible vacuum graphs of a theory governed by } I(\Phi),\text{ with lines representing }\hbar G. \tag{2.10}$$

(In the remainder of Sec. II we use a compact notation where all integrations are suppressed and derivatives are functional.) In (2.10) J^0 is that value of J which makes $\partial W(J,K)/\partial J=\phi$ vanish, i.e., all tadpoles are removed. The normalization factor, frequently omitted since it is a constant, is here explicitly exhibited.

Second, observe that the double Legendre transform (2.4) and (2.5) can also be performed sequentially. Thus we may set, at fixed K,

$$\Gamma^K(\phi)=W(J,K)-\phi J. \tag{2.11}$$

Then we define

$$\frac{\partial\Gamma^K(\phi)}{\partial K}=\tfrac{1}{2}(\phi\phi+\hbar G), \tag{2.12}$$

$$\Gamma(\phi, G)=\Gamma^K(\phi)-\tfrac{1}{2}\phi K\phi-\tfrac{1}{2}\hbar\operatorname{Tr}GK. \tag{2.13}$$

That the definition of $\Gamma(\phi, G)$ in (2.12) and (2.13) is equivalent to (2.4) and (2.5) follows from the equality

$$\frac{\partial\Gamma^K(\phi)}{\partial K}=\frac{\partial W(J,K)}{\partial J}\left.\frac{\partial J}{\partial K}\right|_\phi+\frac{\partial W(J,K)}{\partial K}-\left.\frac{\partial J}{\partial K}\right|_\phi\phi$$
$$=\frac{\partial W(J,K)}{\partial K}. \tag{2.14}$$

To establish (2.9) we use (2.13). According to (2.1) and (2.13), $\Gamma^K(\phi)$ is the effective action for a theory governed by the classical action

$$I^K(\Phi)=I(\Phi)+\tfrac{1}{2}\Phi K\Phi. \tag{2.15}$$

Hence according to previous analysis[4]

$$\Gamma^K(\phi)=I^K(\phi)+\Gamma_1^K(\phi), \tag{2.16a}$$

where $\Gamma_1^K(\phi)$ has the representation

$$\Gamma_1^K(\phi)=-i\hbar\ln\int d\Phi\exp\left\{\frac{i}{\hbar}\left[I^K(\Phi+\phi)-I^K(\phi)-\Phi\frac{\partial I^K(\phi)}{\partial\phi}-\Phi\frac{\partial\Gamma_1^K(\phi)}{\partial\phi}\right]\right\}$$
$$=-i\hbar\ln\int d\Phi\exp\left\{\frac{i}{\hbar}\left[\tfrac{1}{2}\Phi[i\mathfrak{D}^{-1}(\phi)+K]\Phi+I_{\rm int}(\phi,\Phi)-\Phi\frac{\partial\Gamma_1^K(\phi)}{\partial\phi}\right]\right\}. \tag{2.16b}$$

Upon equating (2.9) [which is here viewed as defining $\Gamma_2(\phi, G)$] with (2.13) and using (2.15) and (2.16), we find

$$\Gamma_2(\phi, G)+\text{const}=-\tfrac{1}{2}\hbar\operatorname{Tr}[i\mathfrak{D}^{-1}(\phi)+K]G-\tfrac{1}{2}i\hbar\operatorname{Tr}\operatorname{Ln}G^{-1}+\Gamma_1^K(\phi). \tag{2.17}$$

The proof of (2.9) will now follow, if it can be shown that $\Gamma_2(\phi, G)$, as given by (2.17), is the sum of two-particle irreducible vacuum graphs governed by vertices of $I_{\rm int}(\phi;\Phi)$ and propagators G.

We proceed by eliminating K in (2.17). According to (2.6) and (2.9), K satisfies

$$-\tfrac{1}{2}\hbar K = -\tfrac{1}{2}i\hbar G^{-1} + \tfrac{1}{2}i\hbar\,\mathfrak{D}^{-1}(\phi) + \frac{\partial\Gamma_2(G)}{\partial G}\,. \tag{2.18}$$

Hence (2.17) is

$$\Gamma_2(\phi, G) + \text{const} = \mathrm{Tr}G\frac{\partial\Gamma_2(\phi, G)}{\partial G} - i\hbar\ln\int d\Phi\,\exp\left\{\frac{i}{\hbar}\left[\tfrac{1}{2}\Phi iG^{-1}\Phi + I_{\text{int}}(\phi;\Phi) - \Phi\frac{\partial\Gamma_1^K(\phi)}{\partial\phi} - \frac{1}{\hbar}\Phi\frac{\partial\Gamma_2(\phi, G)}{\partial G}\Phi\right]\right\}$$
$$+ i\hbar\ln\int d\Phi\,\exp\left\{\frac{i}{\hbar}\left[\tfrac{1}{2}\Phi G^{-1}\Phi\right]\right\}. \tag{2.19}$$

Comparing (2.19) with (2.10), we see that $\Gamma_2(\phi, G)$ is precisely the sum of two-particle irreducible vacuum graphs in a theory governed by the action $\frac{1}{2}\Phi i G^{-1}\Phi + I_{\text{int}}(\phi;\Phi)$, since it has already been previously shown[4] that $\partial\Gamma_1^K(\phi)/\partial\phi$ is precisely that value of an external current which makes tadpoles vanish. This completes the proof.

C. Discussion of (2.9)

The series for $\Gamma(\phi, G)$ may also be understood in the following way. According to its definition, $\Gamma(\phi, G)$ is given by

$$e^{i\Gamma(\phi,G)/\hbar} = \int d\Phi\,\exp\left\{\frac{i}{\hbar}\left[I(\Phi) + (\Phi - \phi)J + \tfrac{1}{2}\Phi K\Phi - \tfrac{1}{2}\phi K\phi - \tfrac{1}{2}\hbar\,\mathrm{Tr}GK\right]\right\}$$
$$= \int d\Phi\,\exp\left\{\frac{i}{\hbar}\left[I(\Phi) - (\Phi - \phi)\frac{\partial\Gamma(\phi, G)}{\partial\phi} - \frac{1}{\hbar}(\Phi - \phi)\frac{\partial\Gamma(\phi, G)}{\partial G}(\Phi - \phi) + \mathrm{Tr}G\frac{\partial\Gamma(\phi, G)}{\partial G}\right]\right\}. \tag{2.20a}$$

Upon shifting by ϕ, which removes the one-particle reducible graphs, we find

$$\Gamma(\phi, G) - \mathrm{Tr}G\frac{\partial\Gamma(\phi, G)}{\partial G} = -i\hbar\ln\int d\Phi\,\exp\left\{\frac{i}{\hbar}I(\phi, G;\Phi)\right\},$$
$$I(\phi, G;\Phi) = I(\Phi + \phi) - \Phi\frac{\partial\Gamma(\phi, G)}{\partial\phi} - \frac{1}{\hbar}\Phi\frac{\partial\Gamma(\phi, G)}{\partial G}\Phi. \tag{2.20b}$$

Varying this with respect to G gives

$$-\mathrm{Tr}G\frac{\partial^2\Gamma(\phi, G)}{\partial G\partial G}\int d\Phi\,\exp\left\{\frac{i}{\hbar}I(\phi, G;\Phi)\right\} = -\frac{\partial\Gamma(\phi, G)}{\partial\phi\partial G}\int d\Phi\,\Phi\,\exp\left\{\frac{i}{\hbar}I(\phi, G;\Phi)\right\}$$
$$-\frac{1}{\hbar}\int d\Phi\,\Phi\frac{\partial^2\Gamma(\phi, G)}{\partial G\partial G}\Phi\,\exp\left\{\frac{i}{\hbar}I(\phi, G;\Phi)\right\}. \tag{2.21a}$$

The first term on the right-hand side vanishes, since the expectation of the field Φ in the theory with the action $I(\phi, G;\Phi)$ is zero. [This is a consequence of the shift which was performed in passing from (2.20a) to (2.20b) and was established explicitly previously.[4]] The remaining terms in (2.21a) therefore imply

$$\hbar G = \frac{\int d\Phi\,\Phi\Phi\,\exp\{(i/\hbar)I(\phi, G;\Phi)\}}{\int d\Phi\,\exp\{(i/\hbar)I(\phi, G;\Phi)\}}\,. \tag{2.21b}$$

This means that G is the exact connected propagator of the theory.

Turning now to (2.18), we see that

$$G^{-1} = \mathfrak{D}^{-1}(\phi) - iK - \Sigma(\phi, G), \tag{2.22a}$$

where

$$\Sigma(\phi, G) = \frac{2i}{\hbar}\,\frac{\partial\Gamma_2(\phi, G)}{\partial G}\,. \tag{2.22b}$$

But (2.22a) is just the Schwinger-Dyson equation for the propagator and $\Sigma(\phi, G)$ is to be interpreted as the proper self-energy part, with *no* propagator insertions. However, since Σ is also given by a derivative with respect to G of Γ_2, Γ_2 must be two-particle irreducible. For if Γ_2 has a two-particle reducible contribution of the form $\tilde{\Gamma}GG\tilde{\Gamma}'$, then Σ would have a contribution of the form $\tilde{\Gamma}G\tilde{\Gamma}'$; but such structures do not belong in Σ. The absence of two-particle reducible contributions to Γ_2 is a consequence of the fact that the propagator G has no radiative corrections, and is exact.

D. Concluding remarks

Frequently one is interested only in translation-invariant solutions. In that case, we set $\phi(x)$ to a constant ϕ, and take $G(x,y)$ to be a function only of $x-y$. A generalization of the effective potential may be defined by

$$V(\phi, G)\int d^4x = -\Gamma(\phi, G)\big|_{\text{translation invariant}}\,. \tag{2.23}$$

The series for $V(\phi, G)$ can be easily obtained from (2.9). We define the Fourier-transformed propagators

$$G(p) = \int d^4x\, e^{ip(x-y)} G(x-y), \tag{2.24a}$$

$$\mathfrak{D}\{\phi; p\} = \int d^4x\, e^{ip(x-y)} \mathfrak{D}\{\phi; x-y\}, \tag{2.24b}$$

$$D(p) = \int d^4x\, e^{ip(x-y)} D(x-y). \tag{2.24c}$$

Equation (2.9) reduces to

$$V(\phi, G) = U(\phi) - \tfrac{1}{2} i\hbar \int \frac{d^4p}{(2\pi)^4} \ln \det D(p) G^{-1}(p) - \tfrac{1}{2} i\hbar \int \frac{d^4p}{(2\pi)^4} \operatorname{tr}[\mathfrak{D}^{-1}\{\phi; p\} G(p) - 1] + V_2(\phi, G). \tag{2.25}$$

The determinant and the trace are no longer functional; they apply to the component degrees of freedom. $U(\phi)$ is the classical potential; $-V_2(\phi, G)$ is the sum of all the two-particle irreducible vacuum graphs of the theory with vertices given by $I_{\text{int}}(\phi; \Phi)$ and propagator $G(p)$. The vertices still depend on ϕ, but this is now a constant parameter. Since translation invariance is maintained, an overall factor of space-time volume must be removed.

In terms of $V(\phi, G)$, the stationarity requirements become

$$\frac{\partial V(\phi, G)}{\partial \phi} = 0, \tag{2.26a}$$

$$\frac{\partial V(\phi, G)}{\partial G} = 0. \tag{2.26b}$$

$V(\phi, G)$ is a function of ϕ and a functional of $G(p)$. Hence only the second derivative in (2.26) is functional.

The sequences of Legendre transforms may be continued. In this way an effective action can be defined which depends functionally not only on ϕ and G, but also on irreducible 3-point, 4-point, etc. functions. The obvious generalization of (1.1) is the requirement that the effective action is stationary with respect to independent variations of any irreducible Green's function. Details of this will be given elsewhere.[12]

III. AN O(N)-INVARIANT SPINLESS MODEL

As an illustration of the general formalism, we now present an analysis of an O(N)-invariant spinless model, governed by the Lagrangian

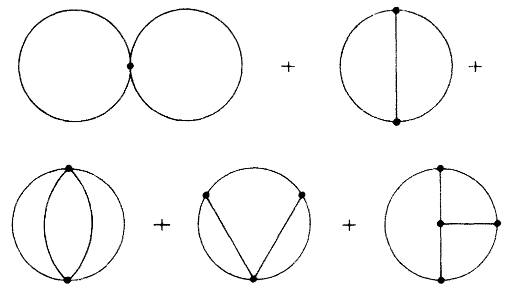

FIG. 1. Two-particle irreducible graphs contributing to $\Gamma_2(\phi, G)$ up to the three-loop level in a $\lambda\Phi^4$ theory. The solid line represents the propagator $\hbar G(x, y)$. There are two kinds of vertices: a three-point vertex proportional to $\lambda\phi$ and a four-point vertex.

$$\mathfrak{L} = \tfrac{1}{2}\partial_\mu \Phi_a \partial^\mu \Phi_a - \tfrac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4!N}\Phi^4,$$

$$\Phi^2 = \Phi_a \Phi_a, \quad \Phi^4 = (\Phi^2)^2, \quad a = 1, \ldots, N. \tag{3.1}$$

The propagator $i\mathfrak{D}^{-1}(\phi)$ is

$$i\mathfrak{D}_{ab}^{-1}\{\phi; x, y\} = -\left[\Box + m^2 + \frac{\lambda}{6N}\phi^2(x)\right]\delta_{ab}\delta^4(x-y) - \frac{\lambda}{3N}\phi_a(x)\phi_b(x)\delta^4(x-y). \tag{3.2}$$

Vertices of the shifted theory are given by the interaction Lagrangian

$$\mathfrak{L}_{\text{int}}(\phi; \Phi) = -\frac{\lambda}{6N}\phi_a(x)\Phi_a(x)\Phi^2(x) - \frac{\lambda}{4!N}\Phi^4(x). \tag{3.3}$$

Consequently the diagrams contributing to $\Gamma_2(\phi, G)$ are as depicted in Fig. 1, up to three-loop contributions. Each line represents the propagator $\hbar G_{ab}(x, y)$, and there are two kinds of vertices:

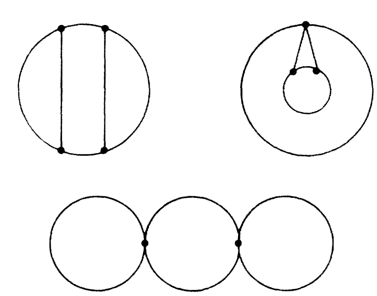

FIG. 2. Two-particle reducible graphs which do not contribute to $\Gamma_2(\phi, G)$ in a $\lambda\Phi^4$ theory. [These graphs do contribute to the ordinary effection action $\Gamma(\phi)$. In that instance the solid line represents the propagator $\hbar\mathfrak{D}\{\phi; x, y\}$.]

a four-point vertex proportional to λ and a three-point vertex, arising from the shift, proportional to $\lambda\phi_a(x)$. The numerical factors are not indicated; they are determined in the usual fashion by Wick's theorem.

[If we were computing the ordinary effective action $\Gamma(\phi)$, then the lines would represent the propagator $\hbar\mathfrak{D}_{ab}\{\phi;x,y\}$ and there would be additional contributions which are two-particle reducible. On the three-loop level these are depicted in Fig. 2.]

We shall evaluate $\Gamma(\phi, G)$ in the Hartree-Fock approximation, which corresponds to retaining only that contribution to $\Gamma_2(\phi, G)$ which is lowest-order in coupling constant. The relevant graph is the first entry, the double bubble, of Fig. 1:

$$\Gamma(\phi, G)+\text{const}=I(\phi)+\tfrac{1}{2}i\hbar\,\mathrm{Tr}\,\mathrm{Ln}G^{-1}+\tfrac{1}{2}i\hbar\,\mathrm{Tr}\mathfrak{D}^{-1}(\phi)G-\frac{\lambda}{4!N}\hbar^2\int d^4x[G_{aa}(x,x)G_{bb}(x,x)+2G_{ab}(x,x)G_{ba}(x,x)]. \tag{3.4}$$

From this $\Gamma(\phi)$ can be obtained by solving for $G_{ab}(x,y)$:

$$\frac{\delta\Gamma(\phi,G)}{\delta G_{ab}(x,y)}=0$$

$$=-\tfrac{1}{2}i\hbar G_{ab}{}^{-1}(x,y)+\tfrac{1}{2}i\hbar\,\mathfrak{D}_{ab}{}^{-1}\{\phi;x,y\}-\frac{\lambda\hbar^2}{12N}[\delta_{ab}G_{cc}(x,x)+2G_{ab}(x,x)]\,\delta^4(x-y), \tag{3.5}$$

$$G_{ab}{}^{-1}(x,y)=\mathfrak{D}_{ab}{}^{-1}\{\phi;x,y\}+\frac{i\lambda\hbar}{6N}[\delta_{ab}G_{cc}(x,x)+2G_{ab}(x,x)]\,\delta^4(x-y).$$

Eliminating $\mathfrak{D}_{ab}{}^{-1}$ between (3.4) and (3.5) finally gives, apart from constants,

$$\Gamma(\phi)=I(\phi)+\tfrac{1}{2}i\hbar\,\mathrm{Tr}\,\mathrm{Ln}G^{-1}+\frac{\lambda\hbar^2}{4!N}\int d^4x[G_{aa}(x,x)G_{bb}(x,x)+2G_{ab}(x,x)G_{ba}(x,x)], \tag{3.6}$$

where $G_{ab}(x,y)$ satisfies the equation (3.5).

It is of course impossible to solve (3.5) for $G_{ab}(x,y)$ exactly. However, a simplification occurs if we consider the large-N limit, and keep terms dominant in N. In that limit ϕ_a is to be taken to be $O(\sqrt{N})$, hence $\mathfrak{D}_{ab}$ and G_{ab} are $O(1)$. We decompose

$$G_{ab}(x,y)=\delta_{ab}g(x,y)+\tilde{G}_{ab}(x,y), \tag{3.7}$$

where $\tilde{G}_{ab}(x,y)$ is traceless. To retain the $O(1)$ parts of (3.5) it is sufficient to keep only the first term in the bracket. Thus

$$g^{-1}(x,y)=i\left[\Box+m^2+\frac{\lambda}{6N}\phi^2(x)+\tfrac{1}{6}\lambda\hbar g(x,x)\right]\times\delta^4(x-y). \tag{3.8}$$

Upon defining

$$\chi(x)=m^2+\frac{\lambda}{6N}\phi^2(x)+\tfrac{1}{6}\lambda\hbar g(x,x) \tag{3.9a}$$

we find that

$$g^{-1}(x,y)=i[\Box+\chi(x)]\,\delta^4(x-y) \tag{3.9b}$$

and

$$\Box+\chi=\Box+m^2+\frac{\lambda}{6N}\phi^2-\tfrac{1}{6}i\lambda\hbar(\Box+\chi)^{-1}. \tag{3.10}$$

From (3.6)

$$\begin{aligned}\Gamma(\phi)&=\int d^4x\left(\tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a-\tfrac{1}{2}m^2\phi^2-\frac{\lambda}{4!N}\phi^4\right)+\tfrac{1}{2}i\hbar N\,\mathrm{Tr}\,\mathrm{Ln}(\Box+\chi)+\frac{3N}{2\lambda}\int d^4x\left(\chi-m^2-\frac{\lambda}{6N}\phi^2\right)^2\\&=\int d^4x\left(\tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a+\frac{3N}{2\lambda}\chi^2-\frac{3N}{\lambda}m^2\chi-\tfrac{1}{2}\phi^2\chi\right)+\tfrac{1}{2}i\hbar N\,\mathrm{Tr}\,\mathrm{Ln}(\Box+\chi).\end{aligned} \tag{3.11}$$

The result (3.11) has been previously obtained.[1-3] The power of the present formalism is now apparent: Only one graph has to be evaluated. Our method in this example is related to the combinatorial trick which has been previously utilized in the analysis of this problem.[3]

IV. DYNAMICAL SYMMETRY BREAKING

We construct the Hartree-Fock approximation to the generalized effective potential for an Abelian gauge theory of fermions and vector mesons which has recently been studied as an example of dynamical symmetry violation.[5] A gap equation is derived; in linearized form it coincides with the Bethe-Salpeter ladder equation which was previously solved.[5] A Rayleigh-Ritz procedure is developed to study the nonlinear aspects of the problem.

A. Effective potential

The Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m - g_A\gamma_\mu A^\mu - g_B\tau_2\gamma_\mu B^\mu)\psi - \tfrac{1}{4}A_{\mu\nu}A^{\mu\nu} - \tfrac{1}{4}B_{\mu\nu}B^{\mu\nu},$$
$$A^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$
$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu, \qquad (4.1)$$

where $\psi = (\psi_1, \psi_2)$ is a two-component field in "isospin" space and τ_2 is the usual Pauli matrix. When the gauge symmetry

$$\psi \to e^{i\theta\tau_2}\psi,$$
$$B_\mu \to B_\mu - \frac{1}{g_B}\partial_\mu\theta \qquad (4.2)$$

is spontaneously broken, the B meson picks up a mass M_B, and the masses of the fermions split from the symmetric value m by an amount $\pm\delta m$. There is a gauge symmetry for the A meson, also, which remains unbroken.

The generalized effective action for this problem will depend only on the complete propagators of the theory: $G(x,y)$ for the fermions and $\Delta_i^{\mu\nu}(x,y)$, $i = A, B$, for the vector mesons. A field dependence is not included, since we do not expect that any of the fields acquire a vacuum expectation value in the absence of sources. The formula (2.9) is now applicable, with the field variable eliminated and with the following modification which reflects Fermi statistics: All factors of $-\frac{1}{2}$ appearing in (2.9) are replaced by 1.[13] Moreover, since we seek a translation-invariant solution, only the effective potential is of interest.

Thus for our problem we have

$$V(G,\Delta_i) = -i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}[\ln S^{-1}(p)G(p) - S^{-1}(p)G(p) + 1] + \tfrac{1}{2}i\sum_{i=A,B}\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}[\ln D^{-1}(p)\Delta_i(p) - D^{-1}(p)\Delta_i(p) + 1] + V_2(G,\Delta_i). \qquad (4.3)$$

Space-time indices on the boson propagators have been suppressed, and in Sec. IV $\hbar = 1$. $S(p)$ and $D^{\mu\nu}(p)$ are the free propagators:

$$S(p) = \frac{i}{\not{p} - m},$$
$$D^{\mu\nu}(p) = \frac{-i}{p^2}P^{\mu\nu}, \qquad (4.4)$$
$$P^{\mu\nu} = g^{\mu\nu} - p^\mu p^\nu/p^2.$$

As is discussed below, the Landau gauge must be used for consistency with the subsequent analysis. In the Hartree-Fock approximation $V_2(G,\Delta_i)$ is given by the graphs of Fig. 3, where the solid lines represent $G(p)$, the wavy line represents $\Delta_A^{\mu\nu}(p)$, and the zigzag line represents $\Delta_B^{\mu\nu}(p)$. The analytic expression is

$$V_2(G,\Delta_i) = \tfrac{1}{2}i\sum_{i=A,B}\int\frac{d^4p\,d^4k}{(2\pi)^8}\,\mathrm{tr}\,\Gamma_i^\mu G(p)\Gamma_i^\nu G(p+k) \times\Delta_{i,\mu\nu}(k),$$
$$\Gamma_A^\mu = g_A\gamma^\mu, \quad \Gamma_B^\mu = g_B\gamma^\mu\tau_2. \qquad (4.5)$$

The reason for using the Landau gauge can now be explained. Our Hartree-Fock approximation replaces the complete $B^\mu\psi\bar{\psi}$ vertex by the free vertex. But in a spontaneously broken theory, this vertex has a Goldstone pole, which certainly is not in the bare vertex. The approximation therefore makes sense only in the Landau gauge, since then the Goldstone pole is annihilated in all vacuum graphs.

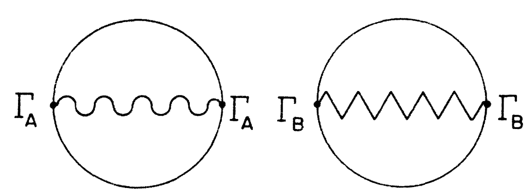

FIG. 3. Hartree-Fock approximation to $V_2(G,\Delta_i)$. The solid line is the fermion propagator G; the wavy line is the boson propagator Δ_A; the zigzag line is the boson propagator Δ_B. Γ_A and Γ_B represent the fermion-boson interactions and are defined in (4.5).

B. The gap equation

Demanding that $V(G,\Delta_i)$ be stationary against variations of G gives from (4.3) and (4.5)

$$G^{-1} = S^{-1} + \sum_{i=A,B}\Gamma_i G\Gamma_i\Delta_i\,. \qquad (4.6)$$

For notational simplicity, all integrations are suppressed. Equation (4.6) is also represented pictorially in Fig. 4. The symmetry-breaking part of G^{-1} is proportional to τ_3. Since S^{-1} has no such contribution, the symmetry-breaking part

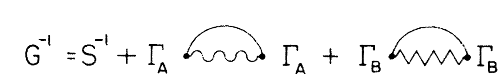

FIG. 4. Equation satisfied by the fermion propagator. S is the free fermion propagator, given in (4.4).

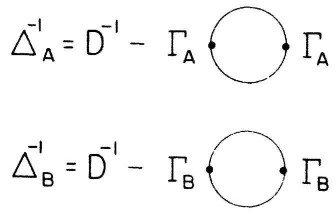

FIG. 5. Equations satisfied by the boson propagators. D is the free boson propagator, given in (4.4).

satisfies a homogeneous, nonlinear equation. Even when the meson propagators are replaced with their free form, Eq. (4.6) remains nonlinear and intractable. [In that approximation (4.6) is the gap equation proposed years ago by Johnson and others.[14]] When the equation is further simplified by linearizing the Fermi propagator, i.e., setting $G^{-1}=S^{-1}-\Sigma$, $G=S+S\Sigma S$, the symmetry-violating part of (4.6) coincides with the ladder Bethe-Salpeter equation, which, as has been shown previously,[5] possesses symmetry-breaking as well as symmetry-preserving solutions.

The equations for the Bose propagators are obtained by varying $V(G,\Delta_i)$ with respect to Δ_i. One finds

$$\Delta_i^{-1}=D^{-1}-\Gamma_i G\Gamma_i G. \tag{4.7}$$

Graphically this is represented in Fig. 5. In its linearized form, the equation, together with (4.6), implies that the B meson acquires a mass (see below).

We shall need the results of the linearized theory for the subsequent analysis. Hence we summarize them now.

C. Summary of linearized theory

The linearized theory is analyzed for $g_A{}^2, g_B{}^2 \ll 1$. For the fermion propagator, we define

$$G^{-1}=S^{-1}-\Sigma, \tag{4.8a}$$

$$\Sigma=\Sigma_N+\Sigma_S+\Sigma_V. \tag{4.8b}$$

Σ_V is the symmetry-violating part, proportional to τ_3. $\Sigma_N+\Sigma_S$ is symmetric, where Σ_N is that part of the self-energy which is also present in the normal solution while Σ_S, though symmetric, arises from the symmetry-violating properties of the theory. The linearized equation for Σ_V, which follows from (4.6), is given below and in Fig. 6, where the insertion V represents Σ_V (dashed lines are free propagators):

$$\Sigma_V=-\sum_{i=A,B}\Gamma_i S\Sigma_V S\Gamma_i D. \tag{4.9}$$

The solution to (4.9) is[5]

$$\Sigma_V(p^2)\underset{|-p^2|\gg m^2}{=}-i\delta m\left(\frac{-p^2}{m^2}\right)^{-\epsilon}\tau_3, \tag{4.10a}$$

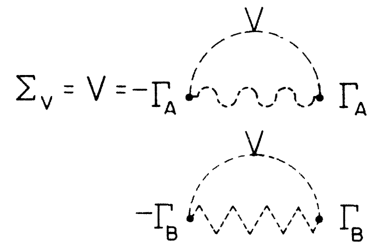

FIG. 6. Linearized gap equation for Σ_V, the symmetry-breaking part of the proper fermion self-energy. Σ_V is also denoted by V, and dashed lines represent free propagators.

$$\epsilon=\frac{3}{16\pi^2}(g_A{}^2-g_B{}^2)+O(g_A{}^2,g_B{}^2). \tag{4.10b}$$

Because ϵ is small, it is a good approximation to take

$$\Sigma_V(p^2)\underset{|-p^2|\leq m^2}{=}-i\delta m\tau_3. \tag{4.10c}$$

(This ignores various threshold effects, which in any event are washed out in subsequent integrations.) The normal part Σ_N is quadratic in the coupling constant and is set to zero. Before discussing Σ_S, we turn to the meson propagators.

We define

$$\Delta_i^{\mu\nu}(p)=\frac{-iP^{\mu\nu}}{p^2-\Pi^i(p^2)}. \tag{4.11}$$

For the A propagator, which has no symmetry breaking, $\Pi^A(p^2)$ is dropped since it is $O(g_A{}^2)$. For the B propagator, G in (4.7) is expanded in powers of Σ, from (4.8a). We seek the symmetry-violating part, hence only Σ_V is kept. It is sufficient to go to second order in Σ_V. All other terms which are not kept are quadratic in the coupling constant. It was found[5] that the symmetry-violating part of $\Pi^B(p^2)$, $\Pi_V^B(p^2)$, behaves like a constant near $p^2=0$; and more recently it was shown that $\Pi_V^B(p^2)$ decreases at infinity[15]:

$$\Pi_V^B(p^2)\underset{|-p^2|\gg m^2}{=}M_B{}^2\left(\frac{-p^2}{m^2}\right)^{-2\epsilon},$$
$$\Pi_V^B(p^2)\underset{|-p^2|\leq m^2}{=}M_B{}^2. \tag{4.12}$$

The B-meson mass is also calculable[5]:

$$M_B{}^2=\frac{g_B{}^2}{2\pi^2\epsilon}(\delta m)^2+O(g_A{}^2,g_B{}^2). \tag{4.13}$$

We shall not use this formula for $M_B{}^2$; rather it shall be derived below. [The approximation scheme which yields (4.12) is not gauge-invariant in that the vacuum-polarization tensor for the B meson is not transverse. However, since we are in the Landau gauge, only the $g^{\mu\nu}$ part of that tensor survives when the complete propagator (4.11) is formed. The final results are gauge-invariant.[15]]

Note that the symmetry-violating objects Σ_V and $\Pi_V^B(p^2)$ are of zeroth order in the coupling constant.

Finally, Σ_S is determined. When (4.8) is inserted in (4.6) and iterated, one finds[16]

$$\Sigma_S = \Sigma_S^{(1)} + \Sigma_S^{(2)},$$

$$\Sigma_S^{(1)} = -\sum_{i=A,B} \Gamma_i S \Sigma_V S \Sigma_V S, \tag{4.14}$$

$$\Sigma_S^{(2)} = -\Gamma_B S \Gamma_B D \Pi_V^B D.$$

The graphical representation is in Fig. 7, where the large dot represents the symmetry-breaking B-meson self-energy given by (4.12). Note that Σ_S is proportional to the symmetry-breaking parameters $(\delta m)^2$ and $M_B{}^2$ and is second order in the coupling. For large p, $\Sigma_S(p)$ decreases like $(\not{p}/p^2)(-p^2/m^2)^{-2\epsilon}$.

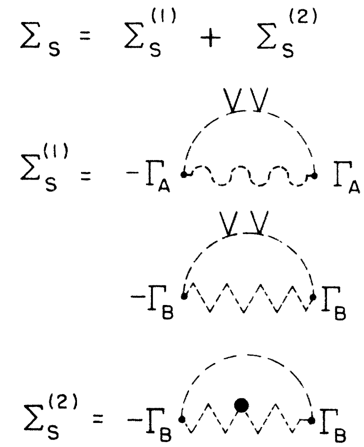

FIG. 7. Equations giving Σ_S, the contribution of symmetry breaking to the symmetric part of the proper fermion self-energy. The large black circle represents Π_V^B.

D. Rayleigh-Ritz approximation

By performing an arbitrary variation on $V(G,\Delta_i)$ nonlinear equations emerge which can only be solved in a linearized approximation. We now develop a strategy for analyzing the nonlinear aspects of the problem. Rather than performing arbitrary variations, we evaluate $V(G,\Delta_i)$ with specific, parameter-dependent expressions for G and Δ_i, and then vary these parameters. The forms for the propagators that we shall use are solutions of the linear theory,[5] summarized in Sec. IV C. They depend on the symmetry-breaking quantities δm, $M_B{}^2$, and ϵ. Our goal therefore is to write down an effective potential which depends on the numbers δm, $M_B{}^2$, and ϵ, and whose minimum approximately determines them. [Clearly, δm plays the role of some suitably regularized expectation value $\langle\bar\psi(x)\tau_3\psi(x)\rangle$.] However, we must insure that our formula is free of divergences. *A priori* one can expect to encounter quartic, quadratic, and logarithmic divergences. In fact all these divergences are absent for the following reasons.

The quartic divergences disappear if we subtract from $V(G,\Delta_i)$ the same expression evaluated at *symmetric* forms for the propagators, denoted by the subscript N. [In scalar field theories this is accomplished by setting $V(\phi)|_{\phi=0}=0$.] We therefore consider

$$\Omega = V(G,\Delta_i) - V(G_N,\Delta_{iN}). \tag{4.15}$$

It was shown previously that the quadratic divergences are also absent, provided the propagators solve the linearized theory; specifically provided that ϵ is given by (4.10b).[17] [It is unnecessary to require that $M_B{}^2$ be given by (4.13).] This is because any quadratic divergence in Ω is proportional to $(\delta m)^2$ or $M_B{}^2$. (Ω has no dependence on odd powers of δm or Σ_V since $\mathrm{tr}\,\tau_3=0$.) It is easy to show that these terms vanish when the propagators satisfy the equations of the linearized theory:

$$\Omega = \tfrac{1}{2}\delta m \frac{\partial\Omega}{\partial\delta m}\bigg|_{M_B{}^2=0} + M_B{}^2 \frac{\partial\Omega}{\partial M_B{}^2}\bigg|_{M_B{}^2=0;\,\delta m=0}$$

$$+\text{higher powers of the masses.} \tag{4.16a}$$

In $\partial\Omega/\partial\delta m|_{M_B{}^2=0}$ only terms linear in δm are kept. But it is also true that

$$\frac{\partial\Omega}{\partial\delta m}\bigg|_{M_B{}^2=0} = \int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\frac{\partial G(p)}{\partial\delta m}\frac{\delta V(G,\Delta_i)}{\delta G(p)}\bigg|_{M_B{}^2=0}, \tag{4.16b}$$

$$\frac{\partial\Omega}{\partial M_B{}^2}\bigg|_{M_B{}^2=0;\,\delta m=0} = \int\frac{d^4p}{(2\pi)^4}\left[\frac{\partial\Delta_B^{\mu\nu}(p)}{\partial M_B^2}\frac{\delta V(G,\Delta_i)}{\delta\Delta_B^{\mu\nu}(p)}\bigg|_{M_B^2=0;\,\delta m=0} + \mathrm{tr}\frac{\partial G(p)}{\partial M_B^2}\frac{\delta V(G,\Delta_i)}{\delta G(p)}\bigg|_{M_B^2=0;\,\delta m=0}\right]. \tag{4.16c}$$

Since only terms linear in δm matter in (4.16b), the symmetry-breaking part of $G(p)$ must satisfy only the linearized equation for $\delta V(G,\Delta_i)/\delta G(p)$ to vanish. In (4.16c) all symmetry-breaking parameters are set to zero. Hence that term will vanish provided $\Delta_B^{\mu\nu}(p)$ satisfies its equation up to symmetry-breaking terms. Therefore when ϵ is fixed by (4.10b), and is *not* viewed as a variational parameter, there are no $(\delta m)^2$ or M_B^2 terms and no quadratic divergences.

Finally, the logarithmic divergences are removed by the power-law falloff in the mass operators (4.10) and (4.12). What could be a possible logarithmic infinity becomes a finite term proportional to an inverse power of ϵ. Such terms necessarily involve $(\delta m)^4$, M_B^4, and $(\delta m)^2 M_B^2$. It is the inverse powers of ϵ that spoil ordinary perturbation theory, since ϵ is $O(g_A^2, g_B^2)$. However, once they have been isolated, the weak-coupling limit may be taken with impunity.

According to (4.3) and (4.5), Ω is given by the following symbolic expression:

$$\Omega(\delta m, M_B^2) = -i\,\mathrm{tr}[\ln G_N^{-1}G - G_N^{-1}G + 1] + \tfrac{1}{2}i \sum_{i=A,B} \mathrm{tr}[\ln \Delta_{iN}^{-1}\Delta_i - \Delta_{iN}^{-1}\Delta_i + 1]$$
$$+ i\,\mathrm{tr}\Sigma_N[G - G_N] - \tfrac{1}{2}i \sum_{i=A,B} \mathrm{tr}\Pi_N^i[\Delta_i - \Delta_{iN}] + \tfrac{1}{2}i \sum_{i=A,B} \mathrm{tr}[\Gamma_i G \Gamma_i G \Delta_i - \Gamma_i G_N \Gamma_i G_N \Delta_{iN}]. \qquad (4.17)$$

The exact variational equations for the *normal* propagators have been used to eliminate S^{-1} and D^{-1} in (4.3) in favor of the complete, normal propagators G_N and Δ_{iN}. These equations are, of course,

$$G_N^{-1} = S^{-1} - \Sigma_N\,, \qquad (4.18a)$$

$$\Sigma_N = -\sum_{i=A,B} \Gamma_i G_N \Gamma_i \Delta_{iN}\,, \qquad (4.18b)$$

$$\Delta_{iN}^{-1} = D^{-1} - \Pi_N^i\,, \qquad (4.18c)$$

$$\Pi_N^i = \Gamma_i G_N \Gamma_i G_N\,. \qquad (4.18d)$$

The last three terms in (4.17) may be combined. Observe that the last term may be rewritten, with the help of (4.18), as

$$\tfrac{1}{2}i \sum_{i=A,B} \mathrm{tr}[\Gamma_i G \Gamma_i G \Delta_i - \Gamma_i G_N \Gamma_i G_N \Delta_{iN}]$$
$$= -i\,\mathrm{tr}\Sigma_N[G - G_N] + \tfrac{1}{2}i \sum_{i=A,B} \mathrm{tr}\Pi_N^i[\Delta_i - \Delta_{iN}] + \bar{\Omega}\,, \qquad (4.19)$$
$$\bar{\Omega} = \tfrac{1}{2}i \sum_{i=A,B} \mathrm{tr}[\Gamma_i G \Gamma_i G - \Gamma_i G_N \Gamma_i G_N][\Delta_i - \Delta_{iN}]$$
$$+ \tfrac{1}{2}i \sum_{i=A,B} \mathrm{tr}\Gamma_i[G - G_N]\Gamma_i[G - G_N]\Delta_{iN}\,.$$

Hence the last three terms in (4.17) can be replaced by $\bar{\Omega}$. [It is important to appreciate that it is entirely legitimate to use Eqs. (4.18) to simplify the dependence of Ω on *normal* propagators. It would be illegitimate to make corresponding reductions on the symmetry-breaking propagators, since they contain variational parameters.]

Equations (4.17) to (4.19) for Ω are exact (in the Hartree-Fock approximation). They will now be approximated. First we set $G_N = S$, $\Sigma_N = 0$, $\Delta_{iN} = \Delta_A = D$, $\Pi^A = 0$. Consequently,

$$\Omega(\delta m, M_B^2) = -i\,\mathrm{tr}[\ln S^{-1}G - S^{-1}G + 1] + \tfrac{1}{2}i \sum_{i=A,B} \mathrm{tr}\Gamma_i[G - S]\Gamma_i[G - S]D$$
$$+ \tfrac{1}{2}i\,\mathrm{tr}[\ln D^{-1}\Delta_B - D^{-1}\Delta_B + 1] + \tfrac{1}{2}i\,\mathrm{tr}[\Gamma_B G \Gamma_B G - \Gamma_B S \Gamma_B S][\Delta_B - \Delta]\,. \qquad (4.20)$$

The evaluation proceeds by inserting Eq. (4.8) for G, with $\Sigma_N = 0$, and Σ_V and Σ_S given by (4.10) and (4.14), respectively, while Δ_B is set equal to (4.11), with $\Pi^B(p^2) = \Pi_V^B(p^2)$ given by (4.12). We keep only terms that are proportional to inverse powers of the coupling (these come from inverse powers of ϵ), as well as terms of zeroth order in ϵ and coupling. That is, we set ϵ to zero everywhere as long as no divergence arises; if $\epsilon = 0$ is not allowed, (4.10) and (4.11) are used. Moreover, even if a divergence is present at $\epsilon = 0$, terms that are multiplied by higher powers of the coupling constant are dropped.

Rather than integrating immediately, it is convenient to simplify. We rewrite the logarithm of the first term in (4.20):

$$i\,\mathrm{tr}\ln G^{-1}S = i\,\mathrm{tr}\ln(1 - \Sigma_V S - \Sigma_S S)\,. \qquad (4.21a)$$

This is expanded in powers of Σ_S. Only the first power is significant; higher powers give contributions $O(g_A^2, g_B^2)$. Thus we may replace (4.21a) by

$$i\,\mathrm{tr}\ln(1 - \Sigma_V S) - i\,\mathrm{tr}\frac{\Sigma_S S}{1 - \Sigma_V S}$$
$$= i\,\mathrm{tr}\ln(1 - \Sigma_V S) - i\,\mathrm{tr}\Sigma_S G\,. \qquad (4.21b)$$

Next we analyze the second term in (4.20), which is a two-loop integral. Note that the Γ_i's provide two powers of the coupling strength. Thus for present purposes it is sufficient to keep only terms which diverge when $\epsilon=0$ (it is easy to verify that no ϵ^{-2} terms are present):

$$\begin{aligned}\tfrac{1}{2}i\sum_{i=A,B}\mathrm{tr}\Gamma_i[G-S]\Gamma_i[G-S]D &= i\sum_{i=A,B}\mathrm{tr}[G-S]\Gamma_i[G-S]\Gamma_i D-\tfrac{1}{2}i\sum_{i=A,B}\mathrm{tr}[G-S]\Gamma_i[G-S]\Gamma_i D\\ &= i\sum_{i=A,B}\mathrm{tr}[G-S]\Gamma_i[S\Sigma_V S+S\Sigma_V S\Sigma_V S]\Gamma_i D-\tfrac{1}{2}i\sum_{i=A,B}\mathrm{tr}[S\Sigma_V S]\Gamma_i[S\Sigma_V S]\Gamma_i D\\ &\quad -\tfrac{1}{2}i\sum_{i=A,B}\mathrm{tr}[S\Sigma_V S\Sigma_V S]\Gamma_i[S\Sigma_V S\Sigma_V S]\Gamma_i D\,. \end{aligned}\qquad(4.22\text{a})$$

In expanding $G-S$, Σ_S has been ignored compared to Σ_V, and only the first two powers of Σ_V are kept. All terms that are dropped lead to convergent integrals at $\epsilon=0$, hence they are at least quadratic in the coupling. Also, only even powers of Σ_V survive the trace. Equations (4.9) and (4.14) allow us to recognize that (4.22a) is equivalently

$$\begin{aligned}&-i\,\mathrm{tr}[G-S][\Sigma_V+\Sigma_S^{(1)}]+\tfrac{1}{2}i\,\mathrm{tr}S\Sigma_V S\Sigma_V+\tfrac{1}{2}i\,\mathrm{tr}S\Sigma_V S\Sigma_V S\Sigma_S^{(1)}\\ &\qquad=-i\,\mathrm{tr}G[S^{-1}-G^{-1}-\Sigma_S^{(2)}]+i\,\mathrm{tr}S\Sigma_S^{(1)}+\tfrac{1}{2}i\,\mathrm{tr}S\Sigma_V S\Sigma_V+\tfrac{1}{2}i\,\mathrm{tr}S\Sigma_V S\Sigma_V S\Sigma_S^{(1)}\,.\end{aligned}\qquad(4.22\text{b})$$

Consequently the first two terms in (4.20) reduce to the following, with the help of (4.21) and (4.22):

$$\begin{aligned}&i\,\mathrm{tr}\ln(1-\Sigma_V S)+\tfrac{1}{2}i\,\mathrm{tr}\Sigma_V S\Sigma_V S+i\,\mathrm{tr}\Sigma_S^{(2)}G\\ &\quad-i\,\mathrm{tr}\Sigma_S G+i\,\mathrm{tr}\Sigma_S^{(1)}S+\tfrac{1}{2}i\,\mathrm{tr}S\Sigma_V S\Sigma_V S\Sigma_S^{(1)}\,.\end{aligned}\qquad(4.23\text{a})$$

The last four terms in (4.23a) combine to

$$\begin{aligned}&i\,\mathrm{tr}\Sigma_S^{(1)}[S-G]+\tfrac{1}{2}i\,\mathrm{tr}S\Sigma_V S\Sigma_V S\Sigma_S^{(1)}\\ &\qquad=-\tfrac{1}{2}i\,\mathrm{tr}S\Sigma_V S\Sigma_V S\Sigma_S^{(1)}\,.\end{aligned}\qquad(4.23\text{b})$$

Therefore the final, simplified expression for the first two terms in Ω is

$$\begin{aligned}&i\,\mathrm{tr}\ln(1-\Sigma_V S)+\tfrac{1}{2}i\,\mathrm{tr}\Sigma_V S\Sigma_V S\\ &\quad+\tfrac{1}{2}i\sum_{i=A,B}\mathrm{tr}[S\Sigma_V S\Sigma_V S]\Gamma_i[S\Sigma_V S\Sigma_V S]\Gamma_i D.\end{aligned}\qquad(4.24)$$

The last term in (4.20) is also expanded in powers of Σ_V. The significant contribution is

$$\begin{aligned}&\tfrac{1}{2}i\,\mathrm{tr}[S\Sigma_V S]\Gamma_B[S\Sigma_V S]\Gamma_B[\Delta_B-D]\\ &\quad+i\,\mathrm{tr}[S]\Gamma_B[S\Sigma_V S\Sigma_V S]\Gamma_B[\Delta_B-D]\,.\end{aligned}\qquad(4.25)$$

We thus arrive at a completely reduced formula for Ω:

$$\begin{aligned}\Omega(\delta m,M_B^2)&=i\,\mathrm{tr}\ln(1-\Sigma_V S)+\tfrac{1}{2}i\,\mathrm{tr}\Sigma_V S\Sigma_V S+\tfrac{1}{2}i\sum_{i=A,B}\mathrm{tr}[S\Sigma_V S\Sigma_V S]\Gamma_i[S\Sigma_V S\Sigma_V S]\Gamma_i[D]\\ &\quad+\tfrac{1}{2}i\,\mathrm{tr}[\ln D^{-1}\Delta_B-D^{-1}\Delta_B+1]+\tfrac{1}{2}i\,\mathrm{tr}[S\Sigma_V S]\Gamma_B[S\Sigma_V S]\Gamma_B[\Delta_B\Pi_V^B D]\\ &\quad+i\,\mathrm{tr}[S]\Gamma_B[S\Sigma_V S\Sigma_V S]\Gamma_B[\Delta_B\Pi_V^B D]\,.\end{aligned}\qquad(4.26)$$

We have replaced Δ_B-D by $\Delta_B\Pi_V^B D$. A graphical representation for (4.26) is given in Fig. 8.

The integrations are straightforward, subject to a minor ambiguity mentioned below. The final result is

$$\begin{aligned}\Omega(\delta m,M_B^2)&=\frac{(\delta m)^4}{32\pi^2\epsilon}-\frac{m^4}{8\pi^2}\left\{\left(1+\frac{\delta m}{m}\right)^4\ln\left(1+\frac{\delta m}{m}\right)+\left(1-\frac{\delta m}{m}\right)^4\ln\left(1-\frac{\delta m}{m}\right)-7\left(\frac{\delta m}{m}\right)^2\right\}\\ &\quad+\frac{3M_B^4}{128\pi^2\epsilon}\left[1-4\epsilon\ln\frac{M_B^2}{m^2}-2\epsilon\right]-\frac{3g_B^2M_B^2(\delta m)^2}{128\pi^4\epsilon^2}\left[1-4\epsilon\ln\frac{M_B^2}{m^2}+\epsilon\right]+O(g_A^2,g_B^2)\,.\end{aligned}\qquad(4.27)$$

All divergences are absent, as anticipated by our general argument. The first two terms in (4.27) come from the first two terms in (4.26). The third term in (4.26) gives no significant value. The $O(M_B^4)$ term in (4.27) arises from the fourth term in (4.26). The $O(M_B^2(\delta m)^2)$ contribution to Ω results from the last term in (4.26). The ambiguity occurs in the terms that are zeroth order in the coupling. They receive a contribution both from asymptotically large and from finite momenta. This ambiguity could be resolved by specifying more exactly the transition between the power-law behavior (4.10a), (4.12a) for Σ_V,Π_V^B and the low-energy behavior (4.10c), (4.12b). However, various reasonable transition behaviors make only a small difference in the numerical coefficients.

Variation of M_B^2 yields

$$\begin{aligned}&M_B^2=\tfrac{8}{3}\frac{(\delta m)^2}{r^2-1}(1-\epsilon),\\ &r^2=g_A^2/g_B^2>1\,.\end{aligned}\qquad(4.28)$$

In deriving (4.28), we have assumed that $|4\epsilon\ln(M_B^2/m^2)|\ll 1$. This means that M_B^2 cannot become arbitrarily small or large compared to m^2. Equation (4.28) coincides [up to $O(\epsilon)$ terms] with (4.13), giving us the promised derivation of that result. We substitute the value (4.28) for M_B^2 into (4.27) and obtain an effective potential which depends only on δm:

$$\Omega(\delta m)=\frac{(\delta m)^4}{32\pi^2\epsilon}-\frac{m^4}{8\pi^2}\left\{\left(1+\frac{\delta m}{m}\right)^4\ln\left(1+\frac{\delta m}{m}\right)+\left(1-\frac{\delta m}{m}\right)^4\ln\left(1-\frac{\delta m}{m}\right)-7\left(\frac{\delta m}{m}\right)^2\right\}$$
$$-\frac{(\delta m)^4}{6\pi^2\epsilon(r^2-1)^2}\left\{1-4\epsilon\ln\frac{\frac{8}{3}}{r^2-1}\left(\frac{\delta m}{m}\right)^2\right\}. \qquad (4.29)$$

Because the underlying physical model is unrealistic, we shall not pursue a detailed study of (4.29), beyond noting the form for small $\delta m/m$ ($|\delta m/m|$ must be less than 1; otherwise Ω becomes complex):

$$\Omega(\delta m)\approx\frac{(\delta m)^4}{32\pi^2\epsilon}-\frac{25(\delta m)^4}{48\pi^2}+\frac{(\delta m)^6}{120\pi^2 m^2}-\frac{(\delta m)^4}{6\pi^2\epsilon(r^2-1)^2}\left[1-4\epsilon\ln\frac{\frac{8}{3}}{r^2-1}\left(\frac{\delta m}{m}\right)^2\right]. \qquad (4.30)$$

This has a minimum at

$$\left(\frac{\delta m}{m}\right)^2=\frac{125}{3}\left[1-\frac{0.06}{\epsilon}+\frac{0.32}{\epsilon(r^2-1)^2}-\frac{0.64}{(r^2-1)^2}\right] \qquad (4.31)$$

provided that quantity is positive; also it must be less than 1. Furthermore, we have dropped $4\epsilon\ln[\frac{8}{3}/(r^2-1)](\delta m/m)^2$ compared to 1 in (4.31); this approximation was already appealed to in the calculation of M_B^2. There are two kinds of solution to (4.31). First, when r is large compared to 1, the last two terms in the brackets of (4.31) may be ignored and ϵ comes out to be ≈ 0.06. In this case $M_B^2\ll m^2$. As r approaches 1, the second term in the brackets becomes negligible compared to the third. Then $(r^2-1)^2\approx 0.64(1-1/2\epsilon)$ and it must be that $\epsilon\geq\frac{1}{2}$. It is not clear whether such a large ϵ is really consistent with our approximation scheme. In any case for these solutions $M_B^2\approx m^2$. [The requirement that $|4\epsilon\ln(M_B^2/m^2)|\ll 1$ is not very restrictive and can be easily met.]

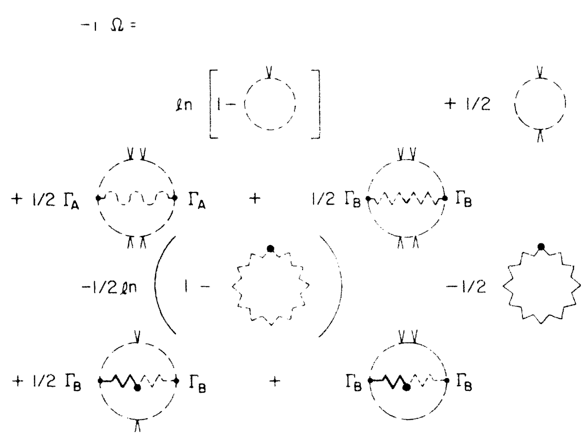

FIG. 8. Formula for $\Omega(\delta m, M_B^2)$.

E. Discussion

Observe that the form (4.29) is reminiscent of the effective potential in a theory with scalar mesons, where δm plays the role of the scalar fields.[18] There is a quartic term in δm, followed by a logarithmic one. The main differences are the appearance of inverse powers of the coupling constant, and the fact that Ω becomes complex for $|\delta m/m|>1$.

It is interesting to contrast the present results with the linear theory. Now δm is calculable and there are constraints on the coupling constants (ϵ and r). In the linear theory, neither of the two conditions is present.

In a further investigation, we hope to survey graphs that contribute beyond the Hartree-Fock approximation. It is most important to ascertain whether it continues to be possible to select the dominant contribution, for small coupling, and whether our present results are stable against higher-order corrections.

V. STATIC, POSITION-DEPENDENT SOLUTIONS

There is considerable interest in finding solutions to field theory which correspond to an energy eigenstate $|\psi\rangle$ in which the expectation of the field Φ is nonvanishing and non-translation-invariant,

$$\langle\psi|\Phi(x)|\psi\rangle=\phi(x). \qquad (5.1)$$

Since $|\psi\rangle$ is an energy eigenstate, $\phi(x)$ is time-independent, $\phi(x)=\phi(\vec{x})$. Although at the present time the physical interpretation of these states has not been firmly fixed, physical intuition suggests that they correspond to coherent excitations, not unlike the familiar Thomas-Fermi nucleus of conventional theory. One might suppose that the observed, physical particles correspond to such states, while the excitations associated with the underlying fields are trapped for some marvelous, as yet ununderstood reason.

We demonstrate that the formalism, which we have here developed, provides a natural framework for a study of these questions. We first show that $\Gamma(\phi, G)$, for static ϕ and G, determines the stationary expectation value of the Hamiltonian H in a normalized state $|\psi\rangle$ for which

$$\langle \psi|\Phi(x)|\psi\rangle = \phi(\vec{x}), \tag{5.2a}$$

$$\langle \psi|\Phi(x)\Phi(y)|\psi\rangle|_{x_0=y_0} = \phi(\vec{x})\phi(\vec{y}) + \hbar G(\vec{x},\vec{y}), \tag{5.2b}$$

$$\langle \psi|H|\psi\rangle = E(\phi, G). \tag{5.3}$$

(The operators are all evaluated at a fixed time x^0.)

For static solutions, $\Gamma(\phi, G)$ is time-translation-invariant, and has an overall factor of time "volume." The relation between $\Gamma(\phi, G)$ and $E(\phi, G)$ will be shown to be

$$-E(\phi, G)\int dt = \Gamma(\phi, G)|_{\text{static}} . \tag{5.4}$$

[The precise meaning of $\Gamma(\phi, G)|_{\text{static}}$ is spelled out below.] Thus we see that the stationary requirements on $\Gamma(\phi, G)$, (1.1), are merely instances of the quantum-mechanical variation principle.

$E(\phi, G)$ is computed for a scalar self-interacting field theory in the Hartree-Fock approximation and the equations for ϕ and G are derived. Next, following Kuti,[6] a Schrödinger picture is introduced, and the abstract variational principle is realized in a functional Schrödinger equation. Kuti's Rayleigh-Ritz method of solving this equation[6] is shown to be equivalent to our Hartree-Fock approximation.

A. Physical interpretation of $\Gamma(\phi,G)|_{\text{static}}$

The problem of finding all the bound states of a general quantum theory, and specifically of a field theory, may of course be formulated as a variational principle. One seeks the states $|\psi\rangle$ which make $\langle \psi|H|\psi\rangle$ stationary against arbitrary variation of $|\psi\rangle$, subject to the normalization constraint $\langle \psi|\psi\rangle = 1$. Rather than performing the arbitrary variation in one fell swoop, it is convenient first to perform a restricted variation, where certain quantities are held fixed, and then to vary these quantities.

We choose to require additional constraints: The expectations of $\Phi(x)$ and of $\Phi(x)\Phi(y)|_{x_0=y_0}$ are fixed. Imposing all conditions with the help of Lagrange multipliers, one is led to consider the variation of

$$\langle \psi|H|\psi\rangle - \epsilon\langle \psi|\psi\rangle - \int d\vec{x}\, J(\vec{x})\langle \psi|\Phi(x)|\psi\rangle - \tfrac{1}{2}\int d\vec{x}\, d\vec{y}\, K(\vec{x},\vec{y})\langle \psi|\Phi(x)\Phi(y)|\psi\rangle|_{x_0=y_0}. \tag{5.5}$$

Clearly $|\psi\rangle$ satisfies the equation

$$\left[H - \int d\vec{x}\, J(\vec{x})\Phi(x) - \tfrac{1}{2}\int d\vec{x}\, d\vec{y}\, K(\vec{x},\vec{y})\Phi(x)\Phi(y)|_{x_0=y_0}\right]|\psi\rangle = \epsilon|\psi\rangle . \tag{5.6}$$

Thus $|\psi\rangle$ is an energy eigenstate of a theory governed by a modified Hamiltonian or, equivalently, by an action with source terms added to it,

$$\int d^4x\, \mathcal{L}(x) + \int d^4x\, J(x)\Phi(x) + \tfrac{1}{2}\int d^4x\, d^4y\, K(x,y)\Phi(x)\Phi(y),$$
$$J(x) = J(\vec{x}), \quad K(x,y) = \delta(x_0 - y_0)K(\vec{x},\vec{y}) . \tag{5.7}$$

The energy eigenvalue of this problem is ϵ. We restrict the discussion to the lowest energy eigenstate.

It is known that the energy of the lowest state is also given by

$$W(J,K) = -\epsilon\int dt, \tag{5.8}$$

where $W(J,K)$ is defined in (2.1). (The time infinity may be removed if the sources are considered to be acting over a large but finite time interval.) Hence we conclude that

$$-\frac{1}{\tau}W(J,K) = \langle \psi|H|\psi\rangle - \int d\vec{x}\, J(\vec{x})\langle \psi|\Phi(x)|\psi\rangle - \tfrac{1}{2}\int d\vec{x}\, d\vec{y}\, K(\vec{x},\vec{y})\langle \psi|\Phi(x)\Phi(y)|\psi\rangle|_{x_0=y_0},$$
$$\tau = \int dt. \tag{5.9}$$

Varying this with respect to J and K and recalling that $|\psi\rangle$ is a normalized eigenstate of the modified Hamiltonian, we find

$$\frac{\delta W(J,K)}{\delta J(\vec{x})} = \tau\langle \psi|\Phi(x)|\psi\rangle = \tau\phi(\vec{x}),$$
$$\frac{\delta W(J,K)}{\delta K(\vec{x},\vec{y})} = \tfrac{1}{2}\tau\langle \psi|\Phi(x)\Phi(y)|\psi\rangle|_{x_0=y_0} = \tfrac{1}{2}\tau[\phi(\vec{x})\phi(\vec{y}) + \hbar G(\vec{x},\vec{y})] \tag{5.10a}$$

Thus

$$\langle\psi|H|\psi\rangle = E(\phi,G)$$
$$= -\frac{1}{\tau}\left\{W(J,K) - \int d^4x\,J(x)\phi(\vec{x}) - \tfrac{1}{2}\int d^4x\,d^4y\,K(x,y)[\phi(\vec{x})\phi(\vec{y}) + \hbar G(\vec{x},\vec{y})]\right\}. \tag{5.10b}$$

The right-hand side of (5.10b) is $-(1/\tau)\Gamma(\phi,G)|_{\text{static}}$ [compare with (2.4) and (2.5)]. The precise meaning of $\Gamma(\phi,G)|_{\text{static}}$ is as follows. First $\Gamma(\phi,G)$ is evaluated with time-translation-invariant forms $\phi(x)=\phi(\vec{x})$; $G(x,y)=G(x_0-y_0;\vec{x},\vec{y})$. This does not yet give $\Gamma(\phi,G)|_{\text{static}}$, since we still must express $G(x_0-y_0;\vec{x},\vec{y})$ in terms of $G(\vec{x},\vec{y})$. From (2.21b) and (5.10a) we see that $G(\vec{x},\vec{y})=G(0;\vec{x},\vec{y})$. The desired relation is obtained from the equation

$$\frac{\delta\Gamma(\phi,G)}{\delta G(x_0-y_0;\vec{x},\vec{y})} = -\tfrac{1}{2}\hbar K(x,y)$$
$$= -\tfrac{1}{2}\hbar\delta(x_0-y_0)K(\vec{x},\vec{y}).$$

Once $G(x_0-y_0;\vec{x},\vec{y})$ is known in terms of $G(\vec{x},\vec{y})$, $\Gamma(\phi,G)$ can be expressed as a functional of $\phi(\vec{x})$ and $G(\vec{x},\vec{y})$. This then is $\Gamma(\phi,G)|_{\text{static}}$.[19]

The arbitrary variation is now completed by varying $E(\phi,G)$ with respect to ϕ and G, and determining values for ϕ and G which render $E(\phi,G)$ stationary. For a physically sensible system, there will always be a solution with constant ϕ corresponding to the vacuum expectation value of $\Phi(x)$.

There may also be a solution with a position-dependent $\phi(\vec{x})$. This then corresponds to a non-translation-invariant energy eigenstate $|\psi\rangle$, with eigenvalue $E(\phi,G)$. $|\psi\rangle$ is *not* the vacuum, since we do not expect translation invariance to be spontaneously broken. Since the underlying theory is translationally invariant, the state $|\psi\rangle$ is infinitely degenerate with respect to energy. One can construct other states by an application of the momentum operator $\vec{P}$,

$$|\psi;\vec{a}\rangle = \exp\left(-\frac{i}{\hbar}\vec{a}\cdot\vec{P}\right)|\psi\rangle. \tag{5.11}$$

Each of the states $|\psi,\vec{a}\rangle$ has the same energy, and

$$\langle\psi;\vec{a}|\Phi(x)|\psi;\vec{a}\rangle = \phi(\vec{x}+\vec{a}). \tag{5.12}$$

Momentum eigenstates may also be formed:

$$|\vec{q}\rangle = \int d\vec{a}\,\exp\left(\frac{i}{\hbar}\vec{q}\cdot\vec{a}\right)|\psi;\vec{a}\rangle. \tag{5.13}$$

B. Sample calculation

We compute $E(\phi,G)$ for the self-interacting Bose field considered in Sec. III. For simplicity, and to make contact with other work, we consider only one field, $N=1$. From (3.4)

$$\Gamma(\phi,G) + \text{const} = I(\phi) + \tfrac{1}{2}i\hbar\,\mathrm{Tr}\,\mathrm{Ln}G^{-1} + \tfrac{1}{2}i\hbar\,\mathrm{Tr}\,\mathfrak{D}^{-1}(\phi)G - \tfrac{1}{8}\lambda\hbar^2\int d^4x\,G(x,x)G(x,x), \tag{5.14}$$

where

$$i\mathfrak{D}^{-1}\{\phi;x,y\} = -[\Box + m^2 + \tfrac{1}{2}\lambda\phi^2(\vec{x})]\,\delta^4(x-y). \tag{5.15}$$

The equation

$$\frac{\delta\Gamma(\phi,G)}{\delta G(x,y)} = -\tfrac{1}{2}\hbar\delta(x_0-y_0)K(\vec{x},\vec{y})$$

implies

$$G^{-1}(x,y) = -i\delta(x^0-y^0)K(\vec{x},\vec{y}) + \mathfrak{D}^{-1}\{\phi;x,y\} + \tfrac{1}{2}i\lambda\hbar G(x,x)\delta^4(x-y). \tag{5.16}$$

Time-translation-invariant solutions to (5.16) are clearly of the form

$$G^{-1}(x_0-y_0;\vec{x},\vec{y}) = i\delta''(x_0-y_0)\delta(\vec{x}-\vec{y}) + i\delta(x_0-y_0)f(\vec{x},\vec{y}). \tag{5.17}$$

Hence if we define the Fourier transform

$$\tilde{G}^{-1}(\omega;\vec{x},\vec{y}) = \int_{-\infty}^{\infty} dx_0\,e^{i\omega x_0}G^{-1}(x_0;\vec{x},\vec{y}),$$

we find

$$\tilde{G}^{-1}(\omega;\vec{x},\vec{y}) = -i\omega^2\delta(\vec{x}-\vec{y}) + if(\vec{x},\vec{y}), \tag{5.18a}$$

$$\tilde{G}(\omega;\vec{x},\vec{y}) = \frac{i}{\omega^2-f}(\vec{x},\vec{y}), \tag{5.18b}$$

where the inverse in (5.18b) is taken in the functional sense in the $\vec{x},\vec{y}$ variables. It follows that

$$G(\vec{x},\vec{y}) = G(0;\vec{x},\vec{y}) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{i}{\omega^2-f}(\vec{x},\vec{y}) = \tfrac{1}{2}f^{-1/2}(\vec{x},\vec{y})$$

or

$$f(\vec{x},\vec{y}) = \tfrac{1}{4}G^{-2}(\vec{x},\vec{y}). \tag{5.19}$$

Hence $G(x_0-y_0;\vec{x},\vec{y})$ is expressed in terms of $G(\vec{x},\vec{y})$ by

$$G^{-1}(x_0-y_0;\vec{x},\vec{y}) = i\delta''(x_0-y_0)\delta(\vec{x}-\vec{y}) + i\delta(x_0-y_0)\tfrac{1}{4}G^{-2}(\vec{x},\vec{y}), \tag{5.20a}$$

$$\tilde{G}^{-1}(\omega;\vec{x},\vec{y}) = -i\omega^2\delta(\vec{x}-\vec{y}) + \tfrac{1}{4}iG^{-2}(\vec{x},\vec{y}). \qquad (5.20b)$$

We can now evaluate $\Gamma(\phi, G)|_{\text{static}}$. The first term on the right-hand side of (5.14) gives simply

$$-\int dx_0 \int d\vec{x}\left(\tfrac{1}{2}[\vec{\nabla}\phi(\vec{x})]^2 + \tfrac{1}{2}m^2\phi^2(\vec{x}) + \frac{\lambda}{4!}\phi^4(\vec{x})\right). \qquad (5.21a)$$

The second term is evaluated in the Fourier representation:

$$\tfrac{1}{2}i\hbar\int dx_0\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\operatorname{Tr}\operatorname{Ln}(-\omega^2+\tfrac{1}{4}G^{-2}) = -\tfrac{1}{4}\hbar\int dx_0 \operatorname{Tr}G^{-1}(\vec{x},\vec{y}) = -\tfrac{1}{4}\hbar\int dx_0\int d\vec{x}\,G^{-1}(\vec{x},\vec{x}). \qquad (5.21b)$$

Constants have been dropped and both Tr and Ln refer to functional operations in $\vec{x}, \vec{y}$ variables. The third term is

$$\begin{aligned}
\tfrac{1}{2}i\hbar\int d^4x\,d^4y\,\mathfrak{D}^{-1}\{\phi;\vec{x},\vec{y}\}G(x_0-y_0;\vec{x},\vec{y})
&= \tfrac{1}{2}\hbar\int d^4x\left\{-\frac{\partial^2}{\partial x_0^2}G(x_0;\vec{x},\vec{y})\Big|_{x_0=0} + \vec{\nabla}_x^2G(0;\vec{x},\vec{y})|_{\vec{x}=\vec{y}} - [m^2+\tfrac{1}{2}\lambda\phi^2(\vec{x})]G(0;\vec{x},\vec{x})\right\}\\
&= \tfrac{1}{2}\hbar\int dx_0\int d\vec{x}\left\{\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2\tilde{G}(\omega;\vec{x},\vec{x}) + \vec{\nabla}_x^2G(\vec{x},\vec{y})|_{\vec{x}=\vec{y}} - [m^2+\tfrac{1}{2}\lambda\phi^2(\vec{x})]G(\vec{x},\vec{x})\right\}\\
&= \tfrac{1}{2}\hbar\int dx_0\int d\vec{x}\{\tfrac{1}{4}G^{-1}(\vec{x},\vec{x}) + \vec{\nabla}_x^2G(\vec{x},\vec{y})|_{\vec{x}=\vec{y}} - [m^2+\tfrac{1}{2}\lambda\phi^2(\vec{x})]G(\vec{x},\vec{x})\}. \qquad (5.21c)
\end{aligned}$$

Again constants have been dropped when (5.20b) was used to evaluate the ω integration. Finally, the last term in (5.14) is

$$-\tfrac{1}{8}\lambda\hbar^2\int dx_0\int d\vec{x}\,G(0;\vec{x},\vec{x})G(0;\vec{x},\vec{x}) = -\tfrac{1}{8}\lambda\hbar^2\int dx_0\int d\vec{x}\,G(\vec{x},\vec{x})G(\vec{x},\vec{x}). \qquad (5.21d)$$

Collecting all the terms in (5.21) we get $\Gamma(\phi, G)|_{\text{static}}$, hence $E(\phi, G)$:

$$\begin{aligned}
E(\phi,G) = \int d\vec{x}\Big\{&\tfrac{1}{2}[\vec{\nabla}\phi(\vec{x})]^2 + \tfrac{1}{2}m^2\phi^2(\vec{x}) + \frac{\lambda}{4!}\phi^4(\vec{x})\\
&+\tfrac{1}{2}\hbar[-\vec{\nabla}_x^2G(\vec{x},\vec{y})|_{\vec{x}=\vec{y}} + m^2G(\vec{x},\vec{x}) + \tfrac{1}{2}\lambda\phi^2(\vec{x})G(\vec{x},\vec{x})] + \tfrac{1}{8}\hbar G^{-1}(\vec{x},\vec{x}) + \tfrac{1}{8}\lambda\hbar^2G(\vec{x},\vec{x})G(\vec{x},\vec{x})\Big\}. \qquad (5.22)
\end{aligned}$$

The equations which are obtained from varying $\phi(\vec{x})$ and $G(\vec{x},\vec{y})$ in $E(\phi, G)$ are

$$0 = -\vec{\nabla}^2\phi(\vec{x}) + m^2\phi(\vec{x}) + \tfrac{1}{6}\lambda\phi^3(\vec{x}) + \hbar\tfrac{1}{2}\lambda\phi(\vec{x})G(\vec{x},\vec{x}), \qquad (5.23a)$$

$$\tfrac{1}{4}G^{-2}(\vec{x},\vec{y}) = [-\vec{\nabla}^2 + m^2 + \tfrac{1}{2}\lambda\phi^2(\vec{x}) + \hbar\tfrac{1}{2}\lambda G(\vec{x},\vec{x})]\,\delta(\vec{x}-\vec{y}). \qquad (5.23b)$$

Consequently the energy of the physical state $|\psi\rangle$ is

$$E = -\tfrac{1}{4}\int d\vec{x}[\tfrac{1}{6}\lambda\phi^4(\vec{x}) + \hbar\lambda\phi^2(\vec{x})G(\vec{x},\vec{x}) - \hbar G^{-1}(\vec{x},\vec{x}) + \hbar^2\tfrac{1}{2}\lambda G(\vec{x},\vec{x})G(\vec{x},\vec{x})], \qquad (5.24)$$

where $\phi(\vec{x})$ and $G(\vec{x},\vec{y})$ satisfy (5.23).

In the classical limit $\hbar\to 0$, $\phi(\vec{x})$ is a solution to the classical equation of motion, and $G(\vec{x},\vec{y})$ is the classical propagator. The quantum corrections are the self-consistent corrections of the Hartree-Fock approximation which modify the mass term m^2 by the position-dependent quantity $\hbar\tfrac{1}{2}\lambda G(\vec{x},\vec{x})$.

C. Schrödinger representation

An entirely different approach to the study of unconventional solutions of a field theory has been developed by Kuti.[6] Here we review his development and show that it leads to the same results—Eqs. (5.22), (5.23), and (5.24)—as our method.

An abstract quantum-mechanical state may be realized by a "wave function." For a field theory involving the field operator $\Phi(x)$, the wave function is a functional of a c number $\Phi(\vec{x})$ (the time is fixed, hence suppressed):

$$|\psi\rangle \to \Psi\{\Phi\}. \qquad (5.25a)$$

The action of the operator $\Phi(x)$ on $|\psi\rangle$ is realized by multiplying $\Psi\{\Phi\}$ by $\Phi(\vec{x})$:

$$\Phi(x)|\psi\rangle \to \Phi(\vec{x})\Psi\{\Phi\}. \qquad (5.25b)$$

The only other independent operator in the theory is the canonical momentum $\Pi(x)$. The action of that operator on $|\psi\rangle$ is realized by functional differentiation

$$\Pi(x)|\psi\rangle \to \frac{\hbar}{i}\,\frac{\delta}{\delta\Phi(\vec{x})}\,\Psi\{\Phi\}\,. \tag{5.25c}$$

Finally, the inner product is defined by functional integration:

$$\langle\psi_1|\psi_2\rangle \to \int d\Phi\,\Psi_1^*\{\Phi\}\Psi_2\{\Phi\}\,. \tag{5.25d}$$

More precisely what is being done is introducing eigenstates at fixed time of the field $\Phi(x)$. When these states are denoted by $|\Phi\rangle$, the wave function $\Psi\{\Phi\}$ is merely

$$\langle\Phi|\psi\rangle = \Psi\{\Phi\}.$$

The analogy with ordinary quantum mechanics is clear.

Energy eigenstates satisfy the Schrödinger equation

$$\int d\vec{x}\,\mathcal{H}\left\{\frac{\hbar}{i}\,\frac{\delta}{\delta\Phi(\vec{x})},\Phi(\vec{x})\right\}\Psi\{\Phi\} = E\Psi\{\Phi\}\,, \tag{5.26}$$

where $\mathcal{H}\{\Pi(x),\Phi(x)\}$ is the Hamiltonian density. The time development is

$$\Psi\{\Phi;t\} = e^{-iEt/\hbar}\,\Psi\{\Phi\}\,.$$

For example, in our Bose model (5.26) is

$$\int d\vec{x}\left[-\frac{\hbar^2}{2}\,\frac{\delta^2}{\delta\Phi(\vec{x})\delta\Phi(\vec{x})} + \tfrac{1}{2}[\vec{\nabla}\Phi(\vec{x})]^2 + \tfrac{1}{2}m^2\Phi^2(\vec{x}) + \frac{\lambda}{4!}\Phi^4(\vec{x})\right]\Psi\{\Phi\} = E\Psi\{\Phi\}\,. \tag{5.27}$$

A direct solution of the functional integro-differential equation is of course impossible. Let us return, however, to the variational principle which can be used to derive (5.26). We demand that

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\int d\Phi\int d\vec{x}\,\Psi^*\{\Phi\}\mathcal{H}\Psi\{\Phi\}}{\int d\Phi\,|\Psi\{\Phi\}|^2} \tag{5.28}$$

be stationary against arbitrary variations of $\Psi\{\Phi\}$. In the usual fashion, this yields (5.26). The form of $\langle\psi|H|\psi\rangle$ can be computed as follows. We define

$$\phi(\vec{x}) = \frac{\int d\Phi\,\Phi(\vec{x})\,|\Psi\{\Phi\}|^2}{\int d\Phi\,|\Psi\{\Phi\}|^2}\,, \tag{5.29a}$$

$$\phi(\vec{x})\phi(\vec{y}) + \hbar G(\vec{x},\vec{y}) = \frac{\int d\Phi\,\Phi(\vec{x})\Phi(\vec{y})\,|\Psi\{\Phi\}|^2}{\int d\Phi\,|\Psi\{\Phi\}|^2}\,. \tag{5.29b}$$

It follows that

$$\int d\Phi\,\Phi(\vec{x})\,|\Psi\{\Phi+\phi\}|^2 = 0\,, \tag{5.30a}$$

$$\int d\Phi\,\Phi(\vec{x})\Phi(\vec{y})\,|\Psi\{\Phi+\phi\}|^2 = \hbar G(\vec{x},\vec{y})\int d\Phi\,|\Psi\{\Phi\}|^2 \tag{5.30b}$$

Hence $\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle$ for our Bose model is given by

$$\begin{aligned}\int d\vec{x}\Big\{&\tfrac{1}{2}[\vec{\nabla}\phi(\vec{x})]^2 + \tfrac{1}{2}m^2\phi^2(\vec{x}) + \frac{\lambda}{4!}\phi^4(\vec{x}) + \tfrac{1}{2}\hbar[-\vec{\nabla}_x{}^2G(\vec{x},\vec{y})|_{\vec{x}=\vec{y}} + m^2G(\vec{x},\vec{x}) + \tfrac{1}{2}\lambda\phi^2(\vec{x})G(\vec{x},\vec{x})]\Big\}\\ &+\left[\int d\Phi\,|\Psi\{\Phi\}|^2\right]^{-1}\int d\Phi\int d\vec{x}\left\{\tfrac{1}{2}\hbar^2\left|\frac{\delta\Psi\{\Phi\}}{\delta\Phi(\vec{x})}\right|^2 + \frac{\lambda}{4!}[\Phi^4(\vec{x}) + 4\phi(\vec{x})\Phi^3(\vec{x})]\,|\Psi(\Phi+\phi)|^2\right\}\,.\end{aligned} \tag{5.31}$$

[We recognize that the first integral in the exact formula (5.31) coincides with the corresponding terms in our approximate expression for $E(\phi,G)$ in (5.22). The second integral in (5.31) is approximated in (5.22) by the last two terms in that equation.]

Rather than applying the variational principle in an arbitrary way, which would merely reproduce the intractable exact equation (5.27), a Rayleigh-Ritz type ansatz is made. Following Kuti[6] we take as a trial function

$$\Psi\{\Phi\} = \exp\left\{-\frac{1}{4\hbar}\int d\vec{x}\,d\vec{y}\,[\Phi(\vec{x})-\phi(\vec{x})]\times G^{-1}(\vec{x},\vec{y})[\Phi(\vec{y})-\phi(\vec{y})]\right\} \tag{5.32}$$

and view ϕ and G as variational parameters. With this choice (5.30) is obviously satisfied, while the second integral in (5.31) gives

$$\int d\vec{x}\,[\tfrac{1}{8}\hbar G^{-1}(\vec{x},\vec{x}) + \hbar^2\tfrac{1}{8}\lambda G(\vec{x},\vec{x})G(\vec{x},\vec{x})]\,. \tag{5.33}$$

This reproduces the last two terms in (5.22). Hence Kuti's Rayleigh-Ritz approximation is entirely equivalent to our Hartree-Fock calculation.

This set of equations has been studied in two-dimensional space-time by Kuti,[6] Dashen, Hasslacher, and Neveu.[20]

D. Comments

Although the two approaches to the problem of static solutions in a field theory yield the same equations for $\phi(\vec{x})$ and $G(\vec{x},\vec{y})$, the following differences should be noted. It is difficult to study the systematics of the corrections to the Rayleigh-Ritz method. In our method, the effective action $\Gamma(\phi, G)$ and the energy $E(\phi, G)$ can be expanded in the series (2.9). On the other hand, Kuti's method gives an explicit expression for the wave function. Hence one can evaluate off-diagonal matrix elements of arbitrary operators, not just diagonal expectations of products of fields, which is all that the effective action provides.

VI. CONCLUSION

We have illustrated how the generalized effective action compactly probes the nonlinear structure of field theory. Moreover, by a suitable parametrization, the exact nonlinear integral equations can be replaced by approximate ordinary equations involving numerical parameters. The former give a precise point-by-point description of the theory, but are intractable. The latter summarize only average properties of the theory, but can be studied by conventional techniques, just as the exact Schrödinger equation can be well analyzed by the Rayleigh-Ritz variational principle. That this can be done for field theory in principle has been known since Schwinger's work in the early fifties, but as far as we know the present work contains the only application of this method.

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[1]L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974).

[2]H. Schnitzer, Phys. Rev. D 10, 1800 (1974).

[3]S. Coleman, R. Jackiw, and H. D. Politzer, this issue, Phys. Rev. D 10, 2491 (1974).

[4]R. Jackiw, Phys. Rev. D 9, 1686 (1974).

[5]R. Jackiw and K. Johnson, Phys. Rev. D 8, 2386 (1973); J. M. Cornwall and R. E. Norton, *ibid.* 8, 3338 (1973).

[6]J. Kuti, unpublished.

[7]T. D. Lee and C. N. Yang, Phys. Rev. 117, 22 (1960).

[8]J. M. Luttinger and J. C. Ward, Phys. Rev. 118, 1417 (1960). P. Martin and C. De Dominicis, J. Math. Phys. 5, 14 (1964); 5, 31 (1964).

[9]H. D. Dahmen and G. Jona-Lasinio, Nuovo Cimento 52A, 807 (1967); A. N. Vasil'ev and A. K. Kazanskii, Teor. Mat. Fiz. 12, 352 (1972) [Theoret. Math. Phys. 12, 875 (1972)]. Also T. D. Lee presented, in the course of a seminar at M.I.T., a relativistic generalization of the results of Ref. 7. One of us (J.M.C.) has been informed by P. Mannheim that he is working on an effective potential for composite operators in quantum electrodynamics.

[10]A graph is said to be "two-particle irreducible" if it does not become disconnected upon opening two lines. Otherwise it is "two-particle reducible."

[11]In this connection it is important to note the relationship that exists between derivatives of $\Gamma(\phi, G)$ and $W(J,K)$. That formula is a generalization of the result relevant to the ordinary effection action $\Gamma(\phi)$:

$$\int d^4z \frac{\delta^2\Gamma(\phi)}{\delta\phi(x)\delta\phi(z)} \frac{\delta^2 W(J)}{\delta J(z)\delta J(y)} = -\delta^4(x-y) .$$

Using a self-explanatory compact notation the formulas are

$$\left(\Gamma_{\phi\phi} - \frac{2}{\hbar}\Gamma_G - \frac{4}{\hbar}\phi\Gamma_{\phi G} + \frac{4}{\hbar^2}\phi^2\,\Gamma_{GG}\right)W_{JJ} + \left(\frac{2}{\hbar}\Gamma_{\phi G} - \frac{4}{\hbar^2}\phi\,\Gamma_{GG}\right)W_{JK} = -1 ,$$

$$\left(\frac{2}{\hbar}\Gamma_{\phi G} - \frac{4}{\hbar^2}\phi\,\Gamma_{GG}\right)W_{KJ} + \frac{4}{\hbar^2}\Gamma_{GG}W_{KK} = -1 ,$$

$$\left(\frac{2}{\hbar}\Gamma_{\phi G} - \frac{4}{\hbar^2}\phi\,\Gamma_{GG}\right)W_{JJ} + \frac{4}{\hbar^2}\Gamma_{GG}W_{JK} = 0 ,$$

$$\left(\Gamma_{\phi\phi} - \frac{2}{\hbar}\Gamma_G - \frac{4}{\hbar}\phi\Gamma_{\phi G} + \frac{4}{\hbar^2}\phi\phi\,\Gamma_{GG}\right)W_{KJ} + \left(\frac{2}{\hbar}\Gamma_{\phi G} - \frac{4}{\hbar^2}\phi\,\Gamma_{GG}\right)W_{KK} = 0 .$$

[12]R. E. Norton and J. M. Cornwall, unpublished. See also Dahmen and Jona-Lasinio, Ref. 9.

[13]This is most easily understood by recalling that for fermions the functional integral $\int d\psi\, d\bar{\psi} \exp(\frac{1}{2}i\,\bar{\psi}A\psi)$ gives $\mathrm{Det}A$, while for a boson $\int d\phi \exp(\frac{1}{2}i\phi A\phi) = \mathrm{Det}^{-1/2}A$.

[14]K. Johnson, in proceedings of the Seminar on Unified Theories of Elementary Particles, 1963, Rochester, N.Y. (unpublished); Th. A. J. Maris, V. E. Herscovitz, and G. Jacob, Phys. Rev. Lett. 12, 313 (1964).

[15]J. M. Cornwall, Phys. Rev. D 10, 500 (1974).

[16]We are using a similar notation, Π, for two different objects. In order to avoid confusion we explain in detail. When an argument is explicitly indicated, viz. $\Pi(k^2)$, as in (4.11) and (4.12), we mean the Lorentz-invariant part of the vacuum-polarization tensor. In symbolic equations like (4.14), Π occurs without an argument; in that case it denotes the full Lorentz-covariant vacuum-polarization tensor. The connection is $\Pi \to \Pi^{\mu\nu}(k) = iP^{\mu\nu}\Pi(k^2)$.

[17]Cornwall and Norton, Ref. 5.

[18]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888

(1973); S. Weinberg, *ibid.* 7, 2887 (1973); R. Jackiw, Ref. 4.
[19]The analysis of the physical interpretation of $\Gamma(\phi, G)|_{\text{static}}$ is an adaptation to the present context of the corresponding argument for $\Gamma(\phi)|_{\text{static}}$. That discussion is due to K. Symanzik, Commun. Math. Phys. 16, 48 (1970). We learned it from S. Coleman, in proceedings of the Lectures given at the International Summer School of Physics "Ettore Majorana," 1973 (unpublished).
[20]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D (to be published).

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Confinement of quarks*

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A mechanism for total confinement of quarks, similar to that of Schwinger, is defined which requires the existence of Abelian or non-Abelian gauge fields. It is shown how to quantize a gauge field theory on a discrete lattice in Euclidean space-time, preserving exact gauge invariance and treating the gauge fields as angular variables (which makes a gauge-fixing term unnecessary). The lattice gauge theory has a computable strong-coupling limit; in this limit the binding mechanism applies and there are no free quarks. There is unfortunately no Lorentz (or Euclidean) invariance in the strong-coupling limit. The strong-coupling expansion involves sums over all quark paths and sums over all surfaces (on the lattice) joining quark paths. This structure is reminiscent of relativistic string models of hadrons.

I. INTRODUCTION

The success of the quark-constituent picture both for resonances and for deep-inelastic electron and neutrino processes makes it difficult to believe quarks do not exist. The problem is that quarks have not been seen. This suggests that quarks, for some reason, cannot appear as separate particles in a final state. A number of speculations have been offered as to how this might happen.[1]

Independently of the quark problem, Schwinger observed many years ago[2] that the vector mesons of a gauge theory can have a nonzero mass if vacuum polarization totally screens the charges in a gauge theory. Schwinger illustrated this result with the exact solution of quantum electrodynamics in one space and one time dimension, where the photon acquires a mass $\sim e^2$ for any nonzero charge e [e has dimensions of $(\text{mass})^{1/2}$ in this theory]. Schwinger suggested that the same effect could occur in four dimensions for sufficiently large couplings.

Further study of the Schwinger model by Lowenstein and Swieca[3] and Casher, Kogut, and Susskind[4] has shown that the asymptotic states of the model contain only massive photons, not electrons. Nevertheless, as Casher *et al.* have shown in detail, the electrons are present in deep-inelastic processes and behave like free pointlike particles over short times and short distances. The polarization effects which prevent the appearance of electrons in the final state take place on a longer time scale (longer than $1/m_\gamma$, where m_γ is the photon mass).

A new mechanism which keeps quarks bound will be proposed in this paper. The mechanism applies to gauge theories only. The mechanism will be illustrated using the strong-coupling limit of a gauge theory in four-dimensional space-time. However, the model discussed here has a built-in ultraviolet cutoff, and in the strong-coupling limit all particle masses (including the gauge field masses) are much larger than the cutoff; in consequence the theory is far from covariant.

The confinement mechanism proposed here is soft (long-time scale). However, in the model discussed here the cutoff spoils the possibility of free pointlike behavior for the quarks.

The model discussed in this paper is a gauge theory set up on a four-dimensional Euclidean lattice. The inverse of the lattice spacing a serves as an ultraviolet cutoff. The use of a Euclidean space (i.e., imaginary instead of real times) instead of a Lorentz space is not a serious restriction; the energy eigenstates (including scattering states) of the lattice theory can be determined from the "transfer-matrix" formalism as has been discussed by suri[5] and reviewed by Wilson and Kogut.[6] A brief discussion of the