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Massless electrodynamics in the one-photon-mode approximation

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We discuss single-fermion-loop vacuum-polarization processes in massless quantum electrodynamics in the one-photon-mode approximation, in which the fermion self-interacts (to all orders in perturbation theory) by the exchange of virtual photons in a single virtual-photon eigenmode. The isolation of one photon mode is made possible by using the O(5)-covariant formulation of massless QED introduced in two earlier papers, in which the photon wave operator has a discrete, rather than a continuous, spectrum. The amplitude integral formalism introduced previously expresses the one-mode radiative-corrected vacuum polarization in terms of the uncorrected vacuum amplitude in the presence of a one-mode external field. By exploiting the residual SO(3) × O(2) symmetry of the one-mode external-field problem, which permits separation of variables, we reduce the external-field problem to a set of two coupled ordinary first-order differential equations. We show that when the two independent solutions to these equations are suitably standardized, their Wronskian gives (up to a constant factor) the external-field-problem Fredholm determinant. We study the distribution of zeros and asymptotic behavior of the Fredholm determinant, relate these properties to the coupling-constant analyticity of the one-mode vacuum polarization, and conclude by giving a brief list of unresolved questions.

I. INTRODUCTION

We begin in this paper the analysis of a simple, nonperturbative approximation to single-fermion-loop vacuum-polarization processes in massless quantum electrodynamics. In our approximation, the virtual fermion in the vacuum-polarization loop self-interacts to all orders of perturbation theory only by the exchange of virtual photons in a single virtual-photon eigenmode. The isolation of one photon mode is made possible by using the O(5)-covariant formulation of massless QED introduced in two earlier papers,[1,2] in which the photon wave operator has a discrete, rather than a continuous, spectrum. Specifically, our approximation is obtained by replacing the full effective photon propagator

$$D^{(0)}_{ab}(\eta_1,\eta_2)_{\rm eff}=\sum_{n,m}\frac{Y^{(1)}_{nma}(\eta_1)Y^{(1)}_{nmb}(\eta_2)}{(n+1)(n+2)} \tag{1.1}$$

by the simple, factorizable form

$$\hat{D}_{ab}(\eta_1,\eta_2)=\tfrac{1}{6}\,Y^{(1)}_{1Ma}(\eta_1)Y^{(1)}_{1Mb}(\eta_2)\,, \tag{1.2}$$

which results when the sum in Eq. (1.1) is truncated to contain only *one* of the 10 modes in the smallest ($n=1$) photon representation of O(5). Specifically, the one mode which we retain has the form

$$Y^{(1)}_{1Ma}(\eta)=\left(\frac{15}{16\pi^2}\right)^{1/2}(v_{1a}\eta\cdot v_2-v_{2a}\eta\cdot v_1)\,, \tag{1.3}$$

where v_1 and v_2 are arbitrary, orthogonal five-dimensional unit vectors,

$$v_1{}^2=v_2{}^2=1\,,\quad v_1\cdot v_2=0\,, \tag{1.4}$$

and where η is the five-dimensional coordinate.

As was shown in Ref. 2, the radiative-corrected single-fermion-loop vacuum functional in the one-mode approximation (denoted by $W_1[A']$) is given by the amplitude integral formula

$$W_1[a'Y^{(1)}_{1M}/e]=\int_{-\infty}^{\infty}da\left(\frac{6}{2\pi e^2}\right)^{1/2}\exp\left(\frac{-3a^2}{e^2}\right)\times W^{(0)}[(a+a')Y^{(1)}_{1M}]\,, \tag{1.5}$$

where $W^{(0)}[A]$ is the single-fermion-loop vacuum functional in the presence of an external electromagnetic potential A, with *no* internal-virtual-photon radiative corrections (and with the dependence on the electric charge e eliminated by a rescaling of the electromagnetic potential). Formally, $W^{(0)}[A]$ is given by the expression

$$\begin{aligned}W^{(0)}[A]&=\tfrac{1}{2}\operatorname{Tr}\ln h_T\,,\\ h_T&=2-L\cdot S-i\alpha\cdot\eta\alpha\cdot A\,,\end{aligned} \tag{1.6}$$

with the anticommuting matrices α and the O(5) angular momentum and spin L and S defined as in Ref. 2. If we introduce the eigenvalues μ of h_T (which, as we shall see, occur in quadruples μ, μ, $-\mu$, $-\mu$) and define the external-field-problem Fredholm determinant

$$\Delta[A]=\Bigg(\prod_{\text{all eigenvalues}}\mu\Bigg)^{1/4}\,, \tag{1.7}$$

then $W^{(0)}$ can be written as

$$W^{(0)}[A]=2\ln\Delta[A]\,. \tag{1.8}$$

As is evident from Eqs. (1.5)–(1.8) and as was developed in detail in Ref. 2, the analyticity properties of W_1 as a function of coupling e^2 are determined by the asymptotic behavior of $W^{(0)}[aY^{(1)}_{1M}]$ for large external-field amplitude a, or, what is essentially equivalent, by the distribution of zeros of the Fredholm determinant Δ in the complex a plane.

Let us now spell out more specifically the connection between the e^2 analyticity of W_1 and the a dependence of $W^{(0)}$. In order to make Eq. (1.5) unambiguous, we must specify the integration contour to be used in evaluating the a integral. In Ref. 2 we argued that this contour should be taken to be along the real a axis, or possibly (and very conjecturally) along the imaginary a axis. Equation (1.5) with real integration contour will be well defined if Δ has no zeros (and hence $W^{(0)}$ no singularities) for a real. If $W^{(0)}$ is asymptotically weaker than an increasing Gaussian in a as a becomes infinite along the real axis, then Eq. (1.5) defines an analytic function of e^2 in the right-hand e^2 half plane. If, moreover, Δ has no singularities in the wedge-shaped sectors $|\mathrm{Re}a|>|\mathrm{Im}a|$ and the vacuum amplitude $W^{(0)}$ is asymptotically weaker than a Gaussian in these sectors, the integration contour can be freely deformed within these sectors from its original position along the real axis, implying that W_1 is an analytic function of e^2 in the entire e^2 plane, apart from a branch cut along the negative real axis from $e^2=0$ to $e^2=-\infty$. Thus, for real integration contour the questions at stake are:

(i) Is Δ zero-free for a real?

(ii) Is $W^{(0)}$ asymptotically weaker than a Gaussian as $a\to\pm\infty$ along the real axis?

(iii) Is Δ zero-free in the sectors $|\mathrm{Re}a|>|\mathrm{Im}a|$?

(iv) Is $W^{(0)}$ asymptotically weaker than a Gaussian as $|a|\to\infty$ within the sectors?

In the following sections we present analytic arguments which answer questions (i), (ii), and (iv) in the affirmative, and we present numerical results (but no proofs) which also suggest an affirmative answer for question (iii). Next let us consider the speculative possibility of an imaginary integration contour. Such a contour is allowed only if *two* conditions are satisfied: Δ must have no zeros for purely imaginary a, and $W^{(0)}$ must vanish as a decreasing Gaussian (or faster) as $a\to\infty$ along the imaginary axis. As shown in Ref. 2, if $W^{(0)}$ oscillates along the imaginary axis with a decreasing Gaussian envelope, then the imaginary contour yields a strong-coupling electrodynamics in which W_1 exists for large enough e^2 and can develop an infinite-order zero as e^2 approaches a positive $e_0^{\,2}$ from above. Thus, the questions at issue for a possible imaginary integration contour are:

(v) Is Δ zero-free for a imaginary?

(vi) What is the asymptotic behavior of $W^{(0)}$ as $|a|\to\infty$ along the imaginary axis?

The analytic arguments which follow answer question (v) affirmatively. With respect to question (vi) we can only give limited numerical results, these show no signs of decreasing asymptotic behavior, but, because the asymptotic region may not have been reached, do not conclusively resolve question (vi).

The material which follows is organized so that a knowledge of the O(5) formalism is needed only to read Sec. II, in which we consider the wave equation determining the eigenvalues μ of h_T,

$$[2-L\cdot S-ia\alpha\cdot\eta\alpha\cdot Y^{(1)}_{1M}(\eta)]\psi=\mu\psi\,, \tag{1.9}$$

and show that separation of variables with respect to the SO(3)×O(2) subgroup of O(5) reduces Eq. (1.9) to a pair of coupled ordinary first-order differential equations within each separable subspace. In the remaining sections, which can be read independently of Sec. II, we study the properties of this differential-equation system. In Sec. III we recapitulate the results of Sec. II and argue directly from the differential equations that Δ has no zeros for a in strips containing the real and imaginary axes. In Sec. IV we construct the Green's function of the one-dimensional system, and use it to establish a connection between the Wronskian of the two independent solutions of the differential equations (suitably standardized) and the Fredholm determinant Δ. In Sec. V we use this connection, combined with WKB estimates, to determine the order of growth of Δ for large $|a|$. In Sec. VI we construct series solutions for the two independent solutions of the differential equation, and use them to study $\Delta(a)$ numerically. Finally, in Sec. VII we briefly summarize the many remaining unresolved questions. In Appendix A we explicitly calculate the Green's function in the free case, and in Appendix B we give the details of the WKB calculation used in Sec. V.

II. REDUCTION OF THE ONE-MODE PROBLEM

In this section we carry out the separation of variables which reduces the partial differential equation (1.9) to a pair of coupled ordinary first-order differential equations. In Sec. II A we determine the conserved quantum numbers of Eq. (1.9), and show that the eigenvalue problem diagonalizes with respect to an SO(3)×O(2) subgroup of O(5). In Sec. II B we introduce a representation of the O(5) generators which facilitates reduction of the eigenvalue problem with respect to the conserved subgroup. The reduction itself is carried out in Sec. II C. In Sec. II D, we perform a check by solving the free ($a=0$) case and verifying the eigenvalue degeneracies found in Ref. 2. We also work out the boundary conditions appropriate to the separated equations in both the free and the interacting cases. Finally, in Sec. II E we make a transformation which simplifies the equations in the interacting case, and construct the external field problem Fredholm determinant introduced in Sec. I.

A. Conserved quantum numbers

To analyze the conserved quantum numbers of Eq. (1.9) we choose axes in the five-dimensional space so that the 1 and 2 axes lie, respectively, along v_1 and v_2. The Hamiltonian in Eq. (1.9) then takes the form

$$h_T = h_T^{(0)} + V\,,$$
$$h_T^{(0)} = 2 - L\cdot S\,,\quad V = i\lambda\alpha\cdot\eta(\alpha_1\eta_2 - \alpha_2\eta_1)\,, \tag{2.1}$$
$$\lambda = -a(15/16\pi^2)^{1/2}\,.$$

Introducing the O(5) generators

$$J_{ab} = L_{ab} + S_{ab} = \eta_a\frac{\partial}{\partial\eta_b} - \eta_b\frac{\partial}{\partial\eta_a} + \tfrac{1}{4}[\alpha_a, \alpha_b] \tag{2.2}$$

we obviously have

$$[J_{ab}, h_T^{(0)}] = 0 \tag{2.3}$$

since the free Hamiltonian $h_T^{(0)}$ is rotationally invariant. Furthermore, since

$$[L_{ab}, \eta_c] = \eta_a\delta_{bc} - \eta_b\delta_{ac}\,,$$
$$[S_{ab}, \alpha_c] = \alpha_a\delta_{bc} - \alpha_b\delta_{ac}\,, \tag{2.4}$$

we find, as expected, that $\alpha\cdot\eta$ is also rotationally invariant,

$$[J_{ab}, \alpha\cdot\eta] = 0\,. \tag{2.5}$$

Hence the generators J_{ab} which commute with h_T will be just the ones which commute with the factor $\alpha_1\eta_2 - \alpha_2\eta_1$ in the potential term. From Eq. (2.4) we find trivially that

$$[J_{34}, \alpha_1\eta_2 - \alpha_2\eta_1] = [J_{35}, \alpha_1\eta_2 - \alpha_2\eta_1] = [J_{45}, \alpha_1\eta_2 - \alpha_2\eta_1] = 0\,, \tag{2.6}$$

indicating that h_T is invariant under the SO(3) subgroup generated by J_{34}, J_{35}, and J_{45}. In addition, we have

$$[J_{12}, \alpha_1\eta_2 - \alpha_2\eta_1] = -\alpha_2\eta_2 - \alpha_1\eta_1 + \alpha_1\eta_1 + \alpha_2\eta_2 = 0\,, \tag{2.7}$$

so that h_T is also invariant under the O(2) subgroup generated by J_{12}. The other generators J_{ab} do not commute with h_T. In addition to the SO(3) $\times$O(2) invariance group which we have just found, there are also two discrete invariances of h_T. Defining a coordinate inversion generator P,

$$P\eta P^{-1} = -\eta\,,\quad P^2 = 1\,, \tag{2.8}$$

we see immediately that

$$[P, h_T] = 0\,. \tag{2.9}$$

Finally, letting α_6 be the α matrix which anticommutes with $\alpha_1, \ldots, \alpha_5$, we have

$$[\alpha_6, h_T] = 0\,. \tag{2.10}$$

This last invariance permits us to split the eight-component spinor eigenvalue problem of Eq. (1.9) into two identical decoupled four-component problems. Diagonalizing the four-component spinor with respect to the conserved quantum numbers, we write

$$\psi = \psi_{jm\xi\epsilon}\,,$$
$$({J_{34}}^2 + {J_{35}}^2 + {J_{45}}^2)\psi_{jm\xi\epsilon} = -j(j+1)\psi_{jm\xi\epsilon}\,,$$
$$J_{45}\psi_{jm\xi\epsilon} = im\psi_{jm\xi\epsilon}\,, \tag{2.11}$$
$$J_{12}\psi_{jm\xi\epsilon} = i\xi\psi_{jm\xi\epsilon}\,,$$
$$P\psi_{jm\xi\epsilon} = \epsilon\psi_{jm\xi\epsilon}\,.$$

As we will see in detail below, the separation constants take the values

$$j = \tfrac{1}{2}, \tfrac{3}{2}, \ldots\,,$$
$$m = -j, -j+1, \ldots, j\,,$$
$$\xi = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots\,, \tag{2.12}$$
$$\epsilon = \pm 1\,.$$

Our task in the succeeding sections will be to find the form taken by the eigenvalue problem of Eq. (1.9) when restricted to the subspace of Eq. (2.11).

B. Explicit representation of the O(5) generators

We introduce now an explicit representation of the O(5) generators which facilitates the reduction of the eigenvalue problem with respect to the SO(3)$\times$O(2) subgroup. We begin with the spin operator $S_{ab} = \frac{1}{4}[\alpha_a, \alpha_b]$. Letting $\sigma_{1,2,3}$, $\tau_{1,2,3}$, and $\rho_{1,2,3}$ be three commuting sets of 2×2 Pauli spin matrices, we represent the 8×8 matrices $\alpha_1, \ldots, \alpha_6$ in the form

$$\alpha_1 = \sigma_1\tau_2\,,\quad \alpha_2 = \sigma_2\tau_2\,,\quad \alpha_3 = \rho_3\sigma_3\tau_2\,,$$
$$\alpha_4 = \rho_1\sigma_3\tau_2\,,\quad \alpha_5 = \rho_2\sigma_3\tau_2\,,\quad \alpha_6 = \tau_3\,, \tag{2.13}$$

so that the spin matrices become

$$S_{12}=\tfrac{1}{2}i\sigma_3\,,\quad S_{14}=-\tfrac{1}{2}i\rho_1\sigma_2\,,$$
$$S_{24}=\tfrac{1}{2}i\rho_1\sigma_1\,,\quad S_{53}=\tfrac{1}{2}i\rho_1\,,$$
$$S_{15}=-\tfrac{1}{2}i\rho_2\sigma_2\,,\quad S_{25}=\tfrac{1}{2}i\rho_2\sigma_1\,, \tag{2.14}$$
$$S_{34}=\tfrac{1}{2}i\rho_2\,,\quad S_{13}=-\tfrac{1}{2}i\rho_3\sigma_2\,,$$
$$S_{23}=\tfrac{1}{2}i\rho_3\sigma_1\,,\quad S_{45}=\tfrac{1}{2}i\rho_3\,.$$

Since the Hamiltonian h_T is even in the α matrices $\alpha_1,\ldots,\alpha_5$, it is a unit matrix in the space of the τ Pauli matrices. As noted above, this immediately reduces Eq. (1.9) to two identical decoupled four-component eigenvalue problems.

To represent the orbital angular momentum L_{ab}, we parametrize the coordinates $\eta_1,\ldots,\eta_5$ in the form

$$\eta_1=\sin\theta_1\cos\phi_1\,,\quad \eta_2=\sin\theta_1\sin\phi_1\,,$$
$$\eta_3=\cos\theta_1\cos\theta_2\,,\quad \eta_4=\cos\theta_1\sin\theta_2\cos\phi_2\,,$$
$$\eta_5=\cos\theta_1\sin\theta_2\sin\phi_2\,, \tag{2.15}$$
$$0\le\theta_1\le\tfrac{1}{2}\pi\,,\quad 0\le\phi_1\le2\pi\,,$$
$$0\le\theta_2\le\pi\,,\quad 0\le\phi_2\le2\pi$$

corresponding to an O(2) (angle ϕ_1) and an SO(3) (angles θ_2,ϕ_2) combined with mixing angle θ_1. In terms of these angular parameters, the coordinate inversion operation is

$$\eta\to-\eta \Longleftrightarrow \begin{cases}\theta_1\to\theta_1\,, & \phi_1\to\phi_1+\pi\,,\\ \theta_2\to\pi-\theta_2\,, & \phi_2\to\phi_2+\pi\,.\end{cases} \tag{2.15$'$}$$

The hyperspherical surface element becomes

$$d\Omega_\eta=\det\begin{bmatrix}\eta_1 & \dfrac{\partial\eta_1}{\partial\theta_1} & \dfrac{\partial\eta_1}{\partial\phi_1} & \dfrac{\partial\eta_1}{\partial\theta_2} & \dfrac{\partial\eta_1}{\partial\phi_2}\\ \eta_2 & & & & \\ \cdot & & & & \cdot\\ \cdot & & & & \cdot\\ \cdot & & & & \cdot\\ \eta_5 & \dfrac{\partial\eta_5}{\partial\theta_1} & \cdots & & \dfrac{\partial\eta_5}{\partial\phi_2}\end{bmatrix} d\theta_1\,d\phi_1\,d\theta_2\,d\phi_2$$
$$=\cos^2\theta_1\sin\theta_1\,d\theta_1(d\phi_1)(\sin\theta_2\,d\theta_2\,d\phi_2)\,, \tag{2.16}$$

which, not surprisingly, has a mixing-angle factor, an O(2) factor ($d\phi_1$), and an SO(3) factor ($\sin\theta_2\,d\theta_2\,d\phi_2$). By dint of considerable algebra one can express the orbital angular momenta in terms of derivatives with respect to the angles of Eq. (2.15). To write the results in a compact form, we introduce auxiliary operators M_j, N_j, P_j, $j=1,\ldots,3$, as follows:

$$M_1=-\sin\phi_1\frac{\partial}{\partial\theta_1}-\cot\theta_1\cos\phi_1\frac{\partial}{\partial\phi_1}\,,$$
$$M_2=\cos\phi_1\frac{\partial}{\partial\theta_1}-\cot\theta_1\sin\phi_1\frac{\partial}{\partial\phi_1}\,,$$
$$M_3=\frac{\partial}{\partial\phi_1}\,,$$
$$N_1=-\sin\phi_2\frac{\partial}{\partial\theta_2}-\cot\theta_2\cos\phi_2\frac{\partial}{\partial\phi_2}\,,$$
$$N_2=\cos\phi_2\frac{\partial}{\partial\theta_2}-\cot\theta_2\sin\phi_2\frac{\partial}{\partial\phi_2}\,, \tag{2.17}$$
$$N_3=\frac{\partial}{\partial\phi_2}\,,$$
$$P_1=\cos\theta_2\cos\phi_2\frac{\partial}{\partial\theta_2}-\csc\theta_2\sin\phi_2\frac{\partial}{\partial\phi_2}\,,$$
$$P_2=\cos\theta_2\sin\phi_2\frac{\partial}{\partial\theta_2}+\csc\theta_2\cos\phi_2\frac{\partial}{\partial\phi_2}\,,$$
$$P_3=-\sin\theta_2\frac{\partial}{\partial\theta_2}\,.$$

These satisfy the commutation relations and identities

$$\left.\begin{aligned}&[M_i,M_j]=-M_k\,,\quad [M_i,N_j]=[M_i,P_j]=0\,,\\ &[N_i,N_j]=-N_k\,,\\ &[P_i,P_j]=N_k\,,\quad [N_i,P_j]=-P_k\,,\end{aligned}\right\}\ i,j,k \text{ cyclic} \tag{2.18}$$
$$\vec{N}^2=\vec{P}^2=\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial}{\partial\theta_2}\right)+\frac{1}{\sin^2\theta_2}\frac{\partial^2}{\partial\phi_2^{\,2}}\,.$$

In terms of the auxiliary operators, the orbital angular momentum operators take the form

$$L_{12}=M_3\,,\quad L_{53}=N_1\,,$$
$$L_{34}=N_2\,,\quad L_{45}=N_3\,,$$
$$L_{14}=-\sin\theta_2\cos\phi_2M_2+\tan\theta_1\cos\phi_1P_1\,,$$
$$L_{15}=-\sin\theta_2\sin\phi_2M_2+\tan\theta_1\cos\phi_1P_2\,,$$
$$L_{13}=-\cos\theta_2M_2+\tan\theta_1\cos\phi_1P_3\,, \tag{2.19}$$
$$L_{24}=\sin\theta_2\cos\phi_2M_1+\tan\theta_1\sin\phi_1P_1\,,$$
$$L_{25}=\sin\theta_2\sin\phi_2M_1+\tan\theta_1\sin\phi_1P_2\,,$$
$$L_{23}=\cos\theta_2M_1+\tan\theta_1\sin\phi_1P_3\,,$$

and by using Eqs. (2.17) and (2.18) it is straightforward to verify that the expressions in Eq. (2.19) satisfy the O(5) commutation relations

$$[L_{ab},L_{cd}]=\delta_{ac}L_{db}-\delta_{ad}L_{cb}+\delta_{bc}L_{ad}-\delta_{bd}L_{ac}\,. \tag{2.20}$$

Using Eqs. (2.14) and (2.19), it is a simple matter to express the Hamiltonian h_T and the conserved generators J_{12}, J_{53}, ... in terms of the angular parameters. We find

$$h_T^{(0)} = 2 - i[M_3\sigma_3 + N_1\rho_1 + N_2\rho_2 + N_3\rho_3 + (M_1\sigma_1 + M_2\sigma_2)(\rho_1 \sin\theta_2 \cos\phi_2 + \rho_2 \sin\theta_2 \sin\phi_2 + \rho_3 \cos\theta_2) + i \tan\theta_1 \sigma_3(\sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1)(P_1\rho_1 + P_2\rho_2 + P_3\rho_3)] ,$$
$$V = \lambda \sin\theta_1[\sigma_3 \sin\theta_1 - \cos\theta_1(\sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1)(\rho_1 \sin\theta_2 \cos\phi_2 + \rho_2 \sin\theta_2 \sin\phi_2 + \rho_3 \cos\theta_2)] , \quad (2.21)$$

and

$$U_3 \equiv -iJ_{12} = -iM_3 + \tfrac{1}{2}\sigma_3 ,$$
$$T_1 \equiv -iJ_{53} = -iN_1 + \tfrac{1}{2}\rho_1 ,$$
$$T_2 \equiv -iJ_{34} = -iN_2 + \tfrac{1}{2}\rho_2 , \quad (2.22)$$
$$T_3 \equiv -iJ_{45} = -iN_3 + \tfrac{1}{2}\rho_3 .$$

C. Reduction of the eigenvalue problem

The first step in the reduction of the eigenvalue problem with respect to the SO(3)×O(2) subgroup is to find the eigenvalues and eigenfunctions of the conserved generators in Eq. (2.22). This is, of course, just a standard angular momentum problem. For the O(2) subgroup we find two eigenfunctions with opposite inversion parity for each eigenvalue ξ of U_3,

$$U_3 u_\pm = \xi u_\pm ,$$
$$P u_\pm = (-1)^{\xi \mp 1/2} u_\pm ,$$
$$u_+ = e^{i(\xi - 1/2)\phi_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_\sigma , \quad (2.23)$$
$$u_- = e^{i(\xi + 1/2)\phi_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_\sigma .$$

The subscript σ on the spinors indicates that they are acted on by the Pauli matrices σ_j. Because the orbital angular momentum $-iM_3$ must have integral eigenvalues, the eigenvalues of U_3 must be half-integral; hence the allowed values of ξ are

$$\xi = \pm\tfrac{1}{2}, \ \pm\tfrac{3}{2}, \ \ldots . \quad (2.24)$$

For the SO(3) subgroup we again find two eigenfunctions with opposite inversion parity for each pair of $\vec{T}$ eigenvalues j, m,

$$\vec{T}^2 v_\pm = j(j+1) v_\pm , \quad T_3 v_\pm = m v_\pm ,$$
$$P v_\pm = (-1)^{j + 2m \mp 1/2} v_\pm , \quad (2.25)$$
$$v_+ = \begin{bmatrix} (j - m + 1) P_{j+1/2}^{m-1/2}(\cos\theta_2) e^{i(m-1/2)\phi_2} \\ P_{j+1/2}^{m+1/2}(\cos\theta_2) e^{i(m+1/2)\phi_2} \end{bmatrix}_\rho ,$$
$$v_- = \begin{bmatrix} (j + m) P_{j-1/2}^{m-1/2}(\cos\theta_2) e^{i(m-1/2)\phi_2} \\ -P_{j-1/2}^{m+1/2}(\cos\theta_2) e^{i(m+1/2)\phi_2} \end{bmatrix}_\rho ,$$

with $P_L^M(z)$ the usual associated Legendre polynomial. The allowed values of j, m are the usual ones for spin-$\frac{1}{2}$ coupled to an orbital angular momentum,

$$j = \tfrac{1}{2}, \tfrac{3}{2}, \ldots ,$$
$$m = -j, \ -j+1, \ldots, j , \quad (2.26)$$

and the subscript ρ indicates that the spinors are acted on by the Pauli matrices ρ_j. In terms of the O(2) and SO(3) eigenfunctions which we have just found, the general decomposition of $\psi_{jm\xi\pm}$ is

$$\psi_{jm\xi\epsilon} = A_+(\theta_1) v_+ u_+ + C_+(\theta_1) v_- u_- ,$$
$$\psi_{jm\xi-\epsilon} = A_-(\theta_1) v_+ u_- + C_-(\theta_1) v_- u_+ , \quad (2.27)$$
$$\epsilon = (-1)^{\xi + j + 2m - 1} .$$

The next step is to substitute Eq. (2.27) into Eq. (1.9), using the expression of Eq. (2.21) for h_T. To find the action of the various terms of h_T on $u_\pm$ and $v_\pm$ we use the following identities, which may be verified by straightforward calculation:

$$\sigma_3 u_\pm = \pm u_\pm ,$$
$$(\sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1) u_\pm = u_\mp ,$$
$$(2 - iM_3\sigma_3) u_\pm = (\tfrac{3}{2} \pm \xi) u_\pm ,$$
$$-i(M_1\sigma_1 + M_2\sigma_2) u_\pm = \pm u_\mp \left[\frac{d}{d\theta_1} + (\tfrac{1}{2} \mp \xi)\cot\theta_1\right] ; \quad (2.28)$$
$$(\rho_1 \sin\theta_2 \cos\phi_2 + \rho_2 \sin\theta_2 \sin\phi_2 + \rho_3 \cos\theta_2) v_\pm = v_\mp ,$$
$$-i\vec{N}\cdot\vec{\rho} v_+ = -(j + \tfrac{3}{2}) v_+ , \quad -i\vec{N}\cdot\vec{\rho} v_- = (j - \tfrac{1}{2}) v_- ,$$
$$\vec{P}\cdot\vec{\rho} v_+ = (j + \tfrac{3}{2}) v_- , \quad \vec{P}\cdot\vec{\rho} v_- = -(j - \tfrac{1}{2}) v_+ .$$

Hence we get

$$
\begin{aligned}
h_T\psi_{jm\xi\epsilon} =& (\tfrac{3}{2}+\xi)A_+v_+u_+ + (\tfrac{3}{2}-\xi)C_+v_-u_- - (j+\tfrac{3}{2})A_+v_+u_+ + (j-\tfrac{1}{2})C_+v_-u_- \\
&+\left[\frac{d}{d\theta_1}+(\tfrac{1}{2}-\xi)\cot\theta_1\right]A_+v_-u_- - \left[\frac{d}{d\theta_1}+(\tfrac{1}{2}+\xi)\cot\theta_1\right]C_+v_+u_+ \\
&-\tan\theta_1(j+\tfrac{3}{2})A_+v_-u_- - \tan\theta_1(j-\tfrac{1}{2})C_+v_+u_+ + \lambda\sin^2\theta_1 A_+v_+u_+ - \lambda\sin^2\theta_1 C_+v_-u_- \\
&-\lambda\sin\theta_1\cos\theta_1 A_+v_-u_- - \lambda\sin\theta_1\cos\theta_1 C_+v_+u_+ \\
=& \mu\psi_{jm\xi\epsilon} \\
=& \mu A_+v_+u_+ + \mu C_+v_-u_-\,,
\end{aligned}
$$

$$
\begin{aligned}
h_T\psi_{jm\xi\,-\epsilon} =& (\tfrac{3}{2}-\xi)A_-v_+u_- + (\tfrac{3}{2}+\xi)C_-v_-u_+ - (j+\tfrac{3}{2})A_-v_+u_- + (j-\tfrac{1}{2})C_-v_-u_+ \\
&-\left[\frac{d}{d\theta_1}+(\tfrac{1}{2}+\xi)\cot\theta_1\right]A_-v_-u_+ + \left[\frac{d}{d\theta_1}+(\tfrac{1}{2}-\xi)\cot\theta_1\right]C_-v_+u_- \\
&+\tan\theta_1(j+\tfrac{3}{2})A_-v_-u_+ + \tan\theta_1(j-\tfrac{1}{2})C_-v_+u_- - \lambda\sin^2\theta_1 A_-v_+u_- + \lambda\sin^2\theta_1 C_-v_-u_+ \\
&-\lambda\sin\theta_1\cos\theta_1 A_-v_-u_+ - \lambda\sin\theta_1\cos\theta_1 C_-v_+u_- \\
=& \mu\psi_{jm\xi\,-\epsilon} \\
=& \mu A_-v_+u_- + \mu C_-v_-u_+\,.
\end{aligned}
\tag{2.29}
$$

Equating coefficients of like terms then gives us the following two sets of coupled first-order differential equations for $A_\pm(\theta_1)$ and $C_\pm(\theta_1)$:

$$
\begin{aligned}
&(\xi-j)A_+ - \left[\frac{d}{d\theta_1}+(\tfrac{1}{2}+\xi)\cot\theta_1+(j-\tfrac{1}{2})\tan\theta_1\right]C_+ + \lambda\sin\theta_1(A_+\sin\theta_1 - C_+\cos\theta_1) = \mu A_+\,, \\
&(j+1-\xi)C_+ + \left[\frac{d}{d\theta_1}+(\tfrac{1}{2}-\xi)\cot\theta_1-(j+\tfrac{3}{2})\tan\theta_1\right]A_+ - \lambda\sin\theta_1(C_+\sin\theta_1 + A_+\cos\theta_1) = \mu C_+\,;
\end{aligned}
\tag{2.30a}
$$

$$
\begin{aligned}
&-(\xi+j)A_- + \left[\frac{d}{d\theta_1}+(\tfrac{1}{2}-\xi)\cot\theta_1+(j-\tfrac{1}{2})\tan\theta_1\right]C_- - \lambda\sin\theta_1(\sin\theta_1 A_- + \cos\theta_1 C_-) = \mu A_-\,, \\
&(j+1+\xi)C_- - \left[\frac{d}{d\theta_1}+(\tfrac{1}{2}+\xi)\cot\theta_1-(j+\tfrac{3}{2})\tan\theta_1\right]A_- + \lambda\sin\theta_1(\sin\theta_1 C_- - \cos\theta_1 A_-) = \mu C_-\,.
\end{aligned}
\tag{2.30b}
$$

These two sets of equations can be further reduced to just one set of coupled differential equations by exploiting the fact that

$$\alpha\cdot\eta h_T = -h_T\alpha\cdot\eta\,. \tag{2.31}$$

Since $\alpha\cdot\eta$ has odd inversion parity, Eq. (2.31) tells us that if $\psi_{jm\xi\,-\epsilon}$ is an eigenfunction of h_T with eigenvalue μ, then $\alpha\cdot\eta\psi_{jm\xi\,-\epsilon}$ is an eigenfunction of h_T with eigenvalue $-\mu$, quantum numbers j, m, ξ unaltered, but (reversed) inversion parity $+\epsilon$. Specifically, writing[3]

$$
\begin{aligned}
\alpha\cdot\eta\psi_{jm\xi\,-\epsilon} =& [(\sigma_1\cos\phi_1+\sigma_2\sin\phi_1)\sin\theta_1+\sigma_3(\rho_1\sin\theta_2\cos\varphi_2+\rho_2\sin\theta_2\sin\varphi_2+\rho_3\cos\theta_2)\cos\theta_1] \\
&\times[A_-(\theta_1)v_+u_- + C_-(\theta_1)v_-u_+] \\
=& \hat{A}_+(\theta_1)v_+u_+ + \hat{C}_+(\theta_1)v_-u_-\,,
\end{aligned}
\tag{2.32}
$$

we find from the relations of Eq. (2.28) that

$$
\begin{aligned}
\hat{A}_+ &= A_-\sin\theta_1 + C_-\cos\theta_1\,, \\
\hat{C}_+ &= -A_-\cos\theta_1 + C_-\sin\theta_1\,.
\end{aligned}
\tag{2.33}
$$

From the differential equations [Eqs. (2.30*b*)] satisfied by A_- and C_-, we find that $\hat{A}_+$ and $\hat{C}_+$ satisfy the coupled differential equations

$$
\begin{aligned}
&(\xi-j)\hat{A}_+ - \left[\frac{d}{d\theta_1}+(\tfrac{1}{2}+\xi)\cot\theta_1+(j-\tfrac{1}{2})\tan\theta_1\right]\hat{C}_+ \\
&\qquad + \lambda\sin\theta_1(\hat{A}_+\sin\theta_1 - \hat{C}_+\cos\theta_1) = -\mu\hat{A}_+\,, \\
&(j+1-\xi)\hat{C}_+ + \left[\frac{d}{d\theta_1}+(\tfrac{1}{2}-\xi)\cot\theta_1-(j+\tfrac{3}{2})\tan\theta_1\right]\hat{A}_+ \\
&\qquad - \lambda\sin\theta_1(\hat{C}_+\sin\theta_1 + \hat{A}_+\cos\theta_1) = -\mu\hat{C}_+\,.
\end{aligned}
\tag{2.34}
$$

As expected, these are identical to Eqs. (2.30a), apart from the reversal in sign of the eigenvalue. Thus, we need only study the one set of equations in Eq. (2.30a).

To find the measure with respect to which two eigensolutions of Eqs. (2.30a) with different eigenvalues μ, μ' are orthogonal, we start from the hyperspherical orthonormality condition

$$\int d\Omega_\eta \psi^\dagger_{jm\xi\epsilon}\psi'_{jm\xi\epsilon}=0\ ,\quad \mu\neq\mu'\ . \tag{2.35}$$

Using the expression for $d\Omega_\eta$ in Eq. (2.16), and the fact that

$$\begin{aligned} u_-^\dagger u_- &= u_+^\dagger(\sigma_1\cos\phi_1+\sigma_2\sin\phi_1)^2u_+ \\ &= u_+^\dagger u_+\,, \\ v_-^\dagger v_- &= v_+^\dagger(\rho_1\sin\theta_2\cos\phi_2+\rho_2\sin\theta_2\sin\phi_2 \\ &\quad +\rho_3\cos\theta_2)^2 v_+ \\ &= v_+^\dagger v_+\,, \end{aligned} \tag{2.36}$$

Eq. (2.35) reduces to

$$\int_0^{\pi/2}\cos^2\theta_1\sin\theta_1\,d\theta_1(A_+^*A_+'+C_+^*C_+')=0\ ,\quad \mu\neq\mu' \tag{2.37}$$

which identifies the measure for Eqs. (2.30a).

Now that the eigenvalue problem has been reduced to a single set of coupled first-order differential equations, the subscripts used in the above analysis are no longer needed. To expedite the subsequent discussion, let us change notation as follows:

$$\begin{aligned} &\theta_1\to\theta\,, \\ &A_+(\theta_1)\to a(\theta)\,, \\ &C_+(\theta_1)\to c(\theta)\,. \end{aligned} \tag{2.38}$$

The differential equations which we must study thus are

$$\begin{aligned} &(\xi-j)a-\left[\frac{d}{d\theta}+(\tfrac12+\xi)\cot\theta+(j-\tfrac12)\tan\theta\right]c \\ &\qquad +\lambda\sin\theta(a\sin\theta-c\cos\theta)=\mu a\ , \\ &(j+1-\xi)c+\left[\frac{d}{d\theta}+(\tfrac12-\xi)\cot\theta-(j+\tfrac32)\tan\theta\right]a \\ &\qquad -\lambda\sin\theta(c\sin\theta+a\cos\theta)=\mu c\ , \end{aligned} \tag{2.39}$$

with the measure for orthogonality

$$\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta(a^*a'+c^*c')=0\ ,\qquad \mu\neq\mu'\ . \tag{2.40}$$

D. Solution of the free ($\lambda=0$) case and check on eigenvalue counting

Let us now check the reduction leading to Eq. (2.39) by solving the differential equations in the case of vanishing interaction and comparing the energy spectrum with the free-particle spectrum calculated in Ref. 2. When $\lambda=0$, the differential equations simplify to

$$\begin{aligned} &(\xi-j)a-\left[\frac{d}{d\theta}+(\tfrac12+\xi)\cot\theta+(j-\tfrac12)\tan\theta\right]c=\mu a\,, \\ &(j+1-\xi)c+\left[\frac{d}{d\theta}+(\tfrac12-\xi)\cot\theta-(j+\tfrac32)\tan\theta\right]a=\mu c\,. \end{aligned} \tag{2.41}$$

Changing the independent variable to $u=\cos^2\theta$ and eliminating either c or a, we find that a satisfies a second-order differential equation of standard Riemann type,[4]

$$\frac{d^2a}{du^2}+\left(\frac32\frac1u+\frac1{u-1}\right)\frac{da}{du}+\left[-\frac{(j+\frac32)(j+\frac12)}{4}\frac1{u^2}-\frac{(\xi-\frac12)^2}{4}\frac1{(u-1)^2}+\frac{(j+\frac32)(j+\frac12)+(\xi-\frac12)^2+2+\mu(1-\mu)}{4}\frac1{u(u-1)}\right]a=0\,, \tag{2.42}$$

and c satisfies a similar equation obtained from Eq. (2.42) by the replacements $j\to j-1$, $\xi\to\xi+1$. The characteristic exponents of Eq. (2.42) at the regular singular points at $u=0$ and $u=1$ are given in Table I. Equation (2.42) can be solved in terms of Jacobi polynomials, giving the following four series of eigenfunctions and eigenvalues.

$\xi\geq\frac12$:

$$a=f(\cos\theta)^{j+1/2}(\sin\theta)^{\xi-1/2}\times P_n^{(j+1,\xi-1/2)}(1-2\cos^2\theta),$$
$$c=(\cos\theta)^{j-1/2}(\sin\theta)^{\xi+1/2}\times P_n^{(j,\xi+1/2)}(1-2\cos^2\theta).$$

Series 1. (2.43)

$$\mu=2n+2+j+\xi,\quad n=0,1,2,\ldots$$
$$f=-(n+\xi+\tfrac{1}{2})/(n+j+1).$$

Series 2.

$$\mu=-(2n+1+j+\xi),\quad n=0,1,2,\ldots$$
$$f=1.$$

$\xi\leq-\frac{1}{2}$:

$$a=f(\cos\theta)^{j+1/2}(\sin\theta)^{1/2-\xi}\times P_n^{(j+1,1/2-\xi)}(1-2\cos^2\theta),$$
$$c=(\cos\theta)^{j-1/2}(\sin\theta)^{-1/2-\xi}\times P_{n+1}^{(j,-1/2-\xi)}(1-2\cos^2\theta).$$

Series 3. (2.44)

$$\mu=2n+3+j-\xi,\quad n=-1,0,1,\ldots$$
$$f=-1.$$

Series 4.

$$\mu=-(2n+2+j-\xi),\quad n=0,1,2,\ldots$$
$$f=(n+j-\xi+\tfrac{3}{2})/(n+1).$$

These solutions can be verified by direct substitution into Eq. (2.41), using the following four identities satisfied by the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$[5]:

$$(1-x)\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=\alpha P_n^{(\alpha,\beta)}(x)-(n+\alpha)P_n^{(\alpha-1,\beta+1)}(x),$$

$$(1+x)\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=(n+\beta)P_n^{(\alpha+1,\beta-1)}(x)-\beta P_n^{(\alpha,\beta)}(x),\tag{2.45}$$

$$\beta(1-x)P_n^{(\alpha,\beta)}(x)-(n+\alpha)(1+x)P_n^{(\alpha-1,\beta+1)}(x)=-2(n+1)P_{n+1}^{(\alpha-1,\beta-1)}(x),$$

$$2\frac{d}{dx}P_{n+1}^{(\alpha,\beta)}(x)=(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(x).$$

Let us now count the total degeneracy with which the eigenvalue $\mu=k+2$ occurs. Remembering that we have reduced our problem to a four-component spinor, the expected degeneracy of the eigenvalue $\mu=k+2$ is

$$\deg(k+2)=\dim(k+\tfrac{1}{2},\tfrac{1}{2})=\tfrac{2}{3}(k+1)(k+2)(k+3),\quad k=0,1,2,\ldots.\tag{2.46}$$

For each eigenfunction with eigenvalue μ and inversion parity ϵ obtained from Eqs. (2.43) and (2.44), there is another eigenfunction with eigenvalue $-\mu$ and opposite inversion parity obtained by inverting the transformation of Eq. (2.33) to give

$$A_-=a\sin\theta-c\cos\theta,\quad C_-=a\cos\theta+c\sin\theta.\tag{2.47}$$

Hence the positive eigenvalues of h_T are

$$\left.\begin{array}{l}2n+2+j+|\xi|\\2n+1+j+|\xi|\end{array}\right\}\text{ twice each}$$
$$n=0,1,2,\ldots,\quad j=\tfrac{1}{2},\tfrac{3}{2},\ldots,\tag{2.48}$$
$$m=-j,\ldots,j,\quad |\xi|=\tfrac{1}{2},\tfrac{3}{2},\ldots,$$

and the degeneracy of the eigenvalue $k+2$ is

$$\deg(k+2)=2\sum_{\substack{n,j,|\xi|\\2n+j+|\xi|=k}}(2j+1)+2\sum_{\substack{n,j,|\xi|\\2n+j+|\xi|=k+1}}(2j+1).\tag{2.49}$$

The right-hand side of Eq. (2.49) is obviously a cubic polynomial in k, which by direct enumeration of the two sums, takes the values 4, 16, 40, 80 for $k=0,1,2,3$, respectively. Hence it is equal to $\frac{2}{3}(k+1)(k+2)(k+3)$, and the eigenvalue-counting checks. In group-theoretic language, what we have done is to exhibit the decomposition of the $(k+\frac{1}{2},\frac{1}{2})$ representation of O(5) in terms of states labeled by the quantum numbers of the SO(3)×O(2) subgroup.

From Eqs. (2.43) and (2.44), we see that in the free case the two-component wave function

$$\psi=\begin{pmatrix}a\\c\end{pmatrix}\tag{2.50a}$$

satisfies the finiteness boundary condition

TABLE I. Characteristic exponents of the differential equations for a and c at $u=\cos^2\theta=0,1$. [See the discussion following Eq. (2.50).]

Singular point: $u=0$, $\theta=\frac{1}{2}\pi$			Singular point: $u=1$, $\theta=0$		
$a\sim u^{\sigma_a}=(\cos\theta)^{2\sigma_a}$			$a\sim(1-u)^{\chi_a}=(\sin\theta)^{2\chi_a}$		
$c\sim u^{\sigma_c}=(\cos\theta)^{2\sigma_c}$			$c\sim(1-u)^{\chi_c}=(\sin\theta)^{2\chi_c}$		
Characteristic exponents			Characteristic exponents		
	Solution 1	Solution 2		Solution 1	Solution 2
σ_a	$\frac{1}{2}(j+\frac{1}{2})$	$-\frac{1}{2}(j+\frac{3}{2})$	χ_a	$\frac{1}{2}(\xi-\frac{1}{2})$	$-\frac{1}{2}(\xi-\frac{1}{2})$
σ_c	$\frac{1}{2}(j-\frac{1}{2})$	$-\frac{1}{2}(j+\frac{1}{2})$	χ_c	$\frac{1}{2}(\xi+\frac{1}{2})$	$-\frac{1}{2}(\xi+\frac{1}{2})$

$$\psi \sim \text{finite at } \theta = 0,\ \theta = \tfrac{1}{2}\pi, \tag{2.50b}$$

and an examination of the characteristic exponents in Table I shows that Eq. (2.50b) is equivalent to the square-integrability boundary condition

$$\int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta(|a|^2 + |c|^2) < \infty . \tag{2.50c}$$

Since the interaction term in Eq. (2.39) is nonsingular at $\theta = 0$, $\theta = \frac{1}{2}\pi$, the characteristic exponents of the differential equation system at $\theta = 0$, $\theta = \frac{1}{2}\pi$ are the same in the interacting case as in the noninteracting case. Hence the boundary condition in Eq. (2.50), which we inferred from the free solution, is appropriate to the interacting case as well.

E. Reduction of the interacting case and construction of the Fredholm determinant

It is convenient, for the work which follows, to reduce the coupled differential equations of Eq. (2.39) to a somewhat simpler form. We work with the two-component spinor notation of Eq. (2.50a), and write Eq. (2.39) in the matrix form

$$H\psi = \mu\psi . \tag{2.51}$$

Introducing Pauli matrices τ_1, τ_2, τ_3 which act on the spinor ψ, it is easy to see that H may be written as

$$\begin{aligned} H = \tfrac{1}{2} &- (j - \xi - \lambda \sin^2\theta + \tfrac{1}{2})\tau_3 \\ &- [\xi \cot\theta + (j + \tfrac{1}{2})\tan\theta + \lambda \sin\theta \cos\theta]\tau_1 \\ &- i\left(\frac{d}{d\theta} + \tfrac{1}{2}\cot\theta - \tan\theta\right)\tau_2 . \end{aligned} \tag{2.52}$$

We now make a similarity transformation on Eqs. (2.51) and (2.52), writing

$$\begin{aligned} \psi &= S\psi_R , \\ H &= SH_R S^{-1} , \\ S &= (\cos\tfrac{1}{2}\theta - i\tau_2 \sin\tfrac{1}{2}\theta)[(\sin\theta)^{1/2}\cos\theta]^{-1} . \end{aligned} \tag{2.53}$$

The transformed eigenvalue problem is

$$\begin{aligned} H_R\psi_R &= \mu\psi_R , \\ H_R &= -[\xi(\sin\theta)^{-1} + \lambda\sin\theta]\tau_1 \\ &\quad - (j + \tfrac{1}{2})(\cos\theta)^{-1}\tau_3 - i\frac{d}{d\theta}\tau_2 . \end{aligned} \tag{2.54}$$

The measure for orthogonality is now

$$\int_0^{\pi/2} d\theta\, \psi_R^\dagger \psi_R' = 0, \quad \mu \neq \mu' \tag{2.55}$$

and the boundary condition is

$$\psi_R \sim (\sin\theta)^{1/2}\cos\theta \times \text{finite at } \theta = 0,\ \theta = \tfrac{1}{2}\pi , \tag{2.56a}$$

or equivalently

$$\int_0^{\pi/2} d\theta |\psi_R|^2 < \infty . \tag{2.56b}$$

To construct the external-field-problem Fredholm determinant, we display the parameter dependence of the eigenvalue μ by writing

$$\mu = \mu_{\xi j}(\lambda) , \tag{2.57}$$

so that the Fredholm determinant within the separable subspace takes the form

$$\begin{aligned} \Delta_{\xi j}(\lambda) &= \prod_{\substack{\text{all eigenvalues} \\ \text{in subspace}}} \mu_{\xi j}(\lambda) \\ &= \det[H_R] . \end{aligned} \tag{2.58}$$

Remembering that for each eigenvalue $\mu_{\xi j}(\lambda)$ there is an eigenvalue $-\mu_{\xi j}(\lambda)$ coming from eigenfunctions with opposite inversion parity [see the discussion following Eq. (2.31)], and that there is an additional duplication of eigenvalues when we reconstruct back to eight-component spinors, we find that

$$\prod_{\text{all eigenvalues}} \mu = \prod_{j=1/2,3/2,\ldots} \ \prod_{\xi = \pm 1/2, \pm 3/2, \ldots} \times \prod_{\substack{\text{all eigenvalues} \\ \text{in subspace}}} [\mu_{\xi j}(\lambda)^4]^{2j+1} . \tag{2.59}$$

Thus, comparing with Eq. (1.7), we see that the full external-field-problem Fredholm determinant is given by

$$\Delta[A] = \prod_{j=1/2,3/2,\ldots} \ \prod_{\xi = \pm 1/2, \pm 3/2, \ldots} \Delta_{\xi j}(\lambda)^{2j+1} . \tag{2.60}$$

One further transformation of this formula proves to be useful. From Eq. (2.54), we see that if

$$H_R\psi = \mu_{\xi j}(\lambda)\psi , \tag{2.61}$$

then

$$H_R \Big|_{\substack{\xi \to -\xi \\ \lambda \to -\lambda}} \tau_1\psi = -\mu_{\xi j}(\lambda)\tau_1\psi . \tag{2.62}$$

Since $\tau_1{}^2 = 1$, we conclude that the sets of numbers $\{-\mu_{\xi j}(\lambda)\}, \{\mu_{-\xi j}(-\lambda)\}$ are identical. Hence

$$\prod_{\substack{\text{all eigenvalues} \\ \text{in subspace}}} \mu_{-\xi j}(\lambda) = \prod_{\substack{\text{all eigenvalues} \\ \text{in subspace}}} (-)\mu_{\xi j}(-\lambda) , \tag{2.63}$$

permitting us to eliminate the negative $-\xi$ factors in Eq. (2.60). Dividing out $\Delta[0]$ to eliminate an irrelevant (and infinite) constant factor, we get finally

$$\frac{\Delta[A]}{\Delta[0]} = \prod_{j=1/2,3/2,\ldots} \ \prod_{\xi=1/2,3/2,\ldots} \left[\frac{\Delta_{\xi j}(\lambda)\Delta_{\xi j}(-\lambda)}{\Delta_{\xi j}(0)^2}\right]^{2j+1} . \tag{2.64}$$

Equation (2.64) is still a formal expression, in that renormalizations have not yet been made. In Sec. V below we discuss the modification of Eq. (2.64) which is made necessary by renormalization subtractions, and which guarantees convergence of the infinite product.

III. ZERO-FREE STRIPS

For the benefit of the reader who has skipped Sec. II, we briefly recapitulate the principal results derived there. In terms of the effective external-field amplitude

$$\lambda = -a\left(\frac{15}{16\pi^2}\right)^{1/2}, \tag{3.1}$$

we found that the external-field problem could be reduced to the two-component eigenvalue problem ($\tau_{1,2,3}$ = Pauli matrices)

$$\begin{aligned} &H\psi = \mu_{\xi j}(\lambda)\psi\ , \\ &H = -[\xi(\sin\theta)^{-1} + \lambda\sin\theta]\tau_1 \\ &\qquad - (j+\tfrac{1}{2})(\cos\theta)^{-1}\tau_3 - i\frac{d}{d\theta}\tau_2\ , \end{aligned} \tag{3.2}$$

with the measure for orthogonality

$$\int_0^{\pi/2} d\theta\, \psi^\dagger\psi' = 0, \quad \mu \neq \mu' \tag{3.3}$$

and the boundary condition

$$\psi \sim (\sin\theta)^{1/2}\cos\theta \times \text{finite at } \theta = 0, \tfrac{1}{2}\pi\ . \tag{3.4}$$

Defining the Fredholm determinant corresponding to Eq. (3.2) by

$$\Delta_{\xi j}(\lambda) = \prod_{\text{all eigenvalues}} \mu_{\xi j}(\lambda)\ , \tag{3.5}$$

we found that the full external-field-problem Fredholm determinant introduced in Eq. (1.7) is given (up to renormalization subtractions) by

$$\frac{\Delta[A]}{\Delta[0]} = \prod_{j=1/2,3/2,\ldots}\ \prod_{\xi=1/2,3/2,\ldots} \left[\frac{\Delta_{\xi j}(\lambda)\Delta_{\xi j}(-\lambda)}{\Delta_{\xi j}(0)^2}\right]^{2j+1}\ . \tag{3.6}$$

The reminder of this paper is devoted to a study of the mathematical properties of Eqs. (3.2)–(3.6).

We begin by showing that $\Delta_{\xi j}(\lambda)$ cannot vanish in strips in the λ plane containing the real and imaginary axes. From Eq. (3.5) we see that zeros of $\Delta_{\xi j}(\lambda)$ occur at values of λ where Eq. (3.2) has a vanishing eigenvalue, that is, where

$$H\psi = 0 \tag{3.7}$$

for nonvanishing, normalizable ψ. To get our first restriction on the locations of zeros, we multiply Eq. (3.7) by $\psi^\dagger\tau_1$ and integrate, giving

$$\begin{aligned} &-\int_0^{\pi/2} [\xi(\sin\theta)^{-1} + \lambda\sin\theta]\psi^\dagger\psi d\theta + iR_1 = 0\ , \\ &R_1 = (j+\tfrac{1}{2})\int_0^{\pi/2} (\cos\theta)^{-1}\psi^\dagger\tau_2\psi d\theta \\ &\qquad + \int_0^{\pi/2} \psi^\dagger\tau_3\left(-i\frac{d}{d\theta}\right)\psi d\theta\ . \end{aligned} \tag{3.8}$$

Using the boundary condition of Eq. (3.4) to integrate by parts, we readily see that R_1 is pure real. Hence taking the real part of Eq. (3.8) gives the relation

$$\frac{-\text{Re}\lambda}{\xi} = \frac{\int_0^{\pi/2}(\sin\theta)^{-1}\psi^\dagger\psi d\theta}{\int_0^{\pi/2}\sin\theta\,\psi^\dagger\psi d\theta} \geq 1\ . \tag{3.9}$$

We learn from this relation that $\Delta[A]$ has no zeros for λ in the strip $|\text{Re}\lambda| \leq \frac{1}{2}$, and in particular no zeros on the imaginary axis. To get a second restriction on the locations of zeros, we multiply Eq. (3.7) by $\psi^\dagger\tau_3$ and integrate, giving

$$\begin{aligned} &-i\int_0^{\pi/2} \lambda\sin\theta\,\psi^\dagger\tau_2\psi d\theta \\ &\qquad - (j+\tfrac{1}{2})\int_0^{\pi/2}(\cos\theta)^{-1}\psi^\dagger\psi d\theta + iR_2 = 0\ , \end{aligned} \tag{3.10}$$

$$R_2 = -\xi\int_0^{\pi/2}(\sin\theta)^{-1}\psi^\dagger\tau_2\psi d\theta - \int_0^{\pi/2}\psi^\dagger\tau_1\left(-i\frac{d}{d\theta}\right)\psi d\theta\ .$$

Again, the boundary condition of Eq. (3.4) implies that R_2 is real, so taking the real part of Eq. (3.10) gives the second relation

$$\frac{\text{Im}\lambda}{j+\frac{1}{2}} = \frac{\int_0^{\pi/2}(\cos\theta)^{-1}\psi^\dagger\psi d\theta}{\int_0^{\pi/2}\sin\theta\,\psi^\dagger\tau_2\psi d\theta}\ . \tag{3.11}$$

Since τ_2 has eigenvalues ± 1, we have the inequality $|\psi^\dagger\tau_2\psi| \leq \psi^\dagger\psi$, and so Eq. (3.11) implies the inequality

$$\frac{|\text{Im}\lambda|}{j+\frac{1}{2}} \geq \frac{\int_0^{\pi/2}(\cos\theta)^{-1}\psi^\dagger\psi d\theta}{\int_0^{\pi/2}\sin\theta\,\psi^\dagger\psi d\theta} \geq 1\ . \tag{3.12}$$

Thus $\Delta[A]$ can have no zeros for λ in the strip $|\text{Im}\lambda| \leq 1$, and in particular no zeros on the real axis. Combining the restrictions of Eqs. (3.9) and (3.12), we get the regions in the λ plane where $\Delta_{\xi j}(\lambda)$ is allowed to have zeros, as illustrated in Fig. 1. Note that the absolute value sign in Eq. (3.12) cannot be removed. In fact, since the Hamiltonian H is Hermitian for real λ, $\Delta_{\xi j}(\lambda)$ is a real analytic function of λ and satisfies the reflection principle

$$\Delta_{\xi j}(\lambda)^* = \Delta_{\xi j}(\lambda^*)\ . \tag{3.13}$$

Hence for each zero λ of $\Delta_{\xi j}(\lambda)$, there is a corresponding zero at the complex-conjugate point λ^*.

IV. WRONSKIAN FORMULA FOR THE FREDHOLM DETERMINANT

We proceed next to derive a connection between the Fredholm determinant in each separable subspace and the Wronskian of two suitably standardized independent solutions of Eq. (3.7). In Sec. IV A we construct the Green's function for H, introduce the standard solutions, and discuss their analyticity and rate of growth in λ. In Sec. IV B we prove the connection between the Wronskian and the Fredholm determinant.

A. Green's function and standard solutions

Let $H=H(\theta)$ be the Hamiltonian of Eq. (3.2), and let $S=H^{-1}$ be the Green's function satisfying

$$H(\theta_1)S(\theta_1,\theta_2)=\delta(\theta_1-\theta_2)1\ ,\qquad 0\leq\theta_1,\ \theta_2\leq\tfrac{1}{2}\pi\ ,\tag{4.1}$$

with 1 the 2×2 unit matrix. To construct an explicit expression for S, we introduce the solutions ψ_1,ψ_2 of Eq. (3.7) which are regular at $\theta=0$, $\theta=\frac{1}{2}\pi$, respectively:

$$H\psi_1=H\psi_2=0\ ,$$
$$\psi_1=\begin{pmatrix}a_1\\ c_1\end{pmatrix}\sim(\sin\theta)^{1/2}\times\text{finite at }\theta=0\ ,\tag{4.2}$$
$$\psi_2=\begin{pmatrix}a_2\\ c_2\end{pmatrix}\sim\cos\theta\times\text{finite at }\theta=\tfrac{1}{2}\pi\ .$$

We also need the Wronskian of the two solutions, defined by (the superscript T denotes transpose)

$$w(\lambda)=\psi_2^T i\tau_2\psi_1=a_2(\theta)c_1(\theta)-a_1(\theta)c_2(\theta)\ .\tag{4.3}$$

Since

$$\frac{dw}{d\theta}=\psi_2^T\left(i\tau_2\frac{d}{d\theta}\psi_1\right)-\left(i\tau_2\frac{d}{d\theta}\psi_2\right)^T\psi_1$$
$$=-\psi_2^T\{[\xi(\sin\theta)^{-1}+\lambda\sin\theta]\tau_1+(j+\tfrac{1}{2})(\cos\theta)^{-1}\tau_3\}\psi_1+\psi_2^T\{[\xi(\sin\theta)^{-1}+\lambda\sin\theta]\tau_1+(j+\tfrac{1}{2})(\cos\theta)^{-1}\tau_3\}^T\psi_1=0\ ,\tag{4.4}$$

the Wronskian is θ-independent. Applying the method of variation of parameters,[6] we then find the following expression for S:

$$S(\theta_1,\theta_2)=w^{-1}\times\begin{cases}\psi_1(\theta_1)\psi_2^T(\theta_2)\ , & \theta_1<\theta_2\\ \psi_2(\theta_1)\psi_1^T(\theta_2)\ , & \theta_1>\theta_2\ .\end{cases}\tag{4.5}$$

To verify Eq. (4.5), we note that

$$H(\theta_1)S(\theta_1,\theta_2)=0,\quad \theta_1<\theta_2,\ \theta_1>\theta_2\ ;$$
$$\int_{\theta_2-\epsilon}^{\theta_2+\epsilon}d\theta_1 H(\theta_1)S(\theta_1,\theta_2)\underset{\epsilon\to 0}{\longrightarrow}-i\tau_2 w^{-1}\times[\psi_2(\theta_2)\psi_1^T(\theta_2)-\psi_1(\theta_2)\psi_2^T(\theta_2)]\tag{4.6}$$
$$=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}w^{-1}\begin{pmatrix}0 & w\\ -w & 0\end{pmatrix}$$
$$=1\ ,$$

as required. In Appendix A, as an illustration of this construction, we give a formula for S in the noninteracting ($\lambda=0$) case.

Up to this point the normalization of ψ_1 and ψ_2 has not been specified, and it is obviously immaterial for the construction of Eq. (4.5). However, for future use we now standardize the normalization by requiring that

$$\frac{\partial\psi_1}{\partial\lambda}(\psi_1^\dagger\psi_1)^{-1/2}\to 0\quad\text{as }\theta\to 0\ ,$$
$$\frac{\partial\psi_2}{\partial\lambda}(\psi_2^\dagger\psi_2)^{-1/2}\to 0\quad\text{as }\theta\to\tfrac{1}{2}\pi\ ,\tag{4.7}$$

conditions which, as we shall see explicitly below, can be satisfied by taking the leading terms in the series developments of ψ_1 (ψ_2) about $\theta=0$ ($\theta=\frac{1}{2}\pi$) to be λ-independent constants.[7] Equation (4.7) uniquely specifies the λ dependence of ψ_1, ψ_2,

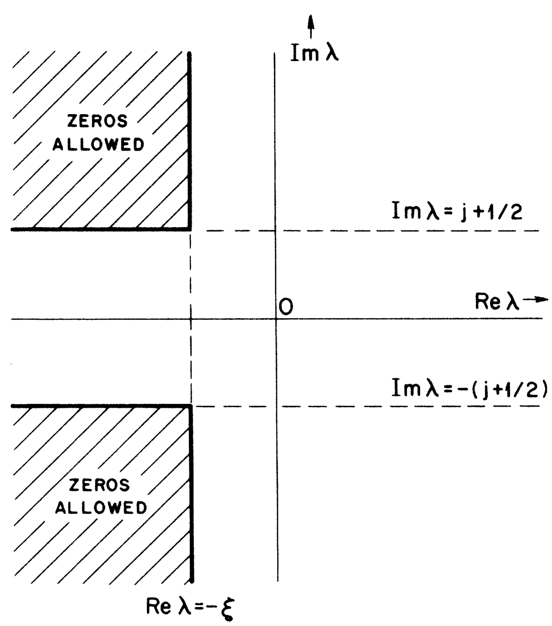

FIG. 1. Regions in which $\Delta_{\xi j}(\lambda)$ can have zeros according to the inequalities of Eqs. (3.9) and (3.12). We assume $\xi>0$.

and w, leaving arbitrary only a λ-independent normalization factor. It is now possible, by straightforward majorization arguments, to prove the following result: The standardized solutions $\psi_{1,2}$ are entire functions of λ, bounded for large λ by $e^{c|\lambda|}$, with c an appropriate constant.

B. Proof of the connection

To connect the Fredholm determinant $\Delta_{\xi j}(\lambda)$ with the Wronskian, we start from the formal relation[8]

$$\begin{aligned}\ln\Delta_{\xi j}(\lambda) &= \mathrm{Tr}\ln H\\ &= \mathrm{Tr}\ln\Big\{-[\xi(\sin\theta)^{-1}+\lambda\sin\theta]\tau_1\\ &\qquad -(j+\tfrac{1}{2})(\cos\theta)^{-1}\tau_3 - i\frac{d}{d\theta}\tau_2\Big\},\end{aligned} \tag{4.8}$$

from which we get by differentiation

$$\Delta_{\xi j}(\lambda)^{-1}\frac{d\Delta_{\xi j}(\lambda)}{d\lambda} = \mathrm{Tr}\left[\frac{\partial H}{\partial\lambda}H^{-1}\right]. \tag{4.9}$$

Substituting Eq. (4.5) for $S=H^{-1}$ and evaluating the trace, we find

$$\Delta_{\xi j}(\lambda)^{-1}\frac{d\Delta_{\xi j}(\lambda)}{d\lambda} = w(\lambda)^{-1}\int_0^{\pi/2} d\theta\,\psi_2^T\frac{\partial H}{\partial\lambda}\psi_1. \tag{4.10}$$

We next show that the numerator on the right-hand side of Eq. (4.10) is just equal to $dw(\lambda)/d\lambda$ when ψ_1 and ψ_2 are taken to be the standard solutions. To see this, we start from Eq. (4.3) for w, which yields

$$\frac{dw(\lambda)}{d\lambda} = \frac{\partial\psi_2^T}{\partial\lambda}i\tau_2\psi_1 + \psi_2^T i\tau_2\frac{\partial\psi_1}{\partial\lambda}. \tag{4.11}$$

Letting $\theta\to\frac{1}{2}\pi$ and using Eq. (4.7), only the second term on the right-hand side of Eq. (4.11) survives, giving

$$\frac{dw(\lambda)}{d\lambda} = \psi_2^T i\tau_2\frac{\partial\psi_1}{\partial\lambda}\bigg|_{\theta\to\pi/2}. \tag{4.12}$$

To proceed we consider the integral appearing in the numerator of Eq. (4.10),

$$\int_0^{\pi/2} d\theta\,\psi_2^T\frac{\partial H}{\partial\lambda}\psi_1 = \lim_{\substack{\theta_1\to 0\\ \theta_2\to\pi/2}}\int_{\theta_1}^{\theta_2} d\theta\,\psi_2^T\frac{\partial H}{\partial\lambda}\psi_1. \tag{4.13}$$

By differentiating the equation $H\psi_1=0$ with respect to λ we get

$$\frac{\partial H}{\partial\lambda}\psi_1 = -H\frac{\partial\psi_1}{\partial\lambda} \tag{4.14}$$

and substituting this into Eq. (4.13), using the explicit form of H and integrating by parts, we find

$$\begin{aligned}\int_0^{\pi/2} d\theta\,\psi_2^T\frac{\partial H}{\partial\lambda}\psi_1 &= \lim_{\substack{\theta_1\to 0\\ \theta_2\to\pi/2}} -\int_{\theta_1}^{\theta_2} d\theta\,\psi_2^T H\frac{\partial\psi_1}{\partial\lambda}\\ &= \lim_{\substack{\theta_1\to 0\\ \theta_2\to\pi/2}} -\int_{\theta_1}^{\theta_2} d\theta\,\psi_2^T\Big\{-i\frac{d}{d\theta}\tau_2 - [\xi(\sin\theta)^{-1}+\lambda\sin\theta]\tau_1 - (j+\tfrac{1}{2})(\cos\theta)^{-1}\tau_3\Big\}\frac{\partial\psi_1}{\partial\lambda}\\ &= \lim_{\substack{\theta_1\to 0\\ \theta_2\to\pi/2}}\left[i\psi_2^T\tau_2\frac{\partial\psi_1}{\partial\lambda}\bigg|_{\theta_1}^{\theta_2} - \int_{\theta_1}^{\theta_2} d\theta (H\psi_2)^T\psi_1\right]\\ &= i\psi_2^T\tau_2\frac{\partial\psi_1}{\partial\lambda}\bigg|_{\theta\to\pi/2}\\ &= \frac{dw(\lambda)}{d\lambda},\end{aligned} \tag{4.15}$$

giving the desired result. Substituting Eq. (4.15) into Eq. (4.10) we get, finally,

$$\Delta_{\xi j}(\lambda)^{-1}\frac{d\Delta_{\xi j}(\lambda)}{d\lambda} = w(\lambda)^{-1}\frac{dw(\lambda)}{d\lambda}, \tag{4.16}$$

which on integration gives the connection between the Fredholm determinant and the Wronskian,

$$\frac{\Delta_{\xi j}(\lambda)}{\Delta_{\xi j}(0)} = \frac{w(\lambda)}{w(0)}. \tag{4.17}$$

Since ψ_1, ψ_2 are entire functions of λ, we conclude that $\Delta_{\xi j}(\lambda)$ is also entire, as expected for a Fredholm determinant. Obviously, $\Delta_{\xi j}(\lambda)$ will also have exponentially bounded growth at infinity; the precise asymptotic form of $\Delta_{\xi j}(\lambda)$ will be given below. Equation (4.17) will be of great utility in the subsequent sections, where it will allow us to study $\Delta_{\xi j}(\lambda)$ by applying WKB and series expansion methods to the solutions of Eq. (3.7).

V. ORDER OF GROWTH OF $\Delta_{\xi j}(\lambda)$ AND $\tilde{\Delta}[A]$

In this section we give more precise results concerning the large-λ asymptotic behavior of

$\Delta_{\xi j}(\lambda)$ and of the full external-field-problem Fredholm determinant $\Delta[A]$. In Sec. V A we present a WKB formula (derived in Appendix B) giving the asymptotic behavior of $\Delta_{\xi j}(\lambda)$. Using this formula, we determine the asymptotic distribution of zeros of $\Delta_{\xi j}(\lambda)$. In Sec. V B we discuss the renormalization subtractions needed to make the infinite product for $\Delta[A]$ convergent. Using our knowledge of the distribution of zeros of Δ, combined with results from the theory of entire functions, we determine the order of growth of the renormalized determinant $\tilde{\Delta}[A]$ for large external-field amplitude λ. Combining this estimate with the absence of zeros in a strip containing the real axis, we show that the real amplitude integral contour discussed in Sec. I yields a function of e^2 analytic in the right-hand e^2 half plane.

A. Asymptotic behavior of $\Delta_{\xi j}(\lambda)$

As we have seen above, $\Delta_{\xi j}(\lambda)$ is given by the Wronskian of two suitably standardized independent solutions of the differential equation $H\psi=0$. In the limit when $|\lambda|$ is large, or more specifically, when the inequalities

$$\epsilon_1=\frac{\xi}{|\lambda|}\ll 1\,,\qquad \epsilon_2=\frac{j}{|\lambda|}\ll 1 \tag{5.1}$$

are satisfied, we can apply WKB methods to calculate approximate solutions of the differential equations, and hence to get the asymptotic form of $\Delta_{\xi j}(\lambda)$. The calculation, which is outlined in Appendix B, gives the result (valid for $\xi>0$)

$$\frac{\Delta_{\xi j}(\lambda)}{\Delta_{\xi j}(0)} \underset{\substack{\epsilon_1\ll 1\\ \epsilon_2\ll 1}}{\approx} \frac{\Gamma(2(j+1))}{\Gamma(j+\frac{3}{2})}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(j+1)}\,\frac{\Gamma(j+\xi+1)}{\Gamma(\xi+\frac{1}{2})}\,2^{-2(j+1/2)}\lambda^{-(j+1/2)} \times[e^{\lambda}+e^{-\lambda}(-1)^{j+\xi+1}2^{-2(\xi+1/2)}(j+\tfrac{1}{2})\Gamma(\xi+\tfrac{1}{2})\lambda^{-(\xi+1/2)}]\,, \tag{5.2}$$

showing that the entire function $\Delta_{\xi j}(\lambda)$ is of exponential type. One special case of Eq. (5.2) is worth noting. When $j\to-\frac{1}{2}$, Eq. (5.2) reduces to

$$\left.\frac{\Delta_{\xi j}(\lambda)}{\Delta_{\xi j}(0)}\right|_{j\to-1/2} \underset{\substack{\epsilon_1\ll 1\\ \epsilon_2\ll 1}}{\approx} e^{\lambda}\,; \tag{5.3}$$

we will show in Sec. VI A below that this is an exact, and not just an asymptotic, result. From Eq. (5.2), we can calculate the asymptotic distribution of zeros of $\Delta_{\xi j}(\lambda)$ by solving the equation

$$0=e^{\lambda}+e^{-\lambda}(-1)^{j+\xi+1}2^{-2(\xi+1/2)}(j+\tfrac{1}{2}) \times\Gamma(\xi+\tfrac{1}{2})\lambda^{-(\xi+1/2)}\,, \tag{5.4a}$$

which we rewrite in the form

$$\begin{aligned} &e^{2\lambda}(-\lambda)^{c_1}=e^{c_2}\\ &c_1=\xi+\tfrac{1}{2}\,,\\ &c_2=-2(\xi+\tfrac{1}{2})\ln 2+\ln(j+\tfrac{1}{2})\\ &\qquad+\ln\Gamma(\xi+\tfrac{1}{2})-i\pi(j+\tfrac{3}{2})\,. \end{aligned} \tag{5.4b}$$

Neglecting terms which vanish for large λ, the solution is

$$\begin{aligned} \lambda&\approx\pi ni+\tfrac{1}{2}c_2-\tfrac{1}{2}c_1\ln(-\pi ni)\\ &\approx\pi ni+\tfrac{1}{2}(\xi+\tfrac{1}{2})\ln\left(\frac{\xi+\frac{1}{2}}{|n|}\right)+O(\xi,j)\,,\\ \mathrm{Re}\lambda&\approx\tfrac{1}{2}(\xi+\tfrac{1}{2})\ln\left(\frac{\xi+\frac{1}{2}}{|n|}\right)+O(\xi,j)\,,\\ \mathrm{Im}\lambda&\approx\pi n+O(\xi,j)\,. \end{aligned} \tag{5.5}$$

In the region of validity of Eq. (5.5), where $|n|\gg\xi$, we see that $\mathrm{Re}\lambda$ is asymptotically negative, as required by the inequality of Eq. (3.9). The occurrence of zeros in complex-conjugate pairs is also apparent from Eq. (5.5).

For application in Sec. V B, it is convenient to give the zeros of $\Delta_{\xi j}(\lambda)$ an index k which arranges them in order of increasing magnitude:

$$\begin{aligned} &\lambda_k^{\xi j}=\text{general zero of }\Delta_{\xi j}(\lambda)\,,\\ &|\lambda_1^{\xi j}|\le|\lambda_2^{\xi j}|\le|\lambda_3^{\xi j}|<\cdots\,. \end{aligned} \tag{5.6}$$

For large k the index defined this way can be identified (up to a factor of two, since the zeros occur in complex-conjugate pairs) with the positive integer $|n|$ appearing in Eq. (5.5). Since the effective expansion parameters in the WKB procedure are thus

$$\frac{\xi}{|\lambda_k^{\xi j}|}\sim\frac{\xi}{k}\,,\quad\frac{j}{|\lambda_k^{\xi j}|}\sim\frac{j}{k}\,, \tag{5.7}$$

we expect the following bounds on $|\lambda_k^{\xi j}|$ to hold uniformly in ξ and j:

$$A_1\le\frac{|\lambda_k^{\xi j}|}{(\pi^2k^2+\frac{1}{4}(\xi+\frac{1}{2})^2\{\ln[(\xi+\frac{1}{2})/k]\}^2)^{1/2}}\le A_2\,,\qquad k\ge k_0=C(j^2+\xi^2)^{1/2}\,; \tag{5.8a}$$

$$(j^2+\xi^2)^{1/2}\le|\lambda_k^{\xi j}|\le A_3(j^2+\xi^2)^{1/2}\,,\quad k\le k_0 \tag{5.8b}$$

for suitable constants $A_{1,2,3}$ and C. [Equation (5.8b) also incorporates the lower bounds of Eqs. (3.9) and (3.12).] We have not constructed a proof of Eq. (5.8), so these inequalities should be considered a conjecture, suggested by the WKB analysis, on which some of the arguments of Sec. V B are based.

B. Order of growth of $\Delta[A]$

We are now ready to examine the asymptotic behavior of the full external-field-problem Fredholm determinant $\Delta[A]$, given by the product formula Eq. (3.6). First we must deal with the question of renormalization subtractions alluded to above. By dividing out $\Delta_{\xi j}(0)^2$ in Eq. (3.6), we have eliminated the most divergent vacuum diagram illustrated in Fig. 2(a). However, the second-order diagram shown in Fig. 2(b) is also divergent, and must be eliminated by a further subtraction. To do this, we write the small-λ expansion

$$\frac{\Delta_{\xi j}(\lambda)\Delta_{\xi j}(-\lambda)}{\Delta_{\xi j}(0)^2}=1+A_{\xi j}\lambda^2+O(\lambda^4)\ , \tag{5.9}$$

and then define the renormalized Fredholm determinant $\tilde{\Delta}[A]$ by writing

$$\tilde{\Delta}[A]=e^{Q(\lambda)}\prod_{j=1/2,3/2,\ldots}\ \prod_{\xi=1/2,3/2,\ldots}\left[\frac{\Delta_{\xi j}(\lambda)\Delta_{\xi j}(-\lambda)}{\Delta_{\xi j}(0)^2}\,e^{-A_{\xi j}\lambda^2}\right]^{2j+1}\ . \tag{5.10}$$

In this expression

$$Q(\lambda)=Q_0+Q_2\lambda^2 \tag{5.11}$$

is a polynomial which expresses the fact that the renormalization counterterms always have an undetermined finite part. To see that Eq. (5.10) is the correct recipe, we note that the renormalized vacuum amplitude, which according to Eq. (1.8) is proportional to

$$\ln\tilde{\Delta}[A]=Q(\lambda)+\ln\Delta[A]-\ln\Delta[0]-\lambda^2\frac{d}{d\lambda^2}\ln\Delta[A]\bigg|_{\lambda^2=0}\ , \tag{5.12}$$

now receives contributions only from the convergent vacuum diagrams illustrated in Fig. 3.

Let us next rewrite Eq. (5.10) in an alternative useful form. Since $\Delta_{\xi j}(\lambda)$ is an entire function of exponential type, we can use the Hadamard factorization theorem[9] to write it as an infinite product in terms of its zeros $\lambda_k^{\xi j}$,

$$\frac{\Delta_{\xi j}(\lambda)}{\Delta_{\xi j}(0)}=e^{\alpha_{\xi j}\lambda}\prod_k\left(1-\frac{\lambda}{\lambda_k^{\xi j}}\right)\exp\left(\frac{\lambda}{\lambda_k^{\xi j}}\right)\ , \tag{5.13}$$

giving

$$\frac{\Delta_{\xi j}(\lambda)\Delta_{\xi j}(-\lambda)}{\Delta_{\xi j}(0)^2}=\prod_k\left[1-\frac{\lambda^2}{(\lambda_k^{\xi j})^2}\right]=1-\lambda^2\sum_k\frac{1}{(\lambda_k^{\xi j})^2}+O(\lambda^4)\ . \tag{5.14}$$

From Eq. (5.14) we identify $A_{\xi j}$ as

$$A_{\xi j}=-\sum_k\frac{1}{(\lambda_k^{\xi j})^2}\ . \tag{5.15}$$

Let us define an additional constant $B_{\xi j}$ by

$$B_{\xi j}=\sum_k\frac{1}{(\lambda_k^{\xi j})^4} \tag{5.16}$$

and combine Eqs. (5.13)–(5.16) to rewrite Eq. (5.10) as

$$\begin{aligned}\tilde{\Delta}[A]&=e^{Q(\lambda)}e^{-B\lambda^4/2}P(\lambda)\ ,\\ P(\lambda)&=\prod_{j=1/2,3/2,\ldots}\ \prod_{\xi=1/2,3/2,\ldots}\ \prod_k\left\{\left[1-\frac{\lambda^2}{(\lambda_k^{\xi j})^2}\right]\exp\left[\left(\frac{\lambda}{\lambda_k^{\xi j}}\right)^2+\frac{1}{2}\left(\frac{\lambda}{\lambda_k^{\xi j}}\right)^4\right]\right\}^{2j+1}\\ &=\prod_{\substack{\text{all zeros}\\ \lambda_\nu\text{ of }\tilde{\Delta}[A]}}\left\{\left(1-\frac{\lambda}{\lambda_\nu}\right)\exp\left[\frac{\lambda}{\lambda_\nu}+\frac{1}{2}\left(\frac{\lambda}{\lambda_\nu}\right)^2+\frac{1}{3}\left(\frac{\lambda}{\lambda_\nu}\right)^3+\frac{1}{4}\left(\frac{\lambda}{\lambda_\nu}\right)^4\right]\right\}\ ,\\ B&=\sum_{j=1/2,3/2,\ldots}\ \sum_{\xi=1/2,3/2,\ldots}(2j+1)B_{\xi j}\ .\end{aligned} \tag{5.17}$$

The constant B is the contribution of the fourth-order graph which appears as the first term in the series of Fig. 3, and hence is finite. The second expression for $P(\lambda)$ in Eq. (5.17) has the form called a *canonical product* in the theory of entire functions[9]; Eq. (5.17) thus expresses $\tilde{\Delta}[A]$ as a canonical product multiplied by the exponential of a fourth-degree polynomial in λ.

FIG. 2. (a) Divergent vacuum diagram which is removed by division by $\Delta_{\xi j}(0)^2$ in Eq. (3.6). (b) Divergent vacuum diagram which is removed by the factor $\exp(-A_{\xi j}\lambda^2)$ in Eq. (5.10).

Let us now introduce some further concepts from the theory of entire functions.[9] Let $f(\lambda)$ be an entire function of the complex variable λ. Its *maximum modulus* $M(r)$ and *minimum modulus* $m(r)$ are defined by

$$M(r) = \max_{0\le\theta\le 2\pi} |f(re^{i\theta})| \,, \qquad m(r) = \min_{0\le\theta\le 2\pi} |f(re^{i\theta})| \,. \tag{5.18}$$

The *order* ρ of $f(\lambda)$ is defined to be

$$\rho = \limsup_{r\to\infty} \frac{\ln \ln M(r)}{\ln r} \,; \tag{5.19a}$$

if f is of order ρ it is asymptotically bounded by

$$|f(\lambda)| \le A e^{B|\lambda|^\rho} \tag{5.19b}$$

for suitable positive constants A and B. Finally, let $\{r_\nu = |\lambda_\nu|\}$ be the sequence of moduli of the zeros λ_ν of $f(\lambda)$, arranged in increasing order. The smallest number σ for which

$$\sum_{\nu=1}^{\infty} \frac{1}{{r_\nu}^\alpha} < \infty, \quad \text{for all } \alpha > \sigma \tag{5.20}$$

is called the *exponent of convergence* of the sequence. According to the theory of entire functions, the order of an entire function is closely related to the exponent of convergence of its zeros.

To determine the order of $\tilde{\Delta}[A]$, we wish then to calculate the exponent of convergence of the zeros λ_ν appearing in Eq. (5.17). Remembering that all zeros $\lambda_k^{\xi j}$ occur with multiplicity $2j+1$, we consider the sum

$$S_\alpha = \sum_{j=1/2,3/2,\ldots} (2j+1) \sum_{\xi=1/2,3/2,\ldots} \sum_k \frac{1}{|\lambda_k^{\xi j}|^\alpha} \,. \tag{5.21}$$

In estimating the convergence properties of S_α, it obviously suffices to replace the sums in Eq. (5.21) by integrals. We first show that Eq. (5.21) is convergent for $\alpha > 4$. Using the lower bounds obtained from Eq. (5.8),

$$\pi k A_1 \le |\lambda_k^{\xi j}|, \quad k \ge k_0 \,; \qquad (j^2+\xi^2)^{1/2} \le |\lambda_k^{\xi j}|, \quad k \le k_0 \tag{5.22}$$

we get the estimate

$$\sum_k \frac{1}{|\lambda_k^{\xi j}|^\alpha} = \sum_{k=1}^{k_0} \frac{1}{|\lambda_k^{\xi j}|^\alpha} + \sum_{k=k_0}^{\infty} \frac{1}{|\lambda_k^{\xi j}|^\alpha} \lesssim \frac{C(j^2+\xi^2)^{1/2}}{(j^2+\xi^2)^{\alpha/2}} + \int_{C(j^2+\xi^2)^{1/2}}^{\infty} \frac{dk}{(\pi k A_1)^\alpha} = \frac{C'}{(j^2+\xi^2)^{(\alpha-1)/2}} \,, \tag{5.23}$$

so that

FIG. 3. Convergent diagrams which contribute to Eq. (5.10).

$$S_\alpha \lesssim \int_{1/2}^{\infty} d\xi \int_{1/2}^{\infty} j\,dj \, \frac{4C'}{(j^2+\xi^2)^{(\alpha-1)/2}} < \frac{4C' 2^{\alpha-4}}{(\alpha-3)(\alpha-4)} < \infty \,, \tag{5.24}$$

as claimed. Next we show that Eq. (5.21) diverges when $\alpha \to 4$. Since $(\ln x)/x \le 1/e$ for $x \ge 1$, the upper bounds in Eq. (5.8) take the form

$$|\lambda_k^{\xi j}| \le A_2 \pi \left(1 + \frac{1}{4e^2}\right)^{1/2} k, \quad k \ge k_0 \,; \qquad |\lambda_k^{\xi j}| \le A_3 (j^2+\xi^2)^{1/2}, \quad k \le k_0 \tag{5.25}$$

giving, by a procedure identical to that in Eqs. (5.23) and (5.24), the estimate

$$S_\alpha \gtrsim \frac{C''}{(\alpha-4)} \,, \quad C'' > 0 \,. \tag{5.26}$$

We conclude that S_α diverges for $\alpha = 4$, and that the exponent of convergence of the zeros of $\tilde{\Delta}[A]$ is $\sigma = 4$.

From the fact that $\sigma = 4$ we can immediately conclude that the order of the canonical product $P(\lambda)$ is 4, and hence that the order of $\tilde{\Delta}[A]$ is less than or equal to 4.[9] If the order of $\tilde{\Delta}[A]$ were actually less than 4, then the sum in Eq. (5.21) would converge[9] for exponents α smaller than 4, which we have seen is not the case. So we conclude that the order of $\tilde{\Delta}[A]$ is precisely 4.

Let us now use these results to determine the convergence properties of the amplitude integral when taken along the real contour. Since $\Delta_{\xi j}(\lambda)$ cannot change sign on the real axis, all of the factors in Eq. (5.10), and hence $\tilde{\Delta}[A]$ itself, are positive for λ real, and so $\ln\tilde{\Delta}[A]$ is real. Since the maximum modulus of $\tilde{\Delta}[A]$ is bounded as in Eq. (5.19b) with $\rho = 4$, we have

$$\ln\tilde{\Delta}[A] < B|\lambda|^4 \tag{5.27}$$

for an appropriate positive constant B. In order to restrict $\ln\tilde{\Delta}[A]$ from below, it is necessary to have a lower bound on the minimum modulus of $\tilde{\Delta}[A]$. We get this by using the following theorem[9]: "Let $P(\lambda)$ be a canonical product of order ρ. About each zero λ_ν ($|\lambda_\nu| > 1$) we draw a circle of radius $1/|\lambda_\nu|^\alpha$, $\alpha > \rho$. Then in the region outside these excluded circles, $|P(\lambda)| > \exp(-r^{\rho+\epsilon})$ for $\epsilon > 0$ and for $r > r_0(\epsilon, \alpha)$." To apply this theorem, we note that the sum of the radii of all the circles is just S_α, and can be made smaller than 1 by choosing α large enough. Since $\tilde{\Delta}[A]$ has no zeros in the strip $|\mathrm{Im}\lambda| \le 1$, the entire real axis then lies in the region outside the excluded circles, and so we learn

$$\ln\bar{\Delta}[A] > -|\lambda|^{4+\epsilon}, \quad |\lambda| > r_0 \tag{5.28}$$

for r_0 appropriately large. Taking Eqs. (5.27) and (5.28) together, we see that $|\ln\bar{\Delta}[A]|$ is polynomial-bounded on the λ-real axis. The Gaussian factor in Eq. (1.5) then guarantees that the amplitude integral converges when taken along the real axis, provided that $\mathrm{Re}\,e^2 > 0$, and thus defines a function of e^2 analytic in the right-hand e^2 half plane. Note that this conclusion does not depend on the fact that $\bar{\Delta}[A]$ is of order 4, but only requires the weaker statement that the order of $\bar{\Delta}[A]$ is finite, which is known to be true[2] independent of the validity of the inequalities in Eq. (5.8).

VI. NUMERICAL RESULTS

We turn next to numerical studies of $\Delta_{\xi j}(\lambda)$ and $\tilde{\Delta}[A]$. In Sec. VI A we derive power-series expansions for the standardized solutions ψ_1 and ψ_2. The circles of convergence of the two series which we obtain overlap, allowing one to compute the Wronskian, and hence $\Delta_{\xi j}(\lambda)$, by picking θ to have any value in the overlap region. In Sec. VI B we numerically study the location of low-lying zeros of $\Delta_{\xi j}(\lambda)$, and find that there are no zeros in the sectors $|\mathrm{Re}\lambda| > |\mathrm{Im}\lambda|$. Consequences of this fact for the coupling-constant analyticity properties of W_1 are discussed. Finally, in Sec. VI C we give numerical results for the behavior of the vacuum amplitude as λ increases along the imaginary axis.

A. Power-series solutions

Substituting

$$\psi = \begin{pmatrix} a \\ c \end{pmatrix} \tag{6.1}$$

into Eq. (3.7) and writing out the coupled differential equations for the two components, we get

$$\frac{da}{d\theta} - [\xi(\sin\theta)^{-1} + \lambda\sin\theta]\,a + (j+\tfrac{1}{2})(\cos\theta)^{-1}c = 0,$$

$$\frac{dc}{d\theta} + [\xi(\sin\theta)^{-1} + \lambda\sin\theta]\,c + (j+\tfrac{1}{2})(\cos\theta)^{-1}a = 0. \tag{6.2}$$

To construct power-series solutions regular around $\theta = 0$ and $\theta = \frac{1}{2}\pi$ we make the following changes of variable, motivated by the form of the noninteracting ($\lambda = 0$) solutions presented in Appendix A.

(1) Solution ψ_1 regular around $\theta = 0$. We substitute

$$a_1 = (\tan\tfrac{1}{2}\theta)^{\xi} f(x),$$
$$c_1 = \tan\theta(\tan\tfrac{1}{2}\theta)^{\xi} g(x), \tag{6.3}$$
$$x = 1 - \frac{1}{\cos\theta}.$$

In terms of the new variables the coupled equations become

$$\frac{df}{dx} + \frac{\lambda f}{(1-x)^2} - (j+\tfrac{1}{2})g = 0,$$
$$x(2-x)\left[\frac{dg}{dx} - \frac{\lambda g}{(1-x)^2}\right] + (2\xi + 1 - x)g + (j+\tfrac{1}{2})f = 0. \tag{6.4}$$

We now look for a power-series solution in the form

$$f = \sum_{n=0}^{\infty} x^n f_n, \quad g = \sum_{n=0}^{\infty} x^n g_n. \tag{6.5}$$

We find that Eqs. (6.4) are satisfied if we take

$$f_n = g_n = 0 \quad (n < 0), \quad f_0 = -2(\xi + \tfrac{1}{2}), \quad g_0 = (j+\tfrac{1}{2}),$$
$$f_{n+1} = \frac{1}{n+1}[(2n-\lambda)f_n - (n-1)f_{n-1} + (j+\tfrac{1}{2})(g_n - 2g_{n-1} + g_{n-2})]$$
$$g_{n+1} = \frac{1}{2n+2\xi+3}[(5n+4\xi+3+2\lambda)g_n - (4n+2\xi-1+\lambda)g_{n-1} + (n-1)g_{n-2} - (j+\tfrac{1}{2})(f_{n+1} - 2f_n + f_{n-1})], \quad n \geq 0. \tag{6.6}$$

(2) Solution ψ_2 regular around $\theta = \frac{1}{2}\pi$. In this case we make the substitution

$$a_2 = \left(\frac{\cos\theta}{1+\sin\theta}\right)^{j+1/2}[h(y) + \cos\theta\, l(y)],$$
$$c_2 = \left(\frac{\cos\theta}{1+\sin\theta}\right)^{j+1/2}[h(y) - \cot\theta\, l(y)], \tag{6.7}$$
$$y = 1 - \frac{1}{\sin\theta}.$$

The coupled differential equations now become

$$\frac{dh}{dy} - \xi l - \frac{\lambda}{(1-y)^2}\,l = 0,$$
$$y(2-y)\frac{dl}{dy} + [2(j+1) - y]\,l + \left[\xi + \frac{\lambda}{(1-y)^2}\right]h = 0. \tag{6.8}$$

Assuming power-series solutions in the form

$$h = \sum_{n=0}^{\infty} h_n y^n, \quad l = \sum_{n=0}^{\infty} l_n y^n, \tag{6.9}$$

we find the solutions

$$h_n = l_n = 0 \quad (n<0), \quad h_0 = -2(j+1), \quad l_0 = \xi + \lambda,$$

$$h_{n+1} = \frac{1}{n+1}[2nh_n - (n-1)h_{n-1} + (\xi+\lambda)l_n + \xi(l_{n-2} - 2l_{n-1})], \tag{6.10}$$

$$l_{n+1} = \frac{1}{2n+2j+4}[(5n+4j+5)\,l_n - (4n+2j)\,l_{n-1} + (n-1)\,l_{n-2} - (\lambda+\xi)h_{n+1} + \xi(2h_n - h_{n-1})], \quad n \geq 0.$$

A number of observations about the above solutions are now in order. First, we note that since

$$\frac{\partial f_0}{\partial\lambda} = \frac{\partial g_0}{\partial\lambda} = \frac{\partial h_0}{\partial\lambda} = 0, \tag{6.11}$$

and since l in Eq. (6.7) appears multiplied by the factor $\cot\theta$, which vanishes at $\theta = \frac{1}{2}\pi$, the standardization conditions of Eq. (4.7) are satisfied. Second, we consider the greatly simplified form of the above equations when $j \to -\frac{1}{2}$. Working directly from Eq. (6.2) we find in this special limit the decoupled equations

$$\begin{aligned} &\frac{da}{d\theta} - [\xi(\sin\theta)^{-1} + \lambda\sin\theta]\,a = 0, \\ &\frac{dc}{d\theta} + [\xi(\sin\theta)^{-1} + \lambda\sin\theta]\,c = 0, \end{aligned} \tag{6.12}$$

which can be immediately integrated, giving

$$\begin{aligned} a_1 &= -2(\tan\tfrac{1}{2}\theta)^{\xi}(\xi+\tfrac{1}{2})\,e^{\lambda(1-\cos\theta)}, \\ c_1 &= 0, \\ a_2 &= -(\tan\tfrac{1}{2}\theta)^{\xi}\,e^{-\lambda\cos\theta}, \\ c_2 &= -(\tan\tfrac{1}{2}\theta)^{-\xi}\,e^{\lambda\cos\theta}. \end{aligned} \tag{6.13}$$

Hence the Wronskian is

$$\begin{aligned} w(\lambda) &= a_2 c_1 - a_1 c_2 \\ &= -2(\xi+\tfrac{1}{2})\,e^{\lambda}, \end{aligned} \tag{6.14}$$

giving for the $j \to -\frac{1}{2}$ limit of the Fredholm determinant the result

$$\frac{\Delta_{\xi\,-1/2}(\lambda)}{\Delta_{\xi\,-1/2}(0)} = e^{\lambda}, \tag{6.15}$$

as was stated in Sec. V A above.

Finally, we discuss the convergence properties of the power-series solutions. Rewriting Eq. (6.4) as a single second-order differential equation we find singular points at $x = 1$, 2, and ∞. Rewriting Eq. (6.8) as a single second-order equation we find singular points at $y = 1$, 2, and ∞, and additionally at

$$y = 1 \pm \left(\frac{-\lambda}{\xi}\right)^{1/2}. \tag{6.16}$$

Since x and y are related by

$$\frac{1}{(1-x)^2} + \frac{1}{(1-y)^2} = 1, \tag{6.17}$$

Eq. (6.16) corresponds to singular points in the x variable at

$$x = 1 \pm \left(1 + \frac{\xi}{\lambda}\right)^{-1/2}, \tag{6.18}$$

which did not appear in the x form of the equation. Hence the singularities in Eq. (6.16) must be removable, and a direct calculation shows this to be the case. We conclude, then, that the power-series solutions for ψ_1 and ψ_2 have the following regions of convergence:

$$\begin{aligned} &\psi_1 \text{ converges for } |x|<1 \Longleftrightarrow 1 \geq \cos\theta > \tfrac{1}{2} \Longrightarrow 0 \leq \theta < \tfrac{1}{3}\pi, \\ &\psi_2 \text{ converges for } |y|<1 \Longleftrightarrow 1 \geq \sin\theta > \tfrac{1}{2} \Longleftrightarrow \tfrac{1}{6}\pi < \theta \leq \tfrac{1}{2}\pi. \end{aligned} \tag{6.19}$$

Thus, in the angular range $\frac{1}{6}\pi < \theta < \frac{1}{3}\pi$ both power series are convergent, and so we can calculate the Wronskian from Eq. (4.3) by taking θ to be any value in this interval. Since the Wronskian is θ-independent, a powerful check on both the programming and the absence of serious roundoff and truncation errors is obtained by calculating W for two different values of θ in the allowed range and then checking that the same answer is obtained. In practice, using double precision on an IBM 360/91, we found we were able to explore the region $\xi \leq 80$, $j \leq 80$, $|\lambda| \leq 20$ in good detail, but for $|\lambda|$ values between 20 and 24, serious roundoff errors started to set in.

B. Low-lying zeros of $\Delta_{\xi j}(\lambda)$

Numerical results for the low-lying zeros of $\Delta_{\xi j}(\lambda)$ in the upper half plane are given in Tables II and III. In Table II we give the locations of the lowest zero (the zero of smallest magnitude $|\lambda|$) for a range of values of ξ and j. In Table III we give the locations of the lowest four zeros for $\xi = j = \frac{1}{2}$. For all of the zeros tabulated, the ratio $|\mathrm{Im}\lambda|/|\mathrm{Re}\lambda|$ is larger than 1. As ξ increases for fixed j, the ratio appears to be approaching 1 from above; as j increases for fixed ξ, the ratio grows, as might be expected from the inequality of Eq. (3.12). For a given ξ, j, the successive higher zeros move up in the imaginary direction with a spacing $\sim\pi$ between the imaginary parts, as is expected from the WKB estimate of Eq. (5.5). The pattern of the numerical results strongly suggests that $|\mathrm{Im}\lambda|/|\mathrm{Re}\lambda| > 1$ for all zeros of

$\Delta_{\xi j}(\lambda)$. If this property were true, the zero-free regions of $\tilde{\Delta}[A]$ would be as indicated in Fig. 4, and a contour of integration in Eq. (1.5) initially along the real axis could be freely deformed to the positions indicated as "# 1" and "# 2." The first (second) contour allows analytic continuation of W_1 into the entire upper (lower) e^2 half plane. Hence, for the distribution of zeros of $\tilde{\Delta}[A]$ shown in Fig. 4 one gets a radiative-corrected vacuum amplitude W_1 which is analytic in the entire e^2 plane except for a branch cut running along the negative real axis from 0 to $-\infty$.

C. Behavior of vacuum amplitude for λ imaginary

As we have stressed repeatedly above, the possibility of taking the contour in Eq. (1.5) to lie along the imaginary axis can be realized only if $W^{(0)}$ decreases as a Gaussian (or faster) as λ becomes infinite along the imaginary axis. Actually, when subtractions are taken into account, the relevant question becomes whether $(d/d\lambda^2)^2 \ln\tilde{\Delta}[A]$ decreases along the imaginary axis. The differentiations just eliminate the arbitrary subtraction polynomial $Q(\lambda)$ which appears in Eq. (5.10); this polynomial is not relevant to the physics, and specifically is not present if we consider (in the one-mode approximation for virtual photons) the set of single-fermion-loop vacuum polarization diagrams shown in Fig. 5. In order to obtain good convergence of the sum over separation parameters ξ, j, we found it necessary to differentiate once more with respect to λ^2. Multiplying (for convenience) by λ^2, we get, finally, as the quantity being studied

$$\tilde{W}^{(0)}(\lambda) \equiv \lambda \frac{d}{d\lambda}\left(\frac{1}{\lambda}\frac{d}{d\lambda}\right)^2 \ln\tilde{\Delta}[A]. \tag{6.20}$$

Results for $\tilde{W}^{(0)}$ versus $-i\lambda$ are shown in Fig. 6. In calculating the points for this curve, we summed on ξ from $\frac{1}{2}$ to $2\frac{1}{2}$ and on j from $\frac{1}{2}$ to $39\frac{1}{2}$; doubling both summation ranges for a subset of the points produced a 6% change for $-i\lambda = 1$ and negligible (< 1%) change for $-i\lambda \geq 5$. In fact, nearly all of the sum for $-i\lambda \geq 5$ came from $\Delta_{\xi j}$'s with $\xi = \frac{1}{2}$, most likely a result of the fact that this is the value of ξ which gives zeros of $\Delta_{\xi j}$ lying closest to the imaginary axis (see Table II). The curve plotted shows no sign of a rapid decrease, but unfortunately the distortions in both the envelope of the oscillations and the wave form suggest that the asymptotic region has not been reached, and so the results are inconclusive. We did not attempt to extend the computations further, because of the roundoff error problem mentioned above.

TABLE II. Lowest-lying zero with Im$\lambda > 0$ for various ξ, j values.

j	ξ	Reλ_1	Imλ_1	$\lvert$Im$\lambda_1\rvert / \lvert$Re$\lambda_1\rvert$
$\frac{1}{2}$	$\frac{1}{2}$	−1.67	7.12	4.26
$\frac{1}{2}$	$\frac{3}{2}$	−3.47	7.36	2.12
$\frac{1}{2}$	$\frac{7}{2}$	−6.46	8.50	1.32
$\frac{1}{2}$	$\frac{11}{2}$	−9.87	12.57	1.27
$\frac{1}{2}$	$\frac{15}{2}$	−13.25	16.65	1.26
$\frac{1}{2}$	$\frac{1}{2}$	−1.67	7.12	4.26
$\frac{3}{2}$	$\frac{1}{2}$	−1.43	8.93	6.24
$\frac{5}{2}$	$\frac{1}{2}$	−1.14	7.73	6.78
$\frac{7}{2}$	$\frac{1}{2}$	−1.10	9.71	8.83
$\frac{9}{2}$	$\frac{1}{2}$	−1.08	11.70	10.83
$\frac{11}{2}$	$\frac{1}{2}$	−1.17	16.60	14.19
$\frac{13}{2}$	$\frac{1}{2}$	−1.14	18.59	16.31
$\frac{1}{2}$	$\frac{1}{2}$	−1.67	7.12	4.26
$\frac{3}{2}$	$\frac{3}{2}$	−2.94	6.27	2.13
$\frac{7}{2}$	$\frac{7}{2}$	−6.13	10.98	1.79
$\frac{11}{2}$	$\frac{11}{2}$	−9.86	18.67	1.89

VII. OPEN QUESTIONS

We conclude by giving a brief recapitulation of the remaining unresolved questions. Within the framework of the one-mode approximation discussed at great length above, some key problems are:

(i) determining the asymptotic behavior of $\tilde{W}^{(0)}(\lambda)$ along the imaginary axis (ruling out a Gaussian decrease would rule out the imaginary contour possibility and hence, as discussed in Ref. 2, would rule out the possibility of obtaining a coupling-constant eigenvalue when only a finite number of photon modes are included),

(ii) proving (or disproving) the distribution of zeros illustrated in Fig. 4,

(iii) if Fig. 4 is correct, finding a simple formula or interpretation for the discontinuity of W_1

TABLE III. First four zeros for $\xi = j = \frac{1}{2}$ with Im$\lambda > 0$. (For each there is a corresponding complex-conjugate zero in the lower half plane.)

Zero number k	Reλ_k	Imλ_k	$\lvert$Im$\lambda_k\rvert / \lvert$Re$\lambda_k\rvert$	Imλ_k − Imλ_{k-1}
1	−1.67	7.12	4.26	
2	−1.86	10.23	5.50	3.11
3	−1.99	13.39	6.73	3.16
4	−2.10	16.52	7.87	3.13

across its cut in the e^2 plane, and

(iv) finding a compact expression for $\Delta_{\xi j}(\lambda)$ in which the parameter θ in the Wronskian has been explicitly eliminated.

Going beyond the one-mode problem to the case when a finite number of photon modes are present, one can ask whether the zero-free regions shown in Fig. 4 persist.[10,11] If so, then the real contour would give cut-plane analyticity in e^2 for any finite number of modes, and the important (and undoubtedly difficult) question of what happens when the limit to an infinite number of modes is taken would be brought to the fore.

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APPENDIX A: FREE GREEN'S FUNCTION

We give here a closed-form expression for the Green's function of Eq. (4.5) in the free ($\lambda = 0$) case. The result is most compactly expressed in terms of the Jacobi functions

$$P_\nu^{(\alpha,\beta)}(z) = \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)\,\Gamma(\alpha+1)} \times F(-\nu,\, \nu+\alpha+\beta+1;\, \alpha+1;\, \tfrac{1}{2}-\tfrac{1}{2}z)\,, \qquad \text{(A1)}$$

where $F(a,b;c;z)$ is the usual hypergeometric function. [The ordinary Jacobi polynomials correspond to the case where ν in Eq. (A1) is a nonnegative integer. We will also use the case where ν is a non-negative half-integer.] We find (for $\xi > 0$)

$$\psi_1^0 = \begin{pmatrix} a_1^0 \\ c_1^0 \end{pmatrix},$$

$$a_1^0 = (\tan\tfrac{1}{2}\theta)^\xi\, P_{j+1/2}^{(\xi-1/2,\,-\xi-1/2)}\left(\frac{1}{\cos\theta}\right),$$

$$c_1^0 = -\tfrac{1}{2}\tan\theta(\tan\tfrac{1}{2}\theta)^\xi\, P_{j-1/2}^{(\xi+1/2,\,-\xi+1/2)}\left(\frac{1}{\cos\theta}\right);$$

$$\psi_2^0 = \begin{pmatrix} a_2^0 \\ c_2^0 \end{pmatrix}, \qquad \text{(A2)}$$

$$a_2^0 = \left(\frac{\cos\theta}{1+\sin\theta}\right)^{j+1/2}\left(1 + \frac{\sin\theta}{\xi}\frac{d}{d\theta}\right) P_\xi^{(j,\,-j-1)}\left(\frac{1}{\sin\theta}\right),$$

$$c_2^0 = \left(\frac{\cos\theta}{1+\sin\theta}\right)^{j+1/2}\left(1 - \frac{\sin\theta}{\xi}\frac{d}{d\theta}\right) P_\xi^{(j,\,-j-1)}\left(\frac{1}{\sin\theta}\right).$$

The Wronskian of the two solutions is easily calculated by taking either the limit $\theta \to 0$ or the limit $\theta \to \frac{1}{2}\pi$, giving

$$w^0 = a_2^0 c_1^0 - a_1^0 c_2^0 = \frac{-\Gamma(j+\xi+1)}{\Gamma(\frac{1}{2})\,\Gamma(\xi+1)\,\Gamma(j+\frac{3}{2})}\,. \qquad \text{(A3)}$$

The free Green's function then immediately follows from the recipe of Eq. (4.5),

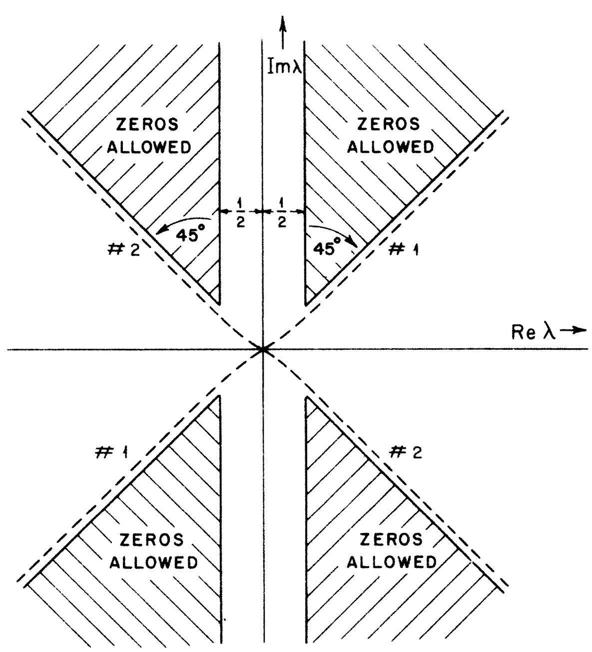

FIG. 4. Conjectured zero-free regions suggested by the numerical results of Sec. VI B and Tables II and III. The dashed lines show permissible deformations of the real-axis contour of integration in Eq. (1.5).

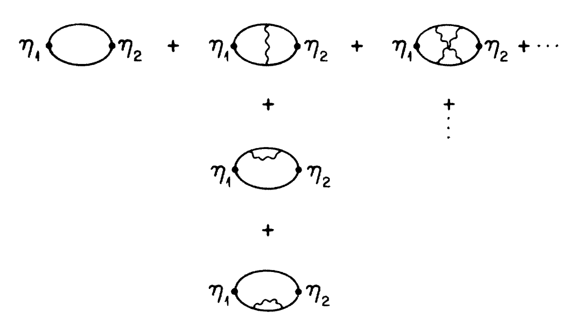

FIG. 5. Single-fermion-loop vacuum-polarization diagrams. This set of diagrams is finite for $\eta_1 \neq \eta_2$, and requires no subtractions. However, if we contract with $Y^{(1)}_{1Ma}(\eta_1)\, Y^{(2)}_{1Mb}(\eta_2)$ and integrate over η_1 and η_2, the short-distance singularity as $\eta_1 \to \eta_2$ leads to a divergence, corresponding to the $A_{\xi j}$ counterterm in Eq. (5.10) and the finite remainder $Q_2\lambda^2$ in Eq. (5.11). This divergence is of no physical significance, and so we differentiate to eliminate Q_2.

$$S^0(\theta_1, \theta_2) = (w^0)^{-1} \times \begin{cases} \psi_1^0(\theta_1)\,\psi_2^{0T}(\theta_2), & \theta_1 < \theta_2 \\ \psi_2^0(\theta_1)\,\psi_1^{0T}(\theta_2), & \theta_1 > \theta_2 \end{cases} . \quad \text{(A4)}$$

We note finally that the solutions ψ_1^0, ψ_2^0 in Eq. (A2) differ by constant factors from the $\lambda = 0$ limit of the power-series solutions for ψ_1, ψ_2 given in Sec. VI.

APPENDIX B: WKB EXPRESSION FOR $\Delta_{\xi j}(\lambda)$

We derive in this Appendix the WKB asymptotic approximation for $\Delta_{\xi j}(\lambda)$ quoted in Sec. V A. Our starting point is the set of coupled differential equations for the components a, c of ψ,

$$\frac{da}{d\theta} - [\xi(\sin\theta)^{-1} + \lambda\sin\theta]\,a + (j+\tfrac{1}{2})(\cos\theta)^{-1}c = 0\,,$$
$$\frac{dc}{d\theta} + [\xi(\sin\theta)^{-1} + \lambda\sin\theta]\,c + (j+\tfrac{1}{2})(\cos\theta)^{-1}a = 0\,. \quad \text{(B1)}$$

These equations are evidently invariant under the interchange

$$\begin{pmatrix} a \\ \xi \\ \lambda \end{pmatrix} \longleftrightarrow \begin{pmatrix} c \\ -\xi \\ -\lambda \end{pmatrix}, \quad \text{(B2)}$$

allowing us to obtain equations satisfied by c by a simple substitution once we have found the corresponding equations satisfied by a. Eliminating c and defining a new variable $x = \cos\theta$, we find the following second-order differential equation satisfied by a ($a' = da/dx$, etc.)

$$a'' + Pa' + Qa = 0\,,$$
$$P = \frac{1-2x^2}{x(1-x^2)}\,,$$
$$Q = \frac{-2\xi\lambda}{1-x^2} - \left[\frac{(j+\frac{1}{2})^2}{x^2(1-x^2)} + \frac{\xi^2}{(1-x^2)^2} + \lambda^2\right] + \frac{\xi}{x(1-x^2)^2} + \frac{\lambda(1-2x^2)}{x(1-x^2)}\,. \quad \text{(B3)}$$

Noting that P is unchanged by the substitution of Eq. (B2), we introduce new dependent variables b and d by writing

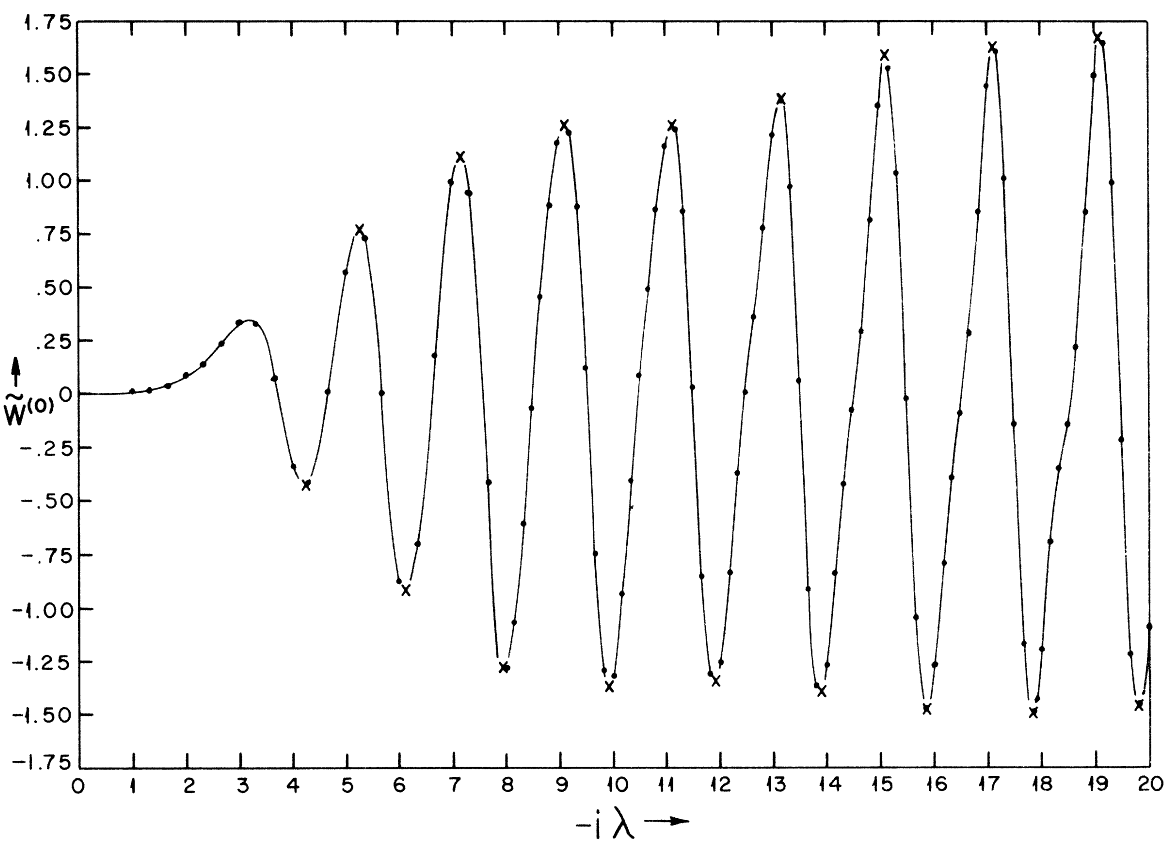

FIG. 6. Results for $\tilde{W}^{(0)}$ versus $-i\lambda$. The dots denote computed points. Maximum and minimum points denoted by × were determined by a polynomial interpolation procedure from the neighboring computed points.

$$a=b\exp\left[-\tfrac{1}{2}\int^x P\,du\right]=bx^{-1/2}(1-x^2)^{-1/4},$$
$$c=d\exp\left[-\tfrac{1}{2}\int^x P\,du\right]=dx^{-1/2}(1-x^2)^{-1/4}. \tag{B4}$$

These satisfy the differential equations

$$\begin{aligned} &b''+k_b^2 b=0, \quad d''+k_d^2 d=0,\\ &k_b^2=t_1+t_2, \quad k_d^2=t_1-t_2,\\ &t_1=\frac{1+2x^2}{4x^2(1-x^2)^2}-\frac{2\xi\lambda}{1-x^2}-\left[\frac{(j+\frac{1}{2})^2}{x^2(1-x^2)}+\frac{\xi^2}{(1-x^2)^2}+\lambda^2\right],\\ &t_2=\frac{\xi}{x(1-x^2)^2}+\frac{\lambda(1-2x^2)}{x(1-x^2)}. \end{aligned} \tag{B5}$$

It is also useful to have the first-order differential equations coupling b and d, which from Eqs. (B1) and (B4) are found to be

$$\begin{aligned} &b'+b\left[\frac{1}{2}\frac{2x^2-1}{x(1-x^2)}+\frac{\xi}{1-x^2}+\lambda\right]-d\frac{(j+\frac{1}{2})}{x(1-x^2)^{1/2}}=0,\\ &d'+d\left[\frac{1}{2}\frac{2x^2-1}{x(1-x^2)}-\frac{\xi}{1-x^2}-\lambda\right]-b\frac{(j+\frac{1}{2})}{x(1-x^2)^{1/2}}=0. \end{aligned} \tag{B6}$$

Finally, in terms of b and d the Wronskian is given by

$$w=a_2c_1-a_1c_2=\frac{1}{x(1-x^2)^{1/2}}(b_2d_1-b_1d_2). \tag{B7}$$

We now proceed to construct approximate, WKB solutions to the above equations when $|\lambda|$ is treated as a large parameter. We begin with the equation for b. We have

$$k_b^2=-\lambda^2+\lambda\left[\frac{-2\xi}{1-x^2}+\frac{1-2x^2}{x(1-x^2)}\right]+O(1), \tag{B8}$$

and hence

$$R\equiv\left|\frac{dk_b/dx}{k_b^2}\right|\ll 1 \tag{B9}$$

for all x except very near the end points at $x=0,1$. Near the end points we find

$$R\sim\frac{1}{2x^2|\lambda|^2},\quad x\approx 0$$
$$R\sim\frac{2\xi+1}{4(1-x)^2|\lambda|^2},\quad x\approx 1 \tag{B10}$$

and so except in the intervals

$$x\approx\frac{1}{|\lambda|},\quad 1-x\approx\frac{\xi^{1/2}}{|\lambda|} \tag{B11}$$

we can use a WKB solution for b and d. Applying the standard lowest-order WKB recipe[12] to the second-order differential equations for b and d, and then imposing the linear equations in Eqs. (B6) which relate b and d, we find the WKB-region solutions

$$\begin{aligned} &b_{\rm WKB}\approx\frac{A}{\lambda}e^{\lambda x}x^{-1/2}(1+x)^{(\xi-1/2)/2}(1-x)^{-(\xi+1/2)/2}+Be^{-\lambda x}x^{1/2}(1+x)^{-(\xi-1/2)/2}(1-x)^{(\xi+1/2)/2},\\ &d_{\rm WKB}\approx\frac{-(j+\frac{1}{2})B}{2\lambda}e^{-\lambda x}x^{-1/2}(1+x)^{-(\xi+1/2)/2}(1-x)^{(\xi-1/2)/2}+\frac{2A}{j+\frac{1}{2}}e^{\lambda x}x^{1/2}(1+x)^{(\xi+1/2)/2}(1-x)^{-(\xi-1/2)/2}. \end{aligned} \tag{B12}$$

In the end-point regions $x\sim 0, 1$ we must join Eq. (B12) on to more accurate approximate solutions. In the vicinity of the end points we find

$$\begin{aligned} &k_b^2=\frac{-j(j+1)}{x^2}+\frac{\lambda+\xi}{x}-(\lambda+\xi)^2+\tfrac{3}{4}-j(j+1)+H_0x+O(x^2),\quad H_0=-\lambda+2\xi,\quad x\approx 0\\ &k_b^2=\frac{-\frac{1}{4}(\xi-\frac{3}{2})(\xi+\frac{1}{2})}{(1-x)^2}-\frac{\xi\lambda+\frac{1}{2}\lambda-\frac{1}{4}\xi+\frac{1}{4}(\xi-\frac{3}{2})(\xi+\frac{1}{2})+\frac{1}{2}j(j+1)}{1-x}\\ &\qquad-\tfrac{5}{4}[j(j+1)-\tfrac{3}{4}]-[\lambda+\tfrac{1}{4}(\xi-\tfrac{5}{2})]^2-\tfrac{1}{8}(\xi-\tfrac{3}{2})^2-\tfrac{1}{8}+H_1(1-x)+O((1-x)^2),\\ &\qquad H_1=-\tfrac{17}{8}j(j+1)+\xi-\tfrac{1}{8}(\xi+\tfrac{3}{2})(\xi-\tfrac{1}{2})+\tfrac{9}{8}\lambda-\tfrac{1}{4}\xi\lambda,\quad x\approx 1. \end{aligned} \tag{B13}$$

For $x\approx j/|\lambda|$, the x^{-2}, x^{-1}, x^0 terms near $x=0$ are of order $|\lambda|^2$, whereas the term H_0x is of order $j\xi$, down by a factor $(j/|\lambda|)(\xi/|\lambda|)$ from the leading terms. Similarly, for $1-x\approx\xi/|\lambda|$, the $(1-x)^{-2}$, $(1-x)^{-1}$, $(1-x)^0$ terms near $x=1$ are of order $|\lambda|^2$, with the term $H_1(1-x)$ of order ξ^2, down by a factor of $(\xi/|\lambda|)^2$ from the leading terms. The terms $O(x^2)$ and $O((1-x)^2)$ can be shown to be as small as the linear terms which we have just evaluated. Hence we identify

$$\epsilon_1=\xi/|\lambda|,\quad \epsilon_2=j/|\lambda| \tag{B14}$$

as the effective smallness parameters in the WKB solution, and proceed to solve the differential equations at the endpoints neglecting the linear and higher terms in x and $1-x$ in Eqs. (B13). Both at $x=0$ and $x=1$, the differential equations can then be reduced to Whittaker's equation

$$\frac{d^2b}{dz^2}+\left(-\tfrac{1}{4}+\frac{\kappa}{z}+\frac{\tfrac{1}{4}-\mu^2}{z^2}\right)b=0, \tag{B15}$$

with the regular solution

$$b=e^{-z/2}z^{1/2+\mu}\Phi(\tfrac{1}{2}+\mu-\kappa,\ 1+2\mu;\ z), \tag{B16}$$

where Φ is the confluent hypergeometric function

$$\Phi(a,c;z)=1+\frac{a}{c}\,\frac{z}{1!}+\frac{a(a+1)}{c(c+1)}\,\frac{z^2}{2!}+\cdots$$
$$=e^z\,\Phi(c-a,\ c;\ -z). \tag{B17}$$

Carrying out the solutions explicitly, we find to the required accuracy the following end-point solutions:

$$\begin{aligned}
&\theta\approx 0,\ x\approx 1:\\
&b_1(z)=\exp\{-[\lambda+\tfrac{1}{4}(\xi-\tfrac{5}{2})]z\}z^{(\xi+1/2)/2}\Phi\left(\xi+\tfrac{1}{2}+\frac{1}{4\lambda}(j+\tfrac{1}{2})^2,\ \xi+\tfrac{1}{2};\ 2[\lambda+\tfrac{1}{4}(\xi-\tfrac{5}{2})]z\right),\\
&d_1(z)=\frac{-(j+\frac{1}{2})}{2^{1/2}(\xi+\frac{1}{2})}\exp\{-[\lambda+\tfrac{1}{4}(\xi+\tfrac{5}{2})]z\}z^{(\xi+3/2)/2}\Phi(\xi+\tfrac{1}{2},\ \xi+\tfrac{3}{2};\ 2[\lambda+\tfrac{1}{4}(\xi+\tfrac{5}{2})]z),\qquad z=1-x;\\
&\theta\approx\tfrac{1}{2}\pi,\ x\approx 0:\\
&b_2(x)=e^{-(\lambda+\xi)x}x^{j+1}\Phi(j+\tfrac{1}{2},\ 2(j+1);\ 2(\lambda+\xi)x),\\
&d_2(x)=e^{-(\lambda+\xi)x}x^{j+1}\Phi(j+\tfrac{3}{2},\ 2(j+1);\ 2(\lambda+\xi)x).
\end{aligned} \tag{B18}$$

Joining the WKB-region solution onto the asymptotic form[13] of the $x\approx 0$ end-point solution, we determine the constants A, B in Eq. (B12) to be

$$\begin{aligned}
\frac{2A}{j+\frac{1}{2}}&=\frac{\Gamma(2(j+1))}{\Gamma(j+\frac{3}{2})}(2\lambda)^{-(j+1/2)},\\
B&=\frac{\Gamma(2(j+1))}{\Gamma(j+\frac{3}{2})}\left(\frac{-1}{2\lambda}\right)^{j+1/2}.
\end{aligned} \tag{B19}$$

This permits us to extend the solution ψ_2 to the region near $\theta\approx 0$, $x\approx 1$, which is the asymptotic region for the $x\approx 1$ end-point solution ψ_1. Substituting the WKB extension of ψ_2 and the asymptotic expansion of ψ_1 into Eq. (B7), we get for the Wronskian

$$\begin{aligned}
w(\lambda)=-\frac{\Gamma(2(j+1))}{\Gamma(j+\frac{3}{2})}&2^{(\xi-1/2)/2}(2\lambda)^{-(j+1/2)}\\
&\times[e^{\lambda}+e^{-\lambda}(-1)^{j+\xi+1}2^{-2(\xi+1/2)}\\
&\quad\times(j+\tfrac{1}{2})\Gamma(\xi+\tfrac{1}{2})\lambda^{-(\xi+1/2)}].
\end{aligned} \tag{B20}$$

To complete the calculation, we must determine the value $w(0)$ corresponding to the normalization of the solutions ψ_1, ψ_2 used in the above analysis. This is most easily done by a comparison with the explicit free solutions given in Appendix A. Writing

$$\begin{aligned}
\begin{pmatrix}a_1\\c_1\end{pmatrix}_{\lambda=0}&=K_1\begin{pmatrix}a_1^0\\c_1^0\end{pmatrix},\\
\begin{pmatrix}a_2\\c_2\end{pmatrix}_{\lambda=0}&=K_2\begin{pmatrix}a_2^0\\c_2^0\end{pmatrix},
\end{aligned} \tag{B21}$$

and letting $\theta\to 0,\ \frac{1}{2}\pi$ to determine K_1, K_2, respectively, we find from Eqs. (B4) and (B18) that

$$\begin{aligned}
K_1&=\frac{2^{(\xi-1/2)/2}\Gamma(\xi+\frac{1}{2})\Gamma(j+\frac{3}{2})}{\Gamma(j+\xi+1)},\\
K_2&=\frac{2^{j+1/2}\Gamma(j+1)\Gamma(\xi+1)}{\Gamma(j+\xi+1)}.
\end{aligned} \tag{B22}$$

Combining with Eq. (A3) we then get

$$w(0)=-K_1K_2\frac{\Gamma(j+\xi+1)}{\Gamma(\frac{1}{2})\Gamma(\xi+1)\Gamma(j+\frac{3}{2})}. \tag{B23}$$

Dividing Eq. (B20) by Eq. (B23) to get $w(\lambda)/w(0)$, and then using Eq. (4.17), gives the final WKB formula quoted in Eq. (5.2) of the text.

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[1]S. L. Adler, Phys. Rev. D 6, 3445 (1972); 7, 3821(E) (1973).

[2]S. L. Adler, Phys. Rev. D 8, 2400 (1973).

[3]We can omit the matrix τ_2 in Eq. (2.32) because the spinors which appear have already been reduced to four-component form.

[4]There is, of course, a third regular singular point at $u=\infty$. For a discussion of the Riemann equation and its solution see G. Birkhoff and G. C. Rota, *Ordinary Differential Equations* (Blaisdell-Ginn, Waltham, Mass., 1969), p. 272 ff.

[5]These may be derived from the identities on pp. 274–276 of Y. L. Luke, *The Special Functions and Their Approximations* (Academic, New York, 1969), Vol. 1.

[6]See, for example, G. Birkhoff and G. C. Rota, *Ordinary Differential Equations* (Ref. 4), p. 47.

[7]It is always possible to find solutions satisfying the standardization conditions because, as stressed in Sec. II D, the boundary conditions at $\theta = 0, \frac{1}{2}\pi$ are λ-independent.

[8]Let tr denote the Pauli matrix trace; then Tr denotes the complete trace $\mathrm{Tr}A = \int_0^{\pi/2} d\theta \langle\theta|\mathrm{tr}A|\theta\rangle$.

[9]A. S. B. Holland, *Introduction to the Theory of Entire Functions* (Academic, New York, 1973). See especially Sec. 1.4, Chap. 4, and Sec. 6.2.

[10]S. Coleman (unpublished) has conjectured this to be the case. Coleman argues that at the 45° sector boundaries in Fig. 4, $\mathrm{Re}(\lambda^2)$ changes sign from positive to negative, corresponding to a transition from "magnetic-field-like" to "electric-field-like" behavior of the external-field problem, and suggesting very different analyticity properties on the two sides of the boundary.

[11]A. S. Wightman (unpublished) has proved, in the Minkowski metric case, that the Fredholm determinant can have no zeros for arbitrary purely real external fields.

[12]See, for example L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), third edition, pp. 270–271.

[13]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, p. 278.

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Exact diagonalization of the Dirac Hamiltonian in an external field

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The Hamiltonian of a Dirac particle in an arbitrary electromagnetic field is exactly diagonalized by a unitary transformation generalizing previous work which was restricted to time-dependent fields. A very simple form is found for the covariant Heisenberg equations which manifestly exhibits the classical correspondence. These results are obtained in a manifestly covariant form using a previously proposed proper-time quantum mechanics with subsequent specialization to a mass eigenstate resulting in the usual theory. The simple theorem used for this diagonalization is also applied to other transformations for helicity and the free-particle Hamiltonian. The source of difficulty in obtaining these results without an intermediate use of proper-time theory is shown.

I. INTRODUCTION

Several interpretational aspects of the free Dirac equation were clarified in the classic paper by Foldy and Wouthuysen[1] in which a unitary transformation was found which diagonalized the Dirac Hamiltonian with respect to positive and negative energies. The application of this transformation to the basic operators of position, momenta, orbital angular momenta, and spin exhibited a separation in the new representation into classical and nonclassical portions. The classical terms obeyed Heisenberg equations formally resembling the equations of classical mechanics, while the nonclassical terms exhibited a rapid oscillatory motion about the classical values *(zitterbewegung)*. When electromagnetic interactions were included, the transformation could not be obtained in closed form. Thus the classical separation could not be effected and the Heisenberg equations were not studied. Furthermore, the general approach was noncovariant. Subsequent work by Eriksen[2] has shown a closed form for the transformation when the electromagnetic field is time-independent and is free of a scalar potential. Chakrabarti[3] has studied a covariant diagonalization, but dealt only with free particles. A general review of these and associated problems can be found in the work of de Vries.[4]

This paper addresses three problems: First, is there a manifestly covariant generalization of the Foldy-Wouthuysen transformation? Second, can this procedure be extended covariantly to include arbitrary electromagnetic interactions in closed form? Third, can a covariant form of the Heisenberg equations be found which explicitly shows the classical form even with an interaction present? An affirmative answer to these questions can be given in the context of a proper-time quantum mechanics which as been previously proposed by one of the authors.[5] Although we utilize the proper-time approach to maintain covariance, the results can be immediately specialized to the usual theory by using mass eigenstates.

We find that the covariance appears mandatory for the diagonalization in arbitrary fields. If one uses the noncovariant Hamiltonian $P^0 = \beta m + \vec{\alpha}\cdot\vec{p}$ and performs the replacement $P^\mu \rightarrow P^\mu - eA^\mu$ one encounters the difficulty pointed out by Sucher[6] that the resulting square-root Klein-Gordon equa-