Diagonal coherent-state representation for polarized light

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We consider the transformation properties of the density operator as well as its diagonal coherent-state representation when a plane monochromatic light is passed through a compensator followed by a rotator. Special cases of unpolarized and polarized light are considered and the general form of the density operator and its coherent-state representation is determined.

In studying the state of polarization of a beam of light, one usually passes the beam through a compensator followed by a polarizer and notes the variations in the intensity of the emerging light. In general the intensity of the emergent light depends on the coherency matrix of the light and is given by[1]

$$I(\delta,\theta)=J_{xx}\cos^2\theta+J_{yy}\sin^2\theta+(J_{xy}e^{-i\delta}+J_{yx}e^{i\delta})\sin\theta\cos\theta\,, \tag{1}$$

where δ is the phase difference introduced by the compensator between the two orthogonal x and y components of the wave field and θ is the angle which the polaroid axis makes with the x axis. J_{ij} is the coherency matrix. We say that the light is unpolarized if one finds that there is no intensity variation when an arbitrary phase change is introduced between the two mutually orthogonal polarization components and the polaroid is then rotated through an arbitrary angle, i.e., when $I(\delta,\theta)$ is a constant and does not depend on θ or δ. Similarly we say that the light is completely polarized if one finds that $I(\delta,\theta)$ is zero for some values of δ and θ. This corresponds to the case when the determinant of J vanishes,

$$J_{xx}J_{yy}-|J_{xy}|^2=0\,. \tag{2}$$

In place of intensity variations, Prakash and Chandra[2] generalized the concept and defined the unpolarized radiation as one when the statistical properties remain unchanged when light is passed through a compensator followed by a rotator. In the present paper, we consider an analogous definition for a completely polarized radiation and study its consequences in terms of the density operator of the radiation field.

Let us consider a plane monochromatic beam of light with two orthogonal polarization modes. Let $a_1, a_1^\dagger$ and $a_2, a_2^\dagger$ be the corresponding pairs of annihilation and creation operators and ρ be the density operator. The action of a compensator can be thought of as a unitary transformation on the density operator

$$\rho_c=U_c^\dagger\rho U_c\,, \tag{3}$$

where

$$U_c=\exp(i\theta_1a_1^\dagger a_1+i\theta_2a_2^\dagger a_2). \tag{4}$$

The unitary transformation (4) corresponds to a phase change of θ_1 in the first polarization mode and a phase change of θ_2 in the second polarization mode. For a stationary field ρ commutes with the total number operator $a_1^\dagger a_1+a_2^\dagger a_2$ and hence ρ_c will depend only on the phase retardation

$$\delta=\theta_2-\theta_1\,, \tag{5}$$

and

$$\rho_c=e^{i\delta a_1^\dagger a_1}\rho e^{-i\delta a_1^\dagger a_1}\,, \tag{6}$$

so that we may effectively write

$$U_c(\delta)=e^{-i\delta a_1^\dagger a_1}\,. \tag{7}$$

In a similar manner the action of a rotator may be represented by a unitary transformation

$$\rho_r=U_r^\dagger\rho U_r\,, \tag{8}$$

where

$$U_r=\exp[\theta(a_1^\dagger a_2-a_2^\dagger a_1)]\,. \tag{9}$$

If the density operator ρ commutes with both U_c and U_r we say that light is unpolarized.[3] By taking the matrix elements between the number states, one may readily show that in this case ρ must be of the form[4]

$$\rho=\sum_m\sum_n\rho_{m+n}|m,n\rangle\langle m,n|\,. \tag{10}$$

We may also write

$$\rho_{m+n}|m,n\rangle=\rho(a_1^\dagger a_1+a_2^\dagger a_2)|m,n\rangle\,, \tag{11}$$

and since the number states form a complete set,

$$\sum_m\sum_n|m,n\rangle\langle m,n|\equiv1\,,$$

we find that the necessary and sufficient condition for a density operator to represent unpolarized radiation is that it is a function of the total number operator,[5]

$$\rho=\rho(a_1^\dagger a_1+a_2^\dagger a_2)\ . \tag{12}$$

We now consider the case of completely polarized light. We define completely polarized light as that in which one of the modes can be brought to the ground state by a suitable phase change between the two orthogonal polarization components followed by a certain rotation. This implies that it is possible to shut off light completely by passing it through a suitable compensator followed by a suitably oriented polaroid. Thus, for a light to be completely polarized, there exists some δ and θ such that

$$U_r^\dagger(\theta)U_c^\dagger(\delta)\rho U_c(\delta)U_r(\theta)=\rho_1(a_1^\dagger, a_1)\rho_{20}\ , \tag{13}$$

where ρ_{20} denotes the density operator for the ground state of the second mode,

$$\rho_{20}=|0\rangle\langle 0| =:e^{-a_2^\dagger a_2}:\ . \tag{14}$$

The colons on the right-hand side of (14) denote the normal-ordering operation. Further, since the density operator must commute with the total number operator $a_1^\dagger a_1+a_2^\dagger a_2$, we find that the functional dependence on $a_1^\dagger$ and a_1 must be through $a_1^\dagger a_1$. We may therefore write

$$U_r^\dagger(\theta)U_c^\dagger(\delta)\rho U_c(\delta)U_r(\theta)=\rho_1(a_1^\dagger a_1):e^{-a_2^\dagger a_2}:\ , \tag{15}$$

or

$$\rho=U_c(\delta)U_r(\theta)\rho_1(a_1^\dagger a_1):e^{-a_2^\dagger a_2}:U_r^\dagger(\theta)U_c^\dagger(\delta)\ . \tag{16}$$

Equation (16) represents the general form of the density operator for a completely polarized radiation.

Before we consider the general form of the diagonal coherent-state representation of a certain state of polarization, let us consider how the diagonal representation varies when light is passed through a compensator followed by a rotator. Let $\phi(z_1, z_2)$ be the initial representation:

$$\rho=\int\int \phi(z_1,z_2)|z_1,z_2\rangle\langle z_1,z_2|\,d^2z_1d^2z_2\ . \tag{17}$$

From Eqs. (6) and (8) we then find that

$$\rho_{cr}=\int\int\phi(z_1,z_2)U_r^\dagger U_c^\dagger|z_1,z_2\rangle\langle z_1,z_2|U_c\,U_r d^2z_1d^2z_2\ . \tag{18}$$

We now write

$$|z_1z_2\rangle\langle z_1z_2|=D_1(z_1)D_2(z_2)|0,0\rangle\langle 0,0|D_1^\dagger(z_1)D_2^\dagger(z_2)\ , \tag{19}$$

where

$$D_k(z)=e^{(a_k^\dagger z-a_k z^*)}\ , \tag{20}$$

and note that

$$U_r^\dagger U_c^\dagger|0,0\rangle\langle 0,0|U_cU_r=|0,0\rangle\langle 0,0|\ , \tag{21}$$

whereas

$$U_r^\dagger U_c^\dagger f(a_1,a_1^\dagger,a_2,a_2^\dagger)U_cU_r =f(e^{-i\delta}(a_1\cos\theta+a_2\sin\theta),e^{i\delta}(a_1^\dagger\cos\theta+a_2^\dagger\sin\theta),(-a_1\sin\theta+a_2\cos\theta),(-a_1^\dagger\sin\theta+a_2^\dagger\cos\theta))\ . \tag{22}$$

On using (21) and (22) in (18) and simplifying we then obtain

$$\rho_{cr}=\int\int\phi(e^{-i\delta}(z_1\cos\theta+z_2\sin\theta),(-z_1\sin\theta+z_2\cos\theta))|z_1,z_2\rangle\langle z_1z_2|\,d^2z_1d^2z_2\ . \tag{23}$$

From (15) and (23) we may now readily determine the most general form of diagonal representation for a completely polarized radiation. Since the diagonal representation of the right-hand side of (15) is of the form

$$\phi_1(|z_1|^2)\delta(z_2)\ ,$$

we must have

$$\phi(e^{-i\delta}(z_1\cos\theta+z_2\sin\theta),(-z_1\sin\theta+z_2\cos\theta)) =\phi_1(|z_1|^2)\delta(z_2)\ , \tag{24}$$

or on making a change of variables, we find that the most general form of the diagonal representation for a completely polarized radiation is given by

$$\phi(z_1,z_2)=\phi(|z_1|^2\sec^2\theta)\,\delta(z_1e^{i\delta}\sin\theta+z_2\cos\theta)\ . \tag{25}$$

The nature of polarization will of course depend on the values of θ and δ. Thus for example if we are considering Cartesian polarization modes and if $\delta=0$ (or π) light is linearly polarized. On the other hand, if $\delta=\pi/2$ and $\theta=\pi/4$ we have circularly polarized radiation, etc.

For thermal radiation the diagonal coherent-state representation is a bivariate complex Gaussian distribution. Noting that the δ function is a limiting case of a Gaussian distribution we find that thermal light is completely polarized if ϕ is of the form

$$\phi_{\text{pol}}(z_1, z_2) = \frac{\lambda}{\pi} \exp(-\lambda |z_1|^2 \sec^2\theta) \times \delta(z_1 e^{i\delta} \sin\theta + z_2 \cos\theta) . \quad (26)$$

On the other hand, completely unpolarized thermal light must be of the form (cf. Ref. 5)

$$\phi_{\text{up}}(z_1, z_2) = \frac{\lambda^2}{\pi^2} \exp[-\lambda(|z_1|^2 + |z_2|^2)] . \quad (27)$$

One may readily verify that the coherency matrix obtained by using (26) or (27) satisfies the properties of completely polarized or completely unpolarized radiation, respectively.

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[1]M. Born and E. Wolf, *Principles of Optics*, 4th edition (Pergamon, Oxford, 1970), p. 547.

[2]H. Prakash and N. Chandra, Phys. Lett. 34A, 28 (1971); Phys. Rev. A 4, 796 (1971).

[3]It is worth noting here that if an operator commutes with $a_1^\dagger a_1$ as well as $a_1^\dagger a_2 - a_2^\dagger a_1$, it must also commute with $a_2^\dagger a_2$, since one has

$$a_2^\dagger a_2 = e^{\pi(a_1^\dagger a_2 - a_2^\dagger a_1)/2}\, a_1^\dagger a_1 e^{-\pi(a_1^\dagger a_2 - a_2^\dagger a_1)/2} .$$

[4]Prakash and Chandra (Ref. 2) have obtained this result after rather laborious calculations.

[5]It is equivalent to saying that the corresponding diagonal coherent-state representation is a function of $|z_1|^2 + |z_2|^2$ only [cf. G. S. Agrawal, Nuovo Cimento Lett. 1, 53 (1971)]:

$$\rho = \int\int \phi(|z_1|^2 + |z_2|^2) |z_1, z_2\rangle \langle z_1, z_2| d^2z_1 d^2z_2 .$$