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PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Gravitational energy*

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(Received 4 March 1974)

The total energy of asymptotically flat, nonsingular gravitational fields is discussed in terms of the initial data on a spacelike hypersurface. The total energy is a surface integral which we relate to a volume integral over "sources," including the contributions of gravitational waves. This relationship follows from a recent formulation of the initial-value equations of general relativity and is free of coordinate conditions. We show that time-symmetric initial-data sets form minima of energy among all initial-data sets on maximal hypersurfaces. Combining this result with a result of Brill, it follows that every nonsingular, axisymmetric, asymptotically flat spacetime admitting at least one maximal slice has non-negative total energy. Negative "interaction energy" contributions are described and a discussion of nonmaximal initial data is given.

I. INTRODUCTION

It is well known that the total energy of an asymptotically flat gravitational field defined on a spacelike hypersurface has two distinctive features: (I) It may always be calculated as an integral over a two-dimensional surface surrounding the sources, including among these gravitational waves; (II) There is, in general, no well-defined local expression for gravitational energy density. These two features are closely related and show that energy is a global rather than a local property of a gravitational field. As a result of (I) the energy of spacetimes with closed spacelike slices may be defined to vanish identically. We shall therefore confine our discussion to nonsingular asymptotically flat spacetimes.

The initial-value data on a spacelike slice form Cauchy data for a spacetime, i.e., define it uniquely and completely for some finite time, and therefore the energy, which is a constant of the motion, should be describable in a natural manner purely in terms of the initial data. This means that we can limit our attention to a given "state" of the gravitational field, defined by the initial data on an asymptotically flat spacelike slice.

Despite the fact that the energy has been studied repeatedly by many authors, a number of its most important features remain unknown. A prime example is whether or not the energy is a positive-definite quantity. Several results exist which strongly suggest that the energy is positive, but a complete proof has yet to be constructed. In this paper we prove a number of results related to the positive-energy question, but we are not able to prove positivity here. We are able, however, to provide in Sec. VI a simple, purely geometrical criterion that suffices for the existence of initial-data sets with negative total energy.

We begin by reviewing in Sec. II the definition of energy given by Arnowitt, Deser, and Misner.[1] Their integral involves explicitly only the intrinsic geometry of the spacelike slice, roughly "half" of the initial data on the slice, but it is assumed that the initial-value equations are satisfied. Viewing this expression in the light of recent work on the initial-value problem[2] leads to the fact that the energy has remarkably simple properties under conformal transformations of the initial data. This leads us to propose a very simple definition of total energy in terms of the asymptotic behavior of the conformal structure of space. We show that this definition is physically natural and general. The basic definition turns out to agree with that proposed by Brill[3] in his work on time-symmetric initial-data sets, but we extend this definition to the general case.[4] Moreover, in the approach presented here, the possibility that "coordinate waves" can affect the energy is manifestly eliminated. This result relies on a certain decomposition of symmetric tensors,[5] by means of which we can also show that every three-metric corresponding to a system with finite mass is conformally related to a metric which becomes asymptotically flat faster than $1/r$, and so contributes nothing to the mass. The results in this paper follow directly and simply from our characterization of the conformal structure of initial-data sets and the Brill-ADM definition of energy.

Using the basic ideas on conformal structure and energy presented in Sec. II, we show in Sec. III that time-symmetric initial-data sets form minima of energy among all data sets defined on maximal spacelike hypersurfaces. (We assume, of course, that the constraints are satisfied.) Combining this result with those of Brill,[3] we obtain the result that every axisymmetric, asymptotically flat spacetime admitting at least one maximal slice has non-negative total energy. The treatment of nonmaximal slices is described in Sec. VI.

The method used in Sec. III is to add momentum to a time-symmetric configuration (vanishing momentum) and to show rigorously that the energy must increase. The "adding" we propose is a mathematical experiment, not a physical one occurring in time, because the total energy is conserved in time. What we are doing is comparing two closely related physical situations, one with momentum and one without, as we explain in the text. In simplistic terms, this result is expected, of course. However, the nonlinearity of gravity suggests that the addition of momentum to an otherwise fixed gravitational configuration can lead to negative as well as positive contributions to the total energy by virtue of the mutual gravitational attraction of the added "parts." One might then wonder whether it might not be possible for these negative contributions to overwhelm the positive ones and lead to a negative total contribution to the energy in some extreme situation. The results of Sec. III show that this cannot happen, at least while we restrict our attention to maximal slices. Of course, we explicitly exclude from consideration any slice which contains a singularity.[6]

In Sec. IV, we derive simple formulas showing precisely how the negative (interaction energy) contributions arise. We show that the total energy is a monotonically increasing function of the "strength" of the momentum. In Sec. V, we give a similar treatment of the increase of energy as nongravitational, locally positive matter/field energy density is "added" to a vacuum configuration. In nearly flat spaces, one then finds a close comparison with Newtonian gravity (in terms of Poisson's equation) together with the added corrections resulting from the nonlinearity of the gravitational field. Section V also contains a treatment of the change of total energy under a constant scaling of the sources.

In Sec. VI, we give a sufficient condition for the existence of negative-energy maximal initial-data sets in terms of a simply stated geometrical criterion. We also discuss properties of the initial-value equations for nonmaximal slices.

II. CONFORMAL PROPERTIES OF ENERGY

Energy and the Hamiltonian

Any gravitational field is fully described by its initial data on a spacelike slice. The initial data are the intrinsic geometry of the slice, i.e., the three-dimensional Riemannian metric g_{ij} of the slice, and a three-dimensional tensor density π^{ij}, which is related to the extrinsic curvature K_{ij} of the slice by

$$\pi^{ij} = \sqrt{g}\,(g^{ij}K - K^{ij})\,. \tag{1}$$

These data cannot be given independently, but must obey four constraints, which in empty space take the form

$$\nabla_j \pi^{ij} = 0\,, \quad \text{momentum constraint} \qquad (2)$$

$$\sqrt{g}\, R = \frac{1}{\sqrt{g}}\,(\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2)\,, \quad \text{Hamiltonian constraint} \qquad (3)$$

where R is the scalar curvature of g_{ij}.

These variables are especially suitable for the construction of Lagrangian and Hamiltonian formulations for general relativity. Starting from the Hilbert action

$$I = \int \sqrt{-{}^{(4)}g}\; {}^{(4)}R\, d^4x\,,$$

and rewriting it in terms of the three-dimensional variables, after removing two irrelevant total divergences,[7] one can express the Lagrangian as

$$L = \int N\sqrt{g}\,(K^{ij}K_{ij} - K^2 + R)\, d^3x\,,$$

where $N = (-\,{}^{(4)}g^{00})^{-1/2}$ is the lapse function.[1] It follows from this Lagrangian that the momentum conjugate to the field variables g_{ij} is π^{ij} defined in (1). It follows that the Hamiltonian can be written as

$$H = \int \left\{ N\left[\frac{1}{\sqrt{g}}\,(\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2) - \sqrt{g}\, R\right] + N_i(-2\nabla_j \pi^{ij}) \right\} d^3x\,,$$

where $N_i = {}^{(4)}g_{0i}$.

This expression vanishes by virtue of the constraints (2) and (3). DeWitt[8] pointed out that this result does not lead to "frozen dynamics" because the Lagrangian contains linearly occurring second derivatives of the field variables. These may be isolated as a total divergence D

$$-D = \int [N\sqrt{g}\; g^{ij}g^{kl}(g_{ik,j} - g_{ij,k})]_{,l}\, d^3x\,, \qquad (4)$$

$$= \oint_\infty N\sqrt{g}\; g^{ij}g^{kl}(g_{ik,j} - g_{ij,k})\, dS_l\,, \qquad (5)$$

$$= \oint_\infty \sqrt{g}\; g^{ij}g^{kl}(g_{ik,j} - g_{ij,k})\, dS_l\,, \qquad (6)$$

where in (6) we assume $N = 1$ at infinity. Therefore, the "real" Hamiltonian H_∞ is obtained after removing this divergence:

$$H_\infty = H - D = H + 16\pi E = 16\pi E\,.$$

This is the quantity that generates the dynamics. The energy E of the gravitational field is the numerical value of H_∞ given by

$$E = \frac{1}{16\pi}\oint_\infty \sqrt{g}\; g^{ij}g^{kl}(g_{ik,j} - g_{ij,k})\, dS_l\,. \qquad (7)$$

If we calculate this integral for the Schwarzschild solution we get the desired result that $E = m$, the Schwarzschild mass. It can be seen that the form of the integrand in (7) is a noncovariant expression that must be evaluated in asymptotically Cartesian coordinates to yield the correct result. Of course, in other asymptotic coordinate systems the form of E will change, while its value will remain invariant. Later in this section we will derive from (7) a coordinate-invariant expression for the energy E [see (18)].

Arnowitt, Deser, and Misner[1] earlier obtained exactly the same expression[7] for the energy by explicitly calculating the generator of time translations at infinity. They also showed that this energy is the quantity that determines the far-field orbits (active gravitational mass), i.e., it is the "Newtonian" mass of the system as seen by an observer who is very far away. These properties require both that the spacelike slice be asymptotically flat and that the slice be properly embedded in a spacetime which is also asymptotically flat. This means that there must exist a coordinate system in which the spacetime metric approaches the Minkowski metric quickly enough, i.e.,

$${}^{(4)}g_{\mu\nu} - \eta_{\mu\nu} = O(1/r). \qquad (8)$$

One also requires an asymptotic bound on ${}^{(4)}g_{\mu\nu,\alpha}$. A possible choice is

$${}^{(4)}g_{\mu\nu,\alpha} = O(1/r^{2+\epsilon}) \text{ for some } \epsilon \geq 0, \qquad (9)$$

but it can be weakened somewhat as we shall see below. These conditions lead to

$$g_{ij} - \delta_{ij} = O(1/r), \quad \pi^{ij} = O(1/r^{2+\epsilon}).$$

We shall later remove the necessity of stating asymptotic conditions in terms of Cartesian coordinates. The asymptotic conformal structure of the initial data is the relevant concept; this is coordinate free.

The energy is effectively the average value of the $1/r$ part of the three-metric at spacelike infinity. ADM[1] showed that this energy is conserved in time and that together with the total momentum of the slice (defined as the generator of spatial translations),[9] it forms a four-vector under Lorentz transformations.[10] An easy way to see the conservation of energy is to consider

$$\partial_t g_{ij} \sim -2NK_{ij} \sim -2K_{ij}\,, \qquad (10)$$

where we have assumed $\nabla_i N_j$ is negligible and $N \to 1$, since we are only interested in the far-field time development of g_{ij} when no asymptotic Lorentz boost is involved. Therefore, so long as K_{ij} (or equivalently π^{ij}) falls off faster than $1/r$, the $1/r$ part of g_{ij}, which contains the energy, is conserved. Below we will propose a definition of energy equivalent to (7) and, in the process, the asymptotic conformal structure of the three-metric will be examined and we will sharpen conditions (8) and (9).

It is obvious from Eq. (7) that the ADM energy depends explicitly only on the intrinsic geometry of the slice, and depends implicitly on the extrinsic geometry π^{ij} only through the fact that the constraints must be satisfied. This fact permits us to compare very simply the energy of two sets of initial data whose intrinsic geometries are conformally related. Equation (7) can be evaluated for any three-metric, of course, but only defines the energy when the metric satisfies the asymptotic conditions and forms part of a valid initial-data set.

Conformal transformation and definition of energy

Let us now compare the total energies of two sets of initial data whose *intrinsic* geometries are conformally related. Therefore, we are given g_{ij}, π^{lm} and g'_{ij}, π'^{lm}, both of which obey the constraints. We shall assume that $g'_{ij} = \phi^4 g_{ij}$, where ϕ is some strictly positive, bounded function which goes to one at infinity. We do not assume that there is any particular relationship between the π^{lm}'s. That is, we know that for a given conformal equivalence class of metrics, there will be many solutions of the constraints, depending on the structure of the π^{lm}'s. The conformal relationship between the three-metrics permits a simple comparison of the total energies. From (7) we get

$$E' = \frac{1}{16\pi} \oint_\infty \sqrt{g'}\, g^{ab\prime} g^{cd\prime}(g'_{bc,d} - g'_{cd,b})\, dS'_a \,, \tag{11}$$

where dS_a is a conformal invariant, $dS'_a = dS_a = (d^2x)_a$. Then

$$\begin{aligned} 16\pi E' &= \oint_\infty \phi^{-2} \sqrt{g}\, g^{ab} g^{cd}[(g_{bc,d} - g_{cd,b})\phi^4 \\ &\qquad + 4\phi^3(\phi_{,d} g_{bc} - \phi_{,b} g_{cd})]\, dS_a \\ &= 16\pi E + 4 \oint_\infty \sqrt{g}\,(1-3)\phi_{,d} g^{ad}\, dS_a \\ &= 16\pi E - 8 \oint_\infty \vec{\nabla}\phi \cdot d\vec{S} \,. \end{aligned} \tag{12}$$

Therefore,

$$16\pi \Delta E = -8 \oint_\infty \vec{\nabla}\phi \cdot d\vec{S} \,, \tag{13}$$

or using Gauss's theorem

$$16\pi \Delta E = -8 \int_V \sqrt{g}\, \nabla^2 \phi\, d^3x \,. \tag{14}$$

A similar expression was obtained by Brill and Deser for the case of infinitesimal conformal mappings.[11] It is especially interesting because the form of ΔE is independent of the choice of coordinates.

This relationship between conformal factor and energy is of particular interest in the case of conformal flatness,[12] i.e.,

$$g_{ij} = f_{ij} \,, \quad g'_{ij} = \phi^4 f_{ij} \,, \tag{15}$$

where f_{ij} is a flat metric expressed in arbitrary coordinates. In this case we can use Eq. (13) or (14) to define the *total* energy of the g'_{ij} system, rather than the change of energy, since the $g_{ij} = f_{ij}$ system has zero energy by definition. For example, we could have $g_{ij} = f_{ij}$, $\pi^{lm} = 0$ and $g'_{ij} = \phi^4 f_{ij}$, $\pi'^{lm} \neq 0$, trace $\pi'^{lm} = 0$. In any event from (15) we obtain

$$16\pi E' = -8 \oint_\infty \vec{\nabla}\phi \cdot d\vec{S} \,, \tag{16}$$

where the integration is carried out in flat space. (In the particular example just cited, one knows that $E' > 0$.[12])

The covariant expression (16) suggests a generalization and a definition of the energy of a *single* initial-data set which is derived from the "comparison" formulas (13) and (14) and which reduces to an expression equivalent to (16). Suppose that for a given solution of the constraints (g_{ij}, π^{ij}), it is true that g_{ij} may be written in the form

$$g_{ij} = \lambda^4[f_{ij} + O(1/r^{1+a})] \,, \qquad a > 0 \,, \quad \lambda - 1 = O(1/r). \tag{17}$$

That is, g_{ij} is conformal to a "base" metric b_{ij}, $g_{ij} = \lambda^4 b_{ij}$ which itself has no $1/r$ part. Of course, in general b_{ij} will not correspond to any solution of the constraints. Moreover, because of the rapid falloff of b_{ij}, all the mass of g_{ij} is thrown into the conformal factor λ^4. Thus, assuming g_{ij} is a solution of the constraints and satisfies (17) for a given b_{ij}, we *define*

$$16\pi E = -8 \oint_\infty \vec{\nabla}\lambda \cdot d\vec{S} \,, \tag{18}$$

where the integration is carried out in the base metric b_{ij}. This is the coordinate-invariant definition alluded to above. It may be considered as a generalization of Brill's definition of E for the case in which $\pi^{ij} = 0$. This definition completely agrees with the Arnowitt-Deser-Misner and with the Landau-Lifshitz definitions of energy when the latter are evaluated in a suitable coordinate frame.

The requirement above says in effect that the three-geometry is asymptotically *conformally* flat. This requirement limits the set of three-geometries on which we can use this formula because it is a stronger requirement than the condition that the geometry be asymptotically flat. However, we argue below that this condition on the three-metric has physical significance, that it is a very reasonable requirement to place on the three-metric, and that we lose no initial-data set of interest when we do impose it. The assumption concerning the existence of a rapidly falling-off base metric is justified by (29).

Degrees of freedom, coordinate waves, and energy

The independent dynamical degrees of freedom of the gravitational field have been defined elsewhere as the conformal intrinsic geometry and the transverse, trace-free part of the momentum.[2] The kinematic degree of freedom, corresponding to a choice of the spacelike hypersurface, is the trace of the momentum. It is natural (following ADM)[1] to define the far-field of any gravitational field as the region in which all the degrees of freedom have been shut off. For simplicity, let us consider vacuum fields and assume that (trace π^{ij})=0. Then this region would have to be conformally flat and have no momentum. Then the only quantity needed to describe the geometry in this region is the conformal factor λ. In this region the momentum constraint is trivially satisfied and the Hamiltonian constraint reduces to $R=0$. But since the region is conformally flat we have $R=-8\lambda^{-5}\nabla^2\lambda$. Therefore, λ obeys Laplace's equation

$$\nabla^2\lambda=0\,.$$

Not only can we calculate the energy at spacelike infinity as

$$16\pi E=-8\oint_\infty \vec{\nabla}\lambda\cdot d\vec{S}\,,$$

but we can also calculate the energy as

$$16\pi E=-8\oint_S \vec{\nabla}\lambda\cdot d\vec{S}\,, \tag{19}$$

where in (19) S is any closed topologically spherical two-surface constructed in the far-field region that surrounds the part of the manifold in which the degrees of freedom are activated. In other words one has a simple flux theorem for the energy in this case.

If we are given an asymptotically flat three-metric g_{ij}, then

$$g_{ij}\to f_{ij} \text{ as } r\to\infty$$

or (20)

$$g_{ij}=f_{ij}+h_{ij}\,,\quad h_{ij}\to o \text{ at } \infty\,.$$

Now h_{ij} is a symmetric tensor and so can be uniquely covariantly decomposed with respect to the flat metric f_{ij}, or with respect to g_{ij}.[5] Here we use f_{ij}. One obtains an orthogonal decomposition into three parts:

$$h_{ij}=h_{ij}^{TT}+(LW)_{ij}+\tfrac{1}{3}hf_{ij}\,,$$

where h_{ij}^{TT} is transverse, trace free with respect to f_{ij};

$$(LW)_{ij}=\nabla_iW_j+\nabla_jW_i-\tfrac{2}{3}f_{ij}\nabla_kW^k$$

is the "conformal Killing form" of the asymptotically vanishing vector W^i, and forms the longitudinal part of h_{ij}. The third term in the decomposition is the trace $h=h_{ij}f^{ij}$.

Hence, we get

$$g_{ij}=f_{ij}+h_{ij}^{TT}+(LW)_{ij}+\tfrac{1}{3}hf_{ij}\,. \tag{21}$$

We will show that the energy of the three-metric g_{ij} is purely the $1/r$ part of the scalar

$$h_T=h-2\nabla_kW^k\,. \tag{22}$$

The scalar h_T supplies another coordinate-invariant definition of the energy, which will be shown to agree with the definition of energy given by (18). In particular, this approach is especially useful in that it shows explicitly how to deal with the problem of "coordinate waves" at infinity.

Since f_{ij} is a flat metric, then

$$f'_{ij}=f_{ij}+\nabla_jM_i+\nabla_iM_j$$

is also a flat metric for any vector M^i, since it corresponds only to a shift of coordinates. Let us therefore pick $M^i=W^i$ and choose as our flat metric

$$\bar{f}_{ij}=f_{ij}+\nabla_iW_j+\nabla_jW_i\,.$$

This is just as good a choice as the original f_{ij} because the two metrics are physically identical. Then we get

$$g_{ij}=\bar{f}_{ij}+\bar{h}_{ij}^{TT}+\tfrac{1}{3}(h-2\nabla_kW^k)\bar{f}_{ij}$$
$$+\text{terms of second order in } h_{ij}, W^i\,,$$

$$\bar{h}_{ij}^{TT}=h_{ij}^{TT}+\text{terms of second order}\,.$$

Since we are only interested in the asymptotic behavior of g_{ij} we need retain only the first-order terms

$$g_{ij}=\bar{f}_{ij}+\bar{h}_{ij}^{TT}+\tfrac{1}{3}h_T\bar{f}_{ij}\,. \tag{23}$$

Now consider the coordinate transformation that goes from $\bar{f}_{ij}$ to the Cartesian metric $f'_{ij}=\delta_{ij}$. Perform this coordinate transformation on g_{ij} (it becomes g'_{ij}), and since each part of the right-hand side of Eq. (23) is a tensor in flat space, we immediately get by the usual coordinate transformation formulas

$$g'_{ij}=f'_{ij}+h'^{TT}_{ij}+\tfrac{1}{3}h_Tf'_{ij}\,.$$

To calculate the mass of this metric, we can substitute immediately in Eq. (7) and obtain

$$E = -\frac{1}{24\pi} \oint_\infty \sqrt{g}\, g^{ij}(h_T)_{;i}\, dS_j \,, \tag{24}$$

which is a coordinate-free expression, since h_T is a scalar. Observe that the energy is contained in the $1/r$ part of h_T and has no explicit dependence on h_{ij}^{TT}. However, there is an implicit dependence, because of course the initial data must obey the constraints. In particular, the Hamiltonian constraint can be written in the form

$$-\tfrac{2}{3}\nabla^2 h_T = [\tfrac{1}{4}(\nabla_k h_{ij}^{TT})^2 + (\pi_{ij}^{TT})^2] + \cdots \,. \tag{25}$$

Therefore, if we have finite energy, i.e., h_T falling off like $1/r$, then $\nabla_k h_{ij}^{TT}$ must fall off faster than $r^{-3/2}$.

Our physical requirement that the space be asymptotically conformally flat, i.e., Schwarzschildian at infinity, reduces to the condition that h_{ij}^{TT} itself falls off faster than $1/r$. These two asymptotic conditions on h_{ij}^{TT} can be satisfied by limiting our attention to those initial-data sets where h_{ij}^{TT} falls off faster than $r^{-3/2}$. This is a reasonable condition because h_{ij}^{TT} contains the gravitational waves and so in general $\nabla_k h_{ij}^{TT}$ only falls off as fast as h_{ij}^{TT}. In addition, we require that the other degrees of freedom of the gravitational field (π_{TT}^{ij}, τ) vanish asymptotically. π_{TT}^{ij} must fall off faster than $r^{-3/2}$, τ faster than r^{-2}.

The method of calculating the mass of an initial-data set as the $1/r$ part of $h_T = h - 2\nabla_k W^k$ is especially appealing because it manifestly eliminates possible effects of "coordinate waves." As an example, let us add a coordinate wave to a metric $g_{ij} = f_{ij} + h_{ij}$. The effect of this wave on g_{ij} will enter in the form $\nabla_i M_j + \nabla_j M_i$ for some vector M^i.

$$\begin{aligned} g^*_{ij} &= g_{ij} + \nabla_i M_j + \nabla_j M_i \\ &= f_{ij} + h_{ij}^{TT} + (LW)_{ij} + \tfrac{1}{3} h f_{ij} + \nabla_i M_j + \nabla_j M_i \,. \end{aligned}$$

But

$$\nabla_i M_j + \nabla_j M_i = (LM)_{ij} + \tfrac{2}{3}(\nabla_k M^k) f_{ij} \,.$$

Therefore,

$$g^*_{ij} = f_{ij} + h_{ij}^{TT} + [L(W+M)]_{ij} + \tfrac{1}{3}(h + 2\nabla_k M^k) f_{ij} \,. \tag{26}$$

The energy will be calculated from $h_T^* = h + 2\nabla_k M^k - 2\nabla_k(W^k + M^k)$. But clearly $h_T^* = h_T$, so the presence of a Killing form $\nabla_i M_j + \nabla_j M_i$, interpreted as the manifestation of a coordinate shift, has no effect on the energy.

Fast falloff of base metric

Another consequence can be deduced from the decomposition of an asymptotic metric. From (23) we have (dropping the primes)

$$\begin{aligned} g_{ij} &= f_{ij} + h_{ij}^{TT} + \tfrac{1}{3} h_T f_{ij} \\ &= (1 + \tfrac{1}{3} h_T) f_{ij} + h_{ij}^{TT} \,, \end{aligned} \tag{27}$$

which shows that, asymptotically, gravitational waves are identified in the metric by the presence of h_{ij}^{TT}, i.e., gravitational degrees of freedom are measured by the deviation of the metric not from simple flatness, but from conformal flatness.[13] This is in accord with our identification of the conformal metric $\tilde{g}_{ij} = g^{-1/3} g_{ij}$ as being the freely specifiable part of the metric in the initial-value problem.

Equation (27) may also be put in the form

$$\begin{aligned} g_{ij} &= (1 + \tfrac{1}{3} h_T)[f_{ij} + (1 + \tfrac{1}{3} h_T)^{-1} h_{ij}^{TT}] \\ &= (1 + \tfrac{1}{3} h_T) b_{ij} \,, \end{aligned} \tag{28}$$

with the asymptotic "base" metric b_{ij} defined by

$$b_{ij} = f_{ij} + (1 + \tfrac{1}{3} h_T)^{-1} h_{ij}^{TT} \,. \tag{29}$$

Since b_{ij} approaches f_{ij} faster than $1/r$, it contributes nothing to the mass integral for g_{ij}. Thus, all the mass in g_{ij} is found in the conformal factor $1 + \frac{1}{3} h_T$. This result shows that every metric with finite mass may be assumed to be conformally related to one that contributes nothing to the mass integral, in full accord with (18). This was also a key assumption in Brill's treatment of axisymmetric waves at a moment of time symmetry.[3]

We may also make the following observation. Since in our approach gravitational radiation is indicated by the presence of spatial conformal curvature and the conjugate variable π_{TT}^{ij}, we may assume that the spacetime of an isolated system that has only been radiating for a finite time can be sliced in such a way that the conformal curvature has a compact domain of support, and likewise for π_{TT}^{ij}, at some instant of time. In this case one may assume that the base metric b_{ij} is exactly equal to f_{ij} outside a compact set and that π_{TT}^{ij} is also zero outside the same set.

The definition of energy as

$$16\pi E = -8 \oint_\infty \vec{\nabla}\lambda \cdot d\vec{S}$$

may be also applied to asymptotically flat initial-data sets, including those which are topologically non-Euclidian (i.e., contain wormholes or singularities). In these cases, however, care must be taken in using Gauss's theorem when going from a surface integral to a volume integral because the surface of integration includes in addition to the surface at infinity, interior surfaces S_{in} surrounding the singularities. Therefore,

$$16\pi E = -8\oint_{\infty} \vec{\nabla}\phi \cdot d\vec{S}$$

$$= -8\int_V \sqrt{g}\ \nabla^2\phi d^3x + 8\sum \oint_{S_{\text{in}}} \vec{\nabla}\phi \cdot d\vec{S}\ .$$

III. TIME-SYMMETRIC INITIAL-DATA SETS AS MINIMA OF ENERGY

We have shown elsewhere[2] that the independent dynamical degrees of freedom of the gravitational field may be given as the conformal intrinsic geometry ($\tilde{g}_{ij}$) and the transverse, trace-free part of the momentum. The trace of the momentum (τ) is a kinematical degree of freedom. The dependent variables are the trace-free longitudinal part of the momentum σ^{ij}_{LL}, which takes the form of the "conformal Killing form" $\nabla^i W^j + \nabla^j W^i - \frac{2}{3} g^{ij}\nabla_k W^k$ of some vector W^i, which is chosen to satisfy constraint Eq. (2), and the conformal factor ϕ which is chosen to satisfy constraint Eq. (3). The constraint equations with this choice of independent and dependent variables form a system of four quasilinear elliptic equations for ϕ and W^i. For an arbitrary choice of the independent variables a solution almost always exists to these equations. If a solution exists, this solution is unique. In view of the close relationship between the conformal factor and energy [Eqs. (13) and (14)], the independent variables uniquely define the energy. Similarly it can be shown that W^i defines the total linear and angular momenta of the system, and so the independent variables also uniquely define the momentum.[9]

In this paper we will not change $\tilde{g}_{ij}$ but will consider the changes of energy arising from changing the other independent variables. In this section we will limit our attention to maximal slices, i.e., we set the trace of the momentum equal to zero. In this case σ^{ij}_{LL} vanishes when we have no source current density and the topology is Euclidean (no "holes"), therefore the momentum is identical to its transverse trace-free part and constraint Eq. (2) is automatically satisfied. We will show that the minimum of energy occurs when we have no momentum (time-symmetric initial-data sets), and any "addition" of momentum or matter cannot but increase the energy. To demonstrate the results we will prove two theorems:

Theorem 1. The intrinsic geometry of any set of initial data on a maximal hypersurface is conformally related to a unique time-symmetric geometry, i.e., one on which $R=0$.

Theorem 2. Among all initial-data sets on maximal hypersurfaces, whose intrinsic geometries are conformally related, the unique time-symmetric geometry has the least energy.

Proof of Theorem 1. Let us limit ourselves for the present to vacuum spaces. We are given a set of initial data with $\pi=0$, i.e., the momentum π^{ij} is a transverse, trace-free tensor density which we will call σ^{ij}. The Hamiltonian constraint shows

$$R=(1/g)\sigma^{ij}\sigma_{ij} \geq 0\,. \tag{30}$$

Under a conformal transformation $g'_{ij}=g_{ij}\phi^4$, the scalar curvature transforms as

$$R' = \phi^{-4}R - 8\phi^{-5}\nabla^2\phi\,. \tag{31}$$

The original space is conformal to a time-symmetric one if $\exists\, \phi>0$ such that

$$R'=0,\ \text{i.e.,}\ 8\nabla^2\phi - R\phi = 0 \qquad (\phi=1 \text{ at } \infty\ ;\ 0<\phi<\infty)\,. \tag{32}$$

Since $R\geq 0$ this equation always has a unique, positive solution (this is an application of a maximum principle[19]). Therefore, the intrinsic geometry of any maximal slice of an asymptotically flat spacetime is conformal to one and only one space that has $R\equiv 0$. To extend the proof to spaces with sources we note that the Hamiltonian constraint becomes

$$R - \frac{1}{g}(\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2) = 16\pi T^*_*\,, \tag{32a}$$

where T^*_* is the local energy density, which we assume non-negative. The momentum need no longer be transverse, because the momentum constraint in the presence of sources becomes

$$\nabla_j \pi^{ij} = 8\pi \mathcal{S}^i\,, \tag{32b}$$

where $\mathcal{S}^i$ is the current density due to the sources. But since we are considering a maximal slice, the momentum will still be trace-free. Therefore, the Hamiltonian constraint reduces to

$$R = \frac{1}{g}(\pi^{ij} - \tfrac{1}{3}g^{ij}\pi)(\pi_{ij} - \tfrac{1}{3}g_{ij}\pi) + 16\pi T^*_* > 0\,, \tag{33}$$

so the proof carries through just as before.

Proof of Theorem 2. Let us have a vacuum, time-symmetric initial-data set, i.e., a metric g_{ab} with $R=0$. Let us have an initial-data set on a maximal slice with intrinsic geometry $\overline{g}_{ij}$. We have

$$\overline{R} = \frac{1}{\overline{g}}\overline{\sigma}^{ij}\overline{\sigma}_{ij} + 16\pi \overline{T}^*_* \geq 0\,. \tag{34}$$

Let the two intrinsic geometries belong to the same conformal class, i.e., there exists a positive bounded function θ such that $\overline{g}_{ij}=\theta^4 g_{ij}$. Equation (31) then shows us that

$$\overline{R} = \theta^{-4}R - 8\theta^{-5}\nabla^2\theta\,. \tag{35}$$

Since $R=0$, this is equivalent to

$$-8\nabla^2\theta = \overline{R}\theta^5 \quad (\theta = 1 \text{ at } \infty)\,. \tag{36}$$

However, Eq. (18) tells us that the difference in energy between the initial-data sets is given by

$$16\pi\Delta E=\int_V(-8\nabla^2\theta)\sqrt{g}\;d^3x\,. \tag{37}$$

Therefore, substituting from Eq. (36)

$$16\pi\Delta E=\int_V(\bar{R}\theta^5)\sqrt{g}\;d^3x\,. \tag{38}$$

But we know $\bar{R}\geqslant 0$, $\theta^5>0$; therefore,

$$16\pi\Delta E>0\,.$$

Therefore, the energy of the maximal initial-data set is strictly larger than the energy of the conformally related time-symmetric set. The only time one gets $\Delta E=0$ is when $\bar{R}\equiv 0$, i.e., $\theta\equiv 1$.

IV. ENERGY AS A FUNCTIONAL OF GRAVITATIONAL MOMENTUM

We have mentioned several times that the transverse, traceless part of the momentum characterizes part of the independent dynamical degrees of freedom. This is because it is a conformal invariant. If σ^{ij} is a transverse, trace-free tensor density with respect to g_{ij} then $\sigma^{ij\,\prime}=\phi^{-4}\sigma^{ij}$ is transverse, tracefree with respect to $g'_{ij}=\phi^4 g_{ij}$. Therefore, if we limit ourselves to maximal slices and choose as momentum any transverse trace-free tensor density, constraint Eq. (2) is automatically satisfied, and all we need to do is find a conformal factor so as to satisfy constraint Eq. (3). This equation reduces to

$$8\nabla^2\phi=-M\phi^{-7}+R\phi\quad(\phi=1\text{ at }\infty\,;\;0<\phi<\infty)\,, \tag{39}$$

where $M=(1/g)\sigma^{ij}\sigma_{ij}\geqslant 0$ and R is the scalar curvature of the original manifold. If this equation has a solution, it will be a unique solution.[14] We can prove the following existence theorem.

Theorem 3. Equation (39) has a solution if and only if

$$8\nabla^2\theta=R\theta\quad(\theta=1\text{ at }\infty\,;\;0<\theta<\infty) \tag{40}$$

has a solution on the same manifold, for the same R. This is equivalent to requiring that the initial three-manifold be conformally related to one on which $R\equiv 0$.

Proof. Theorem 1 proves the necessity part of Theorem 3. As to sufficiency, let θ be the solution of (40). Use θ as a conformal transformation on the original manifold, i.e., $g'_{ij}=\theta^4 g_{ij}$. Equation (39) in the new manifold becomes

$$8\nabla'^2\phi'=-(M\theta^{-12})\phi'^{-7}\qquad(\phi'=\phi/\theta\,;\;\phi'=1\text{ at }\infty\;;\;0<\phi'<\infty)\,. \tag{41}$$

Now solve

$$8\nabla'^2\mu=-M\theta^{-12}\quad(\mu=1\text{ at }\infty)\,. \tag{42}$$

Since $M\theta^{-12}\geqslant 0$ we have $\nabla'^2\mu\leqslant 0$, so we can use a maximum principle[19] to show that μ cannot have any interior minimum. Therefore a solution exists with $1\leqslant\mu<\infty$. Now transform the original manifold using $\theta\mu$ as a conformal factor, i.e., $g''_{ij}=(\theta\mu)^4 g_{ij}$. Equation (39) becomes

$$8\nabla''^2\phi''=-\left[M(\theta\mu)^{-12}\right]\phi''^{-7}-(8\mu^{-5}\nabla'^2\mu)\phi''\,, \tag{43}$$

$$=-M\theta^{-12}\mu^{-12}\phi''^{-7}+M\theta^{-12}\mu^{-5}\phi''\qquad(\text{where }\phi''=\phi/\theta\mu)\,, \tag{44}$$

$$=+M\theta^{-12}\mu^{-12}(-\phi''^{-7}+\mu^7\phi'')\,. \tag{45}$$

The right-hand side of this equation has one and only one positive root at $\phi''=\mu^{-7/8}$. This is sufficient to show that it has a unique positive solution, bounded by $\mu^{-7/8}$ and 1.[14] Therefore, Eq. (39) has a solution and Theorem 3 follows.

We can analyze the restrictions that Theorem 3 places on the existence of a solution to Eq. (39) by appealing to analogous features of Schrödinger's equation. If R is everywhere non-negative, Eq. (40) will have a solution. This is equivalent to zero energy particles scattering off a potential barrier. The wave function is everywhere positive, i.e., no resonances. If R is only "slightly negative," i.e., a shallow potential well, the wave function is again everywhere positive and (40) has a solution. On the other hand, if R is substantially negative, we start getting resonances in the wave function, i.e., places where it vanishes, and we can no longer use it as a conformal factor.

Therefore, the set of all maximal, open (without boundary), mass-free initial-data sets is isomorphic to the set of (all open manifolds in which $R\equiv 0$)$\times$(all transverse trace-free tensors on each manifold).

What we have been doing is holding the conformal manifold fixed, and looking at the changes of energy as the transverse-trace-free tensor density is changed in a nonconformal manner. We obtained the result that the minimum of energy occurs when there is no transverse trace-free tensor density. This is analogous to holding the magnetic field fixed in Maxwell's equations and varying the electric field. The electromagnetic energy in that case is a minimum when there is no electric field. In fact, we can push the analogy further. If we fix the magnetic field and scale the electric field with some constant α, we discover that the energy is a monotonically increasing function of α^2, i.e.,

$$E(\alpha)=\int_V(\vec{B}^2+\alpha^2\vec{E}^2)\,dv\,.$$

Now, if σ^{ij} is a transverse trace-free tensor density with respect to some metric g_{ij}, obviously $\alpha\sigma^{ij}$ is a transverse, trace-free tensor density also with respect to the same metric g_{ij} for a constant α. We can show the following: If (σ^{ij}, g_{ij}) permit a solution of Eq. (39) to exist, i.e., if they can be conformally mapped into a solution of the initial-value equations, then $(\alpha\sigma^{ij}, g_{ij})$ also generate an initial-value data set. for any constant α. In addition, the energy of this initial-value data set is a monotonically increasing function of α^2.

Given that (σ^{ij}, g_{ij}) permit a solution to Eq. (39) to exist, i.e., there exists a positive bounded function ϕ_0 such that ϕ_0 obeys

$$8\nabla^2\phi_0 = -M\phi_0^{-7} + R\phi_0\,, \qquad M = \frac{1}{g}\sigma^{ij}\sigma_{ij}\,, \quad (\phi_0 = 1 \text{ at } \infty)\,. \tag{46}$$

Now consider the set $(\alpha\sigma^{ij}, g_{ij})$ for any α. We will demonstrate that a solution exists to

$$8\nabla^2\theta = -\alpha^2 M\theta^{-7} + R\theta \quad (\theta = 1 \text{ at } \infty)\,. \tag{47}$$

Now let us rewrite Eq. (42), using as our intrinsic geometry the conformally transformed one $\bar{g}_{ij} = \phi_0^4 g_{ij}$. Then Eq. (47) takes the form

$$8\bar{\nabla}^2\mu = -\alpha^2\bar{M}\mu^{-7} + \bar{R}\mu \quad (\mu = 1 \text{ at } \infty)\,, \tag{48}$$

where $\mu = \theta/\phi_0$, $\bar{M} = M\phi_0^{-12}$, $\bar{R}$ is the scalar curvature of $\bar{g}_{ij}$. Equation (46) shows $\bar{R} = \bar{M} = M\phi_0^{-12}$. Equation (47) can be written as

$$8\bar{\nabla}^2\mu = +\bar{M}(-\alpha^2\mu^{-7} + \mu)\,, \quad \bar{M} \geq 0\,. \tag{49}$$

This is a quasilinear elliptic equation. We have discussed elsewhere theorems for the existence and uniqueness of such equations.[14] Let us consider $\alpha^2 > 1$. When $\mu = 1$ we have

$$\bar{M}(-\alpha^2\mu^{-7} + \mu) = (-\alpha^2 + 1)\bar{M} < 0\,, \tag{50}$$

when $\mu = \alpha^{1/4} > 1$ we have

$$\bar{M}(-\alpha^2\mu^{-7} + \mu) = 0\,. \tag{51}$$

Therefore, 1 and $\alpha^{1/4}$ form "lower" and "upper" solutions to the equation. In addition, the boundary value lies in the interval $[1, \alpha^{1/4}]$. Therefore, a solution exists in the interval $[1, \alpha^{1/4})$ to the Eq. (49).

Also, $\bar{M}(-\alpha^2\mu^{-7} + \mu)$ has only one root, and this is enough to show that the solution is unique. Since the solution lies in the interval $[1, \alpha^{1/4})$, we have

$$8\bar{\nabla}^2\mu = -\alpha^2\bar{M}\mu^{-7} + \bar{M}\mu < 0\,. \tag{52}$$

Therefore

$$-8\bar{\nabla}^2\mu > 0\,. \tag{53}$$

But μ is the conformal factor connecting the intrinsic geometry of the solution (σ^{ij}, g_{ij}) to the solution $(\alpha\sigma^{ij}, g_{ij})$. Therefore, the difference in energy is given by [Eq. (18)]

$$16\pi\Delta E = -\int_V 8\bar{\nabla}^2\mu\sqrt{\bar{g}}\,d^3x > 0\,. \tag{54}$$

The energy monotonically increases for increasing α^2. On the other hand, if $\alpha^2 < 1$, we get as "lower" and "upper" solution $\alpha^{1/4}$ and 1. Then the solution lies in the interval $(\alpha^{1/4}, 1]$. In this case we have

$$-8\bar{\nabla}^2\mu < 0 \tag{55}$$

or

$$\Delta E < 0\,. \tag{56}$$

In particular, theorem 3 shows that if we have a three-metric g_{ij} for which $R = 0$, i.e., a moment of time symmetry solution, and any transverse trace-less tensor density σ^{ij} on that manifold, then we can always put them together to form a solution to the initial-value constraints. In this case Eq. (39) reduces to

$$8\nabla^2\phi = -M\phi^{-7} \quad \left(M = \frac{1}{g}\sigma^{ij}\sigma_{ij} \geq 0\right) \tag{57}$$

and we know it has a solution.

This equation shows us that $\Delta E > 0$ (as we expect). In fact we can rewrite this equation in such a way as to demonstrate the existence of the negative "interaction energy" term.

It is possible to show that $\phi \geq 1$. Given a solution ϕ to Eq. (57), we can go by a conformal transformation $\bar{g}_{ij} = \phi^4 g_{ij}$ to a frame in which the Hamiltonian constraint is satisfied.

$$\bar{R} = \frac{1}{\bar{g}}\bar{\sigma}^{ij}\bar{\sigma}_{ij} = \bar{M} = M\phi^{-12} \geq 0\,. \tag{58}$$

Now the conformal transformation that moves us back from this frame to the original frame in which $R = 0$ is $\lambda = 1/\phi$. But from Eqs. (31) and (32) we know that λ obeys the following:

$$8\bar{\nabla}^2\lambda - \bar{R}\lambda = 0 \quad (\lambda = 1 \text{ at } \infty)\,. \tag{59}$$

Since $\bar{R} \geq 0$, this equation obeys a maximum principle, which forces $\lambda \leq 1$. Therefore, $\phi \geq 1$. Equations (57) and (59) permit us to write the change of energy in two ways:

$$16\pi\Delta E = \int_V M\phi^{-7}\sqrt{g}\,d^3x\,, \tag{60}$$

$$16\pi\Delta E = \int_V \bar{M}\lambda\sqrt{\bar{g}}\,d^3x\,. \tag{61}$$

Therefore

$$16\pi\Delta E < \int M\sqrt{g}\,d^3x\,, \tag{62}$$

$$16\pi\Delta E < \int \overline{M}\sqrt{\overline{g}}\, d^3x . \qquad (63)$$

Equation (63) is stronger than Eq. (62) because

$$\overline{M}\sqrt{\overline{g}} = M\sqrt{g}\ \phi^{-6} < M\sqrt{g} . \qquad (64)$$

There are two natural measures of the "local energy density" of the gravational momentum. This involves $M/16\pi$, but it can be measured in either the moment of time-symmetry frame, or in the solution frame. However, in either case we can make the statement

$$\Delta E < \int \text{"local energy density."} \qquad (65)$$

In each case it is possible to write down an expression for the negative "interaction energy." Returning to Eq. (57) we have

$$8\phi^7\nabla^2\phi = -M , \qquad (66)$$

$$-M = 8\vec{\nabla}\cdot(\phi^7\vec{\nabla}\phi) - 56\phi^6(\nabla\phi)^2 . \qquad (67)$$

Therefore

$$-\int M dv = 8\oint_\infty \vec{\nabla}\phi\cdot d\vec{S} - 56\int \phi^6(\nabla\phi)^2 dv , \qquad (68)$$

since $\phi = 1$ at infinity. Therefore

$$16\pi\Delta E = \int_V M\sqrt{g}\ d^3x - 56\int (\phi^3\nabla\phi)^2\sqrt{g}\ d^3x . \qquad (69)$$

In this case $(7/2\pi)\int(\phi^3\nabla\phi)^2\sqrt{g}\ d^3x$ is the negative interaction energy.

On the other hand, if we rewrite Eq. (59) we will get

$$16\pi\Delta E = \int \overline{M}\sqrt{\overline{g}}\ d^3x - 8\int\left(\frac{\overline{\nabla}\lambda}{\lambda}\right)^2\sqrt{\overline{g}}\, d^3x . \qquad (70)$$

V. GEOMETROSTATICS

As an alternative to adding momentum to a time-symmetric geometry, we can add a source, either matter or field energy density T^*_*. To keep the discussion as simple as possible, let us assume that the source current density vanishes. In this case the momentum constraint [Eq. (32a)] is trivially satisfied, and we need only to find a conformal transformation that maps us onto a solution of the Hamiltonian constraint [Eq. (32)]. That is we need to find a positive function ϕ, such that when $\overline{g}_{ij} = \phi^4 g_{ij}$,

$$\overline{R} = 16\pi\overline{T}^*_* . \qquad (71)$$

This reduces to

$$-8\nabla^2\phi = 16\pi\overline{T}^*_*\phi^5 . \qquad (72)$$

The next problem to be solved is to express $\overline{T}^*_*$ in terms of T^*_* and ϕ. The most natural transformation is of the form[15]

$$\overline{T}^*_* = T^*_*\phi^{-8} . \qquad (73)$$

This law may be justified by considering the electromagnetic field as source. The transformation of Eq. (73) is exactly the one required so that the Maxwell constraints remain satisfied. We get a similar result when transforming the neutrino field or from a dimensionality argument. Therefore, Eq. (72) finally becomes

$$-8\nabla^2\phi = 16\pi T^*_*\phi^{-3} \quad (\phi = 1 \text{ at infinity}) . \qquad (74)$$

We will assume that T^*_* obeys the physically reasonable condition $T^*_* \geq 0$. By an argument similar to that for Eq. (57) we can show that a solution exists to Eq. (74) and that $\phi \geq 1$. If we define $\mu = 1/\phi \leq 1$ and set $\overline{g}_{ij} = \phi^4 g_{ij}$, we have parallel to Eqs. (60) and (61)

$$16\pi\Delta E = 16\pi\int_V T^*_*\phi^{-3}\sqrt{g}\ d^3x , \qquad (75)$$

$$16\pi\Delta E = 16\pi\int_V \overline{T}^*_*\mu\sqrt{\overline{g}}\, d^3x , \qquad (76)$$

and therefore

$$\Delta E < \int_V \overline{T}^*_*\sqrt{\overline{g}}\ d^3x < \int_V T^*_*\sqrt{g}\, d^3x . \qquad (77)$$

Similarly we have

$$\Delta E = \int_V T^*_*\sqrt{g}\ d^3x - \frac{3}{2\pi}\int_V (\phi\nabla\phi)^2\sqrt{g}\ d^3x \qquad (78)$$

and

$$\Delta E = \int_V \overline{T}^*_*\sqrt{\overline{g}}\ d^3x - \frac{1}{2\pi}\int_V\left(\frac{\overline{\nabla}\mu}{\mu}\right)^2\sqrt{\overline{g}}\ d^3x . \qquad (79)$$

The definition of energy in terms of the conformal factor ϕ at infinity [Eq. (22)] can be related very closely to Newtonian gravity by writing

$$\phi = 1 - \tfrac{1}{2}\psi \quad (\psi = 0 \text{ at } \infty) \qquad (80)$$

and regarding ψ as the Newtonian gravitational potential. Recall that the gravitational potential V due to a mass distribution ρ satisfies

$$\nabla^2 V = 4\pi\rho \quad (V = 0 \text{ at infinity}) . \qquad (81)$$

Then of course

$$16\pi E = 16\pi\int_V \rho dv = 4\int_V \nabla^2 V dv \quad (\text{Newtonian}) , \qquad (82)$$

$$16\pi E = 4\int_V \nabla^2\psi dv \quad (\text{general relativity}) . \qquad (83)$$

It is of interest to expand Eqs. (69) and (78) in terms of the gravitational potential. We get

$$\Delta E = \int_V \frac{M}{16\pi}\sqrt{g}\, d^3x - \frac{7}{8\pi}\int_V (\nabla\psi)^2\sqrt{g}\ d^3x \qquad (84)$$

and

$$\Delta E = \int_V T^*_* \sqrt{g}\, d^3x - \frac{3}{8\pi}\int_V (\nabla\psi)^2 \sqrt{g}\, d^3x \ . \qquad (85)$$

The reason for the different coefficients on the negative interaction energy terms in Eqs. (84) and (85), is due to the different powers of ϕ on the right-hand side of Eqs. (57) and (74). They in turn arise because in one case we are dealing with a $(\sigma_{TT})^2$ and in the other case with an $\vec{E}^2 + \vec{B}^2$ term. We choose the conformal transformations laws in each case so that they preserve constraints, but in one case we are dealing with a symmetric tensor and in the other with a vector. Let S^{ab}_{TT} be a transverse, trace-free tensor and let E^a be a transverse vector, then under the conformal transformation $\bar{g}_{ij} = \phi^4 g_{ij}$, they transform differently so as to remain transverse.

$$\text{If } \nabla_b S^{ab}_{TT} = 0\ , \text{ let } \bar{S}^{ab} = \phi^{-10} S^{ab};\ \text{then } \bar{\nabla}_b \bar{S}^{ab} = 0 \ . \qquad (86)$$

$$\text{If } \nabla_a E^a = 0,\ \text{let } \bar{E}^a = \phi^{-6} E^a;\ \text{then } \bar{\nabla}_a \bar{E}^a = 0 \ . \qquad (87)$$

These transformation rules can be justified on a physical basis. The rule $\nabla_a E^a = 0$ arises from an integral conservation law, and therefore the flux of $\vec{E}$ through a surface is a physical quantity, which we expect to be unaffected by scale changes in the geometry.[16] Now we have

$$\int \vec{E}\cdot d\vec{S} = \alpha \quad \text{(number of lines of force)} \ . \qquad (88)$$

Under the conformal transformation $\bar{g}_{ij} = \phi^4 g_{ij}$ our measure of area picks up a factor of ϕ^4. Therefore if we wish to leave invariant the number of lines of force, $\vec{E}$ must pick up a factor ϕ^{-4}. Note that $\vec{E}\cdot d\vec{S}$ is the product of (physical component of $\vec{E}$ in normal direction)×(proper area).

Therefore $\vec{E} = E^a \vec{e}_a$, where $\vec{e}_a$ are the basis vectors.

Now

$$\vec{e}_a \cdot \vec{e}_b = \delta_{AB} e^A_a e^B_b = g_{ab} \ . \qquad (89)$$

But

$$\bar{g}_{ab} = \phi^4 g_{ab} \ \text{so} \ \bar{\vec{e}}_a = \phi^2 \vec{e}_a \ . \qquad (90)$$

But we have

$$\bar{\vec{E}} = \bar{E}^a \bar{\vec{e}}_a = \phi^{-4} E^a \vec{e}_a \ . \qquad (91)$$

Therefore

$$\bar{E}^a = \phi^{-6} E^a \ . \qquad (92)$$

Similarly, a transverse trace-free symmetric tensor can give rise to an integral conservation law, but here we require that the space have a symmetry, so as to transform the tensor into a vector. In this case if ξ^a is a conformal Killing vector (including Killing vectors), we have

$$\text{flux} = \text{number of lines} = \int (S^{ab}\xi_b)\sqrt{g}\, dA_a \ . \qquad (93)$$

This is well defined because

$$(S^{ab}\xi_b)_{;a} = 0. \qquad (94)$$

In this case dA_a is a conformal invariant so we need

$$\bar{S}^{ab}\bar{\xi}_b \sqrt{\bar{g}} = S^{ab}\xi_b\sqrt{g} \ . \qquad (95)$$

Since

$$\bar{\xi}_b = \phi^4 \xi_b\ , \quad \sqrt{\bar{g}} = \phi^6 \sqrt{g}, \qquad (96)$$

we get

$$\bar{S}^{ab} = \phi^{-10} S^{ab}. \qquad (97)$$

The reason for $\bar{\xi}_b = \phi^4 \xi_b$ is that if ξ^a is a conformal Killing vector with respect to g_{ij}, it is also one with respect to $\phi^4 g_{ij}$. Therefore,

$$\bar{\xi}^a = \xi^a, \quad \bar{\xi}_b = \phi^4 \xi_b \ .$$

While there is a close relationship between the Newtonian gravitational potential and the conformal factor [Eqs. (80), (82), and (83)], they are not identical. To demonstrate this, consider their behavior under uniform scaling of the source energy-density. A uniform scaling of the source energy-density will induce a constant scaling by the same factor in the Newtonian gravitational potential and hence the total energy of the system will change by exactly the same factor, i.e., if $\nabla^2 \psi_N = 4\pi\rho$, then

$$\nabla^2(\alpha\psi_N) = 4\pi(\alpha\rho) \qquad (98)$$

for α = constant and

$$E(\alpha) = \alpha E(\alpha = 1). \qquad (99)$$

On the other hand, if we scale T^*_* by a constant factor α in Eq. (74) we get a behavior of the energy very similar to the behavior of the energy under uniform scaling of the momentum. The total energy of the system increases monotonically with increasing α, but the rate of increase is always less than unity. In fact it is easy to prove that

$$1 < \frac{E(\alpha)}{E(\beta)} < \frac{\alpha}{\beta}\ , \qquad (100)$$

when $\alpha > \beta$ and α, β are positive constants, by following the procedures in Sec. IV.

A simple example may serve to illustrate this. Let us have a unit sphere of uniform density ρ in flat space. Then the solution to Eq. (74) is

$$\begin{aligned} \phi &= 1 + \pi\rho - \tfrac{5}{2}\pi^2\rho^2 - (\tfrac{1}{3}\pi\rho + \pi^2\rho^2)r^2 - \tfrac{1}{10}\pi^2\rho^2 r^4 \quad (r < 1), \\ \phi &= 1 + (\tfrac{2}{3}\pi\rho - \tfrac{8}{5}\pi^2\rho^2)r^{-1} \quad (r > 1). \end{aligned} \qquad (101)$$

(We have included terms only to order ρ^2.)

Then

$$E = \frac{4}{3}\pi\rho - \frac{16}{5}\pi^2\rho^2 \cdots . \tag{102}$$

The solution to the initial-value equations is a conformally flat space, containing a unit sphere of density $\rho\varphi^{-8}$. The total mass in the solution space is $\frac{4}{3}\pi\rho - \frac{32}{15}\pi\rho^2$. The extra energy loss is the gravitational binding energy of this sphere $\frac{16}{15}\pi^2\rho^2$. It is clear that the correction term grows more quickly under uniform scaling of ρ than the total energy. This detailed analysis is, of course, only correct for $\rho \ll 1$, but, in fact, no matter what the source, or how large the density, it always behaves qualitatively in the same way.

VI. NEGATIVE ENERGY?

The question of the existence of a negative-energy state naturally separates into two subsections: firstly, whether or not there exists a negative-energy initial-data set on a maximal slice, and secondly, whether or not there exists a negative-energy initial-data set when the maximality assumption is dropped. We will discuss these two questions in order.[20]

There exists a number of results about the energy of initial-data sets on maximal slices which makes it unlikely that negative energy exists in this case, but no conclusive proof has yet been discovered. For example, Brill[3] showed that all axially symmetric, moment-of-time-symmetry data sets have positive energy. Upon using the theorems in Sec. III of this paper, we can immediately prove that all axially symmetric, maximal data sets have positive energy.

In fact we can use theorems 1 and 2 to prove that a negative-energy maximal initial-data set exists if and only if a negative-energy moment-of-time-symmetry data set exists. Here we shall produce a sufficient condition for the existence of negative-energy time-symmetric data.

Consider a Riemannian manifold with $R \geq 0$, $R_{ij} \neq 0$ only on a compact domain of support. Therefore the rest of the manifold is flat and we can use Cartesian coordinates outside the region with curvature. It is easy to see

$$\oint_{\delta V} \sqrt{g}\, g^{ab} g^{cd} (g_{bc,d} - g_{cd,b}) dS_a = 0 . \tag{103}$$

Using the results in Sec. II we know that this manifold is conformally related to one in which $\bar{R} \equiv 0$. Therefore we have a positive bounded solution ϕ to

$$8\nabla^2\phi = R\phi \quad (\phi = 1 \text{ at } \infty) . \tag{104}$$

It immediately follows (with $\bar{g}_{ab} = \phi^4 g_{ab}$)

$$\oint_{\delta V} \sqrt{\bar{g}}\, \bar{g}^{ab} \bar{g}^{cd} (\bar{g}_{bc,d} - \bar{g}_{cd,b}) dS_a$$

$$= \oint_{\delta V} \sqrt{g}\, g^{ab} g^{cd} (g_{bc,d} - g_{cd,b}) dS_a - 8\int_V \nabla^2\phi \sqrt{g}\, d^3x \tag{105}$$

$$= -8\int_V \nabla^2\phi\sqrt{g}\, d^3x \tag{106}$$

$$< 0 . \tag{107}$$

Now $\bar{g}_{ab}$ forms the intrinsic geometry of a mass-free initial-data set at a moment of time symmetry since $\bar{R} = 0$. But this initial-data set has negative energy from Eq. (107).

Such manifolds, with a bump of positive scalar curvature, surrounded by flat space, do not exist for two-dimensional manifolds.[17] It is an open question whether or not they exist for three-dimensional manifolds.

Even if it can be proven that all moment-of-time-symmetry data sets have positive energy, one is still left with the problem of nonmaximal initial-data sets. In this case the momentum π^{ij} is no longer trace-free, and therefore the constraints take the form (for empty spacetime)

$$R - (1/g)(\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2) = 0 , \tag{108}$$

$$\nabla_j \pi^{ij} = 0 . \tag{109}$$

If $R \geq 0$ then obviously

$$8\nabla^2\phi = R\phi \quad (\phi = 1 \text{ at } \infty) \tag{110}$$

has a positive, bounded solution, and therefore there exists a moment of time-symmetry initial-data set with less energy than the given nonmaximal data set. These statements follow immediately as in theorems 1 and 2.

If we consider a general nonmaximal data set we can no longer assume $R \geq 0$, but as long as R is "not too negative" we will still have a positive bounded solution to Eq. (110). Therefore, the intrinsic geometry is conformally related to a moment-of-time-symmetry data set; i.e., if $\bar{g}_{ij} = \phi^4 g_{ij}$, then $\bar{R} = 0$. The energy difference between these two data sets is given by

$$16\pi\Delta E = \int_V R\phi\sqrt{g}\, d^3x . \tag{111}$$

If we assume that R is small, then we have $\phi \sim 1$, and so the energy difference is dominated by

$$16\pi\Delta E \approx \int_V R\sqrt{g}\, d^3x . \tag{112}$$

One can show that, at least near flat space, that this integral is positive, even for nonmaximal perturbations.

It has been shown elsewhere[5] that any symmetric tensor S^{ij} on a Riemannian manifold may be decomposed into three parts, a trace, $\frac{1}{3}g^{ij}S$, a transverse-trace-free part S^{ij}_{TT}, and a longitudinal part $(LW)^{ij}$, i.e.,

$$S^{ij} = S^{ij}_{TT} + (LW)^{ij} + \tfrac{1}{3}g^{ij}S\,, \tag{113}$$

where

$$(LW)^{ij} = \nabla^i W^j + \nabla^j W^i - \tfrac{2}{3}g^{ij}\nabla_k W^k\,. \tag{114}$$

These three parts are mutually globally orthogonal. Now we decompose the momentum π^{ij} in the same way

$$\pi^{ij} = \sigma^{ij}_{TT} + \sqrt{g}\,(LW)^{ij} + \tfrac{1}{2}\sqrt{g}\,g^{ij}\tau\,. \tag{115}$$

Constraint Eqs. (107) and (108) can now be expressed as

$$R = (1/g)\sigma^{ij}\sigma_{ij} + (2/\sqrt{g})\sigma^{ij}(LW)_{ij} + (LW)^{ij}(LW)_{ij} - \tfrac{3}{8}\tau^2\,, \tag{116}$$

$$\nabla_j(LW)^{ij} + \tfrac{1}{2}\nabla^i\tau = 0\,. \tag{117}$$

If it could be shown that $\int R dv \geq 0$ by virtue of the constraints, it would then seem unlikely that removing the maximality condition would permit the existence of negative-energy states. The difficulty in proving such a condition is that the curvature of the three-space leads to an integral whose sign in general is difficult to evaluate. By a straightforward integration and use of Gauss's theorem, one can show by using (116) and (117) that[18]

$$\int_V R\sqrt{g}\,d^3x = \int_V [g^{-1}\sigma^{ij}\sigma_{ij} + (LV)^{ij}(LV)_{ij} + R_{ij}\nabla^i\theta\nabla^j\theta]\sqrt{g}\,d^3x\,, \tag{118}$$

where

$$W^i = V^i + \tfrac{1}{2}\nabla^i\theta\,, \quad \nabla^2\theta = -\tfrac{3}{4}\tau\,,$$

and

$$\nabla_j(LV)^{ij} = -R^i{}_j\nabla^j\theta \tag{119}$$

is now equivalent to (117). The only term on the right-hand side of (118) that is not positive-definite involves the Ricci curvature, which is equivalent to the full Riemannian curvature in three-space. The full content of the constraints in *integral* form was employed in obtaining (119). It would seem that a final settling of the positivity question, at least along the lines we have described, awaits a deeper appreciation of the behavior of curvature on asymptotically flat, topologically Euclidean three-spaces.

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*Work supported in part by National Science Foundation under Grants No. GP30799X to Princeton University and GP 43909 to the University of North Carolina.

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[1]R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962); hereafter referred to as ADM.

[2]J. W. York, Phys. Rev. Lett. 26, 1656 (1971); J. Math. Phys. 13, 125 (1972); Phys. Rev. Lett. 28, 1082 (1972); J. Math. Phys. 14, 456 (1973); N. Ó Murchadha and J. W. York, J. Math. Phys. 14, 1551 (1973); Phys. Rev. D 10, 428 (1974); Phys. Rev. D 10, 437 (1974).

[3]D. Brill, Ann. Phys. (N.Y.) 7, 466 (1959).

[4]R. Geroch [J. Math. Phys. 13, 956 (1972)] has proposed an equivalent definition in terms of the structure of initial-data sets at spacelike infinity. Our treatment considers the initial data on the *entire* spacelike surface and relates surface integrals at spacelike infinity to volume integrals over freely specifiable (independent) initial data.

[5]Reference 2, fourth paper.

[6]Of course, singularities may be taken into account by cutting them out of the manifold and adding appropriate surface integrals over the boundaries of the cut-out regions.

[7]Reference 2, third paper.

[8]B. S. DeWitt, Phys. Rev. 160, 1113 (1967).

[9]In our analysis of the initial-value problem, total linear and angular momentum are defined by appropriate surface integrals of the tensor $(LW)^{ij}$ of (115). See Ref. 2, sixth paper, and Bull. Am. Phys. Soc. 19, 509 (1974). A detailed version will be published.

[10]This was also demonstrated by Geroch, Ref. 4.

[11]D. Brill and S. Deser, Ann. Phys. (N.Y.) 50, 548 (1968).

[12]C. W. Misner and J. A. Wheeler, Ann. Phys. (N.Y.) 2, 525 (1957); C. W. Misner, Phys. Rev. 118, 1110 (1960).

[13]Reference 2, first paper.

[14]Reference 2, fifth paper.

[15]Reference 2, fourth and fifth papers.

[16]We thank J. A. Wheeler for suggesting this interpretation of (87).

[17]We assume that the two-space is topologically Euclidean (not "conical").

[18]S. Deser [Ann. Inst. Henri Poincaré 7, 149 (1967)] obtained some expressions similar to (118) without using the orthogonal decomposition (115) of π^{ij}.

[19]See, for example, C. Miranda, *Partial Differential Equations of Elliptic Type* (Springer, New York, 1970), 2nd edition.

[20]Why does the condition trace$\pi^{ij} = 0$ play a pivotal role? Consider the case of nonsingular, asymptotically flat, topologically Euclidean vacuum initial data. Assume that π^{ij}_{TT} falls off faster than r^{-2}, i.e., we have a regular finite collection of gravity waves. Then the *total* momentum of this configuration vanishes. Hence, E is in this case simply the "rest mass" of the self-bound waves as a whole. Thus, a globally regular maximal slice in this situation is sufficient for picking out a generalized global "rest" or "center-of-momentum" frame for the waves. The "center-of-momentum" slice is only asymptotically unique, of course.