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Gauge-free quantization of the linearized equations of general relativity*

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Carmeli's method of gauge-free quantization of the electromagnetic field is extended to the case of the linearized equations of general relativity. The first-order Newman-Penrose equations with a particular choice of frames of reference and tetrad system are formulated in terms of functions over the group SU_2 and all the field variables are expressed in terms of one complex function. The equation for this function is derived and a canonical gauge-free quantization procedure is carried out. Implications of these results to the problem of quantization of the exact (nonlinear) equations of general relativity are discussed.

I. INTRODUCTION

How are the equations of general relativity to be quantized? The first question one encounters in approaching the subject is how to choose the field variables. The obvious choice are the metric tensor components $g_{\mu\nu}$, or, for a weak-field approximation, the quantities

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} , \qquad (1.1)$$

where $\eta_{\mu\nu}$ is the Minkowski metric. This choice gives rise to well-known gauge problems, stemming from the fact that the number of field variables exceeds the number of degrees of freedom.[1] In the weak-field approximation the difficulty turns out to be surmountable.[2] However, difficulties arising in the form of infinities in the expressions for observable quantities are still present.

These problems are similar to the gauge and renormalization problems of quantum electrodynamics, a similarity which was extensively used by most researchers. A comparison with the electromagnetic field indicates that the metric tensor components are not the only possible field variables for the purpose of quantization. The customary way of quantizing Maxwell's equations is to use the vector potential components A_μ as field variables and then impose gauge conditions to reduce the number of independent variables and equate this number with the number of degrees of freedom.[3] Recently, however, an alternative procedure of quantization, which is entirely gauge-free, was carried out by Carmeli.[4] His method is based on formulating Maxwell's equations in terms of functions over the group SU_2 and expressing the electromagnetic field in terms of one complex function.[5] Since the number of field variables (e.g., the real and imaginary parts of the complex function) is equal to the number of degrees of freedom, the canonical quantization procedure is straightforward, and the problem of gauge does not arise at all. Section II contains a summary of Carmeli's work.

The purpose of the present paper is to extend Carmeli's approach to the case of the gravitational field and carry out a gauge-free canonical quantization program for the linearized equations.

The starting point (Sec. III) is the Newman-Penrose null-tetrad formalism for general relativity,[6] specialized to a particular choice of a ten-parametric set of frames of reference and a tetrad system in those frames. (This particular choice was constructed by Robinson and Trautman[7] and Newman and Penrose[6]; it will be referred to as the "Robinson-Trautman frames.") The linearized equations are then formulated in terms of functions over the group SU_2. It is shown that all the field variables can be expressed in terms of one complex function; this function is expanded in the matrix elements of the irreducible representations of the group SU_2 and the equations for the coefficients are derived (Sec. IV). The corresponding Lagrangian density is defined and the canonical quantization procedure is carried out (Sec. V). We conclude with a discussion of the implications of these results to the problem of quantization of the exact (nonlinear) equations of general relativity (Sec. VI).

Throughout the paper three types of coordinates are used: the polar frames, the Robinson-Trautman frames,[7] and a third type which is obtained from the Robinson-Trautman frames by simple linear transformations. All three types are ten-parametric, i.e., they are uniquely determined by a choice of origin (4 parameters) and a choice of tetrad at the point (6 parameters). The fact that a gauge-free quantization procedure can be carried out in these frames agrees with an earlier conjecture, based on general considerations, by Halpern and the present author, concerning the central role of ten-parametric sets of coordinate systems in the problem of quantization of general relativity.[8]

II. CARMELI'S GAUGE-FREE QUANTIZATION OF THE ELECTROMAGNETIC FIELD

Consider Maxwell's equations without sources and introduce the complex vector field

$$\vec{V} = \vec{E} + i\vec{B}. \tag{2.1}$$

Using the notation

$$\begin{aligned} V_\pm &= -2^{-1/2}(V_\phi \pm iV_\theta), \\ V_0 &= V_r \end{aligned} \tag{2.2}$$

we introduce the following functions:

$$\begin{aligned} \eta_\pm &= V_\pm e^{\mp i\phi_2}, \\ \eta_0 &= V_0, \end{aligned} \tag{2.3}$$

where ϕ_2, together with the usual angular variables ϕ, θ, is such that with any value of the variables ϕ, θ, ϕ_2 we can associate a rotation $g \in O_3$, whose Euler angles are $\frac{1}{2}\pi - \phi$, θ, ϕ_2. The functions $\eta_\pm$, η_0 can be considered, therefore, as functions over the group O_3 for each value of the time t and the radius r in polar coordinates. It turns out to be more convenient to consider the functions $\eta_\pm$, η_0 over SU_2, the covering group of O_3, rather than O_3 itself. Euler angles are again employed to describe an element $v \in SU_2$,

$$v = \begin{pmatrix} \cos\frac{1}{2}\theta \exp[\frac{1}{2}i(\phi_1+\phi_2)] & i\sin\frac{1}{2}\theta\exp[-\frac{1}{2}i(\phi_1-\phi_2)] \\ i\sin\frac{1}{2}\theta\exp[\frac{1}{2}i(\phi_1-\phi_2)] & \cos\frac{1}{2}\theta\exp[-\frac{1}{2}i(\phi_1+\phi_2)] \end{pmatrix}, \tag{2.4}$$

where $\phi_1 = \frac{1}{2}\pi - \phi$.

It was shown by Carmeli[5] that Maxwell's equations in free space are equivalent to the following set of equations for the η functions:

$$\begin{aligned} &\frac{1}{\sqrt{2}}\frac{1}{r}\left(\frac{\partial}{\partial r} \pm \frac{\partial}{\partial t}\right)(r^2\eta_0) \mp K_\pm \eta_\mp = 0, \\ &\left(\pm\frac{\partial}{\partial r} + \frac{\partial}{\partial t}\right)(r\eta_\pm) + \frac{1}{\sqrt{2}}K_\pm\eta_0 = 0, \end{aligned} \tag{2.5}$$

where the operators $K_\pm$ are defined by

$$K_\pm = e^{\mp i\phi_2}\left(\pm\cot\theta\frac{\partial}{\partial\phi_2} + i\frac{\partial}{\partial\theta} \mp \csc\theta\frac{\partial}{\partial\phi_1}\right). \tag{2.6}$$

These operators, along with

$$K_3 = i\frac{\partial}{\partial\phi_2}, \tag{2.7}$$

are well known from the theory of representations of SU_2. They satisfy the following relations:

$$\begin{aligned} K_\pm T^j_{mn} &= [(j\pm m+1)(j\mp m)]^{1/2}T^j_{m\pm1,n}, \\ K_3 T^j_{mn} &= mT^j_{mn}, \end{aligned} \tag{2.8}$$

where $T^j_{mn}(v)$ are the matrix elements of the irreducible representation of weight j of the group SU_2.

Since the matrix elements $T^j_{mn}(v)$ of all the irreducible representations of the group SU_2 form a complete orthogonal set over the group, any function $f(v)$, $v \in SU_2$, which satisfies

$$\int |f(v)|^2 dv < \infty \tag{2.9}$$

can be uniquely expanded in the $T^j_{sm}(v)$:

$$f(v) = \sum_j \sum_{m=-j}^{j}\sum_{n=-j}^{j}\beta^j_{mn}T^j_{mn}(v), \tag{2.10}$$

where

$$\beta^j_{mn} = (2j+1)\int f(v)T^{j*}_{mn}(v)dv. \tag{2.11}$$

In Eqs. (2.9) and (2.11), $dv = \frac{1}{16}\pi^{-2}\sin\theta d\phi_1 d\theta d\phi_2$ is the invariant measure over SU_2, normalized so that $\int dv = 1$. $T^{j*}_{mn}(v)$ is the complex conjugate of $T^j_{mn}(v)$. Equation (2.11) follows from the orthogonality relations of the T^j_{sm}:

$$\int T^j_{sm}(v)T^{j*}_{s'm'}(v)dv = \frac{1}{2j+1}\delta_{jj'}\delta_{ss'}\delta_{mm'}. \tag{2.12}$$

If the function $f(v)$ satisfies the equation

$$f(\chi v) = e^{is\alpha}f(v), \tag{2.13}$$

where

$$\chi = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2}\end{pmatrix}, \quad \alpha \text{ real} \tag{2.14}$$

is an element of the group SU_2, and s is an integer or half-integer, then $f(v)$ is called "a quantity of spin weight s," and when it is expanded in the $T^j_{sm}(v)$, the triple sum (2.10) reduces to a double sum:

$$f(v) = \sum_{j=|s|}^{\infty}\sum_{n=-j}^{j}\beta^j_{sn}T^j_{sn}(v). \tag{2.15}$$

It was shown by Carmeli that the functions η_+, η_0, η_- are quantities of spin weight 1, 0, −1 respectively. Therefore they can be expanded as follows:

$$\begin{aligned} \eta_\pm(t,r,v) &= \sum_{j=1}^{\infty}\sum_{n=-j}^{j}\alpha^j_{\pm1,n}(t,r)T^j_{\pm1,n}(v), \\ \eta_0(t,r,v) &= \sum_{j=0}^{\infty}\sum_{n=-j}^{j}\alpha^j_{0,n}(t,r)T^j_{0,n}(v), \end{aligned} \tag{2.16}$$

where the coefficients are given by

$$\begin{aligned} (2j+1)^{-1}\alpha^j_{\pm1,n}(t,r) &= \int \eta_\pm(t,r,v)T^{j*}_{\pm1,n}(v)dv, \\ (2j+1)^{-1}\alpha^j_{0,n}(t,r) &= \int \eta_0(t,r,v)T^{j*}_{0,n}(v)dv. \end{aligned} \tag{2.17}$$

Substituting expansions (2.16) in Eq. (2.5) one can obtain a partial differential equation for the $\alpha^j_{0,m}$ and express $\alpha^j_{\pm 1,m}$ in terms of $\alpha^j_{0,m}$, i.e.,

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}\right)(r^2\alpha^j_{0,m})+j(j+1)\alpha^j_{0,m}=0\,,$$

$$j=0,1,2,\ldots,\quad m=-j,\ldots,+j \tag{2.18}$$

$$\alpha^j_{\mp 1,m}=\pm\frac{1}{[2j(j+1)]^{1/2}}\frac{1}{r}\left(\frac{\partial}{\partial r}\pm\frac{\partial}{\partial t}\right)(r^2\alpha^j_{0,m})\,,$$

$$j=1,2,\ldots,\quad m=-j,\ldots,j\,. \tag{2.19}$$

We thus arrive at the conclusion that the functions $\alpha^j_{0,m}(t,r)$ determine $\alpha^j_{\pm1,m}$ completely, through substitution in Eq. (2.19). The problem of solving Maxwell's equations reduces, therefore, to the solution of Eq. (2.18) for a single scalar complex function $\eta_0(t,r,v)$.

Let us introduce now a new set of functions

$$\alpha^j_m\equiv r^2\alpha^j_{0m}\,. \tag{2.20}$$

Because of Eq. (2.18) the functions α^j_m satisfy the equation

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}\right)\alpha^j_m+\frac{j(j+1)}{r^2}\alpha^j_m=0\,. \tag{2.21}$$

Equation (2.21) will be considered as a complex wave equation and will be quantized by the usual canonical quantization procedure.

A Lagrangian density will be defined as follows:

$$\mathcal{L}=\sum_j\sum_{m=-j}^{j}\omega_j^{-1}\mathcal{L}^j_m\,, \tag{2.22}$$

where

$$\mathcal{L}^j_m=\frac{\partial\alpha^{j*}_m}{\partial t}\frac{\partial\alpha^j_m}{\partial t}-\frac{\partial\alpha^{j*}_m}{\partial r}\frac{\partial\alpha^j_m}{\partial r}-\frac{j(j+1)}{r^r}\alpha^{j*}_m\alpha^j_m \tag{2.23}$$

and the weight factor ω_j is given by

$$\omega_j=2j(j+1)(2j+1)\,. \tag{2.24}$$

α^{j*}_m is the complex conjugate of α^j_m.

Equation (2.21) and its complex conjugate are the Euler-Lagrange equations of the Lagrangian density (2.22); i.e., Eq. (2.21) is obtained as

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}^{j*}_m}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\mathcal{L}}{\partial(\partial\alpha^{j*}_m/\partial r)}\right)-\frac{\partial\mathcal{L}}{\partial\alpha^{j*}_m}=0 \tag{2.25}$$

and the complex conjugate of Eq. (2.21) is obtained as

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}^{j}_m}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\mathcal{L}}{\partial(\partial\alpha^{j}_m/\partial r)}\right)-\frac{\partial\mathcal{L}}{\partial\alpha^{j}_m}=0\,, \tag{2.26}$$

where $\dot{\alpha}^j_m\equiv\partial\alpha^j_m/\partial t$.

The canonical momenta are defined as

$$\Pi^j_m=\frac{\partial\mathcal{L}}{\partial\dot{\alpha}^j_m}=\omega_j^{-1}\dot{\alpha}^{j*}_m\,,\qquad \Pi^{j*}_m=\frac{\partial\mathcal{L}}{\partial\dot{\alpha}^{j*}_m}=\omega_j^{-1}\dot{\alpha}^j_m\,, \tag{2.27}$$

and the equal-time commutation relations are given by

$$[\alpha^j_m(t,r),\Pi^k_n(t,r')]=[\alpha^{j*}_m(t,r),\Pi^{k*}_n(t,r')]=i\delta^{jk}\delta_{mn}\delta(r-r') \tag{2.28}$$

and all the other commutators vanish. $\delta(r-r')$ is the one-dimensional δ function.

The Hamiltonian density is given by

$$\mathcal{H}=-\mathcal{L}+\sum_j\sum_{m=-j}^{j}(\Pi^j_m\dot{\alpha}^j_m+\Pi^{j*}_m\dot{\alpha}^{j*}_m)$$

$$=\sum_j\sum_{m=-j}^{j}\left(\omega_j\Pi^{j*}_m\Pi^j_m+\omega_j^{-1}\frac{\partial\alpha^{j*}_m}{\partial r}\frac{\partial\alpha^j_m}{\partial r}+\omega_j^{-1}\frac{j(j+1)}{r^2}\alpha^{j*}_m\alpha^j_m\right)\,. \tag{2.29}$$

The merit of this method of quantization, as compared to the usual procedure of quantization of the electromagnetic field, is its being gauge free. Since the number of field variables is equal to the number of degrees of freedom, the issue of choice of gauge does not arise at all.

III. LINEARIZED EQUATIONS OF GENERAL RELATIVITY IN THE NULL-TETRAD FORMALISM

The "method of spin coefficients," utilizing a null-tetrad formalism, was introduced by Newman and Penrose[6] as a new approach to the mathematical expression of the ideas of general relativity. The linearized theory of the Newman-Penrose formalism was worked out by Janis and Newman.[9] Higher order approximations to the full nonlinear theory were subsequently worked out by Torrence and Janis,[10] Couch *et al.*,[11] and others; the formalism was successfully applied to problems of gravitational radiation,[6,10,11] and led to the discovery of new conserved quantities.[12]

The null-tetrad formalism is constructed in four-dimensional Riemannian space-time by considering a tetrad system of vectors $l_\mu, m_\mu, m^*_\mu, n_\mu$, where l_μ, n_μ are real null-vectors, m_μ and its complex conjugate m^*_μ are complex null vectors. They satisfy the orthogonality relations

$$l_\mu n^\mu=-m_\mu m^{\mu*}=1\,,\qquad l_\mu m^\mu=l_\mu m^{\mu*}=n_\mu m^\mu=n_\mu m^{\mu*}=0\,. \tag{3.1}$$

In the linearized approximation to general relativity, as formulated utilizing the method of spin coefficients, the field variables are defined as

$$\begin{aligned}
&\Psi_0 \equiv -C_{\mu\nu\rho\sigma} l^\mu m^\nu l^\rho m^\sigma ,\\
&\Psi_1 \equiv -C_{\mu\nu\rho\sigma} l^\mu n^\nu l^\rho m^\sigma ,\\
&\Psi_2 \equiv -C_{\mu\nu\rho\sigma} m^{\mu *} n^\nu l^\rho m^\sigma ,\\
&\Psi_3 \equiv -C_{\mu\nu\rho\sigma} m^{\mu *} n^\nu l^\rho n^\sigma ,\\
&\Psi_4 \equiv -C_{\mu\nu\rho\sigma} m^{\mu *} n^\nu m^{\rho *} n^\sigma ,
\end{aligned} \tag{3.2}$$

where $C_{\mu\nu\rho\sigma}$ is the linearized Weyl tensor.[13] The Ψ_A's are considered a field in flat space-time. A particular choice of a ten-parametric set of coordinate systems in flat space-time will now be made.[7,9] The null polar coordinate system $(u, \tilde{r}, \theta, \phi)$ is defined by $(2u+\tilde{r})/\sqrt{2}$ being ordinary time, $\tilde{r}/\sqrt{2}$ being the radial distance in the usual polar coordinates, and θ, ϕ being the usual polar angles. With this choice of coordinates, u labels null hypersurfaces ($\sqrt{2}u$ is the usual retarded time parameter) and $\tilde{r}$ is an affine parameter along the null geodesics contained in the null hypersurfaces. In these null polar coordinate systems the tetrad system will be chosen as follows:

$$\begin{aligned}
&l^\mu = \delta_1^\mu , \quad n^\mu = \delta_0^\mu - \delta_1^\mu ,\\
&m^\mu = \frac{1}{r}\left(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu\right)
\end{aligned} \tag{3.3}$$

(l^μ is the outward null vector tangent to the hypersurface $u=$const, n^μ is the inward null vector which points towards $\tilde{r}=0$, and m^μ is tangent to the sphere $\tilde{r}=$const contained within the hypersurface $u=$const).

With these choices of coordinate systems and the tetrad system the linearized equations of general relativity take the following form:

$$\frac{\partial\Psi_0}{\partial u} - \frac{1}{2}\frac{\partial\Psi_0}{\partial\tilde{r}} - \frac{1}{2\tilde{r}}\Psi_0 + \frac{\sqrt{2}}{2\tilde{r}}\eth\Psi_1 = 0 , \tag{3.4a}$$

$$\frac{\partial\Psi_1}{\partial u} - \frac{1}{2}\frac{\partial\Psi_1}{\partial\tilde{r}} - \frac{1}{\tilde{r}}\Psi_1 + \frac{\sqrt{2}}{2\tilde{r}}\eth\Psi_2 = 0 , \tag{3.4b}$$

$$\frac{\partial\Psi_2}{\partial u} - \frac{1}{2}\frac{\partial\Psi_2}{\partial\tilde{r}} - \frac{3}{2\tilde{r}}\Psi_2 + \frac{\sqrt{2}}{2\tilde{r}}\eth\Psi_3 = 0 , \tag{3.4c}$$

$$\frac{\partial\Psi_3}{\partial u} - \frac{1}{2}\frac{\partial\Psi_3}{\partial\tilde{r}} - \frac{2}{\tilde{r}}\Psi_3 + \frac{\sqrt{2}}{2\tilde{r}}\eth\Psi_4 = 0 , \tag{3.4d}$$

$$\frac{\partial\Psi_1}{\partial\tilde{r}} + \frac{4}{\tilde{r}}\Psi_1 + \frac{\sqrt{2}}{2\tilde{r}}\eth^*\Psi_0 = 0 , \tag{3.4e}$$

$$\frac{\partial\Psi_2}{\partial\tilde{r}} + \frac{3}{\tilde{r}}\Psi_2 + \frac{\sqrt{2}}{2\tilde{r}}\eth^*\Psi_1 = 0 , \tag{3.4f}$$

$$\frac{\partial\Psi_3}{\partial\tilde{r}} + \frac{2}{\tilde{r}}\Psi_3 + \frac{\sqrt{2}}{2\tilde{r}}\eth^*\Psi_2 = 0 , \tag{3.4g}$$

$$\frac{\partial\Psi_4}{\partial\tilde{r}} + \frac{1}{\tilde{r}}\Psi_4 + \frac{\sqrt{2}}{2\tilde{r}}\eth^*\Psi_3 = 0 . \tag{3.4h}$$

The angular differential operator $\eth$ is defined by its operation on "quantities of spin weight s." A quantity η is defined as having spin weight s if the transformation

$$m^\mu \to m'^\mu = e^{i\phi_2} m^\mu \tag{3.5}$$

in the choice of the tetrad system [see Eq. (3.3)] induces the transformation

$$\eta \to \eta' = e^{is\phi_2}\eta . \tag{3.6}$$

The operators $\eth$ and $\eth^*$ are now defined by

$$\begin{aligned}
&\eth\eta = -(\sin\theta)^s\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)[(\sin\theta)^{-s}\eta],\\
&\eth^*\eta = -(\sin\theta)^{-s}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)[(\sin\theta)^{s}\eta],
\end{aligned} \tag{3.7}$$

where η is a quantity of spin weight s.

It follows from Eqs. (3.2) and (3.6) that the functions Ψ_A are quantities of spin weight $2-A$; and therefore, by the definition (3.7), the last terms on the left-hand side of Eqs. (3.4) are well defined.

The relationship between functions of the polar coordinates (i.e., functions defined on the surface of a sphere) which are "quantities of spin weight s" (Eq. 3.6) and functions over the group SU_2 which are "quantities of spin weight s" [Eq. (2.13)] is described in the following lemma:

Lemma. If $\eta(\theta, \phi)$ is a quantity of spin weight s, i.e., it satisfies Eq. (3.6), then the function over the group SU_2 defined by

$$f(v) = \eta(\theta, \phi)e^{-is\phi_2} \tag{3.8}$$

satisfies Eq. (2.13). (The parameters $\phi_1 = \frac{1}{2}\pi - \phi$, θ, ϕ_2 describe the elements of the group SU_2 [see Eq. (2.4)].)

The proof of the lemma is contained in a paper by Carmeli[14,15] on a group-theoretic approach to the new conserved quantities in general relativity.

IV. THE LINEARIZED EQUATIONS IN TERMS OF FUNCTIONS OVER THE GROUP SU_2

Corresponding to the given field variables Ψ_A [Eq. (3.4)] let us define the following functions ϕ_s:

$$\phi_s(u, \tilde{r}, v) \equiv \Psi_{2-s}(u, \tilde{r}, \theta, \phi)e^{-is\phi_2} , \tag{4.1}$$

where $s = -2, -1, 0, 1, 2$. The ϕ_s are functions of the coordinates u and $\tilde{r}$ as well as the elements v of the group SU_2. By virtue of the lemma at the end of Sec. III they are quantities of spin weight s.

It follows from Eqs. (3.7) and (3.8) that if $\eta(\theta, \phi)$ is a quantity of spin weight s defined over the sphere and $f(v)$ is the corresponding quantity of spin weight s defined over the group SU_2, then

$$\begin{aligned}
&\eth\eta(\theta, \phi) = ie^{+i(s+1)\phi_2}K_+ f(v) ,\\
&\eth^*\eta(\theta, \phi) = ie^{+i(s-1)\phi_2}K_- f(v) ,
\end{aligned} \tag{4.2}$$

where K_+ and K_- are the infinitesimal operators of the group SU_2 [Eq. (2.6)].

Substitution of Eqs. (4.1) and (4.2), with $\eta(\theta, \phi) = \Psi_A(u, \tilde{r}, \theta, \phi)$ and $f(v) = \phi_s(u, \tilde{r}, v)$ in Eqs. (3.6) yields the linearized equations of general relativity in terms of the functions ϕ_s over the group SU_2 (see Ref. 16):

$$\frac{\partial\phi_2}{\partial u} - \frac{1}{2}\frac{\partial\phi_2}{\partial\tilde{r}} - \frac{1}{2\tilde{r}}\phi_2 + i\frac{\sqrt{2}}{2\tilde{r}}K_+\phi_1 = 0, \quad (4.3a)$$

$$\frac{\partial\phi_1}{\partial u} - \frac{1}{2}\frac{\partial\phi_1}{\partial\tilde{r}} - \frac{1}{\tilde{r}}\phi_1 + i\frac{\sqrt{2}}{2\tilde{r}}K_+\phi_0 = 0, \quad (4.3b)$$

$$\frac{\partial\phi_0}{\partial u} - \frac{1}{2}\frac{\partial\phi_0}{\partial\tilde{r}} - \frac{3}{2\tilde{r}}\phi_0 + i\frac{\sqrt{2}}{2\tilde{r}}K_+\phi_{-1} = 0, \quad (4.3c)$$

$$\frac{\partial\phi_{-1}}{\partial u} - \frac{1}{2}\frac{\partial\phi_{-1}}{\partial\tilde{r}} - \frac{2}{\tilde{r}}\phi_{-1} + i\frac{\sqrt{2}}{2\tilde{r}}K_+\phi_{-2} = 0, \quad (4.3d)$$

$$\frac{\partial\phi_1}{\partial\tilde{r}} + \frac{4}{\tilde{r}}\phi_1 + i\frac{\sqrt{2}}{2\tilde{r}}K_-\phi_2 = 0, \quad (4.3e)$$

$$\frac{\partial\phi_0}{\partial\tilde{r}} + \frac{3}{\tilde{r}}\phi_0 + i\frac{\sqrt{2}}{2\tilde{r}}K_-\phi_1 = 0, \quad (4.3f)$$

$$\frac{\partial\phi_{-1}}{\partial\tilde{r}} + \frac{2}{\tilde{r}}\phi_{-1} + i\frac{\sqrt{2}}{2\tilde{r}}K_-\phi_0 = 0, \quad (4.3g)$$

$$\frac{\partial\phi_{-2}}{\partial\tilde{r}} + \frac{1}{\tilde{r}}\phi_{-2} + i\frac{\sqrt{2}}{2\tilde{r}}K_-\phi_{-1} = 0. \quad (4.3h)$$

The functions $\phi_s(u, \tilde{r}, v)$ can be expanded in the matrix elements $T^j_{sm}(v)$ of the irreducible representations of the group SU_2 [Eq. (2.15)]:

$$\phi_s(u, \tilde{r}, v) = \sum_{j=|s|}^{\infty}\sum_{m=-j}^{j} a^j_{sm}(u, \tilde{r})T^j_{sm}(v). \quad (4.4)$$

Substitution of Eq. (4.4) in Eq. (4.3) yields, by virtue of Eq. (2.8), the equations for the coefficients $a^j_{sm}(u, \tilde{r})$:

$$\frac{\partial a^j_{2m}}{\partial u} - \frac{1}{2}\frac{\partial a^j_{2m}}{\partial\tilde{r}} - \frac{1}{2\tilde{r}}a^j_{2m} + \frac{\sqrt{2}}{2\tilde{r}}[(j-1)(j+2)]^{1/2}a^j_{1m} = 0, \quad (4.5a)$$

$$\frac{\partial a^j_{1m}}{\partial u} - \frac{1}{2}\frac{\partial a^j_{1m}}{\partial\tilde{r}} - \frac{1}{\tilde{r}}a^j_{1m} + \frac{\sqrt{2}}{2\tilde{r}}[j(j+1)]^{1/2}a^j_{0m} = 0, \quad (4.5b)$$

$$\frac{\partial a^j_{0m}}{\partial u} - \frac{1}{2}\frac{\partial a^j_{0m}}{\partial\tilde{r}} - \frac{3}{2\tilde{r}}a^j_{0m} + \frac{\sqrt{2}}{2\tilde{r}}[j(j+1)]^{1/2}a^j_{-1m} = 0, \quad (4.5c)$$

$$\frac{\partial a^j_{-1m}}{\partial u} - \frac{1}{2}\frac{\partial a^j_{-1m}}{\partial\tilde{r}} - \frac{2}{\tilde{r}}a^j_{-1m} + \frac{\sqrt{2}}{2\tilde{r}}[(j-1)(j+2)]^{1/2}a^j_{-2m} = 0, \quad (4.5d)$$

$$\frac{\partial a^j_{1m}}{\partial\tilde{r}} + \frac{4}{\tilde{r}}a^j_{1m} + \frac{\sqrt{2}}{2\tilde{r}}[(j-1)(j+2)]^{1/2}a^j_{2m} = 0, \quad (4.5e)$$

$$\frac{\partial a^j_{0m}}{\partial\tilde{r}} + \frac{3}{\tilde{r}}a^j_{0m} + \frac{\sqrt{2}}{2\tilde{r}}[j(j+1)]^{1/2}a^j_{1m} = 0, \quad (4.5f)$$

$$\frac{\partial a^j_{-1m}}{\partial\tilde{r}} + \frac{2}{\tilde{r}}a^j_{-1m} + \frac{\sqrt{2}}{2\tilde{r}}[j(j+1)]^{1/2}a^j_{0m} = 0, \quad (4.5g)$$

$$\frac{\partial a^j_{-2m}}{\partial\tilde{r}} + \frac{1}{\tilde{r}}a^j_{-2m} + \frac{\sqrt{2}}{2\tilde{r}}[(j-1)(j+2)]^{1/2}a^j_{-1m} = 0. \quad (4.5h)$$

The orthogonality property of the functions $T^j_{sm}(v)$ [Eq. (2.12)] was used in deriving Eq. (4.5).

Equations (4.5c), (4.5d), (4.5g), and (4.5h) can be used to express the a^j_{2m}, a^j_{1m}, a^j_{-1m}, and a^j_{-2m} in terms of the a^j_{0m}:

$$a^j_{-1m} = -\left(\frac{2}{j(j+1)}\right)^{1/2}\left(\tilde{r}\frac{\partial a^j_{0m}}{\partial u} - \tfrac{1}{2}\tilde{r}\frac{\partial a^j_{0m}}{\partial\tilde{r}} - \tfrac{1}{2}a^j_{0m}\right), \quad (4.6a)$$

$$a^j_{-2m} = -\left(\frac{2}{(j-1)(j+2)}\right)^{1/2} \times\left(\tilde{r}\frac{\partial a^j_{-1m}}{\partial u} - \tfrac{1}{2}\tilde{r}\frac{\partial a^j_{-1m}}{\partial\tilde{r}} - \tfrac{1}{2}a^j_{-1m}\right), \quad (4.6b)$$

$$a^j_{1m} = -\left(\frac{2}{j(j+1)}\right)^{1/2}\left(\tilde{r}\frac{\partial a^j_{0m}}{\partial\tilde{r}} + 3a^j_{0m}\right), \quad (4.6c)$$

$$a^j_{2m} = -\left(\frac{2}{(j-1)(j+2)}\right)^{1/2}\left(\tilde{r}\frac{\partial a^j_{1m}}{\partial\tilde{r}} + 4a^j_{1m}\right). \quad (4.6d)$$

The equations for the a^j_{0m} are obtained by eliminating a^j_{-1m} from Eqs. (4.5c) and (4.5g) [or, alternatively, by eliminating a^j_{1m} from Eqs. (4.5b) and (4.5f)]:

$$\tilde{r}^2\frac{\partial^2 a^j_{0m}}{\partial u\partial\tilde{r}} - \tfrac{1}{2}\tilde{r}^2\frac{\partial^2 a^j_{0m}}{\partial\tilde{r}^2} - 3\tilde{r}\frac{\partial a^j_{0m}}{\partial\tilde{r}} + 3\tilde{r}\frac{\partial a^j_{0m}}{\partial u} - 3a^j_{0m} + \tfrac{1}{2}j(j+1)a^j_{0m} = 0. \quad (4.7)$$

Using the commutation relations

$$\left[\tilde{r}^n, \frac{\partial}{\partial\tilde{r}}\right] = -n\tilde{r}^{n-1} \quad (4.8)$$

Eq. (4.7) can be written in the form

$$\left[\frac{\partial^2}{\partial u\partial\tilde{r}} - \frac{1}{2}\frac{\partial^2}{\partial\tilde{r}^2} + \frac{1}{2\tilde{r}^2}j(j+1)\right](\tilde{r}^3 a^j_{0m}) = 0. \quad (4.9)$$

So far the functions $a^j_{sm}(u, \tilde{r})$ were considered in the Robinson-Trautman coordinate system.[7] The transformation to the ordinary polar coordinate system in flat space-time (t is the ordinary time; r, θ, ϕ are the usual polar coordinates) is given by

$$t = (2u + \tilde{r})/\sqrt{2}, \qquad r = \tilde{r}/\sqrt{2}. \quad (4.10)$$

Transforming Eq. (4.9) to the ordinary polar coordinate system we obtain

$$\left[3\frac{\partial^2}{\partial t^2} + 2\frac{\partial^2}{\partial t\partial r} - \frac{\partial^2}{\partial r^2} + \frac{1}{r}j(j+1)\right](r^3 a^j_{0m}) = 0. \quad (4.11)$$

V. QUANTIZATION

The field variables to be quantized will be defined now as

$$a^j_m(t,r)=r^3a^j_{0m}(t,r)\,. \tag{5.1}$$

Because of Eq. (4.11) the a^j_m satisfy the equation

$$\left[3\frac{\partial^2}{\partial t^2}+2\frac{\partial^2}{\partial t\partial r}-\frac{\partial^2}{\partial r^2}+\frac{1}{r}j(j+1)\right]a^j_m=0\,. \tag{5.2}$$

Taking the complex conjugate of Eq. (5.2), we obtain the equation for the complex-conjugate functions a^{j*}_m:

$$\left[3\frac{\partial^2}{\partial t^2}+2\frac{\partial^2}{\partial t\partial r}-\frac{\partial^2}{\partial r^2}+\frac{1}{r}j(j+1)\right]a^{j*}_m=0\,. \tag{5.3}$$

Following Carmeli's notation[4] let us define the Lagrangian density as

$$\mathcal{L}=\sum_{j,m}\omega_j^{-1}\mathcal{L}^j_m, \tag{5.4}$$

where

$$\mathcal{L}^j_m=3\dot a^j_m\dot a^{j*}_m+\dot a^j_m\frac{\partial a^{j*}_m}{\partial r}+\frac{\partial a^j_m}{\partial r}\dot a^{j*}_m-\frac{\partial a^j_m}{\partial r}\frac{\partial a^{j*}_m}{\partial r}-\frac{j(j+1)}{r^2}a^j_m a^{j*}_m\,; \tag{5.5}$$

a dot denotes partial differentiation with respect to time; the weight factor ω_j is given by

$$\omega_j=2j(j+1)(2j+1)\,. \tag{5.6}$$

Equations (5.2) and (5.3) are now obtained as the Euler-Lagrange equations for the Lagrangian (5.4):

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot a^{j*}_m}\right)+\frac{\partial}{\partial r}\left[\frac{\partial\mathcal{L}}{\partial(\partial a^{j*}_m/\partial r)}\right]-\frac{\partial\mathcal{L}}{\partial a^{j*}_m}=0\,,\\&\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot a^{j}_m}\right)+\frac{\partial}{\partial r}\left[\frac{\partial\mathcal{L}}{\partial(\partial a^{j}_m/\partial r)}\right]-\frac{\partial\mathcal{L}}{\partial a^{j}_m}=0\,.\end{aligned} \tag{5.7}$$

The canonical momenta conjugate to a^j_m and a^{j*}_m will be defined as follows:

$$\begin{aligned}\Pi^j_m&\equiv\frac{\partial\mathcal{L}}{\partial\dot a^j_m}=\omega_j^{-1}\left(3\dot a^{j*}_m+\frac{\partial a^{j*}_m}{\partial r}\right),\\\Pi^{j*}_m&=\frac{\partial\mathcal{L}}{\partial\dot a^{j*}_m}=\omega_j^{-1}\left(3\dot a^{j}_m+\frac{\partial a^{j}_m}{\partial r}\right),\end{aligned} \tag{5.8}$$

and the Hamiltonian density is given by

$$\begin{aligned}\mathcal{H}&=\sum_{j,m}(\Pi^j_m\dot a^j_m+\Pi^{j*}_m\dot a^{j*}_m)-\mathcal{L}\\&=\sum_{j,m}\left(\tfrac{1}{3}\omega_j\Pi^j_m\Pi^{j*}_m-\tfrac{1}{3}\Pi^j_m a^j_m-\tfrac{1}{3}\Pi^{j*}_m a^{j*}_m-\frac{4\omega_j^{-1}}{3}\frac{\partial a^j_m}{\partial r}\frac{\partial a^{j*}_m}{\partial r}+\frac{\omega_j^{-1}j(j+1)}{r^2}a^j_m a^{j*}_m\right).\end{aligned} \tag{5.9}$$

The a^j_m, a^{j*}_m, Π^j_m, Π^{j*}_m are now assumed to be operators, satisfying the canonical equal-time commutation relations:

$$\begin{aligned}[a^j_m(t,r),\Pi^{j'}_{m'}(t,r')]&=[a^{j*}_m(t,r),\Pi^{j'*}_{m'}(t,r')]\\&=i\delta^{jj'}\delta_{mm'}\delta(r-r')\,,\end{aligned} \tag{5.10}$$

and all the other commutators vanish. $\delta(r-r')$ in Eq. (5.10) is the one-dimensional δ function.

The following notation will be useful[4]:

$$x^0=t\,,\quad x^1=r\,, \tag{5.11}$$

$$x_\mu=\eta_{\mu\nu}x^\nu\,,\quad \mu=0,1 \tag{5.12}$$

where the summation convention over $\nu=0,1$ is assumed in Eq. (5.12) and $\eta_{\mu\nu}$ is defined as

$$\eta_{\mu\nu}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{5.13}$$

By the usual variational techniques the energy-momentum stress tensor will be defined in complete analogy with the electromagnetic case by

$$T_{\mu\nu}=-\eta_{\mu\nu}\mathcal{L}+\sum_{jm}\left[\frac{\partial\mathcal{L}}{\partial(\partial a^j_m/\partial x_\mu)}\frac{\partial a^j_m}{\partial x^\nu}+\frac{\partial\mathcal{L}}{\partial(\partial a^{j*}_m/\partial x_\mu)}\frac{\partial a^{j*}_m}{\partial x^\nu}\right]. \tag{5.14}$$

The conservation laws

$$\frac{\partial}{\partial x_\mu}T_{\mu\nu}=0 \tag{5.15}$$

follow from the Euler-Lagrange equations (5.7). Comparison of Eqs. (5.9) and (5.14) yields

$$T_{00}=\mathcal{H}. \tag{5.16}$$

Defining the integrated quantities P_ν ($\nu=0,1$) by

$$P_\nu=\int T_{0\nu}dr \tag{5.17}$$

and P^μ ($\mu=0,1$) by

$$P^\mu=\eta_{\mu\nu}P_\nu, \tag{5.18}$$

it follows from the commutation relations (5.10) that

$$\begin{aligned}i[P^\mu,a^j_m]&=\partial a^j_m/\partial x_\mu\,,\\i[P^\mu,a^{j*}_m]&=\partial a^{j*}_m/\partial x_\mu\,.\end{aligned} \tag{5.19}$$

VI. ON THE QUANTIZATION OF THE EXACT (NONLINEAR) EQUATIONS OF GENERAL RELATIVITY

Can our approach be extended and used to quantize the exact, nonlinearized equations of general relativity? In trying to carry out the procedure of Secs. IV and V for the case of the exact Newman-Penrose set of equations one confronts the complexity of the equations. It is unclear, however, whether the difficulty is one of mathematical skill, or of a more fundamental nature.

It seems that the nonlinearity of the field equations is not, in itself, an obstacle to the construction of commutation relations between field variables. For example, in the case of the Yang-Mills theory[17] (which is nonlinear) several different quantization schemes were proposed by Arnowitt and Fickler,[18] Schwinger,[19] DeWitt,[20] Mandelstam,[21] and Loos,[22] and consistency conditions were formulated by Fickler and Russo.[23] Rylov[24] has recently proposed a new method of quantization which enabled him to obtain exact solutions for a simple case of a nonlinear scalar field, without using perturbation theory.

One feature of the present quantization procedure indicates that a generalization to the case of the exact, nonlinear theory, may be possible. It was previously argued on general grounds by Halpern and the present author[8] that a successful quantization program must be carried out within a ten-parametric set of coordinate systems; i.e., given the equations of general relativity, a particular choice of a ten-parametric set of coordinate system must be made; only after the equations are formulated in terms of the chosen coordinates can they be quantized. The present quantization procedure is in full accord with this approach: The $(u, \tilde{r}, \theta, \phi)$ frames used in Secs. III and IV as well as the polar frames (t, r, θ, ϕ), used in Secs. IV and V are, indeed, ten-parametric sets; each is uniquely determined by a choice of origin (four parameters) and a choice of a tetrad of unit vectors at the origin (six parameters). The $(u, \tilde{r}, \theta, \phi)$ frames are easily generalized to the general case of a Riemannian space-time. In fact, they were originally defined by Robinson and Trautman[7] as well as Newman and Penrose[6] in a general Riemannian space-time and then specialized to the case of flat space-time when the first-order equations were considered.

APPENDIX

The equations for the functions a^j_{0m} are derived in Sec. IV for two types of coordinate systems: the Robinson-Trautman coordinates $(u, \tilde{r}, \theta, \phi)$ and the usual polar coordinates (t, r, θ, ϕ). Let us define a third kind, a "primed" coordinate system (u', r', θ, ϕ) as follows:

$$u' = (u + \tilde{r})/\sqrt{2}\,, \qquad r' = \tilde{r}/\sqrt{2}\,; \qquad \text{(A1)}$$

θ and ϕ are, again, the usual polar angles. The physical meaning of the primed coordinate systems is given by the observation that $2u'$ is the usual advanced time and $r' = r$ is the usual radial distance.

A transformation of Eq. (4.7) using Eqs. (A1) yields the equation for the a^j_{0m} in the primed coordinate system:

$$\left(\frac{\partial^2}{\partial u'^2} - \frac{\partial^2}{\partial r'^2}\right)(r'^3 a^j_{0m}) + \frac{j(j+1)}{r'^2}(r'^3 a^j_{0m}) = 0\,. \qquad \text{(A2)}$$

It is interesting to note that when the field variables $a^j_m \equiv r'^3 a^j_{0m}$ [Eq. (5.1)] are substituted in Eq. (A2), the equation is formally identical to the one obtained by Carmeli for the electromagnetic field[4] [Eq. (2.21)]. The coordinate u' in Eq. (A2) is, of course, different from the ordinary time t in Eq. (2.21).

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[1]For review articles which contain references to original papers see P. G. Bergmann, in *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1962), Vol. IV, p. 247; B. S. DeWitt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), p. 266; J. L. Anderson, in *Gravitation and Relativity*, edited by H. Y. Chiu and W. F. Hoffmann, (Benjamin, New York, 1964), p. 279; B. S. DeWitt, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964), p. 587; P. A. M. Dirac, in *Contemporary Physics: Trieste Symposium* 1968, edited by A. Salam (International Atomic Energy Agency, Vienna, Austria, 1969), Vol. 1, p. 539; A. Komar, in *Relativity*, edited by M. Carmeli, S. I. Fickler, and L. Witten (Plenum, New York, 1970), p. 19; P. G. Bergmann, in *Relativity and Gravitation*, edited by C. G. Kuper and A. Peres (Gordon and Breach, New York, 1971), p. 23.

[2]R. P. Feynman, Acta, Phys. Polon. 24, 697 (1963); L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967); B. S. DeWitt, Phys. Rev. 162, 1195 (1967); 171, 1834(E) (1968); S. Mandelstam, *ibid.* 175, 1604 (1968).

[3]S. N. Gupta, Proc. Phys. Soc. Lond. A63, 681 (1950); K. Bleuler, Helv. Phys. Acta 23, 567 (1950).

[4]M. Carmeli, Nuovo Cimento 67B, 103 (1970).

[5]M. Carmeli, J. Math. Phys. 10, 1699 (1969).

[6]E. T. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).

[7]I. Robinson and A. Trautman, Proc. R. Soc. 265, 463 (1962).

[8]M. Halpern and S. Malin, in *Studies of Mathematical Physics*, edited by A. O. Barut (D. Reidel, Dordrecht, Holland, 1974), p. 111. See also M. Halpern and S. Malin, J. Math. Phys. 12, 213 (1971).

[9]A. I. Janis and E. T. Newman, J. Math. Phys. 6, 902

(1965).
[10]R. Torrence and A. Janis, J. Math. Phys. 8, 1355 (1967).
[11]W. E. Couch, R. J. Torrence, A. I. Janis, and E. T. Newman, J. Math. Phys. 9, 484 (1968); W. H. Halliday and A. I. Janis, *ibid.* 11, 578 (1970).
[12]E. T. Newman and R. Penrose, Phys. Rev. Lett. 15, 231 (1965); E. T. Newman and R. Penrose, Proc. R. Soc. A305, 175 (1968).
[13]R. Sachs and P. G. Bergmann, Phys. Rev. 112, 674 (1958).
[14]M. Carmeli, J. Math. Phys. 10, 569 (1969).
[15]Quantities of spin weight s were first introduced by E. T. Newman and R. Penrose [J. Math. Phys. 7, 863 (1966)] as functions defined on the surface of a sphere. These functions were subsequently discussed by J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, and E. C. G. Sudarshan [*ibid.* 8, 2155 (1967)] and shown to be related to the matrix elements T^j_m of the irreducible representations of the rotation group. H. E. Moses [Ann. Phys. (N.Y.) 41, 166 (1967)] defined and discussed "generalized surface harmonics," which are closely related to the Newman-Penrose "spin-s spherical harmonics." The group-theoretical and geometrical interpretation of these functions was established by Carmeli (Ref. 14). The physical interpretation of these functions in terms of eigenfunctions of the radial spin operator was recently derived by the present author, who used quantities of spin weight $s=\pm\frac{1}{2}$ to formulate the Weyl and Dirac equations in terms of functions over the group SU_2 [J. Math. Phys. (to be published)]. See also A. O. Barut, M. Carmeli, and S. Malin, Ann. Phys. (N.Y.) 77, 454 (1973).
[16]When $\Psi_A(u, \tilde{r}, \theta, \phi)$ and $\phi_s(u, \tilde{r}, v)$ are substituted for $\eta(\theta, \phi)$ and $f(v)$ in Eq. (4.2), the coordinates $u, \tilde{r}$ are considered as parameters: for fixed values of $u, \tilde{r}$ the functions Ψ_A are defined over the sphere and the functions ϕ_s are defined over the group SU_2.
[17]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).
[18]R. L. Arnowitt and S. I. Fickler, Phys. Rev. 127, 1821 (1962).
[19]J. Schwinger, Phys. Rev. 125, 1043 (1962).
[20]B. S. DeWitt, Phys. Rev. 162, 1195 (1967); 162, 1239 (1967).
[21]S. Mandelstam, Phys. Rev. 175, 1580 (1968).
[22]H. Loos, Phys. Rev. 188, 2342 (1969).
[23]S. I. Fickler and M. Russo, Phys. Rev. D 3, 1782 (1971).
[24]Yu. A. Rylov, Int. J. Theor. Phys. 6, 181 (1972).

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Gravitational energy*

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The total energy of asymptotically flat, nonsingular gravitational fields is discussed in terms of the initial data on a spacelike hypersurface. The total energy is a surface integral which we relate to a volume integral over "sources," including the contributions of gravitational waves. This relationship follows from a recent formulation of the initial-value equations of general relativity and is free of coordinate conditions. We show that time-symmetric initial-data sets form minima of energy among all initial-data sets on maximal hypersurfaces. Combining this result with a result of Brill, it follows that every nonsingular, axisymmetric, asymptotically flat spacetime admitting at least one maximal slice has non-negative total energy. Negative "interaction energy" contributions are described and a discussion of nonmaximal initial data is given.

I. INTRODUCTION

It is well known that the total energy of an asymptotically flat gravitational field defined on a spacelike hypersurface has two distinctive features: (I) It may always be calculated as an integral over a two-dimensional surface surrounding the sources, including among these gravitational waves; (II) There is, in general, no well-defined local expression for gravitational energy density. These two features are closely related and show that energy is a global rather than a local property of a gravitational field. As a result of (I) the energy of spacetimes with closed spacelike slices may be defined to vanish identically. We shall therefore confine our discussion to nonsingular asymptotically flat spacetimes.

The initial-value data on a spacelike slice form Cauchy data for a spacetime, i.e., define it uniquely and completely for some finite time, and therefore the energy, which is a constant of the motion, should be describable in a natural manner purely in terms of the initial data. This means that we can limit our attention to a given "state" of the gravitational field, defined by the initial data on