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Gravitational red-shift measurements as tests of nonmetric theories of gravity*

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Gravitational red-shift measurements can be powerful tools for testing nonmetric theories of gravity. We show that some nonmetric gravitation theories may predict the "wrong" first-order gravitational red shift even while possessing the "correct" Newtonian limit, whereas metric theories with the "correct" Newtonian limit automatically predict the "correct" red shift. We also argue that *all* nonmetric theories ultimately predict a red shift which depends on the nature of the atomic clock whose red shift is being measured. These statements can be worded in the form of a conjecture: Any complete, self-consistent, and relativistic theory of gravity that embodies the universality of gravitational red shift is necessarily a metric theory. Restricting attention to a certain subclass of nonmetric theories and to electromagnetic atomic clocks residing in static, spherically symmetric gravitational fields, we give a detailed proof of this conjecture. Red-shift experiments can thus distinguish between metric and nonmetric theories of gravity. For weak gravitational fields appropriate to the solar system, we compare the sensitivity of red-shift experiments and Eötvös experiments to nonmetric effects. Particularly important are *null* red-shift measurements (comparison of the red shifts of *different* clocks at the same location in a gravitational field) designed to be sensitive to second-order gravitational red shifts.

I. INTRODUCTION AND SUMMARY

During the past decade, gravitational red-shift experiments have fallen into relative disfavor among tests of theories of gravitation. This disfavor has come about for two reasons: (1) Every "metric" theory of gravity[1] which has the correct Newtonian limit *automatically predicts the "correct" lowest-order red shift*. According to any metric theory of gravity two identical, ideal atomic clocks at rest at different locations, $\vec{x}_1$ and $\vec{x}_2$ in a static gravitational field differ in their frequencies according to

$$z \equiv \frac{\nu_1 - \nu_2}{\nu_2} = [g_{00}(\vec{x}_1)/g_{00}(\vec{x}_2)]^{1/2} - 1 \; , \tag{1}$$

where g_{00} is a particular component of the spacetime metric. This prediction is independent of the nature of the ideal clocks. For weak gravitational fields, agreement at lowest order with Newtonian gravitation theory demands[2]

$$g_{00} = 1 - 2U + O(U^2) \; , \tag{2}$$

where U is the Newtonian gravitational potential. Thus, to first order, every metric theory predicts

$$z = U(\vec{x}_2) - U(\vec{x}_1) \; . \tag{3}$$

(2) For weak fields, a simple argument based on the weak equivalence principle (the composition independence of free fall) and conservation of energy was used[3] to "derive" the first-order gravitational red shift [Eq. (3)], independent of theory. Thus gravitational red-shift measurements came to be viewed as weak tests of gravitation theory.[3,4]

But some authors[5] attempted to bolster the theoretical backing of red-shift experiments by arguing that the gravitational red shift proves that spacetime must be curved, i.e., that the correct theory of gravity must be a metric theory. However, their arguments were not rigorous, merely suggestive. Hoping to find ways to strengthen the gravitational red shift, Thorne[6] challenged relativity theorists to search for a complete, self-consistent gravitation theory which has the correct Newtonian limit yet which gives the "wrong" red-shift prediction. Such a theory would provide a counterexample, or a "foil" against which to gauge the significance of gravitational red-shift measurements.

In this paper we meet this challenge: *The desired "foils" are contained in the class of theories called nonmetric theories of gravity*. These are theories whose mathematical equations cannot be put into metric form.[1] In particular, we show that some nonmetric theories may predict the *wrong* first-order red shift even though they may have the "right" Newtonian limit. Furthermore, *we present evidence indicating that every nonmetric theory necessarily predicts a gravitational red shift which depends on the nature of the clock whose red shift is being measured*. This result can be worded in the form of a conjecture: *Any complete, self-consistent, and relativistic theory of gravity that embodies the universality of grav-*

itational red shift (UGR) is necessarily a metric theory.

This conjecture provides a theoretical link between gravitational red-shift experiments and Eötvös experiments. It has long been known that every metric theory of gravity embodies the weak equivalence principle (WEP), the principle which states that the trajectories of freely falling "test" bodies are independent of their composition. In the early 1960's, Leonard Schiff suggested the converse was true and formulated what we now call Schiff's conjecture[1]: Any complete, self-consistent and relativistic theory of gravity that embodies WEP is necessarily a metric theory. Because of Schiff's conjecture, Eötvös experiments have taken on added importance as direct tests of WEP, and thereby as direct tests of the metric hypothesis. The UGR conjecture does the same for gravitational red-shift experiments. From this point of view, gravitational red-shift experiments can be seen as complementary to Eötvös experiments as tests of the viability of metric and nonmetric theories of gravity.

The contrast between the fundamental postulates of metric and nonmetric theories of gravitation reveals a simple "plausibility argument" supporting the UGR conjecture. Metric theories satisfy three postulates: (i) that there exists a metric g, (ii) that the trajectories of freely falling test bodies are geodesics of g, and (iii) that in local freely falling frames the nongravitational laws of physics are those of special relativity. Because of (iii), the frequencies of atomic clocks as measured in local, freely falling frames are functions only of universal, atomic constants ($\hbar$, e, c, etc.) and are independent of the external gravitational field. Then, the comparison of the frequencies of two identical atomic clocks involves only comparing the trajectories of their local, freely falling frames. But because of (ii) these trajectories are universal; hence the comparison of the frequencies is universal, and hence the gravitational red shift is universal. Nonmetric theories, by definition, violate one or more of postulates (i), (ii), and (iii); hence it is not surprising (and probably necessary) that they should violate UGR, at least for some kinds of clocks.

For the special case of clocks made up of electromagnetically interacting particles (atoms), we make this "proof" of the UGR conjecture more precise (at the expense of restricting its realm of validity somewhat), by employing the "$TH\epsilon\mu$" formalism for analyzing theories of gravity—both metric and nonmetric—developed by Lightman and Lee.[7] Their formalism focuses on a certain class of gravitational theories and characterizes their equations of motion for charged particles in an external, static, spherically symmetric gravitational potential U ($=M/r$) by two arbitrary functions $T(U)$ and $H(U)$, and characterizes the response of electromagnetic fields to the external potential ["gravitationally modified Maxwell (GMM) equations"] by two functions $\epsilon(U)$ and $\mu(U)$. The forms of T, H, ϵ, and μ vary from theory to theory: they satisfy only the condition that as $U\to 0$ (vanishing gravitational field far from source) $T, H, \epsilon, \mu \to 1$. Every metric theory satisfies

$$\epsilon = \mu = (H/T)^{1/2} \tag{4}$$

for all U. Conversely, any theory within this class that satisfies Eq. (4) can have its GMM equations cast into metric form. Thus, with respect to electromagnetism, theories within this class are metric theories if and only if they satisfy Eq. (4). There may exist theories that are metric with respect to electromagnetism, yet nonmetric with respect to nuclear interactions or weak interactions; such theories cannot be fully analyzed using the $TH\epsilon\mu$ framework, and so we ignore them for the purposes of this paper.

Lightman and Lee used their formalism to devise a restricted "proof" of Schiff's conjecture; they analyzed the free fall of test bodies made up of electromagnetically interacting point particles and showed that every nonmetric theory in this class must ultimately violate the weak equivalence principle. They showed that the experimental verification of WEP via Eötvös experiments was sufficient to rule out several nonmetric theories of gravity previously thought to be viable.[7]

We have used a straightforward quantum-mechanical procedure to adapt the Lightman-Lee formalism to the analysis of atomic clocks in various theories of gravity, and have found that, in general, the ticking rates of atomic clocks are affected by a gravitational field ("gravitational red shift") in a manner which depends on their structure.[8] As particular examples we discuss three clocks whose frequencies are defined by three different atomic transitions of hydrogen: (1) transitions between principal levels ("principal clock"), (2) transitions between two fine-structure levels within a principal level ("fine-structure clock"), and (3) transitions between two hyperfine levels ("hyperfine clock," hydrogen-maser clock).

When one of each pair of identical clocks is far from the gravitating source, $U=0$ [compare Eq. (1) with $\vec{x}_1\to\infty$, $g_{00}(\vec{x}_1)\to 1$]; we find that the predicted red shifts are

$$\begin{aligned} z_P &= T^{-1/2}(T\epsilon^2/H) - 1\ , \\ z_F &= T^{-1/2}(T\epsilon^2/H)^2 - 1\ , \\ z_H &= T^{-1/2}(T\epsilon^2/H)(\epsilon/\mu) - 1\ , \end{aligned} \tag{5}$$

where the subscripts P, F, and H denote principal, fine structure, and hyperfine, respectively. Equations (5) indicate that the red shift is universal, independent of the nature of the clock if and only if

$$\epsilon = \mu = (H/T)^{1/2} \tag{6}$$

for all U, that is, if and only if the theory of gravity is a metric theory. Thus every nonmetric theory of gravity within our chosen class violates UGR.

Detailed derivations leading to Eqs. (5) are given in Sec. II. Section III focuses on the weak-field limit appropriate for the solar system, and discusses the significance of particular red-shift experiments. We also point out a candidate for Thorne's "foil": a nonmetric version of Whitehead's theory that has the "right" Newtonian limit yet gives the "wrong" red shift. Concluding remarks are made in Sec. IV. In an appendix we discuss the Dirac equation in the curved spacetime appropriate to metric theories of gravity, and compare the results with those of the $TH\epsilon\mu$ formalism derived in Sec. II.

II. ATOMIC CLOCKS IN THE $TH\epsilon\mu$ FORMALISM

A. Gravitationally modified Dirac equation

The $TH\epsilon\mu$ formalism restricts its attention to those theories of gravity whose equations of motion for a charged particle in a static, spherically symmetric (SSS) gravitational potential U can be derived from the Lagrangian

$$L = \int \mathcal{L}\, dt = \int [m(T - Hv^2)^{1/2} + eA_i v^i]\, dt \ , \tag{7}$$

and whose GMM equations have the form

$$\vec{\nabla}\cdot(\epsilon\vec{E}) = 4\pi\rho \ ,$$
$$\vec{\nabla}\times(\mu^{-1}\vec{B}) = 4\pi\vec{J} + \frac{\partial(\epsilon\vec{E})}{\partial t} \ . \tag{8}$$

The symbols and notation are as in Ref. 7. The vector potential A_i is related to $\vec{E}$ and $\vec{B}$ by

$$\vec{E} = \vec{\nabla}A_0 - \frac{\partial\vec{A}}{\partial t} \ ,$$
$$\vec{B} = \vec{\nabla}\times\vec{A} \ . \tag{9}$$

The $txyz$ coordinate system is assumed to be at rest relative to the SSS source.

Now, to be complete, any theory of gravity must provide a set of mathematical laws to describe quantum-mechanical systems, such as atomic clocks, in the presence of gravity. When gravity is absent, these laws should be consistent with those of special relativistic quantum theory, and when gravity is present they should be consistent with the other nonquantum equations of the theory, at least in an appropriate correspondence limit. Thus, in the particular case of $TH\epsilon\mu$ theories, the quantum laws should be derivable from the Lagrangian density $\mathcal{L}$ using the standard formalism of relativistic quantum mechanics. According to this formalism, the Hamiltonian density $\mathcal{H}$ corresponding to $\mathcal{L}$ is given by

$$\mathcal{H} = v^\alpha \frac{\partial\mathcal{L}}{\partial v^\alpha} - \mathcal{L} \ . \tag{10}$$

After some simplification we obtain from Eqs. (7) and (10)

$$\mathcal{H} = T^{1/2}[m^2 + H^{-1}(\vec{p} - e\vec{A})^2]^{1/2} - eA_0 \ , \tag{11}$$

where

$$p_\alpha = \frac{\partial\mathcal{L}}{\partial v^\alpha} \ . \tag{12}$$

To obtain an expression for the square root in Eq. (11) which is linear in m and $\vec{p}$ we define four-dimensional "Dirac" matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad 1 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{13}$$

where I is the two-dimensional unit matrix, and σ_α are the Pauli spin matrices.[9] Then $\mathcal{H}$ may be written as the matrix

$$\mathcal{H} = T^{1/2}[m\beta + H^{-1/2}\vec{\alpha}\cdot(\vec{p} - e\vec{A})] - eA_0 1 \ . \tag{14}$$

(Hereafter we will omit the unit matrix 1.) The gravitationally modified Dirac (GMD) equation is then

$$\mathcal{H}\psi = i\frac{\partial}{\partial t}\psi \ , \tag{15}$$

where ψ is a four-dimensional column vector and $\vec{p}$ is to be replaced by $-i\vec{\nabla}$.

For the case of metric theories of gravity, we show in an appendix that Eqs. (14) and (15) are the correct GMD equations for SSS fields using isotropic coordinates, where the metric is given by

$$g_{00} = T, \quad g_{0\alpha} = 0, \quad g_{\alpha\beta} = -H\delta_{\alpha\beta} \ . \tag{16}$$

More convenient than Eqs. (14) and (15) for some purposes is their Foldy-Wouthuysen representation[9]:

$$\mathcal{H}'\psi' = i\frac{\partial}{\partial t}\psi' \ ,$$
$$\mathcal{H}' = S\mathcal{H}S^\dagger - iS\frac{\partial S^\dagger}{\partial t}, \quad \psi' = S\psi \tag{17}$$

which decouples the positive- and negative-energy states of ψ and yields a Schrödinger-type Hamiltonian with relativistic corrections. Following the method given by Messiah[9] and using an appropriate form for the operator S we obtain

$$\mathcal{H}' = T^{1/2}\left[m + \frac{1}{2mH}(\vec{p} - e\vec{A})^2 - \frac{1}{8m^3H^2}p^4 - \frac{e}{2mH}\vec{\sigma}\cdot\vec{B}\right]$$
$$- eA_0 + \frac{e}{4m^2H}\vec{\sigma}\cdot(\vec{E}\times\vec{p} - \tfrac{1}{2}i\vec{\nabla}\times\vec{E})$$
$$- \frac{e}{8m^2H}\vec{\nabla}\cdot\vec{E} + O(p^6)\,. \tag{18}$$

Equation (18) contains the familiar terms of the nonrelativistic limit of the Dirac equation, but each term is scaled appropriately by the functions T and H.

In deriving Eqs. (14) and (18) we treated the functions $T(U)$ and $H(U)$ as constants, that is, we ignored spatial variations of the Newtonian gravitational potential U. This assumption is justified, for the effects of such variations on the structure of an atom are negligible in the physical situations of interest. Moreover, even if variations in U are permitted, the GMD equations can be written in a representation in which no terms containing $\vec{\nabla}U$ are present; only terms containing $\partial^2U/\partial x^\alpha\partial x^\beta$ or $(\partial U/\partial x^\alpha)^2$ (tidal gravitational forces, "Riemann tensor" coupling) remain (see Appendix).

B. Hydrogen atomic clocks

We now apply the GMD equations to the hydrogen atom. For a static Coulomb field, A_i has the form [see Eqs. (8) and (9) or Ref. 7]

$$A_0 = -e/\epsilon r, \quad \vec{A} = 0\,. \tag{19}$$

Then the Dirac equation (15) for the stationary states of the hydrogen atom may be written

$$[T^{1/2}(m\beta - H^{-1/2}i\vec{\alpha}\cdot\vec{\nabla}) - e^2/\epsilon r]\psi = E\psi\,. \tag{20}$$

(Here the symbol e refers to the magnitude of the electronic charge.) By rescaling m, e, and E according to

$$\bar{m} = mH^{1/2}\,,$$
$$\bar{e}^2 = e^2(H/T)^{1/2}\epsilon^{-1}\,, \tag{21}$$
$$\bar{E} = E(H/T)^{1/2}\,,$$

we may write Eq. (20) as

$$[\bar{m}\beta - i\vec{\alpha}\cdot\vec{\nabla} - \bar{e}^2/r]\psi = \bar{E}\psi\,. \tag{22}$$

Then the eigenvalues $\bar{E}$ are given by

$$\bar{E} = \bar{m}\left\{1 + \bar{e}^4[S + (K^2 - \bar{e}^4)^{1/2}]^{-2}\right\}^{-1/2}\,, \tag{23}$$

where S and K are integral quantum numbers. Reexpressing in terms of unbarred quantities and expanding in powers of e^4, we obtain finally

$$E = T^{1/2}\left[m - \frac{1}{2}\frac{me^4}{n^2}\left(\frac{H}{T\epsilon^2}\right) - \frac{1}{2}\frac{me^8}{n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\left(\frac{H}{T\epsilon^2}\right)^2 + \cdots\right], \tag{24}$$

where n and j are the usual principal and angular momentum quantum numbers. Identical results to Eq. (24) follow from the nonrelativistic Hamiltonian [Eq. (18)]. Then the energy separation between two principal levels is

$$\Delta E_P = -\tfrac{1}{2}me^4\left[\Delta\left(\frac{1}{n^2}\right)\right]\left(\frac{H}{T\epsilon^2}\right)T^{1/2} + O(e^8)$$
$$= \Delta E_P^{(0)}\left(\frac{H}{T\epsilon^2}\right)T^{1/2}\,, \tag{25}$$

while two fine-structure levels within a principal level are separated by $(\Delta n = 0)$

$$\Delta E_F = -\frac{1}{2}\frac{me^8}{n^3}\left[\Delta\left(\frac{1}{j+\frac{1}{2}}\right)\right]\left(\frac{H}{T\epsilon^2}\right)^2 T^{1/2}$$
$$= \Delta E_F^{(0)}\left(\frac{H}{T\epsilon^2}\right)^2 T^{1/2}\,. \tag{26}$$

Here the superscript (0) denotes the corresponding energy differences in the absence of gravity ($U=0$, $T=H=\epsilon=\mu=1$). We note that the Bohr radius a is given by

$$a = \left(\frac{\epsilon T^{1/2}}{H}\right)\frac{1}{me^2} = \left(\frac{\epsilon T^{1/2}}{H}\right)a^{(0)}\,. \tag{27}$$

To determine the hyperfine splitting of the ground state of hydrogen, we must take into account the magnetic moment of the proton. We first focus attention on the term in Eq. (18) that represents the coupling between the spin of the electron and a magnetic field

$$\mathcal{H}'' = (T^{1/2}/H)(e/2m)\vec{\sigma}\cdot\vec{B}\,, \tag{28}$$

from which the gravitationally modified magnetic moment $\vec{M}_e$ of the electron may be read off:

$$\vec{M}_e = (T^{1/2}/H)(e/2m)\vec{\sigma}\,. \tag{29}$$

It is then reasonable to assume that the electromagnetic contribution to the magnetic moment M_p of the proton behaves similarly, and thus that M_p may be written

$$\vec{M}_p = g_p(T^{1/2}/H)(e/2M)\vec{\sigma}\,, \tag{30}$$

where the gyromagnetic ratio g_p measures corrections to $\vec{M}_p$ due to strong and weak interactions.[10] Now, the magnetic field produced by a classical, spinning point charge at rest at $\vec{r}=0$ is given by

$$\vec{B} = -\tfrac{1}{2}\mu\left[\frac{\vec{M} - 3\vec{r}(\vec{r}\cdot\vec{M})/r^2}{r^3} - \frac{8\pi}{3}\delta(\vec{r})\vec{M}\right] \tag{31}$$

where $\vec{M}$ is the classical magnetic moment. The factor μ comes from the GMM equations (8) (see Ref. 7 for detailed derivations). Substituting $\vec{M}_p$ for $\vec{M}$, and using Eq. (31) in Eq. (28), we obtain for $\mathcal{H}''$

$$\mathcal{H}'' = -\tfrac{1}{2}\mu \frac{T}{H^2}\frac{e^2 g_p}{(2m)(2M)}\vec{\sigma}_e\cdot\left\{[\vec{\sigma}_p - 3\vec{r}(\vec{r}\cdot\vec{\sigma}_p)/r^2]\frac{1}{r^3} - \frac{8\pi}{3}\delta(\vec{r})\vec{\sigma}_p\right\} . \quad (32)$$

For the (gravitationally modified) hydrogen-atom ground state we obtain

$$\Delta E_H = \langle 0|\mathcal{H}''|0\rangle = \frac{4\pi}{3}\mu\frac{T}{H^2}\frac{e^2 g_p}{(2m)(2M)}|\psi(0)|^2\langle 0|\vec{\sigma}_e\cdot\vec{\sigma}_p|0\rangle , \quad (33)$$

where $\psi(\vec{r})$ is the ground-state wave function. But

$$|\psi(0)|^2 = \frac{1}{a^3} = \left(\frac{H}{\epsilon T^{1/2}}\right)^3\frac{1}{(a^{(0)})^3} . \quad (34)$$

Thus

$$\Delta E_H = \Delta E_H^{(0)}\left(\frac{H}{T\epsilon^2}\right)\left(\frac{\mu}{\epsilon}\right)T^{1/2} . \quad (35)$$

From this point of view, the effect of gravitation is to shift the atomic energy levels by amounts which depend on T, H, ϵ, and μ. How these shifts are measured physically is the subject of the next subsection.

C. Photon propagation and measurement of the red shift

Any comparison of the energy levels of two atoms at different locations (gravitational red-shift measurement) involves sending an electromagnetic signal from one to the other; thus we must analyze the emission and propagation of photons in the $TH\epsilon\mu$ framework.

Consider first the vacuum GMM equations. In the geometrical-optics limit (wavelength $\ll$ scale of variation of U) the vector potential A_i can be written as

$$A_i = \tilde{A}_i e^{i\theta} \quad (36)$$

where θ is the phase of the wave. According to the GMM equations, then, the wave vector defined by $k_i \equiv \theta_{,i}$ satisfies

$$k_\alpha k_\alpha - (\epsilon\mu)k_0^2 = 0 . \quad (37)$$

Differentiation of Eq. (37) yields

$$k^i k_{j,i} + \tfrac{1}{2}(\epsilon\mu)_{,j}k_0^2 = 0 , \quad (38)$$

where we define

$$k^\alpha \equiv -k_\alpha, \quad k^0 = (\epsilon\mu)k_0 . \quad (39)$$

Define now an "affine" parameter λ which increases monotonically along the spacetime trajectory of a wave front, characterized by θ = constant. Thus, along the trajectory,

$$0 = \frac{d\theta}{d\lambda} = \frac{d\theta}{dx^i}\frac{dx^i}{d\lambda} = k_i\frac{dx^i}{d\lambda} . \quad (40)$$

But from Eqs. (37) and (39) we see that

$$\frac{dx^i}{d\lambda} = k^i . \quad (41)$$

Thus Eq. (38) takes the form

$$\left(\frac{d}{d\lambda}\right)k_j = -\tfrac{1}{2}(\epsilon\mu)_{,j}k_0^2 . \quad (42)$$

For an SSS source, $(\epsilon\mu)_{,0} = 0$; hence k_0 is conserved along the trajectory of the photon.

Consider now an atom at rest in the SSS field at $\vec{x}_1$, which undergoes an electromagnetic transition. Then Eq. (36) along with the interaction term

$$\mathcal{H}_{\text{interaction}} = -e(T^{1/2}/2mH)(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}) \quad (43)$$

in Eq. (18) reveals that the emitted photon satisfies

$$k_0 = \theta_{,0} = \Delta E(\vec{x}_1) , \quad (44)$$

where $\Delta E(\vec{x}_1)$ is the difference in energy between the two states, at $\vec{x}_1$. Now, an observer at rest at $\vec{x}_2$ equipped with an identical atom receives the photon with the same value of k_0 as that emitted, but the energy levels of his atom are separated by energy $\Delta E(\vec{x}_2)$. Thus the frequencey $\nu_1 = k_0$ of the received photon differs from his standard of frequency $\nu_2 = \Delta E(\vec{x}_2)$ by a red shift given by

$$z = \frac{\nu_1 - \nu_2}{\nu_2} = \frac{\Delta E(\vec{x}_1)}{\Delta E(\vec{x}_2)} - 1 . \quad (45)$$

Substituting the expressions for ΔE for principal, fine-structure, and hyperfine transitions of hydrogen from Eqs. (25), (26), and (35), and setting $\vec{x}_1 = \infty$ for simplicity, we obtain the expressions given in Eqs. (5).

III. SIGNIFICANCE OF SOLAR-SYSTEM GRAVITATIONAL RED-SHIFT EXPERIMENTS

To explore the experimental consequences of these theoretical predictions, we focus attention on the solar system, where gravitation is weak ($U \lesssim 10^{-5}$). For weak gravitational fields ($U \ll 1$), the functions T, H, ϵ, and μ may be expanded according to[7]

$$T = 1 - 2U + 2\beta U^2 + \cdots ,$$
$$H = 1 + 2\gamma U + \tfrac{3}{2}\delta U^2 + \cdots ,$$
$$\epsilon = 1 + \epsilon_1 U + \epsilon_2 U^2 + \cdots , \quad (46)$$
$$\mu = 1 + \mu_1 U + \mu_2 U^2 + \cdots .$$

We remark that the form $T = 1 - 2U + \cdots$ guarantees that the theory agree with Newtonian gravitation at lowest order. Then the red-shift predictions [Eqs. (5)] take the form

$$z_P = (1 - 2\Gamma_0)\,U + (\tfrac{3}{2} - \beta - \Gamma_P)\,U^2 ,$$
$$z_F = (1 - 4\Gamma_0)\,U + (\tfrac{3}{2} - \beta - \Gamma_F)\,U^2 , \quad (47)$$
$$z_H = (1 - 4\Gamma_0 + \Upsilon_1)\,U + (\tfrac{3}{2} - \beta - \Gamma_H)\,U^2 ,$$

where

$$\Gamma_0 = 1 + \gamma - \epsilon_1 ,$$
$$\Gamma_1 = \tfrac{3}{2}\delta - 4\gamma^2 - 2\epsilon_2 - 2\beta + 2\epsilon_1^2 + \gamma\epsilon_1 + \mu_1 - 5\gamma + \epsilon_1 - 1 ,$$
$$\Upsilon_1 = 2\gamma + 2 - \epsilon_1 - \mu_1 ,$$
$$\Upsilon_2 = \tfrac{3}{2}\delta - 2\beta + 4\gamma + 4 - \epsilon_1\mu_1 - \epsilon_2 - \mu_2 , \quad (48)$$
$$\Gamma_P = \Gamma_1 + 3(\gamma + 1)\,\Gamma_0 - 3\Gamma_0^2 + \Gamma_0 + \Upsilon_1 ,$$
$$\Gamma_F = 2\Gamma_P - 4\Gamma_0^2 ,$$
$$\Gamma_H = 2\Gamma_P - 4\Gamma_0^2 - \Upsilon_2 + (2\gamma + 1)\,\Upsilon_1 + \Upsilon_1(4\Gamma_0 - \Upsilon_1) .$$

Each theory will yield certain values for the Γ's and Υ's; for metric theories all Γ's and Υ's vanish identically. Thorne's proposed counterexample is thus provided by a nonmetric theory with a nonzero value for Γ_0 or Υ_1, for instance a nonmetric version of Whitehead's theory, that has $\Gamma_0 = 2$.[7]

These results show that gravitational red-shift measurements can actually be used as direct tests of the validity of nonmetric theories of gravity, if one views a given experiment as measuring or placing a limit on certain of the parameters Γ or Υ. Thus a measurement of the first-order red shift of a principal or fine-structure clock would measure the parameter Γ_0. But Lightman and Lee[7] have shown that high-precision Eötvös experiments impose the limit

$$3\times 10^{-3}\,|\Gamma_0 + 10^{-8}(\Gamma_1 - \tfrac{1}{3}\Upsilon_1)| < \begin{cases} 10^{-11} & \text{(Princeton experiment)} \\ 10^{-12} & \text{(Moscow experiment)} \end{cases} \quad (49)$$

which puts severe constraints on possible values of Γ_0 (and also rules out the Whitehead nonmetric theory). With the exception of the Whitehead theory, we expect that most nonmetric theories will do well enough to have $\Gamma_0 \equiv 0$. Thus red-shift experiments that measure Γ_0 are much weaker tests than Eötvös experiments. On the other hand, a "hyperfine" red-shift experiment would measure the parameter Υ_1 which is not so strongly constrained by Eötvös experiments. Such an experiment is, in fact, planned for mid-1975[11]: A hydrogen maser clock will be flown in a rocket probe to an altitude of about 16 000 km and its red shift measured as a function of altitude. The hoped-for accuracy is of the order of

$$\frac{\Delta z}{z} \sim \frac{\text{stability of H-maser clock during flight}}{\text{anticipated red shift}} \sim \frac{1/10^{14}}{5\times 10^{-10}} \sim 2\times 10^{-5} . \quad (50)$$

Potentially more powerful red-shift experiments would be those that probe the second-order effects that depend on Γ_1 and Υ_2. Here, *null red-shift experiments*—those that compare different atomic clocks in an external gravitational field—*may be more feasible than direct second-order red-shift measurements:* Place two different atomic clocks (each endowed with high-frequency stability) on a spacecraft injected into a very eccentric solar orbit. Measure the constancy of the ratio of the clock frequencies as the spacecraft passes through perihelion (thus eliminating special-relativistic Doppler-shift effects); hence, measure the difference between the second-order Γ parameters for the two clocks, with a precision given by

$$\Delta(\text{difference in second-order } \Gamma\text{'s}) \sim 10^{-2} \times \frac{(\text{clock stability time of mission})/10^{-14}}{(\text{perihelion distance}/10^{-2}\,\text{a.u.})^2} . \quad (51)$$

Such high-precision null-gravitational red-shift experiments would be powerful tests of nonmetric theories of gravitation.

IV. CONCLUDING REMARKS

One potential weakness of our "proof" of the UGR conjecture is its restriction to theories of gravity of the $TH\epsilon\mu$ type. Unfortunately, the techniques used in this paper are not as adaptable to non-$TH\epsilon\mu$ theories as were the techniques used by Lightman and Lee, since our method relied on a particularly simple form for the gravitationally modified Dirac equation. But there are very few nonmetric theories of gravity that are complete and self-consistent, and fewer still that do not fit into the $TH\epsilon\mu$ framework, and the latter can be studied on an individual basis in order to check their red-shift predictions against the UGR conjecture.

If nonmetric theories of gravity violate UGR for static, spherically symmetric gravitational fields, they might also be expected to violate UGR for cosmological red shifts (nonstatic, homogeneous, isotropic fields). This would lead to observable effects in the spectra of "distant" objects, for suppose a given nonmetric theory attributed at least a part of the red shift of a cosmological object to gravitation, the remainder being attributed to ordinary Doppler shift (in the cosmological models appropriate to metric theories of gravity, the two kinds of red shift are synonymous). Then spectral lines of different atomic transitions in the object would be red shifted differently, in disagreement with astronomical observations. Indeed, precise upper limits have been placed on such anomalous relative shifts of spectral lines, in objects with z as high as 0.20, for example.[12] Such observations thus provide a further test which specific nonmetric theories must pass in their efforts to achieve viability.

APPENDIX: DIRAC EQUATION IN METRIC THEORIES OF GRAVITY

The Dirac equation for a particle of mass m and charge e in curved spacetime is given by[13]

$$\left[\gamma^i\left(i\frac{\partial}{\partial x^i}+eA_i+i\Gamma_i\right)-m\right]\psi=0\ , \qquad (A1)$$

where γ_i are spinor fields which satisfy

$$\gamma_i\gamma_j+\gamma_j\gamma_i=2g_{ij}\ , \quad \gamma^i=g^{ij}\gamma_j\ , \qquad (A2)$$

and Γ_i are spin-affine connections related to the Christoffel symbols $\Gamma^i{}_{jk}$ by

$$\begin{aligned}&\gamma_{i,k}-\Gamma_{ik}{}^j\gamma_j-\Gamma_k\gamma_i+\gamma_i\Gamma_k=0\ ,\\ &\text{Trace }\|\Gamma_i\|=0\ .\end{aligned} \qquad (A3)$$

In isotropic coordinates, the static spherically symmetric (SSS) metric has the form

$$g_{00}=T, \quad g_{0\alpha}=0, \quad g_{\alpha\beta}=-H\delta_{\alpha\beta}\ , \qquad (A4)$$

where T and H are functions of $U=M/r$. Then the spinors γ_i are related to the special relativistic Dirac matrices of Sec. II by

$$\gamma_0=T^{1/2}\beta, \quad \gamma_\alpha=-H^{1/2}\beta\alpha_\alpha\ . \qquad (A5)$$

From Eqs. (A3), (A4), and (A5) it follows that

$$\gamma^i\Gamma_i=\tfrac{1}{4}H^{-1/2}\beta\vec{\alpha}\cdot\vec{\nabla}\ln(TH)\ . \qquad (A6)$$

Usings Eqs. (A4), (A5), and (A6) along with the substitutions

$$i\frac{\partial}{\partial t}=\mathcal{H}, \quad -i\vec{\nabla}=\vec{p}\ , \qquad (A7)$$

we put Eq. (A1) into the form

$$\mathcal{H}\psi=\{T^{1/2}[m\beta+H^{-1/2}\vec{\alpha}\cdot(\vec{p}-e\vec{A}+\tfrac{1}{4}i\vec{\nabla}\ln(TH))]-eA_0\}\psi. \qquad (A8)$$

As it stands, Eq. (A8) contains explicit dependence on gradients of T and H. However, these gradients can be removed by rewriting Eq. (A8) in terms of the wave function $\bar{\psi}=(TH)^{-1/4}\psi$. Then

$$\bar{\mathcal{H}}\bar{\psi}=\{T^{1/2}[m\beta+H^{-1/2}\vec{\alpha}\cdot(\vec{p}-e\vec{A})]-eA_0\}\bar{\psi}\ . \qquad (A9)$$

Similar transformations permit the explicit removal of terms linear in gradients of T and H from the Foldy-Wouthuysen representation of Eq. (A8), leaving only terms quadratic in gradients of T and H.

Hence the Dirac equation of metric theories of gravitation has a form identical to Eq. (14).

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[1]For detailed discussion of many of the concepts used in this paper, see K. S. Thorne, D. L. Lee, and A. P. Lightman, Phys. Rev. D 7, 3563 (1973) and C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[2]We use units in which the Newtonian gravitational constant G, the speed of light c, and Planck's constant $\hbar$ are unity; Greek indices take the values 1, 2, 3, and Roman indices take the values 0, 1, 2, 3; commas denote partial differentiation.

[3]R. H. Dicke, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).

[4]L. I. Schiff, Am. J. Phys. 28, 340 (1960); R. H. Dicke, *ibid.* 28, 344 (1960).

[5]A. Schild, in *Evidence for Gravitational Theories*, edited by C. Møller (Academic, New York, 1962); K. S. Thorne and C. M. Will, Astrophys. J. 163, 595 (1971).

[6]K. S. Thorne, invited paper presented at the 1972 American Physical Society meeting in San Francisco (unpublished).

[7]A. P. Lightman and D. L. Lee, Phys. Rev. D 8, 364 (1973).

[8]Throughout, we ignore tidal gravitational forces, which if strong enough could alter the structure and ticking rate of an atomic clock; under such circumstances the

clock ceases to be "ideal."

[9]A. Messiah, *Quantum Mechanics* (Wiley, New York, 1966), Vol. 2.

[10]We ignore the effects of gravity on the strong- and weak-interaction contributions to g_p; these interactions have not been given an adequate mathematical representation even in the absence of gravity, so we are forced to concentrate on electromagnetic effects.

[11]R. F. C. Vessot, in *Experimental Gravitation*, edited by B. Bertotti (Academic, New York, 1974).

[12]J. N. Bahcall and M. Schmidt, Phys. Rev. Lett. 19, 1294 (1967).

[13]D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. 29, 465 (1957).