Kaon nonleptonic decays and chiral-symmetry breaking

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A consistent formulation of current algebra and pion partial conservation of axial-vector current is given for the kaon nonleptonic weak decays $K\to 2\pi$ and $K\to 3\pi$. This includes evaluation of kaon σ terms and $\Delta I=1/2$ and $\Delta I=3/2$ contributions to the weak current-current Hamiltonian. A large meson σ term, $\sigma_{KK}\sim 2m_\pi{}^2$, apparently is necessary to explain the $K_{2\pi}$ and $K_{3\pi}$ measured decay parameters. The hyperon nonleptonic decays are then reexamined and also shown to be compatible with a large value for σ_{KK}. Implications for the $(3,\bar{3})$ chiral-symmetry-breaking model and for octet dominance of the weak Hamiltonian are then discussed.

I. INTRODUCTION

The amplitudes and slopes of the four $K\to 3\pi$ decays have not been completely understood in the context of the algebra of currents and the current-current model for the nonleptonic Hamiltonian, H_W. There are now clear indications that H_W must contain $\Delta I=\frac{3}{2}$ parts along with the dominant $\Delta I=\frac{1}{2}$ contribution, not only in $K\to 3\pi$ decays,[1] but also in the $K_S\to\pi\pi$ decays and $K^+\to\pi^+\pi^0$ decay as well.[2,3] The problem is more than simply finding a best phenomenological fit to the data; the very existence of $\Delta I=\frac{3}{2}$ parts in H_W leads to differing conclusions depending upon the method of analysis employed. One of the important results of this work is that the rapidly-varying-pole method first used by Weinberg in K_{l4} decays[4,5] can consistently and satisfactorily account for the $K\to 2\pi$ and $K\to 3\pi$ decay parameters.

At the same time, and in a seemingly different context, there is growing evidence that the chiral-symmetry-breaking σ terms are much larger than first assumed in the $(3,\bar{3})$-breaking model.[6] This can be seen in low-energy πN scattering,[7-9] KN scattering,[8,9] $\eta'\to\eta\pi\pi$ decay, and in $\eta\to 3\pi$ decays.[9] Given this fact, one is obliged to investigate their consequence in the nonleptonic decays where such σ terms have been previously ignored. We shall show that large meson σ terms are precisely what is necessary to account for the $\Delta I=\frac{3}{2}$ contributions to the $K\to 3\pi$ parameters.

These large meson σ terms will also contribute very little to the seven s-wave hyperon decays $B\to B'\pi$, which are now well understood without such corrections.[10] The $B\to B'\pi$ decays are therefore completely described in a manner exactly analogous with low-energy πN scattering: i.e., as a sum of current commutator terms, rapidly varying spin-$\frac{1}{2}$ baryon and spin-$\frac{3}{2}$ decuplet poles coupled with large baryon σ terms in πN and large meson σ terms (which turn out to contribute insignificantly to $B\to B'\pi$).

The nonleptonic meson decays $K\to 2\pi$ and $K\to 3\pi$ will have a parallel expansion as a sum of current commutator terms and rapidly varying meson poles coupled with the same large meson σ terms. Throughout this paper, we shall stress this rapidly-varying-pole (and nonpole) method.

We emphasize that, along with the postulates and techniques of current algebra and soft-pion PCAC (partially conserved axial-vector current), our only fundamental assumption will be that the nonleptonic weak Hamiltonian has the current-current structure, $H_W\sim(V-A)\cdot(V-A)$, and that the currents are octet operators, so that $H_W(\Delta S=\pm 1)$ contains only $\Delta I=\frac{1}{2}$ and $\Delta I=\frac{3}{2}$ parts ($H_{W,1/2}$ and $H_{W,3/2}$).

In Sec. II we gather together the soft-pion and current-algebra results preliminary to the calculation of the nonleptonic kaon decays. A brief discussion of PCAC and of the Cabibbo–Gell-Mann theorem[11] is included.

In Sec. III we discuss the $\Delta I=\frac{1}{2}$ amplitudes for $K_{2\pi}$ and $K_{3\pi}$, and in Sec. IV we include the $\Delta I=\frac{3}{2}$ effects.

Section V contains an analysis of the combined $K_{2\pi}, K_{3\pi}$ results. Good agreement is found, provided the σ term, σ_{KK}, has the anomalously large value of $\sigma_{KK}=2m_\pi{}^2$.

Section VI contains a brief reanalysis of the hyperon nonleptonic decays ($B\to B'\pi$) in light of the large σ_{KK} value discovered in Secs. III and IV. The large value of σ_{KK} is found to be consistent with the data.

Section VII discusses octet enhancement of the $K_{2\pi}, K_{3\pi}$ decay amplitudes. This effect is shown *not* to be due to the suppression of $H_{W,27}$ in the operator sense or in the single-particle matrix elements, but rather to be due to the large value of σ_{KK}. Somewhat similar arguments are indicated for octet enhancement in $B\to B'\pi$.

Section VIII contains a resume of the conclusions of the paper.

Appendix A analyzes the isotopic spin content of the $K_{3\pi}$ decays, assuming only $\Delta I=\frac{1}{2}$ and $\frac{3}{2}$ transformation properties for H_W.

Appendix B contains an analysis of the $K_{2\pi}, K_{3\pi}$, $\Delta I=\frac{1}{2}$ transitions in the multi-soft-pion limits. This analysis justifies the PCAC assumptions for the $\Delta I=\frac{1}{2}$ amplitudes. Unfortunately, the application to the $\Delta I=\frac{3}{2}$ amplitudes shows that the PCAC smoothness assumption must be extended in this case.

II. SOFT-PION AND CURRENT-ALGEBRA RESULTS

The matrix element for the decay $K\to n\pi$ is defined in terms of the weak Hamiltonian by[12]

$$M(K\to\pi^i\cdots)=-\langle\pi^i(p_i)\cdots|H_W(0)|K\rangle\ . \tag{1}$$

In this expression, there is no intrinsic reason why the momentum of the kaon should be equal to the sum of the momenta of the pions; the Hamiltonian can be considered as a momentum-carrying spurion. Indeed, such a "momentum-nonconserving" approach is useful in defining limits in which all (or all but one) of the pions are soft. It is to be understood, of course, that when used as the amplitude for a physical process, M must be evaluated at the point of "momentum conservation": $k=\sum p_i$.

In deriving the soft-pion limits, we have used the standard SU(2)×SU(2) algebra of currents, supplemented by the σ-term commutators derived from the Gell-Mann–Oakes–Renner (GMOR) model[6] of $(3,\bar{3})$-breaking or the Gell-Mann–Lévy σ model[13]

$$[A_0^i(0),\partial A^j(x)]_{x^0=0}=-i\delta^{ij}\sigma(0)\,\delta^3(\vec{x})\,, \tag{2a}$$

$$[A_0^i(0),\sigma(x)]_{x^0=0}=i\partial A^i(0)\,\delta^3(\vec{x})\,, \tag{2b}$$

where, as throughout this article, isospin indices take the values $1\le i,\ j,\ k\le 3$.

In addition, it is assumed that the strangeness-changing weak Hamiltonian is of the standard current-current form $H_W\sim(V-A)\cdot(V-A)$. When combined with the current algebra, this leads to the convenient property relating the axial-vector to the vector charges

$$[Q_5^i,H_W]=-[Q^i,H_W]\ , \tag{3}$$

and the soft-pion PCAC formula ($f_\pi\approx 94$ MeV)

$$\lim_{p_i\to 0}\ \langle\pi^i|H_W|\rangle=\frac{i}{f_\pi}\langle|[Q^i,H_W]|\rangle\ . \tag{4}$$

In the course of this paper, we will need two strong-interaction amplitudes which can be obtained through the use of soft-pion limits and current algebra. The first of these strong-interaction amplitudes is

$$M^{\rm st}(\pi^l\to\pi^i\pi^j\pi^k)=\frac{1}{f_\pi^2}\,\delta^{il}\delta^{jk}[(p_j+p_k)^2-m_\pi^2]+{\rm cycl}\,, \tag{5}$$

where, as throughout this paper, "cycl" denotes the other two terms generated by the cyclic permutation of the indices i,j,k. This amplitude is, of course, the Weinberg amplitude for $\pi\pi$ scattering,[14] with one of the pions crossed to the final state. The matrix element in Eq. (5) is written for π^l on the mass shell ($p_l^2=m_\pi^2$); it obeys the Adler consistency condition,[15] and embodies the σ term as found in the $(3,\bar{3})$ model[14]:

$$\sigma_{\pi\pi}^{\pi\pi}\equiv\sigma_{\pi\pi}=\langle\pi|\sigma|\pi\rangle=m_\pi^2\ . \tag{6}$$

The other strong-interaction amplitude that will be needed is

$$M^{\rm st}(K(k)\to K\pi^i\pi^j)=\frac{i\epsilon^{ijm}\tau^m}{4f_\pi^2}[2k\cdot(p_i-p_j)-p_i^2+p_j^2]-\frac{1}{f_\pi^2}\,\delta^{ij}F(p_i,p_j)\sigma_{KK}\ , \tag{7}$$

which is just the Weinberg amplitude for $K\pi$ scattering, with the initial π crossed to the final state; both kaons are on the mass shell. The amplitude in Eq. (7) retains the σ term which is usually neglected,

$$\sigma_{KK}^{\pi\pi}=\sigma_{KK}\equiv\langle K|\sigma|K\rangle\ . \tag{8}$$

In the simplest form of the GMOR model, this term has the value $\sigma_{KK}=\frac{1}{2}m_\pi^2$, though present indications are that it may be much larger. This point will be discussed later in this article; for the present, σ_{KK} will be treated as a parameter.

The function $F(p_i,p_j)$ in Eq. (7) is necessary to continue the amplitude to the physical region. It is symmetric in p_i,p_j; in order to have the amplitude satisfy the correct soft limit ($p_i,p_j\to 0$) and the Adler consistency condition ($p_i\to 0$), we must have

$$F(0,0)=1\,, \tag{9a}$$

$$F(p_i=0,p_j^2=m_\pi^2)=0\ . \tag{9b}$$

The two obvious candidates for this function are

$$\text{I.}\quad F(p_i,p_j)=1-(p_i^2+p_j^2)/m_\pi^2\,, \tag{10a}$$

$$\text{II.}\quad F(p_i,p_j)=1-(p_i+p_j)^2/m_\pi^2\,, \tag{10b}$$

though obviously a linear combination of the two is possible. It should be noted that both forms (if applied to the σ term in πN scattering) yield the value $F(p_i, p_j)\sigma_{NN} = -\sigma_{NN}$ at the on-shell Cheng-Dashen point $[p_i^2 = p_j^2 = m_\pi^2,\ (p_i + p_j)^2 = 2m_\pi^2]$. Though we examine the effect of both forms in the ensuing sections of the paper, we prefer form II for several reasons:

(i) Form I would imply that the soft limit is obtained at the point $p_i^2 = 0 = p_j^2$

$$\lim_{p_i^2, p_j^2 \to 0} M^{st}(K \to K\pi^i\pi^j) = -\frac{\delta^{ij}}{f_\pi^2}\sigma_{KK}$$

rather than the more restrictive soft limit

$$\lim_{p_i, p_j \to 0} M^{st}(K \to K\pi^i\pi^j) = -\frac{\delta^{ij}}{f_\pi^2}\sigma_{KK}\,.$$

(ii) The momentum dependence in form II is interchangeable in the pion and kaon momenta, $p_i + p_j = k + k'$, and the soft-kaon limits then dictate that some dependence on form II is necessary. In the SU_3 limit, the exact form of F can be determined through the use of the soft-pion *and* the soft-kaon limits to be $\frac{1}{2}F_{\rm I} + \frac{1}{2}F_{\rm II}$ (where the kaons have been put back on the mass shell). Unfortunately in the real world, $m_K^2 \simeq 13m_\pi^2$ and the power series expansion in the variable $(p_i + p_j)^2$ which is used to approximate $F(p_i, p_j)$ has a radius of convergence of $4m_\pi^2$ (the onset of the $\pi\pi$ cut) about the soft point. Since the region of physical interest lies outside this circle of convergence, we cannot use the soft-kaon limits to determine the exact form of F, but we can use them to say that some dependence upon form II is necessary.[16]

(iii) Our analysis will show that the $\Delta I = \frac{3}{2}$ slope tests rule out form I taken alone. Any linear combination of forms I and II will increase the magnitude of σ_{KK} far beyond the GMOR value and only serve to emphasize our conclusion. Therefore we will adopt form II alone.

For the purposes of this paper, we regard PCAC smoothness as a semiphenomenological hypothesis: The soft-pion limits are of primary importance and are determined by the current algebra, which we take as a fundamental dynamical postulate. In its most naive form, PCAC assumes that matrix elements of, e.g., H_W behave approximately as constants in the extrapolation from the physical region to the soft-pion limits.

While this most naive form of PCAC works for some cases, there are other cases where it is clearly inconsistent with the soft-pion limits: e.g., K_{l4}, K_{l5}, $B \to B'\pi$, $B \to B'\gamma$, and the present case of $K_{2\pi}, K_{3\pi}$. When it is recognized that the strong interactions tend to be "solved" in terms of resonances and bound states, however, a more sophisticated form of PCAC can be postulated: Once the pole terms have been explicitly extracted, the remaining "background" contributions to the matrix elements are approximately constants. With this form of PCAC, it proves possible in most cases to satisfy the soft-pion limits; in addition, good agreement with experimental data can be obtained. It should be kept in mind that PCAC is a hypothesis whose major justification is that it is simple and gives good results.

For the $\Delta I = \frac{3}{2}$ weak nonleptonic kaon decays, however, the situation is not so clear. The weak Hamiltonian lies in an exotic channel, for which the strong interactions appear not to have been solved in terms of resonances. Once again, we find ourselves in a situation in which PCAC smoothness as enunciated is inconsistent with the soft-pion limits; furthermore, this is a channel in which PCAC has not been adequately verified in other processes. We shall try to enunciate a "maximal smoothness" form of PCAC for $\Delta I = \frac{3}{2}$ in Sec. IV, but clearly in this case the assumptions are more vague and less well justified than in most applications.

Finally, we must mention the Cabibbo–Gell-Mann theorem,[11] which is of fundamental importance in the kaon nonleptonic decays. This theorem states simply that the $K \to 2\pi$ CP-conserving decays are forbidden in the limit of exact SU_3 if H_W is of the current-current form. Through the use of PCAC, this theorem can be extended to show that *all* the decay parameters in Secs. III–V are zero in the limit of exact SU_3.

We do not propose to present any specific theory of how SU_3 is broken to give nonzero $K_{2\pi}, K_{3\pi}$ decay parameters. It is simply assumed that such breaking does occur, and in the evaluation of all matrix elements, only SU_2 is used to relate one SU_3-broken quantity to another. [The single exception to this occurs in Eq. (75), where SU_3 is used to estimate the ratio of two amplitudes, both of which are zero in the limit of exact SU_3.] It is possible, however, that SU_3 is broken in $K_{2\pi}$ in a very simple way. It has been shown[17] that "kinematical" SU_3 breaking of the masses $(m_K \approx 13m_\pi^2)$ in the invariant dynamical $K \to 2\pi$ amplitude which is then combined with SU_3 Clebsch-Gordan coefficients in fact voids the Cabibbo–Gell-Mann conclusion.

III. $K_{2\pi}, K_{3\pi}$: $\Delta I = \frac{1}{2}$ AMPLITUDES

It is shown in Appendix A that the data for nonleptonic kaon decays can easily be separated into $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ parts, provided there are no contributions to the decays from parts of the weak Hamiltonian with $\Delta I \geq \frac{5}{2}$. The data support the assumption that there are no $\Delta I \geq \frac{5}{2}$ contributions, and in any case our model contains only $\Delta I = \frac{1}{2}$ and $\frac{3}{2}$; hence it will prove convenient to consider

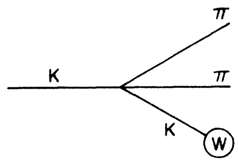

FIG. 1. Rapidly varying K pole in $K_{2\pi}$ decays.

the two cases separately.

It is convenient to define the fundamental strength for the $\Delta I=\frac{1}{2}$ nonleptonic kaon decays through the matrix element[18]

$$\langle\pi^i|H^{\mathrm{PC}}_{W,1/2}|K\rangle=-\frac{1}{\sqrt{2}}L_{1/2}\tau^i\,, \tag{11}$$

so that

$$\langle\pi^0|H^{\mathrm{PC}}_{W,1/2}|K_L\rangle=L_{1/2}\,.$$

A formal soft-pion limit of Eq. (11) yields

$$\langle 0|H^{\mathrm{PV}}_{W,1/2}|K\rangle=-i\sqrt{2}f_\pi L_{1/2}\,. \tag{12}$$

A. $K_{2\pi}$ amplitudes

In the case of pure $\Delta I=\frac{1}{2}$ transitions, Eq. (4) yields the following soft-pion limit for the $K_{2\pi}$ amplitude:

$$\lim_{p_i\to 0} M_{1/2}(K\to\pi^i\pi^j)=-\frac{i}{f_\pi}\langle\pi^j|[Q^i,H_{W,1/2}]|K\rangle \tag{13a}$$

$$=-\frac{i\sqrt{2}\,L_{1/2}}{4f_\pi}(\delta^{ij}+i\epsilon^{ijm}\tau^m). \tag{13b}$$

As is well known, the single soft-pion limit for the $K_{2\pi}$ amplitude depends on the identity of the pion which is soft. Indeed, this is manifest in the last term of Eq. (13b), which changes sign under the interchange of the identity of the two pions.

Following the K_{l4} anaylsis of Weinberg,[4] it is clear that this rapid variation can be accounted for by the kaon pole of Fig. 1. We implement this model by postulating an unsubtracted dispersion relation in q^2, where $q=k-p_i-p_j$ is the momentum carried by the weak Hamiltonian (spurion); clearly at the end of the calculation, we will set $q=0$ to find the physical amplitude. Thus

$$M_{1/2}(K\to\pi^i\pi^j)=M^{\mathrm{pole}}_{2\pi,1/2}+\overline{M}_{2\pi,1/2}\,, \tag{14}$$

where $\overline{M}$ is a background term arising from the continuum contributions. The pole term is readily found to be

$$M^{\mathrm{pole}}_{2\pi,1/2}=\langle 0|H_{W,1/2}|K\rangle M^{\mathrm{st}}(K(k)\to\pi^i\pi^jK(q))/(q^2-m_K^2) \tag{15a}$$

$$=i\sqrt{2}\,\frac{L_{1/2}}{4f_\pi}\,\frac{\{i\epsilon^{ijm}\tau^m[2k\cdot(p_i-p_j)-p_i^2+p_j^2]-4\delta^{ij}F(p_i,p_j)\sigma_{KK}\}}{2k\cdot(p_i+p_j)-(p_i+p_j)^2}\,, \tag{15b}$$

where use has been made of the fact that $q^2=m_K^2$ in the residue of the pole term; the strong-interaction amplitude is taken from Eq. (7).

In accordance with the PCAC hypothesis of maximal smoothness, we assume that the background term $\overline{M}$ is, to a good approximation, a constant. Hence it can be evaluated through the limit

$$\overline{M}_{2\pi,1/2}=\lim_{p_i\to 0}[M_{1/2}(K\to\pi^i\pi^j)-M^{\mathrm{pole}}_{2\pi,1/2}] \tag{16a}$$

$$=-\frac{i\sqrt{2}\,L_{1/2}\delta^{ij}}{4f_\pi}\,. \tag{16b}$$

When we impose on-mass-shell and momentum conservation constraints to evaluate the physical amplitude we find from (14)–(16)

$$M^{\mathrm{phys}}_{1/2}(K\to\pi^i\pi^j)\doteq-\frac{i\sqrt{2}\,L_{1/2}}{4f_\pi}\delta^{ij}[1+4F(p_i,p_j)\sigma_{KK}/m_K^2] \tag{17}$$

or, in terms of the specific charge amplitudes

$$M^{\mathrm{phys}}_{1/2}(K^+\to\pi^+\pi^0)=0\,, \tag{18a}$$

$$M^{\mathrm{phys}}_{1/2}(K_S\to\pi^+\pi^-)=M^{\mathrm{phys}}_{1/2}(K_S\to\pi^0\pi^0)$$
$$=-\frac{iL_{1/2}}{2f_\pi}[1+4F(p_i,p_j)\sigma_{KK}/m_K^2]. \tag{18b}$$

B. $K_{3\pi}$ amplitudes

The traditional parametrization for the amplitude and Dalitz-plot slope parameter for the $K_{3\pi}$ decays is

$$M^{\mathrm{phys}}(K\to\pi^a\pi^b\pi^c)=A^{abc}\left[1+\frac{g^{abc}}{2m_\pi^2}(s_c-s_0)\right]\,, \tag{19}$$

where $s_i=(k-p_i)^2$; s_c is the invariant corresponding to the momentum of the odd pion in the decay; $s_a+s_b+s_c=3s_0=m_K^2+3m_\pi^2$.

Since the days of the original current-algebra-PCAC calculations of the $\Delta I=\frac{1}{2}$ contribution to the $K_{3\pi}$ slope and amplitude,[19,20] there have been four types of approach to the problem:

A. Momentum-conserving approaches. Here one can neglect the rapid variation of the matrix element due to the behavior of the pole terms in the soft-pion limits.

(i) Assuming that $M=a+bs_c$ describes the off-shell as well as the on-shell amplitude, one can calculate the slopes and amplitudes in the following manner: Equation (4) and Bose statistics show that $\lim_{p(\pi^-)\to 0} M(K_L \to \pi^+\pi^-\pi^0)=0$; hence the Dalitz-plot slope is $g^{+-0}=6m_\pi^2/m_K^2=0.47$ which no longer compares favorably with experiment: $g^{+-0}=0.60\pm0.02$ (Ref. 1) or $g^{+-0}=0.73\pm0.02$ (Ref. 21). Taking the momentum of π^0 soft in this matrix element gives an amplitude ratio $A^{+-0}/M(K\to 2\pi)$ which is in reasonable agreement with experiment. This method cannot be applied to $\eta\to 3\pi$ (Ref. 22) or $\eta'\to\eta\pi\pi$ (Ref. 9).

(ii) The amplitude in (i) is rather *ad hoc* and, perhaps, does not adequately take into account off-shell effects. One can take a more general form of the amplitude $M=a+bs_c+cp_c^2+d(p_a^2+p_b^2)$ and try to determine the four coefficients from various soft-pion limits. Unfortunately, the two soft-pion limits are ambiguous in this momentum-conserving approach. This method, however, can be applied to $\eta\to 3\pi$ decays, since the ambiguous terms vanish identically in this case.[23]

B. Momentum-nonconserving (spurion) approaches. Here one must take into account the rapid variation of the matrix element caused by the presence of pole terms.

(iii) Explicitly extracting the rapidly varying pole terms (as in the case of $K_{2\pi}$ above), one evaluates the background term, $\overline{M}$, through single-soft-pion limits. This will be the approach used in this paper.

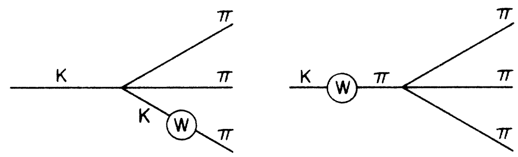

FIG. 2. Rapidly varying K and π poles in $K_{3\pi}$ decays.

(iv) With the same assumptions as in (iii), one demands that multiple-soft-pion limits also be satisfied. This approach displays the full implications of the current algebra and will be outlined in Appendix B; while it gives consistent results for $\Delta I=\frac{1}{2}$, the implementation of PCAC for the $\Delta I=\frac{3}{2}$ amplitudes is obscure.

We therefore consider the momentum-nonconserving amplitude for the matrix element of the $\Delta I=\frac{1}{2}$ part of the weak Hamiltonian:

$$M_{1/2}(K\to\pi^i\pi^j\pi^k)=-\langle\pi^i\pi^j\pi^k|H^{\rm PC}_{W,1/2}|K(k)\rangle\,.$$

Dispersing in the spurion momentum q^2, we are led to an expression analogous to the expression for the case of $K_{2\pi}$ [Eq. (14)]:

$$M_{1/2}(K\to\pi^i\pi^j\pi^k)=M^{K\,\rm pole}_{3\pi,1/2}+M^{\pi\,\rm pole}_{3\pi,1/2}+\overline{M}_{3\pi,1/2}\,, \tag{20}$$

where the pole terms correspond to the diagrams in Fig. 2. The kaon-pole term is evaluated through Eqs. (7) and (11) (with $q^2=m_K^2$ in the residue numerator):

$$M^{K\,\rm pole}_{3\pi,1/2}=\langle\pi^i|H_{W,1/2}|K(q)\rangle M^{\rm st}(K(k)\to\pi^j\pi^k K(q))/(q^2-m_K^2)+\text{cycl} \tag{21a}$$

$$=\frac{\sqrt{2}\,L_{1/2}}{8f_\pi^2}\{[i\epsilon^{ijk}+\delta^{ij}\tau^k-\delta^{ik}\tau^j][2k\cdot(p_j-p_k)-p_j^2+p_k^2]-4\delta^{jk}\tau^iF(p_j,p_k)\sigma_{KK}\}$$
$$\times[2k\cdot(p_j+p_k)-(p_j+p_k)^2]^{-1}+\text{cycl}\,. \tag{21b}$$

The pion-pole term is likewise found to be ($q^2=m_\pi^2$ in the residue numerator)

$$M^{\pi\,\rm pole}_{3\pi,1/2}=M^{\rm st}(\pi^n(q)\to\pi^i\pi^j\pi^k)\langle\pi^n(q)|H_{W,1/2}|K\rangle/(q^2-m_\pi^2) \tag{22a}$$

$$=-\frac{\sqrt{2}\,L_{1/2}}{2f_\pi^2}\{\delta^{ij}\tau^k[(p_i+p_j)^2-m_\pi^2]+\text{cycl}\}[(p_i+p_j+p_k)^2-m_\pi^2]^{-1}\,. \tag{22b}$$

As previously, the basic PCAC hypothesis in the approach is that the background term, $\overline{M}$, is to a good approximation a constant. Thus we can evaluate it in the single soft-pion limit,

$$\overline{M}_{3\pi,1/2}=\lim_{p_i\to 0}[M_{1/2}(K\to\pi^i\pi^j\pi^k)-M^{K\,\rm pole}_{3\pi,1/2}-M^{\pi\,\rm pole}_{3\pi,1/2}]\,. \tag{23}$$

The pole terms are given explicitly in Eqs. (21b) and (22b); the soft limit of $M_{1/2}$ is found by using Eqs. (4), (14), (15b), and (16b); all the momentum-dependent terms cancel, leaving

$$\overline{M}_{3\pi,1/2}=\frac{\sqrt{2}\,L_{1/2}}{8f_\pi^2}(\delta^{ij}\tau^k+\delta^{jk}\tau^i+\delta^{ki}\tau^j)\,. \tag{24}$$

Gathering together the information in Eqs. (20)–

(24) and imposing on-mass-shell and momentum-conservation constraints, we find the physical amplitudes

$$M_{1/2}^{\text{phys}}(K \to \pi^i \pi^j \pi^k) = \delta^{ij}\tau^k \mathfrak{M}_{1/2}^{(k)} + \delta^{jk}\tau^i \mathfrak{M}_{1/2}^{(i)} + \delta^{ki}\tau^j \mathfrak{M}_{1/2}^{(j)}, \tag{25a}$$

$$\mathfrak{M}_{1/2}^{(i)} = -\frac{\sqrt{2}\,L_{1/2}}{8f_\pi^2}\,\frac{[s_i + 4F(p_j,p_k)\sigma_{KK}]}{m_K^2 - m_\pi^2}. \tag{25b}$$

In terms of charge states, this gives[18]

$$M_{1/2}^{\text{phys}}(K^+ \to \pi^0\pi^0\pi^+) = \sqrt{2}\,\mathfrak{M}_{1/2}^{(3)}, \tag{26a}$$

$$M_{1/2}^{\text{phys}}(K^+ \to \pi^+\pi^+\pi^-) = \sqrt{2}\,(\mathfrak{M}_{1/2}^{(1)} + \mathfrak{M}_{1/2}^{(2)}), \tag{26b}$$

$$M_{1/2}^{\text{phys}}(K_L \to \pi^+\pi^-\pi^0) = -\sqrt{2}\,\mathfrak{M}_{1/2}^{(3)}, \tag{26c}$$

$$M_{1/2}^{\text{phys}}(K_L \to \pi^0\pi^0\pi^0) = -\sqrt{2}\,(\mathfrak{M}_{1/2}^{(1)} + \mathfrak{M}_{1/2}^{(2)} + \mathfrak{M}_{1/2}^{(3)}), \tag{26d}$$

where 1, 2, 3 refer to the first, second, and third pion written in the argument of the decay amplitude, $M_{1/2}$.

Rewriting these amplitudes in terms of the conventional amplitudes and slopes of Eq. (19), we find, of course, the $\Delta I = \frac{1}{2}$ relations:

$$A_{1/2}^{000} = 3A_{1/2}^{+-0} = -3A_{1/2}^{00+} = -\tfrac{3}{2}A_{1/2}^{++-}, \tag{27}$$

$$g_{1/2}^{+-0} = g_{1/2}^{00+} = -2g_{1/2}^{++-}. \tag{28}$$

Hence it will suffice to consider explicitly only the quantities $A_{1/2}^{++-}$ and $g_{1/2}^{+-0}$; the other physical parameters are then determined by Eqs. (27) and (28).

The fundamental strength $L_{1/2}$ is eliminated in favor of experimentally measurable quantities through Eq. (18b). The form of the amplitude differs considerably, depending on which form of the function F is used to continue the value of the σ term from the soft limit to the physical region. Some tedious algebra yields, for the two forms given in Eqs. (10),

Form I: $$g_{1/2}^{+-0} = 6m_\pi^2(m_K^2 + 3m_\pi^2 - 12\sigma_{KK})^{-1}, \tag{29a}$$

$$A_{1/2}^{++-} = \frac{-i}{3f_\pi}\,\frac{m_K^2 + 3m_\pi^2 - 12\sigma_{KK}}{(m_K^2 - m_\pi^2)(1 - 4\sigma_{KK}/m_K^2)}\,M_{1/2}(K_s \to \pi^+\pi^-); \tag{29b}$$

Form II: $$g_{1/2}^{+-0} = 6m_\pi^2(1 - 4\sigma_{KK}/m_\pi^2)(m_K^2 + 3m_\pi^2 - 4\sigma_{KK}m_K^2/m_\pi^2)^{-1}, \tag{30a}$$

$$A_{1/2}^{++-} = \frac{-i}{3f_\pi}\,\frac{(m_K^2 + 3m_\pi^2 - 4\sigma_{KK}m_K^2/m_\pi^2)M_{1/2}(K_s \to \pi^+\pi^-)}{(m_K^2 - m_\pi^2)[1 - 4\sigma_{KK}(m_K^2 - m_\pi^2)/(m_K^2 m_\pi^2)]}. \tag{30b}$$

C. Comparison with experiment: $\Delta I = \frac{1}{2}$

As is shown in Appendix A, we can extract the $\Delta I = \frac{1}{2}$ contributions to the slope from the experimental data by

$$g_{1/2}^{+-0} = \tfrac{1}{3}(g^{00+} + g^{+-0} - 2g^{++-}). \tag{31}$$

Using the experimental values[1]

$$g^{00+} = 0.523 \pm 0.023, \tag{32a}$$

$$g^{++-} = -0.214 \pm 0.004, \tag{32b}$$

$$g^{+-0} = 0.604 \pm 0.023, \tag{32c}$$

one obtains $g_{1/2}^{+-0} \simeq 0.52$. Use of the more recent experimental value[21] of $g^{+-0} = 0.73 \pm 0.02$ gives the higher value, $g_{1/2}^{+-0} \simeq 0.56$.

TABLE I. Comparison of $K_{3\pi}$ decay parameters with experiment for various values of σ_{KK}. Form II for the function F [Eq. (10b)] has been used throughout. It should be noted that the transition from $\sigma_{KK} = 0$ to $\sigma_{KK} = \frac{1}{2}m_\pi^2$ is not a continuous one.

	$\sigma_{KK} = 0$	$\frac{1}{2}m_\pi^2$	m_π^2	$2m_\pi^2$	$3m_\pi^2$	Expt
$\Delta I = \frac{1}{2}$ tests						
$\frac{1}{3}(g^{00+} + g^{+-0} - 2g^{++-})$	0.38	0.60	0.50	0.48	0.47	0.52 ± 0.01
$\lvert A^{++-}\rvert \times 10^6$	1.9	1.4	1.5	1.6	1.6	1.92 ± 0.02
$\Delta I = \frac{3}{2}$ tests						
$g^{00+} + 2g^{++-}$	0.10	0.13	0.12	0.12	0.11	0.096 ± 0.024
$g^{00+} - 2g^{++-} - 2g^{+-0}$	0.30	0.20	0.07	−0.20	−0.42	−0.256 ± 0.052
$\frac{1}{2}\frac{\Gamma^{++-}}{\Gamma^{+-0}}\frac{\phi^{+-0}}{\phi^{++-}} - 1$	0.17	0.18	0.11	0.008	−0.11	0.190 ± 0.025

We will consider three different possibilities:

(i) Neglect of σ term: $\sigma_{KK}=0$. In this case, either Eq. (29a) or Eq. (30a) yields $g_{1/2}^{+-0}=0.38$. Thus it is clearly inconsistent to neglect the term σ_{KK}.

(ii) Form I for the function F: In this case, Eq. (29a) yields $g_{1/2}^{+-0}=0.60$ for $\sigma_{KK}=\frac{1}{2}m_\pi^2$, the naive GMOR value; values of $\sigma_{KK}\gtrsim\frac{1}{2}m_\pi^2$ are incompatible with the experimental slope data.

(iii) Form II for the function F: In this case, Eq. (30a) yields $0.60\geq g_{1/2}^{+-0}\geq 0.46$ for $\frac{1}{2}m_\pi^2\leq\sigma_{KK}\leq\infty$ and values of $\sigma_{KK}\lesssim\frac{1}{2}m_\pi^2$ are definitely ruled out for this case.

With regard to the amplitude $A_{1/2}^{++-}$: We will neglect the small $\Delta I=\frac{3}{2}$ contribution to the amplitude for the moment. From the $K_{2\pi}$ decay rate, we deduce the magnitude[24]

$$M_{1/2}(K_S\to\pi^+\pi^-)\simeq M(K_S\to\pi^+\pi^-)\simeq\pm 0.79\times 10^{-6}m_K\,.$$

Thus, we find that the present scheme yields the following results in the same three cases as considered above for the slopes:

(i) Neglect of σ term:

$$|A^{++-}|=1.9\times 10^{-6}\,;$$

(ii) Form I for F: Eq. (29b) yields, with $\sigma_{KK}=\frac{1}{2}m_\pi^2$ as implied by the slopes,

$$|A^{++-}|=1.4\times 10^{-6}\,;$$

(iii) Form II for F: Eq. (30b) gives

$$1.4\times 10^{-6}\leq|A^{++-}|\leq 1.6\times 10^{-6}$$

for $\frac{1}{2}m_\pi^2\leq\sigma_{KK}\leq\infty$. The experimental value is $|A^{++-}|=(1.92\pm 0.02)\times 10^{-6}$. It is felt that PCAC often misscales absolute quantities such as amplitudes by 10–20% (for example, the Goldberger-Treiman value of f_π), and that therefore, when all the uncertainties are taken into acccount, the slope predictions are a more accurate test of our scheme.

In summary, comparison of our model with the $\Delta I=\frac{1}{2}$ slope data indicates that σ_{KK} should not be neglected. Form I of the function F yields acceptable results *only* for $\sigma_{KK}\simeq\frac{1}{2}m_\pi^2$. Form II yields acceptable results for $\sigma_{KK}\gtrsim\frac{1}{2}m_\pi^2$. Detailed results (for form II) are presented in Table I.

IV. $K_{2\pi}, K_{3\pi}$: $\Delta I=\frac{3}{2}$ AMPLITUDES

As was stated in Sec. II, the $\Delta I=\frac{3}{2}$ amplitudes present considerably more difficulty than their $\Delta I=\frac{1}{2}$ counterparts. Many of the pole terms which accounted for the rapid variation of the $\Delta I=\frac{1}{2}$ matrix elements in the soft-pion limits do not contribute in the case of $\Delta I=\frac{3}{2}$, and those pole terms which do appear are not sufficient to account for the rapid variation that occurs. The soft-pion limits are *not* consistent with constant background terms. Accordingly, we will have to make several assumptions in this section that go beyond the usual assumptions of PCAC smoothness.

In analyzing the $\Delta I=\frac{3}{2}$ transitions, we employ the Rarita-Schwinger formalism to describe the $I=\frac{3}{2}$ weak isotopic spurion. Thus, any $\Delta I=\frac{3}{2}$ matrix element is to be multiplied from the left by the vector-spinor isotopic spurion[18]:

$$\Delta S=-1:\;\; \tfrac{1}{2}\sqrt{2}\,\delta^{w,1+i2}(1,0)+\sqrt{2}\,\delta^{w,3}(0,1)\,, \tag{33a}$$

$$\Delta S=1:\;\; -\tfrac{1}{2}\sqrt{2}\,\delta^{w,1-i2}(0,1)+\sqrt{2}\,\delta^{w,3}(1,0)\,, \tag{33b}$$

where $w=1,2,3$.

In analogy with Eq. (11), we define the fundamental matrix element for $\Delta I=\frac{3}{2}$ nonleptonic kaon decays as[18,25]

$$\langle\pi^i|H_{W,3/2}^{\rm PC}|K\rangle=\tfrac{1}{2}L_{3/2}\delta^{wi}\,, \tag{34}$$

so that

$$\langle\pi^0|H_{W,3/2}^{\rm PC}|K_L\rangle=L_{3/2}\,. \tag{35}$$

We also note that, due to conservation of isotopic spin,

$$\langle 0|H_{W,3/2}^{\rm PV}|K\rangle\equiv 0\,. \tag{36}$$

A formal comparison of Eq. (36) and the soft-pion limit of Eq. (34) shows that, even at this level, naive PCAC is badly violated, since $L_{3/2}$ must go to zero in the soft-pion limit.

A. $K_{2\pi}$ amplitudes

For the $K\to 2\pi$ decay, isotopic-spin symmetry [i.e., Eq. (36)] does not allow the pole term of Fig. 1 to contribute to the $\Delta I=\frac{3}{2}$ amplitude. Thus the analog to Eq. (14) is simply

$$M_{3/2}(K\to\pi^i\pi^j)=\overline{M}_{2\pi,3/2}\,. \tag{37}$$

We can, however, no longer assume that the background term $\overline{M}$ is a constant. Explicit calculation of the soft-pion limit yields[25]

$$\lim_{p_i\to 0}\overline{M}_{2\pi,3/2}\equiv\lim_{p_i\to 0}M_{3/2}(K\to\pi^i\pi^j)=\frac{iL_{3/2}}{8f_\pi}(5i\epsilon^{ijw}+\delta^{wj}\tau^i+\delta^{wi}\tau^j)\,. \tag{38}$$

Now the background term can be parametrized in terms of two functions:

$$\bar{M}_{2\pi,3/2} = i\epsilon^{ijw}\bar{M}^{(1)}_{2\pi,3/2} + (\delta^{wj}\tau^i + \delta^{wi}\tau^j)\bar{M}^{(2)}_{2\pi,3/2}, \tag{39}$$

where $\bar{M}^{(I)}_{2\pi,3/2}$ is the amplitude leading to the $I=1$ or 2 2π final state. Although Eq. (38) determines the value of $\bar{M}^{(1)}$ to be $5iL_{3/2}/8f_\pi$ in the limit $p_i \to 0$, Bose statistics demands that at the physical point ($p_i^2 = p_j^2 = m_\pi^2$, $k = p_i + p_j$), it is identically zero. Thus $\bar{M}^{(1)}_{2\pi,3/2}$ is of no immediate interest.

On the other hand, $\bar{M}^{(2)}$ is apparently also momentum-dependent; the results of Appendix B [Eq. (B16)] would indicate that $\bar{M}^{(2)}$ is zero in the two-soft-pion limit ($p_i, p_j \to 0$). In the absence of any specific information on the form of the momentum dependence of this function, we will make the following assumption:

(a) The value of $\bar{M}^{(2)}_{2\pi,3/2}$ at the physical point is closely approximated by its value in the *single-soft-pion limit*. Thus,

$$\bar{M}^{\text{phys}}_{2\pi,3/2} \simeq \lim_{p_i \to 0} \bar{M}^{(2)}_{2\pi,3/2}(\delta^{wi}\tau^j + \delta^{wj}\tau^i) = \frac{iL_{3/2}}{8f_\pi}(\delta^{wi}\tau^j + \delta^{wj}\tau^i). \tag{40}$$

In terms of specific charge amplitudes, the on-shell amplitudes are then

$$M^{\text{phys}}_{3/2}(K^+ \to \pi^+\pi^0) = \tfrac{3}{2}M^{\text{phys}}_{3/2}(K_S \to \pi^+\pi^-) = -\tfrac{3}{4}M^{\text{phys}}_{3/2}(K_S \to \pi^0\pi^0) = \frac{3iL_{3/2}}{8f_\pi}. \tag{41}$$

B. $K_{3\pi}$ amplitudes

For the $\Delta I = \frac{3}{2}$ contribution to the $K \to 3\pi$ decays, the final 3π state can be in either $I=1$ or 2. In the case of the $I=1$ state, the isotopic factors always occur in the form $\delta^{ab}\delta^{wc}$, while for the $I=2$ final state, one always find the combination[26]

$$I^{abc} = i\epsilon^{abl}(\delta^{wl}\tau^c + \delta^{wc}\tau^l). \tag{42}$$

Through use of identities, one can always recast the isotopic factors into one of these two forms.

In the $K \to 3\pi$ decays, the pole diagrams of Fig. 2 contribute to the $\Delta I = \frac{3}{2}$ amplitude, though they do not completely reconcile naive PCAC and the soft-pion limits. Thus, we find

$$M_{3/2}(K \to \pi^i\pi^j\pi^k) = M^{K\text{ pole}}_{3\pi,3/2} + M^{\pi\text{ pole}}_{3\pi,3/2} + \bar{M}_{3\pi,3/2}. \tag{43}$$

In complete analogy with Eqs. (21) and (22), the pole terms can be calculated in terms of previously defined quantities:

$$M^{K\text{ pole}}_{3\pi,3/2} = -\frac{L_{3/2}}{16f_\pi^2}\{(I^{ijk} + \delta^{jk}\delta^{wi} - \delta^{ik}\delta^{wj})[2k\cdot(p_i - p_j) - p_i^2 + p_j^2] - 8\delta^{ij}\delta^{wk}F(p_i,p_j)\sigma_{KK}\}[2k\cdot(p_i+p_j) - (p_i+p_j)^2]^{-1} + \text{cycl}, \tag{44}$$

$$M^{\pi\text{ pole}}_{3\pi,3/2} = \frac{L_{3/2}}{2f_\pi^2}\{\delta^{ij}\delta^{kw}[(p_i+p_j)^2 - m_\pi^2] + \text{cycl}\}[(p_i+p_j+p_k)^2 - m_\pi^2]^{-1}. \tag{45}$$

An important but technical assumption in Eqs. (44) and (45) is that $L_{3/2}$ is taken as a constant defined by Eq. (41); this does not directly contradict the zero soft-pion limit formally implied by Eq. (36) since the values of the arguments of $L_{3/2}$ are different in the two cases.

As previously, we will attempt to evaluate the background term through the single-soft-pion limit. In evaluating this limit, we will maintain momentum conservation ($q \equiv k - p_i - p_j - p_k = 0$). Thus we will be able to avoid taking such unknown quantities as $\bar{M}^{(1)}_{2\pi,3/2}$ into account. This procedure yields

$$\lim_{p_i \to 0}\bar{M}_{3\pi,3/2} \equiv \lim_{p_i \to 0}[M_{3/2}(K \to \pi^i\pi^j\pi^k) - M^{K\text{ pole}}_{3\pi,3/2} - M^{\pi\text{ pole}}_{3\pi,3/2}]$$

$$= \frac{L_{3/2}}{32f_\pi^2}\{-4[1 + \tfrac{4}{3}F(p_j,p_k)\sigma_{KK}/m_K^2](\delta^{ij}\delta^{kw} + \delta^{jk}\delta^{iw} + \delta^{ki}\delta^{jw}) - [9 + \tfrac{16}{3}F(p_j,p_k)\sigma_{KK}/m_K^2][(\delta^{jk}\delta^{iw} - \delta^{ik}\delta^{jw}) + (\delta^{jk}\delta^{iw} - \delta^{ij}\delta^{kw})] + (I^{ijk} + I^{ikj})\}. \tag{46}$$

As is apparent, we will have to make additional assumptions. Assuming that $\bar{M}$ truly behaves as a background term, we would expect that $\bar{M}$ is a constant for the physical matrix element; hence we assume

(b) $\bar{M}_{3\pi,3/2}$ can vary only with p^2 [*some* variation is demanded by Bose statistics and Eq. (46)]. Then the part of $\bar{M}$ that leads to the $I=2$ 3π final state is proportional to the covariant

$$I^{ijk}(p_i^2 - p_j^2) + \text{cycl}$$

and *not* to the covariant

$$I^{ijk}(s_i - s_j) + \text{cycl},$$

which would lead to a nonconstant $\bar{M}$ in the physical region.[27] With this assumption, the I^{ijk} terms in Eq. (46) disappear on the mass shell. Bose statistics and the demand that $\bar{M}$ vary only through p^2 also cause the second term in Eq. (46) to vanish on the mass shell. Finally, we make a smoothness assumption similar to (a):

(c) The totally symmetric part of $\bar{M}_{3\pi,3/2}$ [first term of Eq. (46)] varies little in the single-soft-pion extrapolation.

Since the calculations are becoming complex, we simply note that $F(p_i, p_j)$ of form I [Eq. (10a)] is inconsistent with Eq. (58) below; one obtains a result which is of different sign from the experimental results. Hence, from this point on, we consider *only* form II of F [Eq. (10b)]. With this, we find

$$\bar{M}^{\text{phys}}_{3\pi,3/2} \simeq -\frac{L_{3/2}}{8f_\pi^2}\left[1 - \frac{4}{3}\frac{(m_K^2 - m_\pi^2)}{m_K^2 m_\pi^2}\sigma_{KK}\right](\delta^{ij}\delta^{kw} + \delta^{jk}\delta^{iw} + \delta^{ki}\delta^{jw}). \tag{47}$$

Putting everything together, we have

$$M^{\text{phys}}_{3/2}(K \to \pi^i \pi^j \pi^k) = \delta^{ij}\delta^{kw}\mathfrak{M}^{(k)}_{3/2,1} + \delta^{jk}\delta^{iw}\mathfrak{M}^{(i)}_{3/2,1} + \delta^{ki}\delta^{jw}\mathfrak{M}^{(j)}_{3/2,1} + \mathfrak{M}_{3/2,2}[I^{ijk}(s_i - s_j) + I^{jki}(s_j - s_k) + I^{kij}(s_k - s_i)], \tag{48}$$

where the $I=1$ 3π final-state amplitude is given by

$$\mathfrak{M}^{(i)}_{3/2,1} = \frac{L_{3/2}}{16f_\pi^2(m_K^2 - m_\pi^2)}\left\{11s_i - 9s_0 - \frac{8\sigma_{KK}}{m_\pi^2}[s_i - \tfrac{1}{3}(m_K^2 + m_\pi^2 + m_\pi^4/m_K^2)]\right\} \tag{49}$$

and the $I=2$ 3π final-state amplitude by

$$\mathfrak{M}_{3/2,2} = \frac{L_{3/2}}{16f_\pi^2(m_K^2 - m_\pi^2)}. \tag{50}$$

In terms of charge amplitudes, one has

$$M^{\text{phys}}_{3/2}(K^+ \to \pi^0\pi^0\pi^+) = \mathfrak{M}^{(3)}_{3/2,1} + 9(s_3 - s_0)\mathfrak{M}_{3/2,2}, \tag{51a}$$

$$M^{\text{phys}}_{3/2}(K^+ \to \pi^+\pi^+\pi^-) = \mathfrak{M}^{(1)}_{3/2,1} + \mathfrak{M}^{(2)}_{3/2,1} + 9(s_3 - s_0)\mathfrak{M}_{3/2,2}, \tag{51b}$$

$$M^{\text{phys}}_{3/2}(K_L \to \pi^+\pi^-\pi^0) = 2\mathfrak{M}^{(3)}_{3/2,1}, \tag{51c}$$

$$M^{\text{phys}}_{3/2}(K_L \to \pi^0\pi^0\pi^0) = 2[\mathfrak{M}^{(1)}_{3/2,1} + \mathfrak{M}^{(2)}_{3/2,1} + \mathfrak{M}^{(3)}_{3/2,1}], \tag{51d}$$

where 1, 2, 3 refer to the first, second, and third pion in the argument of $M_{3/2}$.

C. Combined $\Delta I = \frac{1}{2}, \frac{3}{2}$ amplitudes

Anticipating the fact that form I for $F(p_i, p_j)$ is ruled out, we gather together the combined $\Delta I = \frac{1}{2}$ and $\frac{3}{2}$ amplitudes, using form II only [Eq. (10b)].

For $K_{2\pi}$, the traditional parametrization is

$$M(K^+ \to \pi^+\pi^0) = A_{3/2}e^{i\delta_2}, \tag{52a}$$

$$M(K_s \to \pi^+\pi^-) = A_{1/2}e^{i\delta_0} + \tfrac{2}{3}A_{3/2}e^{i\delta_2}, \tag{52b}$$

$$M(K_s \to \pi^0\pi^0) = A_{1/2}e^{i\delta_0} - \tfrac{4}{3}A_{3/2}e^{i\delta_2}, \tag{52c}$$

where $\delta_{0,2}$ are the final state $I=0$, 2 $\pi\pi$ phase shifts. The problem of comparing the real results of current-algebra low-energy theorems with the complex amplitudes yielded by experiments has been discussed elsewhere.[27] We will simply adopt the usual assumption that the (real) current-algebra results are to be multiplied by the final-state phase shift factor. Thus we find, comparing Eqs. (18), (41), and (52),

$$L_{1/2} = 2if_\pi A_{1/2}[1 - 4\sigma_{KK}(m_K^2 - m_\pi^2)/(m_K^2 m_\pi^2)]^{-1}, \tag{53a}$$

$$L_{3/2} = -8if_\pi A_{3/2}/3. \tag{53b}$$

Using these values, we can write the $K_{3\pi}$ amplitudes in the standard form:

$$M(K \to \pi^a\pi^b\pi^c) = A^{abc}\left[1 + \frac{g^{abc}}{2m_\pi^2}(s_c - s_0)\right],$$

with

$$A^{++-}=2A^{00+}=A_{1/2}^{++-}(1+\delta B)\,, \tag{54a}$$

$$A^{+-0}=\tfrac{1}{3}A^{000}=-\tfrac{1}{2}A_{1/2}^{++-}(1-2\delta B)\,, \tag{54b}$$

$$g^{+-0}=g_{1/2}^{+-0}\{1+\delta[-(11-8\sigma_{KK}/m_\pi^2)d+2B]\}\,, \tag{55a}$$

$$g^{00+}=g_{1/2}^{+-0}\{1+\delta[(10-4\sigma_{KK}/m_\pi^2)d-B]\}\,, \tag{55b}$$

$$g^{++-}=-\tfrac{1}{2}g_{1/2}^{+-0}\{1+\delta[(1-4\sigma_{KK}/m_\pi^2)d-B]\}\,, \tag{55c}$$

where $\delta=A_{3/2}/A_{1/2}$ and where $A_{1/2}^{++-}$ and $g_{1/2}^{+-0}$ are given by Eqs. (30) with

$$B=\frac{2}{3}\,\frac{[m_K^2+3m_\pi^2-4\sigma_{KK}(2m_K^2-m_\pi^2)/m_K^2][1-4\sigma_{KK}(m_K^2-m_\pi^2)/m_K^2]}{[m_K^2+3m_\pi^2-4\sigma_{KK}m_K^2/m_\pi^2]}\,, \tag{56a}$$

$$d=\tfrac{2}{3}[1-4\sigma_{KK}(m_K^2-m_\pi^2)/(m_K^2m_\pi^2)]/[1-4\sigma_{KK}/m_\pi^2]\,. \tag{56b}$$

The above expressions are exact only to order δ.

The combinations of the slopes that will be needed below are simply expressed as

$$g^{00+}+2g^{++-}=9d\delta g_{1/2}^{+-0}\,, \tag{57}$$

$$g^{00+}-2g^{++-}-2g^{+-0}=3\delta g_{1/2}^{+-0}[(11-8\sigma_{KK}/m_\pi^2)d-2B]\,. \tag{58}$$

As is noted in Appendix A, Eq. (57) is a measure of the $\Delta I=\frac{3}{2}$ transition to the $I=2$ 3π final state; the prediction depends only on assumptions (a) and (b) above, and is independent of assumption (c). The right-hand side of Eq. (57) comes purely from $M_{3\pi,3/2}^{K\,\mathrm{pole}}$.

On the other hand, Eq. (58) is a measure of the $\Delta I=\frac{3}{2}$ transition to the $I=1$ 3π final state; the prediction depends on assumptions (a), (b), and (c) above.

D. Comparison with experiment: $\Delta I=\frac{3}{2}$

The first thing to be done is to extract the value of $\delta=A_{3/2}/A_{1/2}$ from the $K_{2\pi}$ experimental data. The difference, $\delta_0-\delta_2$, of the $\pi\pi$ phase shifts in Eq. (52) can be determined either from $\pi N\to\pi\pi N$ and $\pi N\to\pi\pi\Delta$ peripheral collisions or from extrapolating the Weinberg $\pi\pi$ threshold amplitude up to $s=m_K^2$ using singly- and doubly-subtracted dispersion relations.[28] In either case the result[2,28] is $\delta_0-\delta_2\approx 47°$ at $s=m_K^2$. Neglecting radiative corrections, the experimental branching ratio of[1] $\Gamma(K_S\to\pi^+\pi^-)/\Gamma(K_S\to\pi^0\pi^0)=2.195\pm0.029$ along with Eqs. (52) implies

$$\delta\approx 0.044 \tag{59}$$

for $\delta_0-\delta_2=47°$. This compares favorably with the magnitude extracted from the experimental ratio of $\Gamma(K^+\to\pi^+\pi^0)/\Gamma(K_S\to\pi\pi)=(0.152\pm0.002)\times10^{-2}$ which is essentially independent of $\delta_0-\delta_2$:

$$|\delta|\approx 0.05\,. \tag{60}$$

If one includes the radiative corrections,[2] the value of δ increases to 0.06. Since we have not estimated the effects of the radiative corrections in $K\to3\pi$ decays, it is perhaps more consistent to neglect them in the $K\to2\pi$ decays as well. Therefore, we use Eq. (59) to estimate the strength of the (real) current-algebra predictions of Eqs. (18), (41), and (53).

The results for the $K_{3\pi}$ decay parameters that measure the $\Delta I=\frac{3}{2}$ transition are presented in Table I. A word of caution is necessary: Almost all the quantities listed in the table pass through a singularity as the value of σ_{KK} is increased from 0 to $\frac{1}{2}m_\pi^2$. Hence, the results with $\sigma_{KK}=0$ and $\sigma_{KK}=\frac{1}{2}m_\pi^2$ are of the same order of magnitude, but the transition is not a continuous one. The reason for this discontinuity is the factor of $(1-4\sigma_{KK}/m_\pi^2)$ in Eq. (53a).

The quantity

$$g^{00+}+2g^{++-}$$

is seen to be in reasonable agreement with experiment for all values of σ_{KK} quoted. The quantity

$$g^{00+}-2g^{++-}-2g^{+-0}$$

is much more sensitive to the value of σ_{KK}, and indicates that $\sigma_{KK}\approx 2m_\pi^2$, much larger than the naive GMOR value of $\sigma_{KK}=\frac{1}{2}m_\pi^2$. This slope test completely rules out form I for the $K\pi$ amplitude. Unfortunately, as noted above, this prediction rests on all three of our assumptions (a), (b), and (c).

With regard to the amplitudes themselves, there is only one relation that is a measure of the $\Delta I=\frac{3}{2}$ part of the weak nonleptonic Hamiltonian. Expressed in terms of rates, this is[1]

$$\frac{1}{2}\,\frac{\Gamma(K^+\to\pi^+\pi^+\pi^-)}{\phi(++-)}\left[\frac{\Gamma(K_L\to\pi^+\pi^-\pi^0)}{\phi(+-0)}\right]^{-1}-1=6\,\delta B\,, \tag{61}$$

where ϕ are phase-space factors that include pion mass differences, the observed slopes, and the final-state Coulomb interaction. It should be noted, however, that corrections as large as 0.1 can be added to the experimental value of the left-hand side of Eq. (61), due to uncertainties arising from mass differences, and from the large slopes.[29]

V. DISCUSSION OF THE $K_{2\pi}$, $K_{3\pi}$ RESULTS

In Secs. III and IV we have discussed the nonleptonic decays of the kaons in terms of the algebra of currents and of the current-current weak Hamiltonian (which implies $\Delta I = \frac{1}{2}$ and $\frac{3}{2}$ *only*). The inclusion of the σ_{KK} term in this analysis is not trivial; in order to make the σ-term contribution have the correct Adler zero, it was necessary to assume a function F [Eq. (10)] to continue the contribution of the σ term to the $K\pi$ scattering amplitude from the soft point to the physical region.

With regard to the $\Delta I = \frac{1}{2}$ contributions to the $K_{3\pi}$ parameters: The slope data seem to indicate fairly unambiguously that one does, indeed, need a σ term $\sigma_{KK} \gtrsim \frac{1}{2} m_\pi^2$. The data on the $K_{3\pi}$ rates, however, which are well fitted for $\sigma_{KK} = 0$, give an amplitude which is too low by 15–25% for $\sigma_{KK} \gtrsim \frac{1}{2} m_\pi^2$. We regard this 20% suppression of the amplitude as an inherent problem of PCAC and the soft-pion limit—analogous to the suppression of the Goldberger-Treiman value of $f_\pi = 86$ MeV compared with the value found from the Cabibbo form for π_{l2} decay, $f_\pi = 94$ MeV.

With regard to the $\Delta I = \frac{3}{2}$ contributions to the $K_{3\pi}$ parameters: One combination of the slopes [Eq. (57)] gives a prediction which depends primarily on the hypothesis that the background term, $\overline{M}_{3\pi, 3/2}$, is a constant in the physical region; the prediction agrees well with the data. The other combination of slopes [Eq. (58)], which depends on a stronger PCAC smoothness assumption, argues unambiguously for form II for the function F [Eq. (10b)] and for a value of $\sigma_{KK} \simeq 2 m_\pi^2$. Finally, the one test on the $\Delta I = \frac{3}{2}$ contribution to the rates [Eq. (61)] seems to be in agreement with any large value of σ_{KK} *except* $\sigma_{KK} \simeq 2 m_\pi^2$; the estimate of the experimental value of the left-hand side of Eq. (61), however, is rather uncertain.[29] In view of the approximations used in Sec. IV, moreover, it is not unreasonable to assume that a more accurate treatment of the problem would reconcile these last two pieces of data.

PCAC has been employed throughout as a semiphenomenological demand for smoothness in the extrapolation from the soft-pion point to the physical point. This has been interpreted as the postulate that, after explicit pole terms have been extracted, the remaining "background" term is a constant. For the $\Delta I = \frac{1}{2}$ transitions, this assumption is completely consistent with the soft-pion limits. For the $\Delta I = \frac{3}{2}$ transitions, on the other hand, this simple PCAC assumption proves inconsistent with the soft-pion limits, and some variation has to be included in the background term to achieve consistency.

Finally, the large value $\sigma_{KK} \simeq 2 m_\pi^2$ is clearly inconsistent with the simplest form of the GMOR model for chiral symmetry breaking

$$H' = u_0 + c u_8 , \tag{62}$$

which implies

$$\sigma_{\pi\pi} = m_\pi^2 , \tag{63a}$$

$$\sigma_{KK} = \tfrac{1}{2} m_\pi^2 , \tag{63b}$$

$$\sigma_{\eta_8\eta_8} = \tfrac{1}{3} m_\pi^2 , \tag{63c}$$

$$c = -1.25 , \tag{63d}$$

and the quadratic meson mass formula. While Eq. (63a) agrees well with experiment [and is incorporated in the $\pi\pi$ scattering amplitude, Eq. (5)], it is valid in the $(3, \overline{3})$ model, independent of c and the structure of the meson mass formula. It has been argued elsewhere, however, that the remaining predictions Eqs. (63b)–(63d) are not so well verified.[9] In fact, uncertainties in the traditional (SU_3-dependent) methods of determining the parameter c allow a value of $c \simeq -1$. Then σ_{KK} in Eq. (63b) is increased by a factor of ~3 which is in agreement with our results of Sec. IV (which are SU_3-independent). It is interesting to note that this change of the value of c also accounts for the discrepancy between the GMOR determination of $\sigma_{NN}^{\pi\pi}$ and the larger value that seems to be indicated by several independent analyses of the data.[8,9,30]

VI. NONLEPTONIC HYPERON DECAYS AND σ TERMS

Having seen that the meson σ terms play an important role in $K_{2\pi}$ and $K_{3\pi}$ decays, we are obliged to investigate their consequences in the seven hyperon decays $B \to B'\pi$. Traditionally one simply ignores the role of any σ term or kaon pole and proceeds to calculate the s- and p-wave amplitudes A and B defined as

$$M(B \to B'\pi) = iA + B\gamma_5 \tag{64}$$

from the current commutator term $M_{cc} = \lim_{q_\pi \to 0} M(B \to B'\pi)$ and the (rapidly varying) baryon pole term of Fig. 3:

$$M(B \to B'\pi) = M_{cc} + M^{B\,\text{pole}} - M^{B\,\text{pole}}(q_\pi = 0) + \overline{M} . \tag{65}$$

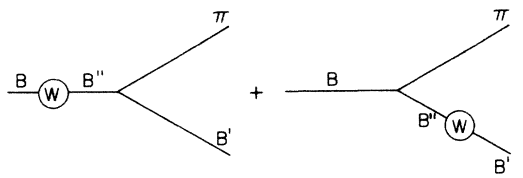

FIG. 3. Rapidly varying octet baryon pole in $B\to B'\pi$ decays.

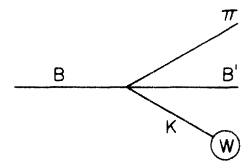

FIG. 4. Rapidly varying K pole in $B\to B'\pi$ decays.

In a recent paper, it was shown that one must also separate out the rapidly varying decuplet poles from $\overline{M}$ in order that the theoretical fits to the data agree with experiment.[10] One then finds that $H_W\sim J\cdot J=(V-A)\cdot(V-A)$ obeys

$$\langle B'|H^{PC}_{W,\underline{8}}|B\rangle=h_1(d_W d^{B'B6}-f_W i f^{B'B6}),\qquad(66a)$$

$$\langle n|H^{PC}_{W,\underline{27}}|\Sigma^0\rangle=\frac{3}{2\sqrt{2}}h_{27},\qquad(66b)$$

with

$$d_W/f_W\simeq-0.9,\qquad(67a)$$

$$h_1 f_W\simeq110\text{ eV},\qquad(67b)$$

$$h_{27}\simeq-1.7\text{ eV},\qquad(67c)$$

and $d_W+f_W=1$.

Indeed, a small $\Delta I=\frac{3}{2}$ part in H_W ($H_{W,\underline{27}}$) of ~2% was even found necessary to improve the fits to Eq. (65). Furthermore, this procedure also correctly describes the weak radiative decay $\Sigma^+\to p\gamma$ rate and the sign of its measured asymmetry parameter.[10] In sum, all of the (two-body) weak hyperon nonleptonic decays are now completely understood in terms of current algebra and there is little room to incorporate any σ-term contributions which have thus far been neglected.

On the other hand, from the standpoint of the present work, we see no reason to ignore the rapidly varying K^0 pole term of Fig. 4 which is the baryon analog of Fig. 1 and corresponds to the σ-term amplitude [which contributes to the s-wave amplitude A in Eq. (64)]:

$$M^\sigma(B\to B'\pi)=-M^\sigma(BK^0\to B'\pi)\langle 0|H^{PV}_{W,1/2}|K^0\rangle/m_K^2\qquad(68)$$

We must regard Eq. (68) as a rapidly varying part of $\overline{M}$ in Eq. (65). In this case $M^\sigma(BK^0\to B'\pi)$ is the strong meson-baryon amplitude evaluated at $q^2=m_K^2$ which obeys

$$\lim_{q_\pi^2\to0,\,q_K\to0} f_K f_\pi M^\sigma(BK^0\to B'\pi)=-\sigma^{\pi K}_{B'B}\qquad(69)$$

where, in the $(3,\overline{3})$ model,

$$\sigma^{\pi^0K^0}_{B'B}=-\frac{1}{2\sqrt{3}}(\sqrt{2}+c)\langle B'|u_{K^0}|B\rangle.\qquad(70)$$

We have taken the $q_\pi^2=0$ limit along with the required $q_K=0$ limit in order to avoid setting the baryon masses m_B and $m_{B'}$ equal. Equations (69) and (70) lead to the soft amplitudes

$$|f_K f_\pi M(BK^0\to B'\pi)|\to\sim10\text{ MeV for }c=-1.25,\qquad(71a)$$

$$\sim30\text{ MeV for }c=-1.0\qquad(71b)$$

for all B, B' except $\sigma^{\pi^+K^0}_{n\Sigma^+}$ which vanishes, as does the entire $\Sigma^+\to n\pi^+$ s-wave amplitude. We shall therefore use Eq. (71) to estimate the strong low-energy amplitude in Eq. (68).

In order to complete the estimate of Eq. (68), we must rely upon our $K_{2\pi}$, $\Delta I=\frac{1}{2}$ amplitude, Eq. (53a). Since

$$|A_{1/2}|\simeq\left|\frac{1}{2f_\pi}L_{1/2}(1-4\sigma_{KK}/m_\pi^2)\right|$$

$$=0.77\times10^{-6}m_K,\qquad(72)$$

we infer from Eq. (11) that

$$|\langle 0|H^{PV}_{W,1/2}|K^0\rangle|=0.42f_\pi m_K^2\times10^{-6}\text{ for }\sigma_{KK}=\tfrac{1}{2}m_\pi^2,\qquad(73a)$$

$$=0.06f_\pi m_K^2\times10^{-6}\text{ for }\sigma_{KK}=2m_\pi^2.\qquad(73b)$$

Combining Eq. (71) with Eq. (73) and $f_K\simeq120$ MeV, our estimate of Eq. (68) is

$$|M^\sigma(B\to B'\pi)|\sim0.04\times10^{-6}\text{ for }\sigma_{KK}=\tfrac{1}{2}m_\pi^2,\qquad(74a)$$

$$\sim0.02\times10^{-6}\text{ for }\sigma_{KK}=2m_\pi^2,\qquad(74b)$$

which must be added to the other s-wave contributions in Eq. (65). The actual magnitude of the s-wave amplitudes is of the order of 0.3×10^{-6}, and it is clear that neither value of σ_{KK} destroys the previous good fits to the data.

VII. THE SIGNIFICANCE OF OCTET DOMINANCE

Finally we return to the original current-current form for H_W, and try to understand the meaning of the resulting octet dominance of the $K_{2\pi}$, $K_{3\pi}$, and $B\to B'\pi$ amplitudes. In the operator sense, the Cabibbo current-current Hamiltonian contains equally large $\Delta I=\frac{1}{2}$ and $\Delta I=\frac{3}{2}$ contributions to H_W and so one usually assumes that dynamics somehow enhances the octet part of H_W. Recently, Pagels and co-workers[31] have discovered the dynamical octet enhancement mechanism for the semistrong mass-breaking Hamiltonian H_{ss}, but the extension to H_W (Ref. 32) leads to dynamical inconsistencies[33] and to fitted parameters not in agreement with the data [Eqs. (66) and (67)].

Another dynamical procedure is to saturate $H_W \sim J\cdot J$ with intermediate single particle and resonance states and sandwich the result between single particle baryon or meson states. Updating the experimental parameters used in Ref. 34 which occur in $\langle B'|J\cdot J|B\rangle$ leads to precisely[35] the fitted parameters, Eq. (67). Unfortunately, the same procedure applied to $\langle\pi|J\cdot J|K\rangle$ involves too many undetermined parameters and does not lead to a convincing explanation for the octet dominance of $\langle\pi|H_W|K\rangle$ (Ref. 36).

On the other hand, our discussion in the last section shows that the preferred value of $\sigma_{KK}\simeq 2m_\pi^2$ suppresses the strength of $\langle\pi|H_{W,1/2}|K\rangle$ by a factor of 7 relative to $\langle\pi|H_{W,3/2}|K\rangle$:

$$\frac{\langle\pi^0|H^{\mathrm{PC}}_{W,3/2}|K_L\rangle}{\langle\pi^0|H^{\mathrm{PC}}_{W,1/2}|K_L\rangle}\equiv\frac{L_{3/2}}{L_{1/2}}$$
$$\simeq -\tfrac{4}{3}\delta(1-4\sigma_{KK}/m_\pi^2)$$
$$\simeq 0.4\,, \qquad (75a)$$

where we have used Eqs. (53), (59), and $\sigma=2m_\pi^2$ to evaluate the right-hand side of Eq. (75a). This result is unaltered if we use standard SU_3 arguments to rewrite Eq. (75a) in terms of matrix elements of the $\underline{8}$ and $\underline{27}$ pieces of the Cabibbo current-current Hamiltonian:

$$\frac{\langle\pi^0|H_{W,27}|K_L\rangle}{\langle\pi^0|H_{W,8}|K_L\rangle}=\frac{9L_{3/2}/L_{1/2}}{10+L_{3/2}/L_{1/2}}\simeq 0.4\,. \qquad (75b)$$

This result is satisfying in that it does *not* imply the octet dominance of H_W in the operator sense, consistent with the discussion above. In conjunction with Eq. (75b), however, the Cabibbo-Gell-Mann theorem[11] should be remembered: In the limit of exact SU_3, all the kaon matrix elements of this paper are identically zero. The Wigner-Eckart theorem for SU_3 has been used in deriving Eq. (75b), basically to compare the relative strength of $I=\frac{1}{2}$ and $\frac{3}{2}$ in $\langle\pi|H_{W,27}|K\rangle$. Hence, we have assumed that the SU_3 Wigner-Eckart theorem gives reasonably good results for the ratio, even though the reduced matrix elements are essentially SU_3-broken.[17]

It is difficult to draw any conclusion about octet dominance of H_W from the hyperon decays. The large values $f_W\sim 10$, $d_W\sim -9$ in Eqs. (67) obscure the "raw strength" of the baryon matrix elements of the octet part of H_W. However, there is no obvious contradiction with the conclusion drawn above.

We therefore conclude that H_W is *not* somehow octet-dominated in the operator or single-particle matrix-element sense. Rather, $H_{W,27}$ is almost as large as $H_{W,8}$, as the ordinary Cabibbo current-current form would indicate; the dynamical values of $\sigma_{KK}=2m_\pi^2$ and $f_W=10$ in effect suppress $H_{W,27}$ for the three-body matrix elements of $K_{2\pi}$ and $B\to B'\pi$.

VIII. CONCLUSION

It has proved possible to give a consistent description of the $K_{2\pi}$ and $K_{3\pi}$ decays through the Cabibbo current-current Hamiltonian, using low-energy theorems and an extended version of PCAC. We have employed the rapidly varying pole method in all cases. In so doing, we have found clear indications that it is necessary to include the effects of the σ terms in order to achieve reasonable agreement with the experimental data.

One of the predictions of our work [Eq. (58)] implies that

$$\sigma_{KK}\equiv\langle K|\sigma^{\pi\pi}|K\rangle$$
$$=\tfrac{1}{3}(\sqrt{2}+c)\langle K|\sqrt{2}\,u_0+u_8|K\rangle \qquad (76)$$

has the anomalously large value of $\sigma_{KK}=2m_\pi^2$. This interesting result reinforces recent speculation that σ terms have values larger than one would expect from the simplest form of the GMOR $(3,\bar{3})$ model.

The hyperon nonleptonic decays have also been reanalyzed by including the kaon pole (Fig. 5) which has previously been neglected. Once again, comparison with the data indicates that a large σ_{KK} term is compatible with the data.

A conclusion of this work is the possibility that octet dominance is *not* to be understood in terms of the vanishing of the $\underline{27}$ piece of the current-current Hamiltonian in the operator sense, *nor* in the suppression of single-particle matrix elements of $H_{W,27}$ with respect to $H_{W,8}$. Rather, the enhancement of $\langle\pi\pi|H_{W,8}|K\rangle$ is due to the large value of σ_{KK}.

FIG. 5. Pole terms contributing to Eq. (B7).

Note added in proof. It has been pointed out to us that B. Holstein [Phys. Rev. 183, 1228 (1969)] has also attempted to find the $\Delta I=\frac{3}{2}$ amplitude in $K_{2\pi}$ and $K_{3\pi}$ by following the rapidly varying pole technique originally advocated by one of us (P.C.M.). Our treatment differs from of Holstein because we demand that the nonpole background amplitude, $\bar{M}$, be constant, at least in the physical region. Furthermore, Holstein's choice of the function F is $\frac{1}{2}F_{\rm I}+\frac{1}{2}F_{\rm II}$.

ACKNOWLEDGMENT

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APPENDIX A

In analyzing the experimental data for $K_{3\pi}$, it is frequently useful to separate explicitly the different contributions to the four charge modes of the decay.[37]

We assume that there are no contributions to the decay from a weak Hamiltonian with $\Delta I \geq \frac{5}{2}$. This seems to be verified from the data, and is implied by the current-current model used in this paper. With this assumption, we can write the amplitudes (neglecting any higher order momentum dependence)

$$\begin{aligned}
M(K^+\to\pi^0\pi^0\pi^+) &= a^{1/2}+b^{1/2}y+a^{3/2}+b^{3/2}y+c^{3/2}y\,,\\
M(K^+\to\pi^+\pi^+\pi^-) &= 2a^{1/2}-b^{1/2}y+2a^{3/2}-b^{3/2}y+c^{3/2}y\,,\\
M(K_L\to\pi^+\pi^-\pi^0) &= -a^{1/2}-b^{1/2}y+2a^{3/2}+2b^{3/2}y\,,\\
M(K_L\to\pi^0\pi^0\pi^0) &= -3a^{1/2}+6a^{3/2}\,,
\end{aligned} \tag{A1}$$

where $y=(s_3-s_0)/(2m_\pi^{\,2})$, as defined in Eq. (19). In these equations, $a^{1/2}$ and $b^{1/2}$ are constants that arise from the $\Delta I=\frac{1}{2}$ piece of the weak Hamiltonian; $a^{3/2}$ and $b^{3/2}$ arise from the $\Delta I=\frac{3}{2}$ contribution to the $I=1$ 3π final state; $c^{3/2}$ describes the $\Delta I=\frac{3}{2}$ contribution to the $I=2$ 3π final state.

Assuming that the $\Delta I=\frac{1}{2}$ contributions are much larger than the $\Delta I=\frac{3}{2}$ contributions ($a^{1/2}, b^{1/2} \gg a^{3/2}, b^{3/2}, c^{3/2}$) one can derive, to lowest order in $a^{3/2}, b^{3/2}, c^{3/2}$ the following:

$$\begin{aligned}
M(K^+\to\pi^0\pi^0\pi^+) &= a^{1/2}\left(1+\frac{a^{3/2}}{a^{1/2}}\right)\left[1+y\frac{b^{1/2}}{a^{1/2}}\left(1+\frac{b^{3/2}}{b^{1/2}}+\frac{c^{3/2}}{b^{1/2}}-\frac{a^{3/2}}{a^{1/2}}\right)\right],\\
M(K^+\to\pi^+\pi^+\pi^-) &= 2a^{1/2}\left(1+\frac{a^{3/2}}{a^{1/2}}\right)\left[1-\frac{1}{2}y\frac{b^{1/2}}{a^{1/2}}\left(1+\frac{b^{3/2}}{b^{1/2}}-\frac{c^{3/2}}{b^{1/2}}-\frac{a^{3/2}}{a^{1/2}}\right)\right],\\
M(K_L\to\pi^+\pi^-\pi^0) &= -a^{1/2}\left(1-2\frac{a^{3/2}}{a^{1/2}}\right)\left[1+y\frac{b^{1/2}}{a^{1/2}}\left(1-2\frac{b^{3/2}}{b^{1/2}}+2\frac{a^{3/2}}{a^{1/2}}\right)\right],\\
M(K_L\to\pi^0\pi^0\pi^0) &= -3a^{1/2}\left(1-2\frac{a^{3/2}}{a^{1/2}}\right).
\end{aligned} \tag{A2}$$

In terms of the parametrization of Eq. (19), we see that the experimental values of the slopes can be combined to yield the following information on the $\Delta I=\frac{1}{2}$ weak interaction:

$$g^{00+}-2g^{++-}+g^{+-0}=3\frac{b^{1/2}}{a^{1/2}}. \tag{A3}$$

To isolate the information on the $\Delta I=\frac{3}{2}$ weak interaction that leads to the $I=2$ 3π final state, the following combination is relevant:

$$g^{00+}+2g^{++-}=2\frac{c^{3/2}}{a^{1/2}}. \tag{A4}$$

Finally, to isolate the $\Delta I=\frac{3}{2}$ weak contribution to the $I=1$ 3π final state, the combination is

$$g^{00+}-2g^{++-}-2g^{+-0}=6\frac{b^{1/2}}{a^{1/2}}\left(\frac{b^{3/2}}{b^{1/2}}-\frac{a^{3/2}}{a^{1/2}}\right). \tag{A5}$$

It is clear that a similar separation can be effected in terms of the amplitudes themselves, as in Eq. (61).

APPENDIX B

It is the purpose of this appendix to show that, in the case of the $\Delta I=\frac{1}{2}$ transitions, the interplay of pole terms, constant background terms $\bar{M}$, and multi-soft-pion limits yields an intricate but consistent picture of $K_{2\pi}$ and $K_{3\pi}$ decays.

Given the assumptions of current algebra (extended to the σ term), the current-current Hamiltonian and the soft-pion limits [Eqs. (2)–(4)], one can use standard T product identities to derive the following soft-pion limits[38]:

$$\lim_{p_i\to 0}\langle \pi^i|H_W|\,\rangle = \frac{i}{f_\pi}\langle\,|[Q^i,H_W]|\,\rangle\,, \tag{B1}$$

$$\lim_{p_i, p_j \to 0} \langle \pi^i \pi^j | H_W | \rangle = -\frac{1}{2f_\pi^2} \langle | \{[Q^i, [Q^j, H_W]] + [Q^j, [Q^i, H_W]]\} | \rangle$$
$$- \frac{\epsilon^{ijl}}{2f_\pi^2} \lim_{p_i, p_j \to 0} \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle | T((p_i - p_j)\cdot V^l(x) H_W(0)) | \rangle$$
$$- \frac{i}{f_\pi^2} \delta^{ij} \lim_{p_i, p_j \to 0} \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle | T(\sigma(x) H_W(0)) | \rangle , \qquad \text{(B2)}$$

$$\lim_{p_i, p_j, p_k \to 0} \langle \pi^i \pi^j \pi^k | H_W | \rangle = -\frac{i}{4f_\pi^3} (3\delta^{ij}\delta^{kl} + 3\delta^{ik}\delta^{jl} + 2\delta^{jk}\delta^{il}) \langle | [Q^l, H_W] | \rangle$$
$$- \frac{i}{4f_\pi^3} \langle | \{[Q^i, [Q^j, [Q^k, H_W]]] + [Q^i, [Q^k, [Q^j, H_W]]]$$
$$+ [Q^j, [Q^k, [Q^i, H_W]]] + [Q^k, [Q^j, [Q^i, H_W]]]\} | \rangle$$
$$- \frac{i\epsilon^{ijl}}{2f_\pi^3} \lim_{p_i, p_j, p_k \to 0} \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle | T((p_i - p_j)\cdot V^l(x) [Q^k, H_W]) | \rangle + \text{cycl}$$
$$+ \frac{\delta^{ij}}{f_\pi^3} \lim_{p_i, p_j, p_k \to 0} \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle | T(\sigma(x) [Q^k, H_W]) | \rangle + \text{cycl} . \qquad \text{(B3)}$$

In the last equation, the first two terms appear unsymmetric in the indices i, j, k; when taken together, however, the resulting matrix element is symmetric. In Eqs. (B2) and (B3), the terms in which the limit has not yet been taken are either singular in the limit or have a value which depends on the path taken to the limiting point.

In Sec. II we demonstrated that [Eq. (16)]

$$\lim_{p_i \to 0} \bar{M}_{2\pi, 1/2} = -\frac{i\sqrt{2}}{4f_\pi} L_{1/2} \delta^{ij} . \qquad \text{(B4)}$$

Now it will be demonstrated that the PCAC assumption that the background term $\bar{M}$ is a constant is consistent with the two-soft-pion limit,

$$\lim_{p_i, p_j \to 0} \bar{M}_{2\pi, 1/2} \equiv \lim_{p_i, p_j \to 0} [M_{1/2}(K \to \pi^i \pi^j) - M^{K\text{pole}}_{2\pi, 1/2}] . \qquad \text{(B5)}$$

The kaon-pole term is given explicitly in Eq. (15), so it only remains to calculate from Eq. (B2)

$$\lim_{p_i, p_j \to 0} M_{1/2}(K \to \pi^i \pi^j) = -\frac{i\sqrt{2}}{4f_\pi} L_{1/2} + \frac{\epsilon^{ijl}}{2f_\pi^2} \lim_{p_i, p_j \to 0} \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle 0 | T((p_i - p_j)\cdot V^l(x) H_{W,1/2}(0)) | K \rangle$$
$$+ \frac{i}{f_\pi^2} \delta^{ij} \lim_{p_i, p_j \to 0} \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle 0 | T(\sigma(x) H_{W,1/2}(0)) | K \rangle . \qquad \text{(B6)}$$

The second term in this expression can be calculated by extracting from the one- and three-body intermediate states the pole term which corresponds to the Feynman diagram in Fig. 5(a); all other contributions to this matrix element go smoothly to zero in the limit $p_i, p_j \to 0$. Thus

$$(p_i - p_j)^\mu \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle 0 | T(V^l_\mu(x) H_{W,1/2}(0)) | K(k) \rangle = (p_i - p_j)^\mu \left[\frac{1}{2} \frac{\sqrt{2} f_\pi L_{1/2} \tau^l (2k - p_i - p_j)_\mu}{(k - p_i - p_j)^2 - m_K^2} \right] + O(p) . \qquad \text{(B7)}$$

Similarly the last term of Eq. (B6) can be calculated by extracting the pole term of Fig. 5(b):

$$\int d^4x\, e^{i(p_i+p_j)\cdot x} \langle 0 | T(\sigma(x) H_{W,1/2}(0)) | K(k) \rangle = \frac{-\sqrt{2} f_\pi \sigma_{KK} L_{1/2}}{2k\cdot(p_i + p_j) - (p_i + p_j)^2} + C + O(p) , \qquad \text{(B8)}$$

where C is a possible constant term. When Eqs. (B6), (B7), and (B8) are substituted into Eq. (B5), all the apparently singular (or ill-defined) terms exactly cancel against similar terms in $M^{K\text{pole}}_{2\pi,1/2}$, leaving

$$\lim_{p_i, p_j \to 0} \bar{M}_{2\pi, 1/2} = -\frac{i\sqrt{2}}{4f_\pi} L_{1/2}\delta^{ij} + \frac{i}{f_\pi^2} C\delta^{ij}. \qquad \text{(B9)}$$

Comparing Eqs. (B4) and (B9), we see that $\bar{M}$ differs from a constant by momentum-dependent terms which give rise to C, and which depend on the detailed dynamics of the weak and strong interactions. Rather than abandon the PCAC assumption that $\bar{M}$ is essentially a constant, we could make either of two different (though not unrelated) assumptions:

(i) C is zero or small for dynamical reasons;

(ii) $\bar{M}$ evaluated at the single-soft-pion limit involves the smallest extrapolation to the physical value, and the momentum-dependent terms in $\bar{M}$ that give rise to C are small on the mass shell. These assumptions are rather similar to those made in the vastly more complex case of $\Delta I = \frac{3}{2}$ in Sec. IV.

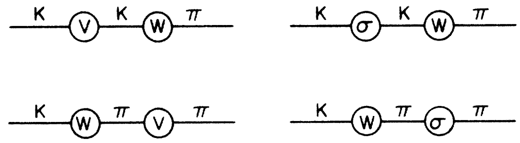

FIG. 6. Pole terms contributing to Eqs. (B12).

For the case of $K_{3\pi}$, we have already derived the one-soft-pion limit of $\bar{M}_{3\pi, 1/2}$ in Eq. (24). For the two-soft-pion limit

$$\lim_{p_i, p_j \to 0} \bar{M}_{3\pi, 1/2} \equiv \lim_{p_i, p_j \to 0} [M_{1/2}(K \to \pi^i\pi^j\pi^k) - M^{K\,\text{pole}}_{3\pi, 1/2} - M^{\pi\,\text{pole}}_{3\pi, 1/2}], \qquad \text{(B10)}$$

we use Eq. (B3) to find

$$\lim_{p_i, p_j \to 0} M_{1/2}(K \to \pi^i\pi^j\pi^k) = -\frac{\sqrt{2}L_{1/2}}{8f_\pi^2}\delta^{ij}\tau^k + \frac{\epsilon^{ijl}}{2f_\pi^2} \lim_{p_i, p_j \to 0} \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle \pi^k | T((p_i - p_j)\cdot V^l(x) H_{W, 1/2}(0)) | K\rangle$$
$$+ \frac{i\delta^{ij}}{f_\pi^2} \lim_{p_i, p_j \to 0} \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle \pi^k | T(\sigma(x) H_{W, 1/2}(0)) | K\rangle . \qquad \text{(B11)}$$

In exactly the same manner as before, the pole terms of Fig. 6 are isolated to give

$$(p_i - p_j)^\mu \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle \pi^k | T(V^l_\mu(x) H_{W, 1/2}(0)) | K(k)\rangle$$
$$= (p_i - p_j)^\mu \left[-\frac{i\sqrt{2}L_{1/2}\tau^k\tau^l(2k - p_i - p_j)_\mu}{(k - p_i - p_j)^2 - m_K^2} + \frac{1}{2}\frac{\sqrt{2}L_{1/2}\epsilon^{klm}\tau^m(2p_k - p_i - p_j)_\mu}{(p_i + p_j + p_k)^2 - m_\pi^2}\right] + O(p), \qquad \text{(B12a)}$$

$$i \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle \pi^k | T(\sigma(x) H_{W, 1/2}(0)) | K(k)\rangle$$
$$= -\frac{\sqrt{2}L_{1/2}\tau^k\sigma_{KK}}{4[2k\cdot(p_i + p_j) - (p_i + p_j)^2]} + \frac{1}{2}\frac{\sqrt{2}L_{1/2}m_\pi^2\tau^k}{(p_i + p_j + p_k)^2 - m_\pi^2} + C'\tau^k + O(p), \qquad \text{(B12b)}$$

where C' is a new constant term that cannot be ruled out *a priori*. Equations (B11) and (B12) are to be substituted into Eq. (B10) to find

$$\lim_{p_i, p_j \to 0} \bar{M}_{3\pi, 1/2} = \frac{\sqrt{2}L_{1/2}}{8f_\pi}(\delta^{ij}\tau^k + \delta^{jk}\tau^i + \delta^{ki}\tau^j) + \frac{C'}{f_\pi^2}\delta^{ij}\tau^k, \qquad \text{(B13)}$$

which differs from the single-soft-pion limit [Eq. (24)] only by the term proportional to C'.

Finally, the three-soft-pion limit can be arrived at, using Eq. (B3) to find

$$\lim_{p_i, p_j, p_k \to 0} M_{1/2}(K \to \pi^i\pi^j\pi^k) = -\frac{3\sqrt{2}}{8f_\pi^2} L_{1/2}(\delta^{ij}\tau^k + \delta^{jk}\tau^i + \delta^{ki}\tau^j)$$
$$+ \frac{i\epsilon^{ijl}}{2f_\pi^3} \lim_{p_i, p_j, p_k \to 0} \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle 0 | T((p_i - p_j)\cdot V^l(x)[Q^i, H_{W,1/2}]) | K\rangle + \text{cycl}$$
$$+ \frac{\delta^{ij}}{f_\pi^3} \lim_{p_i, p_j, p_k \to 0} \int d^4x\, e^{i(p_i+p_j)\cdot x} \langle 0 | T(\sigma(x)[Q^k, H_{W,1/2}]) | K\rangle + \text{cycl}. \qquad \text{(B14)}$$

Evaluating the last two terms by Eqs. (B7) and (B8), we proceed in the usual way to find

$$\lim_{p_i,p_j,p_k\to 0} \bar{M}_{3\pi,1/2} = \left(\frac{\sqrt{2}\,L_{1/2}}{8f_\pi^{\,2}} + \frac{C}{2f_\pi^{\,3}}\right) \times(\delta^{ij}\tau^k + \delta^{jk}\tau^i + \delta^{ki}\tau^j)\,. \tag{B15}$$

In short, we have found that it would be totally consistent with the soft-pion limits to assume that $\bar{M}_{2\pi,1/2}$ and $\bar{M}_{3\pi,1/2}$ are constants, if it were not for the unknown constants C and C' which arise in conjunction with the σ field. It is difficult, *a priori*, to say anything concerning the value of these terms; we can, however, make the assumption that these terms contribute little to the on-mass-shell value of $\bar{M}$.

With regard to the $\Delta I = \frac{3}{2}$ transitions: The case is very complex since it is quite apparent that there must be large variation in the background terms. For instance, isospin conservation [Eq. (36)], the lack of a pole term [Eq. (37)], and Eq. (B2) immediately show

$$\lim_{p_i,p_j\to 0} \bar{M}_{2\pi,3/2} = 0\,. \tag{B16}$$

A careful consideration, moreover, of the multi-soft-pion limits for $\Delta I = \frac{3}{2}$ only leads to a large number of relations among a number of unknown functions, and it has so far proved impossible to solve these equations with any reasonable ansatz.

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