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Statistical description of hadron constituents as a basis for the fluid model of high-energy collisions*

Rudolph C. Hwa
Institute of Theoretical Science and Department of Physics, University of Oregon, Eugene, Oregon 97403
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The pionization process in high-energy collisions is considered in a constituent-fluid model. The fluid picture describes the macroscopic behavior of the hadronic matter under expansion before the formation of physical pions. The constituent picture provides the microscopic basis for the fluid description. Assuming that the hadronic matter is a highly degenerate fermion system, the density and temperature variations during the expansion phase are derived. The single-particle inclusive distribution is shown to be scale-invariant and constant in rapidity.

I. INTRODUCTION

For hadronic collisions at very high energies, the multiperipheral model[1] and the diffractive model[2] have each been successful in describing certain aspects of the experimental data, but neither provides any significant insight on the internal structure of the hadrons. The parton model,[3] on the other hand, does make a specific assertion about the constitution of the hadronic matter, but the recent data on e^+e^- annihilation[4] have cast some doubt on the validity of the details of the model. If Bjorken scaling[5] breaks down at larger values of Q^2, it is not clear what aspects of the parton model can still be salvaged. Landau's hydrodynamical model,[6] as revived recently by Carruthers and Minh Duong-Van,[7] is remarkably successful in fitting a variety of experimental data, and should be taken seriously as a preliminary but realistic description of the high-energy phenomena.

The model that we shall discuss in this paper is similar in spirit to the hydrodynamical model, although in detail quite different. The main points of our model to be emphasized are the following. First, the fluid description of a macroscopic system should have a statistical basis at the microscopic level. Second, we regard a hadron as being made up of many constituents with essentially infinite degrees of freedom. Third, we envisage that the gross features of the hadronic interactions as revealed by high-energy experiments are determined mainly by the statistics that the constituents satisfy, rather than by the nature of the detailed dynamics governing the interaction between the constituents.

We do not pretend to know what the constituents are, let alone their interaction. If they are the same as Feynman's partons,[3] then we should call them partons. However, if the usual properties associated with them as described in the canonical parton model (such as their quarklike nature) turn out to be wrong, it does not invalidate our model here, since our discussion is independent of those properties. Thus, to be free from any prejudices, let them be referred to simply as constituents for now.

The process which we shall consider exclusively here is the pionization process common to all hadronic collisions. We associate this production process with the bulk of the constituent matter. The leading-particle behavior, on the other hand, retains the memory of the quantum numbers of the initial particles, and should therefore be associated, not so much with the bulk, but with the tags, which for example may be the valence quarks,[8] and which usually carry large momenta. In this paper we shall say nothing about the leading-particle behavior.

We regard pionization and diffraction excitation[2] as two extreme situations corresponding to small and large impact parameters, respectively. In the case of large impact parameter, the two initial particles barely overlap at the instant of collision; they emerge from the peripheral configuration in excited states while retaining all the initial quantum numbers. The description of the excitation process in the constituent fluid picture is a problem that remains to be studied. On the other hand, for small impact parameter, the collision is violent and more interaction takes place between the constituents of the two incident particles. Indeed, we regard the central collision as forming one big fireball, a hot fluid which expands in the longitudinal direction; when the local mass density becomes low enough, breakup occurs and pions emerge from the fluid. The aim of this paper is to provide a description of this process both in the macroscopic and in the microscopic pictures.

Our specific goal will be to obtain the single-particle inclusive distribution of the produced pions. This is a modest objective. The significance, we hope, lies in the demonstration of the meaningfulness of the concept that the gross features (at least the pion spectrum) of particle production can be understood in terms of the statistical aspects of the hadronic constituents without the reliance on any knowledge of the dynamics of the interaction among the constituents. However, we anticipate that the two-particle correlation and the transverse-momentum distribution will reveal more about the details of the statistics (such as the correlation length in coordinate space) or even about the nature of the constituent interactions themselves. In that respect this paper may be regarded as setting the language for more detailed studies later.

II. THE MODEL

For simplicity we consider here only the case of zero impact parameter and ignore any transverse motion. We describe first the physical picture of the production process and then its statistical basis.

A. Macroscopic picture

We regard the hadrons as extended objects, which are Lorentz-contracted into thin disks as they impinge upon each other at high momentum. As they coalesce at zero impact parameter, the initial kinetic energy is partly converted into the creation of hadronic matter and partly into the internal energy of that matter. The hypothesis is that the creation of constituents takes place entirely during this initial phase of the collision process, and not later. We have then a disk of high-density, high-"temperature" hadronic matter at some initial time. The details of this formation phase will not concern us in the following.

As time develops, the disk expands longitudinally in accordance to relativistic fluid mechanics.[9] The leading edges of the fluid move out at relativistic velocities with the space in between filled continuously with hadronic matter. During this phase of expansion no hadronic constituents are created. Local density and temperature, defined by appropriate averages over some reasonable spatial extension within the expanding cylindrical volume, decrease with local time.

When the local mass density in its proper frame is lowered to the value corresponding to that of a free pion, that section of the fluid breaks off and a pion is produced. Since different parts of the fluid move at different velocities, the mass density is not uniform, so the breakup process occurs at different times for different parts of the cylinder. Moreover, because of Lorentz contraction different slices of the cylinder that break off have different thicknesses. Evidently, the single-particle inclusive distribution for the pionization process depends on how the local velocity varies with the longitudinal distance.

B. Microscopic picture

We hypothesize that a hadron consists of many constituents. The number of constituents is not Lorentz-invariant because virtual pair production and annihilation cause fluctuation in that number, and the lifetime in those virtual states depends on the frame that they are described in.[3] During the initial formation phase of the collision process many pairs of constituents are created. In the c.m. frame they move out longitudinally at high velocities, $v \approx 1$, during the expansion phase. Owing to time dilation the virtual states are almost real, so it is a good approximation to regard the number of constituents as being fixed during the expansion phase if we stay in the c.m. system. The only exception is for those that are slow in the c.m. system, i.e., Feynman's "wee" region.[3] We shall, however, ignore their perturbation, since they contribute only to an infinitesimal fraction of the total number of constituents. What the total number is need not be specified, so long as they are numerous enough to render statistical considerations meaningful.

It is also unnecessary to specify at this stage the sizes of the constituents, their masses, and their quantum numbers. Since our immediate objective is to describe, in the constituent fluid picture, the pionization process common to all high-energy hadronic collisions, we shall ignore the whereabouts of the few "valence quarks" that tag the initial particles; thus, we shall not attempt to say anything about the leading particles in this paper.

For a free pion that is produced, one can define a momentum distribution function $f(k)$ to describe the probability of finding a constituent in the pion with momentum k_μ. Before the breakup process that produces that pion, those constituents belong to a certain slab in the fluid moving longitudinally in a definite relation to its neighbors, viz. faster than the fluid closer to the center of mass but slower than the fluid farther from the center. Continuity of the fluid ensures a smooth dependence of the constituent density on the average position and velocity of any slab. Let the average position of a slab of fluid be denoted by a 4-vector x_μ. We then define $F(x,k)$ to be the number density of the constituents in that slab located at x_μ with momenta (of the constituents) between $\vec{k}$ and $\vec{k}+d\vec{k}$. At breakup $F(x,k)$ for the pion at x_μ becomes proportional to $f(k)$. Our hypothesis is that there exists a function $F(x,k)$ which interpolates between the macroscopic properties of the fluid in the variables x_μ and the microscopic properties of constituents in the variables k_μ.

We define the flux vector $S_\mu(x)$ by

$$S_\mu(x) = \int k_\mu F(x,k) dw \,, \tag{1}$$

where $dw = d^3k/k_0$. Clearly, S_0 has the meaning of the average number density of constituents at x_μ, and S_i the flux density. Contact with the macroscopic description of the fluid is made via the energy-momentum tensor which is defined by

$$T_{\mu\nu}(x) = \int k_\mu k_\nu F(x,k) dw \,. \tag{2}$$

Since the number of constituents are conserved during the expansion phase, as we have argued, and since energy-momentum must also be conserved, we have the continuity equations

$$\partial_\mu S^\mu = 0 \,, \tag{3}$$

$$\partial_\mu T^{\mu\nu} = 0 \,. \tag{4}$$

They restrict the possible forms that $F(x,k)$ can have, a subject to be discussed in Sec. V.

We have defined the function $F(x,k)$ in the c.m. system. Its meaning in the proper frame of a slab centered at x_μ is hard to make precise because the number of real constituents is frame-dependent. However, the energy-momentum tensor $T_{\mu\nu}$ given in (2) has a well-defined meaning in any frame according to the usual theory of relativistic fluid mechanics.[9] Since we know how k_μ transforms, we may therefore regard (2) as a definition of the *scalar* function $F(x,k)$. When evaluated in the proper frame, we can interpret it as being the number density of real and virtual constituents in that frame such that when used in

$$T'_{\mu\nu} = \int k'_\mu k'_\nu F(x',k') dw' \,, \tag{5}$$

where the primes denote proper frame, it gives the correct energy-momentum tensor which is unambiguously defined in that frame. We shall have no need to use (1) in the proper frame.

III. RELATIVISTIC FLUID MECHANICS

For later usage we collect here some of the basic equations of relativistic fluid mechanics.[9] In the proper frame of an elementary volume of a fluid, the tensor $T'^{\mu\nu}$ is

$$T'^{00} = \rho, \quad T'^{ij} = P\delta_{ij} \,, \tag{6}$$

where ρ is the internal energy density of the fluid (or, equivalently, the mass density of the hadronic matter in our problem), and P is the pressure.

In a general frame, which we take in our present problem to be the over-all c.m. system, and in which the elementary volume of fluid moves at an average velocity v in the longitudinal (z) direction, the energy-momentum tensor becomes

$$T^{\mu\nu} = (\rho + P) v^\mu v^\nu - P g^{\mu\nu} \,, \tag{7}$$

where $v^\mu = (1,0,0,v)\gamma$, and $\gamma = (1-v^2)^{-1/2}$.

There are two equations derivable from (7) that are particularly useful to us. They are

$$T^{00} - vT^{03} = \rho \tag{8}$$

and

$$\frac{T^{00} + T^{33}}{T^{03}} = \frac{1+v^2}{v} \,. \tag{9}$$

Evidently, if we are given $F(x,k)$, we can calculate from $T^{\mu\nu}$ the average velocity of any portion of the fluid and the associated proper density of the hadronic matter.

IV. CONNECTION WITH OBSERVATION

We must now establish an important link between the macroscopic picture of the fluid and the observables accessible to experiments, before we discuss the microscopic behavior of the constituents. First, let us consider the momentum of a pion detected in an inclusive experiment. It comes from a slice of the cylinder of the fluid of thickness R/γ, when the proper density ρ is equal to the free-pion density $\rho_\pi \sim m_\pi R^{-3}$. Here, R is the pion diameter, m_π its mass, and γ the Lorentz factor. The associated momentum (hereafter, always in the c.m. system) is

$$p = m_\pi v\gamma \,. \tag{10}$$

In terms of the energy-momentum tensor, we have first in the proper frame

$$\begin{aligned} m_\pi &= T'^{00}R^3 \\ &= T'^{0\nu}\Delta\sigma'_\nu \,, \end{aligned} \tag{11}$$

where $\Delta\sigma'^\nu = (R^3,0,0,0)$, and then in the c.m. frame

$$p^\mu = T^{\mu\nu}\Delta\sigma_\nu \,, \tag{12}$$

where $\Delta\sigma_\nu$ is related to $\Delta\sigma'_\nu$ by a Lorentz transformation. Indeed, the total energy-momentum of the whole system is

$$P^\mu = \int T^{\mu\nu} d\sigma_\nu \,, \tag{13}$$

where the integration extends over the entire body of the fluid along some spacelike surface. This is to be compared with the energy-momentum sum rule of the inclusive distribution:

$$P^\mu = \int p^\mu \frac{d^3N}{dp^3} d^3p \,. \tag{14}$$

For the sake of simplicity let us ignore hereafter the transverse motion and consider only one space and one time dimension. Then, from (13) and (14) we can identify

$$p^0 dN = T^{0\nu} d\sigma_\nu , \qquad (15)$$

where dN stands for the number of pions detected, corresponding to the breakup of a certain portion of the hadronic fluid signified by $d\sigma_\nu$. This is inclusive, since we do not question what happens to the remainder of the fluid. Let us call that portion of the fluid under consideration a *cell*.

Let the longitudinal dimension of the cell in its proper frame be $\delta z'$, so that

$$d\sigma'^\nu = \begin{pmatrix}\delta z' \\ 0\end{pmatrix}.$$

Then in the c.m. frame, where the cell moves with an average velocity v, we have

$$d\sigma^\nu = \begin{pmatrix}\delta z \\ \delta t\end{pmatrix}, \qquad (16)$$

where $\delta z = \gamma \delta z'$ and $\delta t = v\delta z$. It then follows from (15) that

$$p^0 dN = (T^{00} - vT^{03})\delta z . \qquad (17)$$

Here δz is an interval defined for a fixed time t in the c.m. system. Define the rapidity variable y in the usual way so that

$$\frac{dN}{dy} = p^0 \frac{dN}{dp}$$

and we obtain, using (8) at breakup,

$$\frac{dN}{dy} dp = \rho_\pi \delta z . \qquad (18)$$

In the unit $R = 1$, we have $dp = \rho_\pi \gamma^3 dv$, whence

$$\frac{dN}{dy} = \frac{\delta z}{\gamma^3 dv} . \qquad (19)$$

Now, dv is the difference in velocities between the leading and the trailing edges of the cell, whose average velocity is v. However, δz is not the length of the cell in the c.m. system, as can be seen in Fig. 1. The distance between the boundaries of the cell at a fixed time t is dz, which differs from δz by

$$\delta z - dz = v\delta t = v^2 \delta z ;$$

consequently,

$$\delta z = \gamma^2 dz . \qquad (20)$$

Using this in (19) we obtain finally

$$\frac{dN}{dy} = \left[\gamma \left(\frac{\partial v}{\partial z}\right)_t\right]^{-1} . \qquad (21)$$

This equation relates the observable on the left-hand side to the expansion characteristics of the fluid on the right-hand side. What we need to know next is how the velocity $v(z,t)$ depends on z for fixed t. It is evident from (2) and (9) that v is an average quantity, to be determined by appropriate integrations over the momenta of the constituents. Note, however, that (21) is a general result independent of the details of the microscopic picture.

V. THE DISTRIBUTION FUNCTION

The distribution F is a scalar function that depends on the vectors k_μ and x_μ. It describes the momentum distribution of the constituents in a volume element centered at x_μ. It could, in principle, depend also on $dx_\mu/d\tau$, where $\tau^2 = x^\mu x_\mu$, higher-order derivatives, and other possible vector quantities in the problem. We shall, however, at this point consider just the simple case where F is a function of k_μ and x_μ only. Then the only scalars that can be constructed out of k_μ and x_μ are τ, $\mu^2 = k^\mu k_\mu$, the constituent mass, and $k_\mu x^\mu$, upon which F depends.

The world line of a cell of the fluid may not be straight in the Minkowski space. That is, if $x^\mu = (t, z)$ refers to the center of the cell, and if we define

$$v = \frac{dz}{dt}, \quad u = \frac{z}{t},$$

it is not necessary that $v = u$. In other words, the proper frame (or v frame) of the cell does not necessarily coincide with what we shall call the u frame, reached by a Lorentz transformation by a

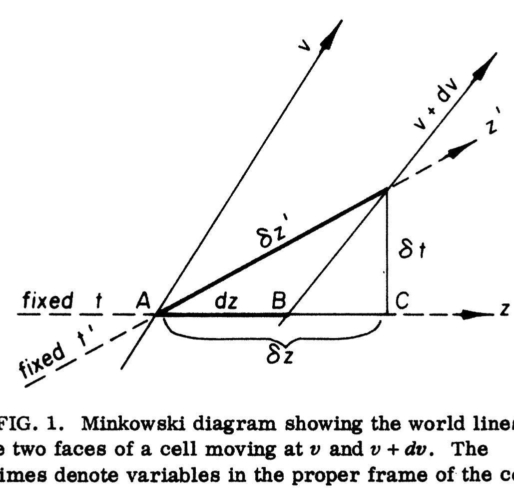

FIG. 1. Minkowski diagram showing the world lines of the two faces of a cell moving at v and $v + dv$. The primes denote variables in the proper frame of the cell. The length AB at fixed t is dz, while AC is δz.

velocity u from the c.m. system. Let us use a double prime to label the variables in the u frame, so

$$x''^{\mu} = (\tau, 0), \quad \tau = (x_\mu x^\mu)^{1/2},$$

$$k''^{\mu} \equiv \begin{pmatrix} k_0'' \\ k'' \end{pmatrix} = \gamma_u \begin{pmatrix} 1 & -u \\ -u & 1 \end{pmatrix} \begin{pmatrix} k_0 \\ k \end{pmatrix},$$

$$\gamma_u = (1-u^2)^{-1/2}.$$

Now, if $F(x, k)$ depends on k_μ only through the variable $k_\mu x^\mu$, as we have assumed, then in terms of the u-frame variables it depends only on $k_0''\tau$ for a cell at x_μ. Consequently, F is an even function of k'', and one expects the average momentum of the cell in the u frame to vanish, whatever form F may take. That implies $v = u$.

More precisely, we calculate v using (9). From (2) we have

$$\begin{aligned} T^{00} + T^{33} &= \int (k_0{}^2 + k^2) F(k\cdot x, \tau) dw \\ &= \gamma_u{}^2 \int [(k_0'' + uk'')^2 + (uk_0'' + k'')^2] \\ &\qquad \times F(k_0''\tau, \tau)\frac{dk''}{k_0''} \\ &= \frac{1+u^2}{1-u^2}\int (k_0''^2 + k''^2) F(k_0''\tau,\tau)\frac{dk''}{k_0''}. \end{aligned}$$

A similar consideration gives

$$T^{03} = \frac{u}{1-u^2}\int (k_0''^2 + k''^2) F(k_0''\tau, \tau)\frac{dk''}{k_0''}.$$

Substitution into (9) yields $v = u$.

The result $v = z/t$ implies that the world lines are straight, as for free particles. It is a consequence of our assumption that F depends on k_μ through $k_\mu x^\mu$ only. In general, v need not be equal to u. However, since most pions detected in high-energy inclusive experiments are relativistic, the only exceptions being the ones in the wee region,[3] most cells of the hadronic fluid are near the light cone where $v = u$ is very nearly exact. Hence, our assumption about F is quite reasonable for an overwhelming majority of the detected pions at high energy. It is in accord with the argument which we used earlier (due originally to Feynman[3]) that in the c.m. frame the fast-moving constituents behave as if they are free.

From $v = z/t$ we immediately get

$$\left(\frac{\partial v}{\partial z}\right)_t = \frac{1}{t} = \frac{1}{\gamma\tau},$$

which when substituted into (21) yields

$$\frac{dN}{dy} = \tau_0, \tag{22}$$

where τ_0 is the proper time that the cell takes from formation to breakup.[10] Note that (22) has no explicit dependence on v. Thus, if τ_0 is the same for all cells of the fluid, then the pion distribution is constant in rapidity.

We remark that if $F(x, k)$ has a different dependence on k_μ than what we have assumed, we would still have $v \approx u \approx 1$ for all cells of interest, but dv/du need not be 1. We expect that dv/du would still exist in the limit of $u \to 1$, and that its value is of $O(1)$. If that is true, then (22) is still correct except for a numerical multiplicative factor. To illustrate this point, consider Landau's hydrodynamical model[6,7] in which

$$\begin{aligned} \gamma^2 &= \frac{1}{1-v^2} \\ &= \frac{1+u}{1-u}; \end{aligned}$$

consequently, for $u \to 1$,

$$\left(\frac{\partial v}{\partial z}\right)_t = \frac{dv}{tdu} \to \frac{1}{4t}.$$

The result thus differs from (22) by only a multiplicative factor of 4.

We now assert that as a consequence of $F(x, k)$ being a function of $x_\mu k^\mu$ and τ only, the breakup time τ_0 is universal, i.e., independent of the location or velocity of any cell. This can be seen by noting that in the proper frame (labeled by a prime) or the u frame, we have $k_\mu x^\mu = k_0'\tau$. It then follows from (5) and (6) that ρ is a function of τ only. When $\rho(\tau_0) = \rho_\pi$, the mass density of a free pion, breakup occurs; consequently, τ_0 is universal. We therefore obtain the result from (22) that the single-particle distribution is constant in rapidity, which is, of course, in agreement with the experimental data on pionization.

It is remarkable that the very simple assumption that $F(x, k)$ depends only on the two vectors x_μ and k_μ is sufficient to lead us to the flat rapidity plateau in the fluid picture for pionization. However, we still need a concrete description of $F(x, k)$, which contains all the general properties we imposed, viz., (3) and (4), and which determines explicitly $\rho(\tau)$ as a function τ so that we can follow the development of the expansion phase of the fluid until breakup. It should be stated from the outset that we have no *a priori* knowledge of $F(x, k)$ until a dynamical theory of the constituents becomes available. For the moment we can only adopt the inductive approach in the hope that some simplifying assumption can help us to catch a glimpse of the ultimate microscopic theory revealing some aspects of its essence.

We have assumed that $F(x, k)$ depends on k_μ through $k_\mu x^\mu$ only. Let us now further assume that its dependence on $k_\mu x^\mu$ and τ can be expressed in terms of one scalar variable ζ, defined by

$$\zeta = k_\mu x^\mu g(\tau)\,. \tag{23}$$

This assumption is made not just for simplicity's sake; it is also sensible in that it is the generalization of $\epsilon/\kappa T$ in nonrelativistic statistical mechanics, since $\zeta \to k_0' \tau g(\tau)$ in the proper frame of the cell at x_μ. Thus $g(\tau)$ gives a measure of the local temperature T that changes with proper time.

With the assumption that $F(\zeta)$ is a function of ζ only, we now make use of the continuity equations (3) and (4) and deduce the constraints on $F(\zeta)$.

(a) $\partial_\mu S^\mu = 0$. From (1) we have

$$\partial_\mu S^\mu = \int k^\mu \frac{\partial \zeta}{\partial x^\mu} F'(\zeta) dw = 0\,,$$

where

$$F'(\zeta) = \frac{dF(\zeta)}{d\zeta}\,,$$

$$\frac{\partial \zeta}{\partial x^\mu} = k_\mu g + (k\cdot x)\frac{x_\mu}{\tau}\frac{dg}{d\tau}.$$

If we define an average over $F'(\zeta)$ by

$$\langle \zeta^n \rangle = \frac{\int \zeta^n F'(\zeta) dw}{\int F'(\zeta) dw}\,, \tag{24}$$

then by substitution we obtain

$$-g^{-3}\frac{dg}{d\tau} = \frac{\mu^2 \tau}{\langle \zeta^2 \rangle}\,, \tag{25}$$

where $\mu^2 = k_\mu k^\mu$. Integration yields

$$g^{-2}(\tau) = \mu^2 \int_0^{\tau^2} \frac{d\tau^2}{\langle \zeta^2 \rangle} + g_0^{-2}\,, \tag{26}$$

where $g_0 = g(0)$ is a constant. It is clear from (23) and (24) that $\langle \zeta^2 \rangle$ can at most be a function of τ only.

(b) $\partial_\mu T^{\mu\nu} = 0$. From (2) we have

$$\int \left(\mu^2 g + \frac{\zeta^2}{\tau g^2}\frac{dg}{d\tau}\right) k^\nu F'(\zeta)\, dw = 0\,.$$

Multiplying by x_ν yields

$$\int \left(\mu^2 \zeta + \frac{\zeta^3}{\tau g^3}\frac{dg}{d\tau}\right) F'(\zeta) dw = 0\,,$$

which implies on account of (25)

$$\langle \zeta^3 \rangle = \langle \zeta^2 \rangle \langle \zeta \rangle\,. \tag{27}$$

A sufficient condition for this to be satisfied is that $F'(\zeta)$ is a sharply peaked function of ζ.

VI. FERMI-DIRAC STATISTICS

In order to introduce the notion of temperature in the present context, we need some contact with nonrelativistic statistics where temperature is well defined. It is not absolutely necessary for us to discuss temperature if we do not relate the breakup process to some critical temperature. We have, in fact, defined breakup to occur when the mass density ρ is equal to the density ρ_π of free pions. However, to calculate ρ we must know $F(x, k)$, which is intimately related to the temperature variation. It suggests that we could just as well have defined the breakup process in terms of T reaching a critical temperature. Indeed, to interpret the pionization process as a phase transition is an interesting speculation.

It is by now widely accepted from electroproduction experiments that the hadronic constituents have spin $\frac{1}{2}$; this property probably survives even if the details of the parton model[3] do not.[4] We assume that their spin is $\frac{1}{2}$; hence, they satisfy the Fermi-Dirac statistics. It has been our hope that the gross features of hadronic interaction do not depend so much upon the nature of the interaction between the constituents as on their statistics. Thus, instead of speculating on the basic interaction, we conjecture that we are dealing with a highly degenerate fermion system. As in the case of a low-temperature, high-density electron gas, many properties of such a system are determined mainly by the states near the Fermi energy. Let us therefore consider the very naive but interesting and soluble model in which $F(x, k)$ is given by

$$F(\zeta) = \frac{F_0}{\exp(\zeta - \zeta_0) + 1}\,, \tag{28}$$

where F_0 and ζ_0 are constants. This distribution has the following attributes. If ζ_0 is large compared to unity, $F(\zeta)$ is very sharply cut off at $\zeta = \zeta_0$ (in a scale normalized by ζ_0), so that $F'(\zeta)$ is sharply peaked at ζ_0. We therefore have $\langle \zeta^n \rangle \approx \zeta_0^n$, which satisfies (27). Thus the necessary condition for the conservation of energy-momentum tensor is satisfied to a high degree of approximation in a rather simple way. It is not hard to see that by restricting ζ to a small neighborhood near ζ_0 the constraint $\partial_\mu T^{\mu\nu} = 0$ becomes equivalent to $\partial_\mu S^\mu = 0$. The latter implies through (26)

$$g^{-2}(\tau) = \left(\frac{\mu\tau}{\zeta_0}\right)^2 + g_0^{-2}\,. \tag{29}$$

Apart from some unspecified parameters, the function $F(x, k)$ is now completely determined.

The other attribute of (28) obviously is that in the proper frame of a cell it reduces to the familiar form of the Fermi-Dirac distribution

$$F_{\rm FD}(\epsilon) = \frac{F_0}{\exp[\beta(\epsilon - \epsilon_F)] + 1} \tag{30}$$

where ϵ is the kinetic energy, ϵ_F the Fermi energy, and $\beta = 1/\kappa T$, κ being the Boltzmann constant and T the temperature. If we substitute for ζ in (28) its form in the proper frame, $\zeta = k_0' \tau g$, and then compare the result with (30), we can make the following identifications:

$$k_0' = \mu + \epsilon,$$
$$\tau g = \beta, \quad \zeta_0 = \beta(\epsilon_F + \mu). \tag{31}$$

We have thus obtained the important equation relating the temperature to the parameters of the fluid; in particular, because of (29), we have the time dependence

$$T(\tau) = \frac{1}{\kappa}\left[\left(\frac{\mu}{\zeta_0}\right)^2 + \left(\frac{1}{\tau g_0}\right)^2\right]^{1/2}. \tag{32}$$

As we have already mentioned, in order that (28) may satisfy at least approximately the constraint imposed by (27), ζ_0 should be large; it should in particular be large compared to the minimum value of ζ corresponding to $k' = 0$, i.e., $\zeta_0 \gg \zeta_{\min} = \mu\beta$. It then follows that the first term in the square bracket in (32) is negligible, so we may write

$$T(\tau) = (\kappa g_0 \tau)^{-1}, \tag{33}$$

to the extent that (28) may be regarded as being consistent with the continuity equations. Thus the temperature decreases as the fluid expands according to a remarkably simple inverse-power dependence on τ until it reaches a critical temperature T_π characteristic of a free pion. The formula should be regarded as meaningless after breakup.

The divergence of T as $\tau \to 0$ is merely a reflection of the fact that if we let the expansion phase of the fluid go backward in time, the fluid would eventually become infinitely dense and hot. In reality, of course, the finite size of the system at the beginning of the expansion phase precludes the applicability of (33) from arbitrarily small values of τ. We shall return to this point later. We note that the condition $\zeta_0 \gg \mu\beta$ implies further that $g(\tau)$, as expressed in (29), changes very little from g_0. Thus, the breakup time τ_0 is long enough for the temperature and density to decrease to their critical values, but is short enough to keep g essentially constant.

Because T is a function of the proper time τ only, independent of the cell location, and since at a fixed t in the c.m. system large $|z|$ corresponds to smaller τ, the leading edges of the expanding fluid are hotter than the center. Hence, breakup occurs in the center first.

The criterion for breakup is $\rho(\tau) = \rho_\pi$. Using (5) and (6) and approximating (28) by

$$F(\zeta) = F_0\,\theta(\zeta - \mu\beta)\,\theta(\zeta_0 - \zeta),$$

we obtain

$$\rho(\tau) = F_0\left(k'k_0' + \mu^2 \ln\frac{k' + k_0'}{\mu}\right)_{k_0' = \zeta_0/\beta}$$
$$= F_0\mu^2\{\alpha(\alpha^2 - 1)^{1/2} + \ln[\alpha + (\alpha^2 - 1)^{1/2}]\},$$

where $\alpha = \zeta_0/\mu\beta$. Using again the condition $\alpha \gg 1$, we have approximately

$$\rho(\tau) = F_0[\zeta_0 \kappa T(\tau)]^2. \tag{34}$$

The mass (or energy) density is therefore proportional to T^2 in contrast to the case of black-body radiation ($\propto T^4$) as assumed in Landau's hydrodynamical model.[6,7] Because of (33) we see that $\rho(\tau)$ varies as an inverse-square power of τ,

$$\rho(\tau) = \rho_\pi(\tau_0/\tau)^2, \tag{35}$$

independent of the location and velocity of the cell.

Regarding the Fermi energy, we note first that the condition $\zeta_0 \gg \mu\beta$, applied to (31), implies

$$\epsilon_F(\tau) = \zeta_0 \kappa T(\tau)$$
$$= \zeta_0/g_0\tau. \tag{36}$$

Since $\zeta_0 \gg 1$, we see that $\kappa T \ll \epsilon_F$. It should come as no surprise that we recover the condition for a highly degenerate system. Moreover, we note that ϵ_F decreases with τ in the same way that T does. That ϵ_F should decrease as a cell expands is reasonable, since the eigenenergies associated with the normal modes in the cell also decrease.[11] That the degree of degeneracy remains unchanged is a consequence of our assumption that ζ_0 is a constant. To the extent that there exists a critical temperature T_π at which the fluid breaks up and produces a pion having mass density ρ_π, we can define a corresponding Fermi energy ϵ_F^π for free pions by means of (36):

$$\epsilon_F^\pi = \zeta_0 \kappa T_\pi.$$

It is a parameter characteristic of a free pion independent of the mode of production. In terms of it the breakup time is

$$\tau_0 = \zeta_0/g_0\epsilon_F^\pi. \tag{37}$$

So far we have not discussed the dependence on s, the square of the c.m. energy. The form (28) for $F(\zeta)$ has no explicit s dependence. We now argue that g_0 should furthermore be independent of s, so $F(\zeta)$ has no implicit dependence on s either through ζ. We have already discussed in connection with (33) that there is an initial τ_i corresponding to an initial temperature T_i at the beginning of the expansion phase. It is related to the initial volume of the fluid. The longitudinal dimension of the initial volume is approximately R/γ_i due to the Lorentz contraction of the incident particles by the factor γ_i. If we regard the (straight) world lines of all constituents to extrapolate back to the tip of the light cone where $\tau=0$, then the initial longitudinal dimension of R/γ_i implies that τ_i is proportional to γ_i^{-1}. Since $\gamma_i \propto \sqrt{s}$, we have $\tau_i \propto s^{-1/2}$. On the other hand, T_i should be proportional to the total c.m. energy available; hence, by (33) we obtain the result that g_0 should be independent of s.

The constancy of g_0 implies by virtue of (37) that τ_0 is indeed a universal constant, not only independent of the locations and velocities of the cells, but also independent of the incident energy. It then follows from (22) that the height of dN/dy is s-independent. It is remarkable that all the experimental features of the single-particle distribution can be deduced from the simple form that we have chosen for $F(x, k)$.

Stodolsky[12] has argued that the height of the central plateau is a measure of the strength of strong coupling. It is of order unity. We have seen that the height is τ_0 in the unit where the pion diameter R is set equal to 1. Since τ_0 is a universal constant in our model, but is expected on physical grounds to be shorter (longer) if the interaction between the constituents is weaker (stronger), we support the interpretation of Stodolsky.

VII. CONCLUSION

The constituent-fluid model that we have described involves three different levels of consideration. The general fluid consideration relates the single-particle distribution to the velocity gradient of the fluid. The constituency aspect of the fluid then relates the velocity gradient to a characteristic (breakup) time of the hadronic matter. Finally, the statistical consideration of the fermion system provides a concrete solution to the hydrodynamical equations, giving not only the time dependences of density and temperature variations of the expanding fluid, but also the scale invariance of the single-particle spectrum.

This model differs from Landau's hydrodynamical model in almost every aspect except the common usage of the continuity equation for $T_{\mu\nu}$ during the expansion phase. Landau treats a massless boson system whose equation of state corresponds to that of black-body radiation ($\rho \propto T^4$). By taking the transverse motion into account, a Gaussian distribution in rapidity is obtained for the single-particle spectrum, which is not scale-invariant. Our consideration of the pionization process has, on the other hand, led to a flat spectrum; moreover, by regarding the hadronic matter as a highly degenerate fermion system, we have obtained the relation $\rho \propto T^2$.

Turning to a comparison with the parton model, it must be said that our model relies heavily on the physical picture that Feynman originally envisaged about the hadron constituents. However, the reasoning for the pionization process is quite different. By incorporating the fluid picture as a macroscopic manifestation of the constituent dynamics, we can discuss pionization without encountering the difficult question of how a parton turns into a hadron. The embarrassing question of why the constituents are not seen remains a problem. Although preliminary and simple minded, our statistical consideration of the fermion system has nevertheless shed some light on the hydrodynamical properties of the hadronic fluid.

The most interesting aspect of our fluid picture is the possibility of relating the momentum of a detected particle to the position and velocity of a portion of the fluid. In this paper we have shown that the single-particle distribution is inversely proportional to the velocity gradient. We envisage that the momentum correlation of two detected particles can be related to the position correlation of the fluid and consequently the range of interaction of the constituents. The implication of this model on large-momentum-transfer processes and lepton-induced reactions are all interesting problems that remain to be investigated.

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[1]L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento 25, 626 (1962); D. Amati, A. Stanghellini, and S. Fubini, *ibid.* 26, 896 (1962); G. F. Chew and A. Pignotti, Phys. Rev. 176, 2112 (1968); G. F. Chew, M. Goldberger, and F. Low, Phys. Rev. Lett. 22, 208 (1969); C. E. DeTar, Phys. Rev. D 3, 128 (1971).

[2]J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. 188, 2159 (1969); R. C. Hwa, Phys. Rev. Lett. 26, 1143 (1971); R. C. Hwa and C. S. Lam, *ibid.* 27, 1098 (1971); M. Jacob and R. Slansky, Phys. Rev. D 5, 1847 (1972).

[3]R. P. Feynman, Phys. Rev. Lett. 23, 1415 (1969);

Photon-Hadron Interactions (Benjamin, Reading, Mass., 1972).
[4]B. Richter, in proceedings of the Conference on Lepton-Induced Reactions, University of California at Irvine, 1973 (unpublished).
[5]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).
[6]L. D. Landau, Izv. Akad. Nauk SSSR 17, 51 (1953); L. D. Landau and S. Z. Belenkij, Usp. Phys. Nauk 56, 309 (1955); Nuovo Cimento Suppl. 3, 15 (1956); *Collected Papers of L. D. Landau*, edited by D. Ter Haar (Gordon and Breach, New York, 1965).
[7]P. Carruthers and Minh Duong-Van, Phys. Rev. D 8, 859 (1973); Phys. Lett. 42B, 597 (1972).
[8]J. Kuti and V. F. Weisskopf, Phys. Rev. D 4, 3418 (1971).
[9]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, Mass., 1959).
[10]After the completion of this work, Dr. F. Cooper informed me that in investigating the e^+e^- annihilation problem in Landau's hydrodynamical model he could obtain the same result.
[11]I am grateful to Professor Feynman for his illumination on this point.
[12]L. Stodolsky, Phys. Rev. Lett. 31, 139 (1973).