PHYSICAL REVIEW D VOLUME 10, NUMBER 7 1 OCTOBER 1974

Equivalent exclusive and inclusive multiperipheral descriptions*

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(Received 11 February 1974)

It is shown that for each exclusive model satisfying a multiperipheral principle, there is an inclusive model satisfying a multiperipheral principle which predicts the same values for all exclusive and inclusive differential cross sections. The converse statement is also true. The relation between these models is thus one to one; we give formal expression for the transformation between one model and the other. These expressions have the form of the nonrelativistic Lippmann-Schwinger equation. The proof is given first for multi-Regge models having poles and cuts as the J-plane singularities. It is then generalized to models with multiparticle thresholds. The physical origin of such thresholds is discussed. Also treated are the further generalizations of the proof to models having N quantum number channels and to models having any general multiperipheral structure.

I. INTRODUCTION

The multiperipheral model (MPM) first proposed by Amati, Bertocchi, Fubini, Stanghellini, and Tonin (ABFST)[1] has been an important source of insight into multiparticle dynamics. It provided a theoretical basis for a qualitative understanding of early cosmic-ray observations of large multiplicities, small average transverse momentum, leading-particle effects, and approximate independence of particle production.[2] These observations have since been confirmed by accelerator experiments.[3] The original MPM has been simplified (such as the Chew-Pignotti multi-Regge model)[4] and modernized (see Chew, Rogers, and Snider,[5] and more recently Dash[6]).

A substantial advance in our understanding of the dynamics of many-particle reactions was made by the introduction of the notion of scaling for inclusive processes.[7] It is notable that this scaling is predicted by the MPM.[2] Mueller,[8] in an attempt to obtain the scaling predictions in a more general framework than the MPM, showed that inclusive processes can be obtained as discontinuities of forward $n \to n$ amplitudes (generalized optical theorem). Thus inclusive processes measure information directly about an amplitude, and demand as much interest in themselves as a single exclusive process.

In exhibiting scaling, Mueller assumed that the (discontinuities of the) $n \to n$ amplitudes had a *leading* Regge behavior. In practice, this assumption is difficult to implement unless some simple rule relates the n-particle inclusive cross section to the $(n-1)$-particle cross section. In *exclusive* multiperipheral models, a rule exists and can be stated generally as: *The O(2, 1) partial-wave projections*[9] *of the cross sections satisfy finite recursion relations*. We shall call this rule the multiperipheral principle (MPP). The recursions are such that when satisfied by exclusive cross sections, the O(2, 1) projection of the total cross section satisfies a Fredholm integral equation. The intuitive consequence of the MPP is that the projection of the n-particle cross section is simply the nth power of some kernel. We nevertheless mean to include the ABFST model, as well as all subsequent MPM's, within this general principle. In particular it should be noted that we include those MPM's which attempt to describe both the large and small subenergy behavior of the cross section. The MPP is then a constraint not only on the leading J-plane singularities, but also on the secondaries (daughters, background integral, etc.).

We now propose to extend the MPP to the discontinuities of the $n \to n$ amplitudes considered by Mueller. Such models, which we define as *inclusive* multiperipheral models, will necessarily lead to Feynman scaling. As with exclusive MPM's, the factorized O(2, 1) structure for inclusive MPM's is assumed to hold not only for the leading trajectory, but also for the daughter trajectories. The models considered are extended to apply over all of phase space. Formally the O(2, 1) projection of the elastic cross section in the inclusive MPM satisfies a Fredholm integral equation. The simplest examples are those in which the projection of the n-particle inclusive cross section is just the nth power of some inclusive kernel. Of course more complicated models are included as well, such as all versions of so-called Mueller-Regge models (MRM's),[10] and all models of this type defined by integral equations.[11] In what follows, we shall at times loosely refer to all inclusive MPM's as MRM's.

Having now formulated the MPP, we can be more precise about the exact connections which exist between inclusive and exclusive models which satisfy the multiperipheral principle. We show in this paper that for each exclusive model satisfying the MPP, there is an inclusive model satisfying the MPP predicting the same values for all exclusive and inclusive differential cross sections. The converse statement is also true. Thus whether one formulates a multiperipheral model for exclusive or for inclusive amplitudes is purely a matter of convenience. In particular, Mueller's original proof of scaling can be interpreted as a MPM proof.

It should be noted that this equivalence has been suspected for a long time.[12] To our knowledge, the first explicit demonstrations of equivalence were given for certain specific models in Ref. 11, and for a general class of N-channel Chew-Pignotti models in Ref. 13. In this paper we have generalized the demonstration to include any class of models satisfying the MPP.

In Sec. II we discuss the general one-channel problem; that is, the case where there is one input trajectory $\alpha_{\text{in}}(t)$ in the exclusive picture. We start from the MPM and explicitly calculate the equivalent MRM. The process is then reversed; it is proven that the connection of MRM to MPM, and MPM to MRM is governed by the same unitary transformation. The general multichannel problem where there are several trajectories simply requires the addition of matrix indices, and the treatment of Pinsky, Snider, and Thomas[13] (PST) can be incorporated here directly. Therefore, any standard multiperipheral model which had a finite recursion relation can be treated by the method of Sec. II. The method also applies to all models of the Chew-Pignotti type and all ABFST models where the elastic amplitude is saturated by Regge exchanges (which is almost always the case). As an example we treat the recent model of Webber and Green.[14] In Sec. III, we consider models that do not have simple J-plane propagators. Such models arise, for example, when thresholds are included.[15] The physical motivation for such models is discussed in some detail and it is shown that all models of this general class obey the same equivalence as the simpler model of Sec. II. The method of Sec. III also applies in principle to general ABFST-type models. Section IV discusses such generalizations.

II. MULTI-REGGE MODELS

Consider a multi-Regge model for the n-particle exclusive cross section, i.e.,

$$\mathcal{L}(s\sigma_{n+2}(s))=\int_{-\infty}^{0}dt_1\cdots\int_{-\infty}^{0}dt_{n+1}D_a(t_1)\frac{1}{J-l(t_1)}G(t_1,t_2)\frac{1}{J-l(t_2)}\cdots\frac{1}{J-l(t_{n+1})}D_b(t_{n+1}),\tag{2.1}$$

where $\mathcal{L}$ implies Laplace transform (J-plane partial-wave projection),

$$l(t)=2\alpha_{\text{in}}(t)-1,$$

$D(t)$ is the square of the end vertex in the multiperipheral chain, $G(t_1,t_2)$ is the square of the center vertex of the chain, and the t_i's are the invariant momentum transfers along the chain. The Toller-angle dependence has already been integrated out, i.e.,[16]

$$G(t_1,t_2)=\int_0^{\pi}d\omega\,G(t_1,t_2,\omega),\tag{2.2}$$

and we have one trajectory being exchanged along the chain.

If we now change $[J-l(t)]^{-1}$ to $\delta(t-t')[J-l(t)]^{-1}$ and introduce a fictitious integration over t', we can cast this equation for $\mathcal{L}(s\sigma_{n+2}(s))$ in the following operator form:

$$\begin{aligned}\mathcal{L}(s\sigma_{n+2}(s))&=Q_n(J)\\&=DF(J)[GF(J)]^nD,\end{aligned}\tag{2.3}$$

where $F(J)=\delta(t-t')/[J-l(t)]$. In this operator formulation, all the algebraic manipulations of PST go through unchanged. However, for completeness, we will repeat them again in the present context. First we construct a generating function

$$\begin{aligned}Q(J)&=\sum_{n=0}^{\infty}Q_n(J)z^n\\&=\sum_{n=0}^{\infty}DF(J)[zGF(J)]^nD\\&=D\frac{1}{F(J)^{-1}-zG}D.\end{aligned}\tag{2.4}$$

This generating function can also be constructed starting from the MRM, in particular:

$$Q(J)=\sum_{n=0}^{\infty}(z-1)^nP_n(J),\tag{2.5}$$

where

$$P_n(J)=\mathcal{L}(s\sigma_n^{\text{inc}}(s)).\tag{2.6}$$

$\sigma_n^{\text{inc}}(s)$ is the inclusive n-particle cross section.

Its Laplace transform can be written as follows in the MRM, where we have explicitly taken into account poles and cuts:

$$P_n(J) = \sum\!\!\!\!\!\!\int d\lambda_1 \cdots d\lambda_{n+1} \Delta^*(\lambda_1)\frac{1}{J-\lambda_1} \times \Gamma(\lambda_1,\lambda_2)\frac{1}{J-\lambda_2}\cdots\frac{1}{J-\lambda_{n+1}}\Delta(\lambda_{n+1}). \tag{2.7}$$

To ensure the reality of (2.7) we write complex conjugate on the left (Δ^*), and assume Γ is Hermitian, $\Gamma(\lambda_1,\lambda_2)^* = \Gamma(\lambda_2,\lambda_1)$. The sum extends over the poles at λ_i, $i=1,\ldots,N$, while the integral extends over the cuts. Again, we can cast this into an operator form by introducing a δ function, $\delta(\lambda,\lambda')$ and an additional sum-integral over λ'. The notation $\delta(\lambda,\lambda')$ implies

$$\delta(\lambda,\lambda') = \begin{cases} \delta_{\lambda\lambda'} & \text{for poles}, \\ \delta(\lambda-\lambda') & \text{for cuts}. \end{cases}$$

Now we introduce the diagonal operator

$$\Phi(J) = \frac{\delta(\lambda,\lambda')}{J-\lambda} \tag{2.8}$$

and $P_n(J)$ becomes

$$P_n(J) = \Delta^*\Phi(J)[\Gamma\Phi(J)]^n\Delta. \tag{2.9}$$

Our explicit demonstration of equivalence now involves casting (2.4) in the form of (2.5), with $P_n(J)$ having the form (2.9): *The only J dependence occurs in the diagonal operator $\Phi(J)$.*

Now from Eq. (2.4), we obtain

$$Q(J) = \sum_{n=0}^{\infty} D[F(J)^{-1} - G]^{-1} \times\{(z-1)G[F(J)^{-1}-G]^{-1}\}^n D. \tag{2.10}$$

This formula for $Q(J)$ almost has the form required by (2.5) and (2.9), except that the operator $[F(J)^{-1}-G]^{-1}$ which should correspond to $\Phi(J)$ is not, in general, diagonal. The problem, therefore, is to diagonalize $[F(J)^{-1}-G]^{-1}$ in a way that will not introduce J dependence elsewhere in the expression for P_n; that is, we must find an invertible and, as it turns out, unitary operator which is independent of J such that

$$S^+[F(J)^{-1}-G]^{-1}S = \Phi(J). \tag{2.11}$$

Then the equivalence is complete and

$$S^+GS = \Gamma, \quad S^+D = \Delta. \tag{2.12}$$

A. Transformation from MPM→MRM by S

Now we turn to the search for S. Formally S is the solution of the following integral equation:

$$[F(J)^{-1}-G]S = S\Phi(J)^{-1}. \tag{2.13}$$

When writing out the operators explicitly, it is convenient to introduce a matrix notation, which separates the continuous and discrete channels. Therefore, $S(t,\lambda)$ will be a row vector since in the exclusive channel which corresponds to the variable t, we assume only a cut, while in λ there may be poles in addition to the cut. In λ, the poles will be the upper entries. In an obvious and convenient notation, we express (2.13) as

$$\int_{-\infty}^{0} dt'\{\delta(t-t')[J-l(t')]-G(t,t')\}(S(t',\lambda_i),\, S(t',\lambda)) = \left(\sum_j S(t,\lambda_j),\, \int d\lambda' S(t,\lambda')\right)\begin{pmatrix}(J-\lambda_j)\delta_{ij} & 0\\ 0 & (J-\lambda')\delta(\lambda-\lambda')\end{pmatrix}$$
$$= ((J-\lambda_i)S(t,\lambda_i),\, (J-\lambda)S(t,\lambda)). \tag{2.14}$$

We see that the J dependence of (2.13) drops out, so that S satisfies a J-independent integral equation, which of course ensures that S is J-independent:

$$\int_{-\infty}^{l_c} dl' G(l,l')[S(l',\lambda_i), S(l',\lambda)] = [(\lambda_i - l)S(l,\lambda_i), (\lambda - l)S(l,\lambda)]. \tag{2.15}$$

We have changed variables from t to l and absorbed the Jacobian in G. It is important to note that this is not a Fredholm equation, so that completeness and orthonormality are not guaranteed for $S(l,\lambda)$ considered for different λ as functions of l. We assume that $G(l,l')$ is square integrable and symmetric, and can therefore be expanded in terms of a complete orthonormal set of functions for which G is the kernel[17]:

$$G(l,l') = \sum_n g_n(l)E_n g_n(l') \equiv g^T(l)Eg(l'), \tag{2.16}$$

where $g(l)$ is a vector of the eigenfunctions and E is the diagonal matrix of the eigenvalues. Whenever G is real, $g(l)$ will be real; for simplicity we assume G real in what follows.

The solution of (2.15) that corresponds to discrete values of λ has the following form:

$$S(l,\lambda_i) = \frac{g^T(l)E\gamma_i}{\lambda_i - l}. \tag{2.17}$$

Substituting (2.17) into Eq. (2.15), we find the following condition on the location of the discrete eigenvalues (poles):

$$[I-f(\lambda_i)E]\gamma_i=0\,, \tag{2.18}$$

where $f(\lambda)$ is the operator

$$f(\lambda)=\int_{-\infty}^{l_c} dl\,\frac{g(l)g^T(l)}{\lambda-l+i\epsilon}.$$

The $i\epsilon$ prescription used to define $f(\lambda)$ for λ real and less than l_c will turn out to be important in order to prove the completeness of $S(l,\lambda)$ [q.v. (2.52) below)]. The location of the poles is determined by the (generally infinite-dimensional) determinant equation

$$\det[I-f(\lambda)E]=0\,. \tag{2.19}$$

The γ_i can be determined up to a normalization by standard eigenfunction techniques. The normalization is fixed by orthonormality of $S(l,\lambda)$. That is,

$$\int dl\, S^\dagger(l,\lambda_i)S(l,\lambda_j)=\delta_{ij}$$
$$=\gamma_i^\dagger E\int dl\,\frac{g(l)g^T(l)}{(\lambda_i-l)(\lambda_j-l)}E\gamma_j\,. \tag{2.20}$$

For $i\neq j$, this is

$$\gamma_i^\dagger E\frac{f(\lambda_j)-f(\lambda_i)}{\lambda_i-\lambda_j}E\gamma_j=0\,. \tag{2.21}$$

For $i=j$, this is

$$-\gamma_i^\dagger E\frac{\partial f(\lambda_i)}{\partial\lambda_i}E\gamma_i=1\,. \tag{2.22}$$

This condition fixes the normalization of γ_i.

The solution corresponding to the continuous spectrum or cut is

$$S(l,\lambda)=\frac{g^T(l)E\gamma(\lambda)}{\lambda-l+i\epsilon}+b(\lambda)\delta(\lambda-l)\,, \tag{2.23}$$

where we define the integrals for λ and l real using the $i\epsilon$ prescription. Substituting (2.23) into Eq. (2.15) we see that the δ function only contributes if $\lambda<l_c=l(0)$. Therefore, we only have a cut for $\lambda<l_c$. Then

$$f(\lambda)E\gamma(\lambda)+g(\lambda)b(\lambda)=\gamma(\lambda)\,. \tag{2.24}$$

Since $\gamma(\lambda)$ and $b(\lambda)$ are free functions, this can clearly be satisfied for any $\lambda\leq l_c$, that is to say, there is a continuous spectrum or a cut in λ from $-\infty$ to l_c. The orthonormality condition on $S(l,\lambda)$ along with (2.24) determines $\gamma(\lambda)$ and $b(\lambda)$; i.e.,

$$\int_{-\infty}^{l_c} dl\,S^\dagger(l,\lambda)S(l,\lambda')=\delta(\lambda-\lambda')\,. \tag{2.25}$$

Hence,

$$\frac{\gamma^\dagger(\lambda)E}{\lambda'-\lambda+i\epsilon}[f(\lambda)^\dagger-f(\lambda')]E\gamma(\lambda')+\frac{b^*(\lambda)g(\lambda^*)^TE\gamma(\lambda')}{\lambda'-\lambda+i\epsilon}+\frac{\gamma^\dagger(\lambda)Eb(\lambda')g(\lambda')}{\lambda-\lambda'-i\epsilon}+|b(\lambda)|^2\delta(\lambda-\lambda')=\delta(\lambda-\lambda')\,. \tag{2.26}$$

Using (2.24) for $b(\lambda)g(\lambda)$, it is easy to show that the left-hand side of (2.26) reduces to $|b(\lambda)|^2\delta(\lambda-\lambda')$; we take therefore $b(\lambda)=1$, yielding the following equation for $\gamma(\lambda)$:

$$[I-f(\lambda)E]\gamma(\lambda)=g(\lambda)\,. \tag{2.27}$$

B. Transformation from MRM→MPM by $S^\dagger$

Let us calculate the single-particle inclusive density from the MPM, assuming that the inverse exists and is equal to $S^\dagger$:

$$\begin{aligned}P_1(J)&=D^TSS^\dagger[F(J)^{-1}-G]^{-1}SS^\dagger GSS^\dagger\\&\quad\times[F(J)^{-1}-G]^{-1}SS^\dagger D\\&=D^TS\Phi(J)S^\dagger GS\Phi(J)S^\dagger D\,.\end{aligned} \tag{2.28}$$

Now

$$\Gamma(\lambda,\lambda')=\int dl\,dl'\begin{pmatrix}S^\dagger(l,\lambda_i)\\S^\dagger(l,\lambda)\end{pmatrix}g^T(l)Eg(l')\times(S(l',\lambda_i'),\ S(l',\lambda'))\,. \tag{2.29}$$

Therefore the central Mueller-Regge coupling has the form

$$\Gamma(\lambda,\lambda')=\begin{pmatrix}\gamma^\dagger(\lambda_i)\\\gamma^\dagger(\lambda)\end{pmatrix}E(\gamma(\lambda_i'),\ \gamma(\lambda'))=\begin{pmatrix}\gamma^\dagger(\lambda_i)E\gamma(\lambda_i') & \gamma^\dagger(\lambda_i)E\gamma(\lambda')\\\gamma^\dagger(\lambda)E\gamma(\lambda_i') & \gamma^\dagger(\lambda)E\gamma(\lambda')\end{pmatrix}. \tag{2.30}$$

The end coupling has not yet been specified but it is usually taken as follows in this type of model:

$$D(l)=g(l)^T\delta\,. \tag{2.31}$$

Therefore, the end couplings to the pole in the inclusive amplitude are

$$\Delta(\lambda_i)=\gamma(\lambda_i)^\dagger\delta\,, \tag{2.32}$$

and to the cut they are

$$\Delta(\lambda)=\gamma(\lambda)^\dagger\delta\,. \tag{2.33}$$

For example in the case where $G(l,l')$ is separable, there will be one pole at $\lambda=\lambda_0$ and a cut starting at l_c. Therefore the single-particle inclusive cross section integrated over $P_\perp^{\,2}$ is

$$\frac{d\sigma}{dy_1}=\frac{\delta}{e^Y}\left[\gamma^2(\lambda_0)e^{(Y/2-y_1)\lambda_0}+\int_{-\infty}^{l_0}d\lambda|\gamma(\lambda)|^2e^{(Y/2-y_1)\lambda}\right]E\left[\gamma^2(\lambda_0)e^{(y_1+Y/2)\lambda_0}+\int_{-\infty}^{l_0}d\lambda'|\gamma(\lambda')|^2e^{(y_1+Y/2)\lambda'}\right]\delta\,. \tag{2.34}$$

Similarly, the total cross section is given by

$$\sigma_T=\frac{\delta^2}{e^Y}\left[\gamma^2(\lambda_0)e^{Y\lambda_0}+\int_{-\infty}^{l_0}d\lambda|\gamma(\lambda)|^2e^{Y\lambda}\right]. \tag{2.35}$$

These inclusive processes display all the expected properties.

Let us now reverse our procedure, start with the MRM, and find the MPM. From (2.5) and (2.9), we see that the generating function is

$$Q(J)=\Delta^*\frac{1}{\Phi(J)^{-1}-(z-1)\Gamma}\Delta \tag{2.36}$$

$$=\sum_{n=0}^{\infty}\Delta^*[\Phi(J)^{-1}+\Gamma]^{-1}\{z\Gamma[\Phi(J)^{-1}+\Gamma]^{-1}\}^n\Delta\,. \tag{2.37}$$

We see that this almost has the required form (2.4); however, it still must be diagonalized in a J-independent way. From (2.12) and (2.13)

$$\Phi(J)^{-1}+\Gamma=S^+[F(J)^{-1}-G+G]S$$
$$=S^+F(J)^{-1}S\,, \tag{2.38}$$

so that S^+ must satisfy

$$S[\Phi(J)^{-1}+\Gamma]S^+=F(J)^{-1}\,. \tag{2.39}$$

Consistency of our scheme requires that S satisfy the following equation:

$$S[\Phi(J)^{-1}+\Gamma]=F(J)^{-1}S\,,$$

or

$$\left[\sum_j S(l,\lambda_j),\int d\lambda' S(l,\lambda')\right]\begin{pmatrix}\delta_{ij}(J-\lambda_i)+\Gamma(\lambda_j,\lambda_i) & \Gamma(\lambda_j,\lambda)\\ \Gamma(\lambda',\lambda_i) & (J-\lambda')\delta(\lambda'-\lambda)+\Gamma(\lambda',\lambda)\end{pmatrix}$$
$$=\int_{-\infty}^{l_c}dl'(J-l')\delta(l-l')[S(l',\lambda_i),\ S(l',\lambda)]\,. \tag{2.40}$$

The J dependence of (2.40) cancels out and this equation reduces to

$$\left[\sum_j S(l,\lambda_j),\ \int S(l,\lambda')d\lambda'\right]\begin{bmatrix}\Gamma(\lambda_j,\lambda_i) & \Gamma(\lambda_j,\lambda)\\ \Gamma(\lambda',\lambda_i) & \Gamma(\lambda',\lambda)\end{bmatrix}=[(\lambda_i-l)S(l,\lambda_i),\ (\lambda-l)S(l,\lambda)]\,. \tag{2.41}$$

If we assume that Γ is square integrable and square summable and Hermitian, then Γ can be uniquely expanded in terms of a complete orthonormal set of eigenfunction for which Γ is the kernel. The general form is given by (2.30), where $\gamma(\lambda)$ will, in general, be complex. Now using the notation of (2.30), we find the discrete and continuous solution for S to be

$$S(l,\lambda_i)=\frac{h^+(l)E\gamma_i}{\lambda_i-l} \tag{2.42a}$$

and

$$S(l,\lambda)=\frac{h^+(l)E\gamma(\lambda)}{\lambda-l+i\epsilon}+\bar{b}(\lambda)^*\delta(\lambda-l)\,. \tag{2.42b}$$

Substituting (2.42) into (2.41), we find

$$h(l)=\phi(l)Eh(l)+\bar{b}(l)\gamma(l)\,,$$

where

$$\phi(l)=\sum_i\frac{\gamma_i\gamma_i^+}{\lambda_i-l}+\int_{-\infty}^{l_c}d\lambda\frac{\gamma(\lambda)\gamma^+(\lambda)}{\lambda-l-i\epsilon}\,. \tag{2.43}$$

Equation (2.42) along with the orthonormality condition on S^+ will be sufficient to determine $\bar{b}$ and h, namely,

$$\left[\sum_i S(l,\lambda_i),\ \int_{-\infty}^{l_c}d\lambda\, S(l,\lambda)\right]\begin{bmatrix}S^+(l',\lambda_i)\\ S^+(l',\lambda)\end{bmatrix}=\delta(l-l')\,, \tag{2.44}$$

which implies, by arguments analogous to those following (2.26),

$$|\bar{b}(l)|^2\delta(l-l')=\delta(l-l')\,. \tag{2.45}$$

Choosing $\bar{b}(l)=1$, we can now solve for $h(l)$:

$$h(l)=[I-\phi(l)E]^{-1}\gamma(l)\,. \tag{2.46}$$

C. Equivalence between MPM and MRM

For consistency, we must have $h(l)=g(l)$. Therefore, using (2.27) the following must be an identity:

$$[I-f(l)E]^{-1}=1-\phi(l)E\,. \tag{2.47}$$

The proof of this identity will be equivalent to the proof of the completeness of $S(l, \lambda)$, and will complete the proof of the equivalence between MPM and MRM for models of the form (2.1).

Let us consider

$$\tilde{\phi}(l)E \equiv -[I-f(l)E]^{-1}f(l)E\,. \qquad (2.48)$$

Consider the function on the right-hand side; clearly, it has poles at $l=\lambda_i$, and a cut starting from l_c. Let us write a dispersion relation for $\tilde{\phi}(l)E$. Consider the pole at λ_i, where $\tilde{\phi}(l)\sim R/(\lambda_i - l)$; in general we have

$$[I-f(\lambda_i)E]R=0\,. \qquad (2.49)$$

The operator R, which is the residue of $\tilde{\phi}$ at λ_i, is real, symmetric, and can in general be expanded in a (possibly infinite) sum of separable terms:

$$R=\sum_m r_m r_m^{\dagger}\mu_m\,,$$

where the vectors r_m are orthogonal, $r_m^{\dagger}r_n=\delta_{mn}$. This implies

$$[I-f(\lambda_i)E]r_m=0\,, \qquad (2.49')$$

using (2.49) for each r_m which contributes to R. This is precisely Eq. (2.18), and hence each r_m is proportional to γ_i. We therefore denote R by $rr^{\dagger}$, where $r\propto\gamma_i$. To obtain the over-all normalization, we note that (2.48) implies

$$f(\lambda_i)E=-\frac{\partial f}{\partial\lambda_i}Err^{\dagger}E\,. \qquad (2.50)$$

Left-multiplying (2.50) by $r^{\dagger}E$ and right-multiplying (2.50) by r we obtain, after simplifying,

$$-r^{\dagger}E\frac{\partial f}{\partial\lambda_i}Er=1\,,$$

which is the normalization equation (2.22) for γ_i. We therefore have the result

$$\tilde{\phi}(l)E\big|_{l\approx\lambda_i}=\frac{\gamma_i\gamma_i^{\dagger}E}{\lambda_i - l}\,. \qquad (2.51)$$

Next we compute the discontinuity across the cut:

$$\begin{aligned}\operatorname{Disc}\tilde{\phi}(\lambda)E &= -[I-f(\lambda)E]^{-1}[\operatorname{Disc}f(\lambda)]\\ &\quad\times[I-Ef(\lambda)^{\dagger}]^{-1}E\\ &= -[I-f(\lambda)E]^{-1}g(\lambda)g(\lambda)^{T}\\ &\quad\times[I-Ef(\lambda)^{\dagger}]^{-1}E\\ &= -\gamma(\lambda)\gamma(\lambda)^{\dagger}E\,. \end{aligned} \qquad (2.52)$$

Therefore,

$$\tilde{\phi}(l)=\sum_i\frac{\gamma_i\gamma_i^{\dagger}}{\lambda_i - l}+\int_{-\infty}^{l_c}d\lambda\frac{\gamma(\lambda)\gamma(\lambda)^{\dagger}}{\lambda - l - i\epsilon} \qquad (2.53)$$

is identical to $\phi(l)$, Eq. (2.43). This proves (2.47) and completes the proof of the equivalence between multi-Regge and Mueller-Regge models.

D. Example

As an example of the above method, we comment briefly on the calculation of Webber and Green.[14] They have calculated the implications of a hard cut and a pole at $J=1$ in the inclusive channel, on the couplings in the exclusive channel. In particular, they have found that the internal-cut coupling exclusively is soft (i.e., vanishing at the branch point). In their model, a specific cut is chosen: We carry through the calculation assuming only that in the inclusive picture $\gamma(1)$ is finite and nonzero.

Now the inclusive coupling matrix in the separable approximation is

$$\Gamma(\lambda,\lambda')=\begin{pmatrix}\gamma_0\\ \gamma(\lambda)^*\end{pmatrix}(e)(\gamma_0,\ \gamma(\lambda'))\,, \qquad (2.54)$$

where $\lambda_0=1$. Then

$$S(l,\lambda_0)=\frac{g(l)e\gamma_0}{\lambda_0 - l} \qquad (2.55)$$

for the discrete channel and

$$S(l,\lambda)=\frac{g(l)^*e\gamma(\lambda)}{\lambda - l + i\epsilon}+\delta(\lambda - l) \qquad (2.56)$$

for the continuous channel. Then

$$\phi(l)=\frac{\gamma_0^{\,2}}{\lambda_0 - l}+\int_{-\infty}^{1}d\lambda'\frac{|\gamma(\lambda')|^2}{\lambda' - l - i\epsilon}\,, \qquad (2.57)$$

$$g(l)=\frac{\gamma(l)}{1-\phi(l)e}\,, \qquad (2.58)$$

and

$$\begin{aligned}G(l,l') &= g(l)^*eg(l')\\ &= \frac{\gamma(l)^*e\gamma(l')}{[1-\phi(l)^*e][1-\phi(l')e]}\,. \end{aligned} \qquad (2.59)$$

Hence the internal exclusive couplings are

$$G(l,l')=\frac{(\lambda_0 - l)\gamma(l)^*e\gamma(l')(\lambda_0 - l')}{\left[\lambda_0 - l - \gamma_0^{\,2}e-(\lambda_0 - l)\int_{-\infty}^{1}d\lambda'\frac{|\gamma(\lambda')|^2}{\lambda' - l + i\epsilon}e\right]\left[\lambda_0 - l' - \gamma_0^{\,2}e-(\lambda_0 - l')\int_{-\infty}^{1}d\lambda'\frac{|\gamma(\lambda')|^2}{\lambda' - l - i\epsilon}e\right]}\,. \qquad (2.60)$$

Since both the pole and the branch point are at the same point, namely $l=\lambda_0=1$, we see that $G(l,l')$ vanishes at that point, making the exclusive cut soft when coupled internally.

It is interesting to note that if one starts from the above MRM, then the MPM coupling $G(l, l')$ is not necessarily real and symmetric. To insure positive cross sections, it is sufficient for G to be Hermitian, which follows if Γ is Hermitian, which is true in this case. The MPM's requiring G real are those, for example, which ignore "cross graphs" in computing the cross section. There exists, therefore, at least one interpretation for a nonreal but Hermitian G; but if we insist (as is usual) that G be real, then we must require that $\gamma(l)/[1-\phi(l)e]$ be real for $l<l_c$. This is of course a strong constraint on $\gamma(l)$.

III. MODELS WITH THRESHOLDS

Any attempt at making a more realistic model will, of course, lead to more complicated behavior than the model illustrated above where the only J dependence occurs in a simple propagator. The equivalence principle can also be proven for more complicated types of models. To illustrate this, we will consider in detail the J-plane complications introduced by attempts to treat energy thresholds correctly. While the effects of thresholds have been discussed previously in the literature,[15,6] we feel they deserve some additional attention.

A. Multiperipheral thresholds

When calculating $\text{Im}\,A_{el}(s)$ through unitarity from $\sigma_n(s)$, clearly the s-channel thresholds of $\sigma_n(s)$ must have significant effects. Consider first the thresholds in $\sigma_2(s)$. There is clearly some value of s below which $\sigma_2(s)$ is zero. That is

$$\sigma_2(s)=f(s)\theta(s-s_2).$$

Therefore the Mellin transform of $\sigma_2(s)$ is of the form

$$\sigma_2(J)=\int_{s_0}^{\infty}\frac{ds}{s}f(s)\theta(s-s_2)\left(\frac{s}{s_0}\right)^J.$$

This expression has two scale parameters s_0 and s_2. s_0 sets the scale for the Mellin transform, while s_2 is the threshold. In the simple models considered in Sec. II, these are taken to be the same. In general, this is not the case. The mismatch of these two parameters introduces an additional complication into the J dependence of the problem. While different models choose these scales differently, they all have this property.

Consider the dependence of $\sigma_2(J)$ in the limit of J large. The large J dependence is clearly dominated by the dependence at small values of the complementary variables $\ln s$. That is,

$$\sigma_2(J)=\int_{\ln(s_2/s_0)}^{\infty}d[\ln(s/s_0)]f(s)e^{-J\ln(s/s_0)}$$

$$\to f(s_2)\frac{e^{-J\ln(s_2/s_0)}}{J}.$$

This illustrates how this mismatch of scale variables introduces additional complications in the J dependence. The additional factor e^{-bJ} is a general property of $\sigma_2(J)$, with b being crudely the logarithm of the ratio of the scale parameters.

If one considers a model with only pions and assumes that the $(s/s_0)^J$ behavior comes from the asymptotic behavior of $P_J(\cos\theta_t)$, then $s_0=2m^2$. In addition, s_2 is the two-pion threshold; that is, $s_2=4m^2$ and therefore $b=\ln 2$. This is what might be naively expected in an ABFST model. A better estimate is computed below.

The same threshold effect occurs generally in the $2\to n$ amplitude, with the cutoff b_n being determined by the n-particle threshold and a scale parameter. In multiperipheral models a special relation holds, namely

$$b_n\sim nb, \tag{3.1}$$

where b is determined by the thresholds of the kernel. Since J is dual essentially to rapidity $Y=\ln s$, (3.1) implies a cutoff when $Y=nb$. An instructive example of this threshold has been given by Chew and Snider.[15] In the ABFST model, the iterated kernel is the off-shell $\pi\pi$ cross section: Let u and v denote the off-shell pion masses, and $K(u,v,s)$ denote the integrated cross section. The ABFST model diagonalizes in the J plane; the partial-wave projections are formally O(2, 1) projections

$$K(u,v,J)=\int_{s_0}^{\infty}ds\; e^{-\beta(s,u,v)J}K(u,v,s), \tag{3.2}$$

with the "boost angle" β defined by

$$\cosh\beta=\frac{s-u-v}{2(uv)^{1/2}}. \tag{3.3}$$

The essential structure of the total cross section for our purposes is given by the trace approximation[17]

$$\sigma_{\text{tot}}(J)\sim\frac{\text{Tr}K}{1-g\,\text{Tr}K}. \tag{3.4}$$

Here

$$\text{Tr}K=\int_{-\infty}^{0}du\,K(u,u,J)F(u), \tag{3.5}$$

where F represents the pion propagators, form factors, off-shell factors, etc. We can now derive the Chew-Snider formula for the multiparticle thresholds.

For large values of J, (3.2) has the behavior $e^{-\beta(s_0,u,v)J}$, but by (3.5) this must be integrated over $u=v$ to obtain the cross sections. Let Δ^2

be the effective cutoff due to $F(u)$ in (3.5); the largest $-u$ gives the smallest β, so we expect

$$\mathrm{Tr}K \sim e^{-\beta(s_0,-\Delta^2,-\Delta^2)J}$$

as $J\to\infty$. Thus the b cutoff for the kernel is

$$b=\beta(s_0,-\Delta^2,-\Delta^2)\,,$$

or equivalently (3.6)

$$\cosh b = 1+\frac{s_0}{2\Delta^2}\,.$$

Because of the structure of (3.4), we see that the n-particle cross section behaves as

$$\sigma_n(J)\sim e^{-nbJ}\quad \text{as } J\to\infty\,,$$

which is the result (3.1). We shall take (3.6) as a general guide for all MPM's even though strictly speaking we have obtained this only in the trace approximation to a class of ABFST models.

The actual magnitude of b will depend upon the average mass $\sqrt{s_0}$ of produced particles along the chain, and upon the strength Δ^2 with which the momentum transfer damps out. For a model producing pions, and a cutoff at $\Delta^2=1(\mathrm{GeV}/c)^2$, we obtain $b\sim m_\pi c/(1\ \mathrm{GeV}/c)=0.14$; if $s_0\sim M_\rho^{\ 2}$ such as in ABFST models producing ρ's and if $\Delta^2\sim M_\rho^{\ 2}$, then we see that $b=\cosh^{-1}(1.5)\cong \ln 2.6=0.96$. (This agrees with estimates by Dash[6].) We speculate that the MPM cutoff (3.1) may be more general, and may depend only on the existence of a transverse-momentum cutoff in the problem. Then b would still depend upon the average P_T and the average mass of the produced particles.

It is interesting to consider the situation where particles occur in clusters. To see the effect of this, consider the case where there are N particles per cluster. Then the threshold in the subenergy channels is $(Nm+Nm)^2=4N^2m^2$, and since we do not expect an appreciable change in Δ^2 for clusters made up of pions, we see from (3.6) for large N that

$$b_{\text{cluster}}\simeq \ln(4N^2m^2/\Delta^2)\,.$$

B. Equivalence for threshold models

Now let us consider the proof of equivalence for models of the above general type. We note that the proof given here of this equivalence is not limited just to models with thresholds, but rather is a proof of equivalence for models with a rather general form of propagator.

As an example of a model with thresholds, consider the case where the propagator is e^{-bJ}/J and the vertex functions are t-independent.[15] Then

$$A(J)=d\frac{e^{-bJ}}{J}E\frac{e^{-bJ}}{J}E\cdots\frac{e^{-bJ}}{J}d\,. \tag{3.7}$$

We can rewrite the propagator in the following form:

$$\frac{e^{-bJ}}{J}=\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{e^{-bl}}{l(J-l)}\frac{dl}{i} \tag{3.8}$$

for $\mathrm{Re}J>c$ and $c>0$. Then

$$A(J)=\int_{c-i\infty}^{c+i\infty}\frac{dl_1}{i}\cdots\frac{dl_n}{i}D(l_1)\frac{1}{J-l_1}$$

$$\times G(l_1,l_2)\frac{1}{J-l_2}\cdots D(l_n)\,, \tag{3.9}$$

where

$$D(l)=g(l)d$$
$$=\frac{e^{-bl/2}}{(2\pi l)^{1/2}}d\,; \tag{3.10}$$

$$G(l_1,l_2)=g(l_1)Eg(l_2)$$
$$=\frac{e^{-bl_1/2}}{(2\pi l_1)^{1/2}}E\frac{e^{-bl_2/2}}{(2\pi l_2)^{1/2}}\,. \tag{3.11}$$

Now $A(J)$ has formally the same algebraic structure as (2.1), the difference being that now the contour is parallel to the imaginary axis. All of the manipulations of Sec. II can be carried out on (3.9).

The operator $S(l,\lambda)$ that diagonalizes the propagator in a J-independent manner satisfies the equation

$$\int_{c-i\infty}^{c+i\infty}\frac{dl'}{i}G(l,l')S(l',\lambda)=(\lambda-l)S(l,\lambda)\,. \tag{3.12}$$

The solution for S corresponding to the cut, that is, for l on the contour of integration, is (2.23) rotated by 90°:

$$S(l,\lambda)=\frac{g(l)E\gamma(\lambda)}{\lambda-l+\epsilon}+i\delta(\lambda-l)\,, \tag{3.13}$$

where an ϵ prescription defines the singular integrals which will occur when λ and l are on the contour. The equation for $\gamma(\lambda)$ is

$$\gamma(\lambda)=\frac{g(\lambda)}{1-f(\lambda)E}\,, \tag{3.14}$$

where

$$f(\lambda)=\int_{c-i\infty}^{c+i\infty}\frac{dl}{i}\frac{g^2(l)}{\lambda-l+\epsilon}$$
$$=\begin{cases}\dfrac{e^{-b\lambda}}{\lambda}, & \mathrm{Re}\lambda+\epsilon>c\\[4pt] 0, & \mathrm{Re}\lambda+\epsilon<c\,.\end{cases} \tag{3.15}$$

For $\mathrm{Re}\lambda+\epsilon>c$, there may be discrete values of λ that satisfy Eq. (3.14) which correspond to poles. In that case, $S(l,\lambda_i)$ is given by

$$S(l,\lambda_i)=\frac{g(l)E\gamma_i}{\lambda_i-l}. \qquad (3.16)$$

The condition for the existence of such a solution is

$$1=f(\lambda_i)E. \qquad (3.17)$$

Those poles that would be exposed for a smaller choice of c are absorbed in the background integral.

The value of γ_i is, as before, set by the normalization condition [cf. (2.22)] on $S(l,\lambda_i)$:

$$\gamma_i{}^2E^2=-\left[\frac{\partial f}{\partial\lambda}(\lambda)\right]^{-1}_{\lambda=\lambda_i}. \qquad (3.18)$$

As in (2.33) and (2.32), we compute the coupling in the Mueller model to be

$$\Delta(\lambda)=\tilde{\gamma}(\lambda)d \qquad (3.19)$$

and

$$\Delta_i=\gamma_i d\,; \qquad (3.20)$$

for the central vertices we obtain in analogy to (2.30)

$$\Gamma(\lambda,\lambda')=\tilde{\gamma}(\lambda)E\gamma(\lambda'),$$
$$\Gamma(\lambda,\lambda_i)=\tilde{\gamma}(\lambda)E\gamma_i, \qquad (3.21)$$
$$\Gamma(\lambda_i,\lambda_j)=\gamma_iE\gamma_j.$$

Here (in analogy to $\gamma^\dagger$) $\tilde{\gamma}$ is the transpose of γ, with ϵ replaced by $-\epsilon$; in particular this means $\tilde{\gamma}(\lambda)=g(\lambda)$ due to (3.15). It is a straightforward calculation to show that if S takes us from MPM to MRM, then $\tilde{S}$ defined by S transposed with $\epsilon\to-\epsilon$ takes us from the MRM back to the original MPM. The proof of this as well as the completeness of $S(l,\lambda)$ relies on the truth of the following identity:

$$1-\int_{c-i\infty}^{c+i\infty}\frac{g^2(l)E}{\lambda-l+\epsilon}\frac{dl}{i}=\left[1-\int_{c-i\infty}^{c+i\infty}\frac{dl}{i}\frac{\gamma(l)\tilde{\gamma}(l)E}{l-\lambda-\epsilon}-\sum_i\frac{\gamma_i{}^2E}{\lambda_i-\lambda}\right]^{-1}. \qquad (3.22)$$

One proof of this identity parallels that of (2.48) and will not be given here. In carrying out this proof it is important to note that the analogs to the discontinuity formulas are $\mathrm{Disc}F(x)=F(x+\epsilon)-F(x-\epsilon)$ when x is on the complex contour. A simple proof of (3.22) is as follows. Without loss of generality we choose the contour such that no explicit poles appear on the right-hand side of (3.22). Then for $\mathrm{Re}\lambda+\epsilon>c$, the identity is proved by performing the contour integration and using the definitions. For $\mathrm{Re}\lambda+\epsilon<c$ both sides of (3.22) are unity and the identity is trivial.

We now make some general comments on the "threshold" models. Let us consider for a moment σ_T as calculated from the MPM using the techniques in this section. We may take the contour far enough to the right so that no pole solutions explicitly appear. Then the total cross section is

$$P_0(J)=d\int_{c-i\infty}^{c+i\infty}\frac{d\lambda}{i}\frac{\gamma(\lambda)\tilde{\gamma}(\lambda)}{J-\lambda}d$$
$$=\mathcal{L}(s\sigma_T(s)). \qquad (3.23)$$

Now consider pushing the contour to the left so that the poles can explicitly appear. To do this, we use our identity (3.22) to rewrite this as

$$P_0(J)=d\int_{c-i\infty}^{c+i\infty}\frac{g^2(l)}{J-l}\frac{dl}{i}d\left[1-\int_{c-i\infty}^{c+i\infty}\frac{Eg^2(l)}{J-l}\frac{dl}{i}\right]^{-1}$$
$$=\frac{df(J)f}{1-f(J)E}. \qquad (3.24)$$

Clearly as c decreases, $f(J)E$ can take on the value 1, and $P_0(J)$ develops a simple pole. If, for example, we take E to be e^b, following Chew and Snider, the model will have its first pole at $J=1$. If we were to take c slightly below 1, the pole appears explicitly in (3.23). The residue of the pole at 1, of course, agrees with that in (3.24). It is straightforward now to push the contour further to the left and pick up the additional complex poles.

IV. GENERALIZATIONS

The method that we have illustrated above can be used for large classes of models having the form

$$A(J)=\int_{-\infty}^{0}dt_1\cdots dt_nD(t_1)F(J,t_1)G(t_1,t_2)\times F(J,t_2)\cdots D(t_n). \qquad (4.1)$$

To apply the method of Sec. III, we first write $D(t)$ and $G(t,t')$ in terms of the eigenvectors and the diagonal matrix of eigenvalues of the kernel $G(t,t')$ as in Sec. II:

$$G(t,t')=g^T(t)Eg(t'),$$
$$D(t)=d^Tg(t) \qquad (4.2)$$
$$=g(t)^Td.$$

Then

$$A(J)=d^T\int_{-\infty}^{0}dt_1g(t_1)F(J,t_1)g^T(t_1)\times E\int_{-\infty}^{0}dt_2g(t_2)F(J,t_2)g^T(t_2)E\cdots\times E\int_{-\infty}^{0}dt_ng(t_n)F(J,t_n)g^T(t_n)d$$

or

$$A(J)=d^T\hat{F}(J)E\hat{F}(J)\cdots d\,, \tag{4.3}$$

where now $\hat{F}(J)$ is the operator

$$\hat{F}(J)=\int_{-\infty}^{0} g(t)F(J,t)g^T(t)dt\,. \tag{4.4}$$

We have $A(J)$ in the same form as (3.7). Analogous to (3.8) we can represent $\hat{F}(J)$ by an integral over a complex contour $c-i\infty$ to $c+i\infty$ provided that there is a contour to the right of which $\hat{F}(J)$ has no singularity and also provided that $\hat{F}(J)$ falls rapidly enough to zero to allow the closing of the contour at infinity. Since $\hat{F}(J)$ is in general a (possibly infinite-dimensional) matrix, it may be convenient to define $G(l_1, l_2)$ in a somewhat unsymmetric fashion in obtaining a form for $A(J)$ analogous to (3.9) [note that to preserve an earlier notation, we use the symbols G, g, and D below to denote different quantities than in (4.1)–(4.4)]:

$$\begin{aligned}&G(l,l')=\hat{F}(l)E/2\pi i\,,\\&D_a(l)^T=d^T\,,\\&D_b(l)=\hat{F}(l)d/2\pi i\,.\end{aligned} \tag{4.5}$$

Then (4.3) is identical to

$$A(J)=\int dl_1\cdots dl_n D_a^T(l_1)\frac{1}{J-l_1}G(l_1,l_2)\cdots \times G(l_{n-1},l_n)\frac{1}{J-l_n}D_b(l_n)\,. \tag{4.6}$$

For nonsymmetric $G(l,l')$, the proof of equivalence can be formally carried out as in Sec. II, or as above for e^{-bJ}/J. The main changes are that in general $G(l,l')$ will have two sets of eigenfunctions $g(l)$ and $h(l')$ corresponding to operations on the left and on the right, and a corresponding expansion[17]

$$G(l,l')=g(l)^TEh(l')\,. \tag{4.7}$$

Moreover, S will no longer be unitary (or rotated unitary): S and S^{-1} can be separately calculated and to prove equivalence is to prove the existence of S and S^{-1} separately. Although this method is more cumbersome, its advantage is that it does not require one to take square roots of complex functions [as was done in (3.10) and (3.11)] in order to prove equivalence for threshold models and more general models.

We close this section by remarking on what additional complications can arise if one considers models of the ABFST type. First we note that such models contain quantum-number exchanges. For example, models which iterate $\pi\pi$ scattering must alternate odd and even G-parity exchanges. This implies that one needs a two-channel formalism to describe both kinds of exchanges. This is in addition to the fictitious channel used in Sec. II to distinguish the pole from the cut contribution.

To be somewhat more explicit, after the partial-wave [for forward scattering, this is an O(3, 1)] projection is taken, the ABFST equation is[9]

$$A_J(u,v)=I_J(u,v)+\frac{1}{J+1}\int_{-\infty}^{0}du'\rho(u')I_J(u,u')A_J(u',v)\,, \tag{4.8}$$

where the off-shell pion masses are u and v, $\rho(t)$ contains the pion propagator squared times constants, and $I_J(u,v)$ is the projection of the off-shell $\pi\pi$ elastic cross section. The elastic cross section has *even* G-parity exchanges, whereas the factor $1/(J+1)$ originates from π exchange. By formal devices, we can reexpress (4.8) in one of the forms we have previously discussed. First note that we can write

$$I_J(u,v)=g(J,u)^TE(J)g(J,v)\,, \tag{4.9}$$

which allows us to express (4.8) as the series

$$A_J=g^TEg+g^TEE_\pi Eg+g^TEE_\pi EE_\pi Eg+\cdots\,, \tag{4.10}$$

where

$$E_\pi(J)\equiv\int_{-\infty}^{0}du'\rho(u')\frac{g(J,u')g(J,u')^T}{J+1}\,. \tag{4.11}$$

The series (4.10) is formally equivalent to g^TAg, where A satisfies the coupled set of equations

$$\begin{aligned}&A=E+EB\,,\\&B=E_\pi A\,.\end{aligned} \tag{4.12}$$

Let us define

$$\mathfrak{A}(J)=\begin{pmatrix}A(J) & D(J)\\ B(J) & C(J)\end{pmatrix},\quad \mathfrak{F}(J)=\begin{pmatrix}E(J) & 0\\ 0 & E_\pi(J)\end{pmatrix},$$
$$\mathfrak{G}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}; \tag{4.13}$$

then Eqs. (4.12) are contained in

$$\mathfrak{A}(J)=\mathfrak{F}(J)+\mathfrak{F}(J)\mathfrak{G}\mathfrak{A}(J)\,. \tag{4.14}$$

Indeed (4.8) is obtained from

$$A_J(u,v)=\mathfrak{D}^T\mathfrak{A}(J)\mathfrak{D}\,, \tag{4.15}$$

where $\mathfrak{D}^T=(g(J,u)^T,\ 0)$. But a general term in the expansion of (4.14) is of the form (4.3). Therefore ABFST models contain only one new feature, the addition of quantum-number channels. This complication has been treated by PST, and in principle presents no difficulty in showing equivalence.

V. SUMMARY AND DISCUSSION

By considering successively more and more complicated classes of models, we believe we have shown the complete equivalence of MRM's to MPM's and the converse. In Sec. II, the multi-Regge models considered are characterized by some end couplings $D(l)$ and an internal coupling matrix $G(l, l')$; the Mueller-Regge models are characterized by end couplings $\Delta(\lambda)$ and an internal coupling matrix $\Gamma(\lambda, \lambda')$. The proof of equivalence involved showing the existence of a unitary matrix S such that [Eq. (2.12)]

$$\Gamma = S^{+}GS\,,\qquad \Delta = S^{+}D\,. \tag{5.1}$$

The technique of the proof involved the observation that in general $G(l, l')$ has the representation

$$G(l, l') = g(l)^T E g(l')\,, \tag{5.2}$$

so that if S exists we expect

$$\Gamma(\lambda, \lambda') = \gamma(\lambda)^{+} E \gamma(\lambda')\,, \tag{5.3}$$

where

$$\gamma(\lambda) \equiv \int dl\, g(l) S(l, \lambda)\,. \tag{5.4}$$

The end coupling is treated analogously:

$$D(l) = g(l)^T \delta\,,\qquad \Delta(\lambda) = \gamma(\lambda)^{+}\delta\,. \tag{5.5}$$

An intuitive motivation for the existence of a unitary S is the observation that the equation [q.v. Eq. (2.23)]

$$S(l, \lambda) = \delta(l - \lambda) + \frac{g(l)^T E \gamma(\lambda)}{\lambda - l + i\epsilon} \tag{5.6}$$

has an analog in nonrelativistic quantum mechanics. If we consider $g(l)$ to be the noninteracting wave function, $\gamma(\lambda)$ to be the outgoing interacting wave function, E the potential, and l the analog energy, then (5.6) is equivalent to a Lippmann-Schwinger equation for $\gamma(\lambda)$ (Ref. 18):

$$\gamma(\lambda) = g(\lambda) + \frac{1}{\lambda - K + i\epsilon} E\gamma(\lambda)\,, \tag{5.7}$$

where K is the (diagonal) free Hamiltonian,

$$Kg(l) = lg(l)$$

and

$$(K + E)\gamma(\lambda) = \lambda\gamma(\lambda)\,.$$

We recover (5.6) by multiplying (5.7) by $g(l)^T$, and use $g(l)^T g(\lambda) = \delta(l - \lambda)$, and $g(l)^T\gamma(\lambda) = S(l, \lambda)$. If the noninteracting wave functions $g(l)$ provide a complete set of states, then physically we expect the $\gamma(\lambda)$, supplemented by the bound- (and partially bound-) state wave functions (γ_i) to form a complete set of interacting states. When this is true, S will exist, have an inverse, and be unitary.

In Secs. III and IV we generalized the proof of equivalence to include a much larger class of models than multi-Regge. One new point was that because of s-channel thresholds for multiperipheral phase space, we expect an exponential decrease of the partial-wave projections as $J \to \infty$. Such models can be treated by using a complex contour from $c - i\infty$ to $c + i\infty$ to represent the J-plane propagator. The arguments from Sec. II then apply after allowance is made for the 90° rotation of the contour. An additional point was also made that ABFST models could be treated by the techniques described here, generalized to include quantum-number channels (q.v. PST). There is no difficulty proving equivalence for such generalizations, nor indeed for any model whose rapidity structure involves no more than a fixed number of nearest-neighbor interactions.

It is possible to make the proof of equivalence even more general than has been done here. We have discussed models after integrating over transverse momentum (or Toller angles). We conjecture that a generalization is possible for the differential cross sections. Perhaps the functional techniques of Brown[19] would be helpful in such a computation.

In conclusion, it is our belief that every multiperipheral (exclusive or inclusive) model can be treated by one of the methods discussed here, or by their generalization. The essential physics of such models is their (partial-wave projected) structure; this structure is inherited by the inclusive cross sections if it is assumed for the exclusive cross sections, and conversely. It is in this sense that we prove equivalence between MPM and MRM.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge several useful conversations with R. Mills, D. Snider, J. Dash, and B. Webber during the course of this work.

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*Work performed under the auspices of the U. S. Atomic Energy Commission.

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PHYSICAL REVIEW D VOLUME 10, NUMBER 7 1 OCTOBER 1974

Proof of the Reading-Bassichis conjecture in high-energy potential scattering at backward angles*

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(Received 13 June 1974)

A proof is given for the Reading-Bassichis conjecture in the Nth-order Born approximation for high-energy potential scattering at backward angles. The conjecture states that under the stationary-phase approximation the hard scattering acts in such a way that "the radius vector bisects the angle through which the particle is scattered."

Reading and Bassichis[1] reemphasized[2] the importance of stationary-phase contribution to the backward scattering in the Born approximation. By an Nth-order scattering process is meant a particle undergoing N-step "hard" scattering while all the "soft" (essentially forward) scatterings to all orders are summed to an eikonal phase.

Reading and Bassichis have made a very interesting observation. Under the stationary-phase approximation, the hard scattering acts in such a way that "the radius vector bisects the angle through which the particle is scattered."[3] That is, the radius vector $\vec{r}_j$ (connecting from the origin to the jth vertex) bisects the angle between the two vectors $(\vec{r}_j-\vec{r}_{j-1})$ and $(\vec{r}_j-\vec{r}_{j+1})$. An equivalent statement is that all the perpendiculars drawn from the origin to $\vec{r}_j-\vec{r}_{j+1}$ and to (momentum) $\vec{k}$ are equal in distance. This suggests that the particle when turned by the potential somehow manages to maintain its impact parameter. Reading and Bassichis have verified the above statement in quotation marks for $N=2$ and 3, and conjectured that this holds in general.

The purpose of this note is to give a proof of the RB conjecture for arbitrary N. Since the essential ingredient in the RB conjecture pertains to the N-step *hard* scattering, the soft (eikonal) scattering part, which does not really affect the outcome of the conjecture, will be suppressed. Thus we shall simply work in the ordinary Nth-order Born approximation. For spherically symmetric potentials (which are the cases treated by RB), we have the scattering amplitude

$$f_{BN}(\pi)=\text{const}\times\int\cdots\int d\vec{r}_1\cdots d\vec{r}_N\prod_{j=1}^{N}U(r_j)\,e^{i\vec{k}\cdot(\vec{r}_1+\vec{r}_N)}\prod_{l=2}^{N}\frac{e^{ik|\vec{r}_{l-1}-\vec{r}_l|}}{|\vec{r}_{l-1}-\vec{r}_l|}\,. \tag{1}$$