New theory of nuclear forces. Relativistic origin of the repulsive core

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A relativistic, three-dimensional wave equation which restricts one of the two interacting particles to its mass shell is applied to the study of nuclear forces. In the nonrelativistic limit, the equation reduces to a Schrödinger equation with effective potentials composed of two parts. One part, with longer range, is similar to nonrelativistic potentials obtained from other theories, while a second short-range part, which arises solely from the relativistic nature of this equation, is repulsive and dominates over the longer-range part at short distances. The resulting soft-core potentials are evaluated numerically for a simple one-particle-exchange model limited to the exchange of π, ρ, ω, and a neutral spinless meson. By adjusting four of the parameters we obtain good fits to the Reid soft-core potentials, especially in the S states. The couplings obtained are very reasonable, and the results are compared with other recent models. The general features of the theory and the quantitative details of the model are thoroughly discussed.

I. INTRODUCTION AND SUMMARY

In this paper a relativistic wave equation introduced previously is applied to the study of nuclear forces. The equation can be used eventually to calculate relativistic nucleon-nucleon scattering amplitudes and relativistic deuteron wave functions, but this paper is limited to an examination of the behavior of this equation in the nonrelativistic limit. Our prescription for obtaining the nonrelativistic limit is to expand the momentum-space kernels of the wave equation in powers of p/M and retain the lowest-order terms, a procedure which is of doubtful validity at short range. We find that the effective potentials we obtain are composed of two parts. One part, with longer range, is similar to nonrelativistic potentials obtained from other theories while a second short-range part is repulsive and dominates over the longer-range part at short distances. The precise *size* of this second repulsive part of the potential depends on the details of the dynamics, but the *existence* of such a repulsion is independent of the dynamics which govern the intermediate and long ranges, and can be traced to the relativistic structure of our particular equation when applied to two spin-$\frac{1}{2}$ particles. This new theory, therefore, suggests that the short-range repulsion in nuclear forces can be viewed primarily as a relativistic phenomenon, and not as due to the exchange of massive vector mesons as required by most theories.

As a quantitative test of these general ideas we examine a simple model in which the nuclear force is represented by the exchange of four mesons, the π, ρ, ω, and a fictitious α. The α is an isoscalar-spin scalar meson intended to represent phenomenologically the major contributions of the two-pion-exchange potential and the massive ϵ meson. By adjusting some of the couplings and taking the nonrelativistic limit we achieve a satisfactory fit to the phenomenological soft-core potentials obtained by Reid.[1]

This paper does not discuss how this theory can be extended to the relativistic treatment of the many-body problem, but such an extension is currently under investigation and appears to be complicated but straightforward. Nonrelativistically, the connection with many-body theory can be made in the usual way through the nonrelativistic potentials.

In this section we present our major results and give a detailed discussion of the theory. The mathematical details and most of the specific formulas are presented in the remaining sections of the paper.

A. The relativistic wave equation

The wave equation we use together with physical motivations for introducing such an equation were presented in a previous paper.[2] Briefly, we employ a covariant integral equation in which the full two-body Green's function is replaced by the Green's function for the propagation of one *free* particle (on the mass shell) and one virtual particle (off the mass shell). The condition that one of the particles be restricted to its positive-energy mass shell eliminates the integration over the relative energy, leaving only the three-dimensional integration over the relative three-momentum. Our equation is therefore a covariant three-

dimensional equation and as such bears a resemblance to other quasipotential equations introduced by Blankenbecler and Sugar[3] and Logunov and Tavkhelidze[4] (BSLT) and also by Todorov.[5] We will make a few comments on these other approaches after we have discussed our equations.

In order to study the dynamics of our equation without solving for the phase shifts, we took the nonrelativistic limit (i.e., the adiabatic limit where all momenta are assumed small compared to the nucleon mass) and obtained a Schrödinger equation with an effective potential which could be compared with Reid's phenomenological potentials. This limiting process is very well defined, but its accuracy is certainly in doubt, particularly at short distances. Hence, a more stringent test of the ideas presented here must await a numerical solution of the fully relativistic equations, which we are encouraged to undertake by the success of the nonrelativistic limit.[6]

The relativistic wave equation and one-particle-exchange potentials for the nucleon-nucleon system are written in detail in Sec. II.[7] In Sec. III we take the nonrelativistic limit. The equations reduce to a coupled set of Schrödinger-type wave equations for two companion wave function ψ^+ and ψ^-, which in position space become

$$-\left(\frac{\nabla^2}{M}+\epsilon\right)\psi^+(x)=-V^{++}(x)\psi^+(x)-V^{+-}(x)\psi^-(x), \tag{1.1a}$$

$$-2M\psi^-(x)=-V^{-+}(x)\psi^+(x)-V^{--}(x)\psi^-(x). \tag{1.1b}$$

We have suppressed two-component spin indices; the potentials above are actually matrices in two-component spin space. The existence of the ψ^- wave function is due to the fact that one of the particles (particle 2 in this paper) is off-shell, and hence propagates as a mixture of a physical (on-shell) particle and a physical antiparticle (with the opposite momentum). This superposition is expressed quantitatively in the following decomposition of the one-particle Green's function:

$$\frac{M+\not{p}}{M^2-p^2-i\epsilon}\equiv\frac{M}{E_{\vec{p}}}\left[\frac{u(\vec{p})\bar{u}(\vec{p})}{E_{\vec{p}}-p_0-i\epsilon}-\frac{v(-\vec{p})\bar{v}(-\vec{p})}{E_{\vec{p}}+p_0-i\epsilon}\right], \tag{1.2}$$

where $E_{\vec{p}}=(M^2+\vec{p}^2)^{1/2}$ and u and v are ordinary Dirac spinors.[8] The identity (1.2) shows that as the energy of the off-shell particle approaches $+E_{\vec{p}}$ it propagates almost fully as a particle, while if p_0 approaches $-E_{\vec{p}}$ the propagation is almost fully as an antiparticle (with opposite three-momentum as required by conservation of baryon number). The decomposition (1.2) is used in the covariant integral equation to express the covariant wave function in terms of two noncovariant but coupled pieces, ψ^+ coming from the first term of (1.2) and ψ^- from the last term of (1.2).

The relativistic normalization condition provides a heuristic justification for the interpretation of the wave functions ψ^+ and ψ^- as probability amplitudes and the equations (1.1) as coupled Schrödinger equations. When the relativistic potentials are independent of the total energy (which is true for one-particle exchanges) this reduces to

$$1=\int d^3x\left[\,|\psi^+(x)|^2+|\psi^-(x)|^2\right]. \tag{1.3}$$

The potentials on the right-hand side (RHS) of (1.1) can now be viewed as a matrix potential for one large Schrödinger coupled-channel system. Thus we see that V^{++} and V^{--} are Hermitian and the off-diagonal potentials are related by

$$V^{+-}=(V^{-+})^\dagger. \tag{1.4}$$

Note that in the asymptotic region where all potentials are zero, the structure of the left-hand side (LHS) of (1.1) forces ψ^- to be zero also. Hence only the ψ^+ component contributes to the asymptotic wave function, and if we can obtain a Schrödinger equation for ψ^+ alone, then the effective potential which enters that equation is the correct one to compare with phenomenological potentials. Now $V^{--}(x)$ is local in our approximation, and hence ψ^- can be eliminated almost trivially:

$$\psi^-=(2M-V^{--})^{-1}V^{-+}\psi^+, \tag{1.5a}$$

$$-\left(\frac{\nabla^2}{M}+\epsilon\right)\psi^+=-\left[V^{++}+V^{+-}(2M-V^{--})^{-1}V^{-+}\right]\psi^+. \tag{1.5b}$$

The correct effective potential includes not only V^{++}, but a term quadratic in the off-diagonal potentials. In the BSLT method there is no ψ^- wave function and the V^{+-} potentials do not occur, so that the extra term in the right-hand side of (1.5b) is missing. But this extra term has some very exciting properties, which we now discuss.

B. Dynamical origin of the repulsive core

Using (1.4) we see that the effective potential becomes

$$V_{\rm eff}=V^{++}+V^{+-}(2M-V^{--})(V^{+-})^\dagger \tag{1.6a}$$

$$\cong V^{++}+\frac{1}{2M}\,|V^{+-}|^2, \tag{1.6b}$$

where in (1.6b) we have used the fact that $V^{--}\ll 2M$ in the intermediate region. Hence the quadratic terms are repulsive. Furthermore, in the one-particle-exchange cases discussed below, the

matrix potentials are typically of the form

$$V^{++} \to \frac{-g_\alpha{}^2}{4\pi} m_\pi \frac{e^{-\alpha x}}{x}\,,$$
$$V^{+-} \to \frac{-g_\alpha{}^2}{4\pi} \frac{m_\pi{}^2}{2M} \frac{e^{-\alpha x}}{x}\left(\alpha + \frac{1}{x}\right), \tag{1.7}$$

so that the quadratic term is of shorter range than V^{++} and more singular at small distances. This shows how these quadratic terms can dominate at short distances, providing a short-range repulsion which is present even for interactions which are fundamentally attractive. Our fits show that this attractive qualitative explanation of the repulsive core can be made to work quantitatively as well. It is not yet known how these cores will be affected when the calculation is done relativistically and when important two-pion-exchange effects are included.

Perhaps the most interesting aspect of the theory proposed in this paper is the possibility that the repulsive core will be a natural consequence of the Lorentz invariance of the two-nucleon interaction. In this theory, potentials calculated from the exchange of any single particle, no matter what its behavior at intermediate distances, are always repulsive at short distances, and the range of the repulsive-core contributions from a particle of mass m are always $(2m)^{-1}$, half the range of its contribution to the direct potential V^{++}. Hence we conjecture that the situation for the repulsive core may be similar to that at intermediate range—the full potential is a sum of particle exchanges, the lowest-mass particles tending to be more important because of their longer range but the details depending as usual almost as much on the strength of the coupling as on the range.

In our fits the one-pion exchange (OPE) dominates the repulsive core, primarily because of its long range. This is a new role for the pion. To obtain these results we used a πNN coupling of the form

$$g_\pi \tau^\alpha [\gamma^5 \lambda + (\not{q}/2M)\gamma^5(1-\lambda)]\,, \tag{1.8}$$

where $q = p_f - p_i$ so that $\not{q}\gamma^5 = 2M$ between positive-energy spinors. Hence the coupling (1.8) is a linear combination of pseudoscalar and pseudovector couplings with λ adjusting the relative amounts of each coupling in such a way that the coupling between positive-energy spinors is *independent of* λ. We fixed g_π to agree with Reid[1]:

$$\frac{g_\pi{}^2}{4\pi} = 14.0\,, \tag{1.9}$$

so that the long-range part of the OPE calculated from (1.8) is identical to that of Reid. As it turns out, the quadratic part of the OPE depends primarily on the γ^5 part of (1.8), the pseudovector coupling making a very small contribution to the V^{+-} potential. Hence by adjusting λ we can change the amount of repulsion produced by the pion without changing its long-range potential V^{++}. In our final fits we took $\lambda = 0.41$, although there is some flexibility and $\lambda = 0.5$ could have been used also. If one insists on $\lambda = 1$, fits are possible but the cores tend to be too tough, and for $\lambda = 0$ the cores would be much too soft. A more detailed discussion of this point awaits careful calculations of the two-pion-exchange (TPE) contribution and numerical fits to low-energy parameters using the exact equations in momentum space. It appears that the nuclear-force problem may give insight into the off-shell structure of the πNN coupling by determining λ.

Our discussion until now has ignored the potential V^{--}. This potential is also of the Yukawa type, and at a short distance will exceed $2M$. Hence, the effective potential has singularities at short distances due to the $(2M - V^{--})^{-1}$ factor in (1.6). It is in the spirit of our discussion to take these singularities seriously, in which case our potential is technically a *hard*-core potential. (The singularity turns out to be a double pole.) We have examined the singularities, and they present no serious difficulties. However, the question is really not very important because the singularities are inside of the distance $x = 0.3 m_\pi{}^{-1}$, so that they are masked by the soft repulsive cores which reach considerably outside of this distance. As a result we have felt justified in fitting the Reid soft-core potentials. Actually, at such short distances our adiabatic approximation is no longer quantitatively reliable, and the detailed behavior of the solutions at short distances must await a numerical solution of the momentum-space equations. Here the singularities should offer no serious difficulty because the individual matrix potentials are regular.

Before we turn to a detailed discussion of the fits to the Reid potentials we present a brief comparison of our treatment with other current work on the nuclear force.

C. Current status of theories of the nuclear force

Until now, the BSLT equation has received almost exclusive attention in modern relativistic analyses of the nuclear force problem. It has been used by Partovi and Lomon[9] and by Chemtob, Durso, and Riska,[10] both of which groups calculate the important TPE contributions. It has also been used by Gersten, Thompson, and Green[11] in a recent one-boson-exchange fit to the phase shifts, and a related equation has been used by

Schierholz[12] for the same purpose. An equation like ours has been discussed extensively by Fronsdal and collaborators.[13]

A principal argument given in favor of the BSLT approach is that it is covariant, depends on a relative three-momentum only, and satisfies two-body unitarity. However, these advantages are common to all equations of the quasipotential type, and are also enjoyed by the equation discussed in this paper and the one proposed by Todorov[5] which we referred to earlier. In fact, an infinite number of equations can be easily constructed which enjoy these properties,[3,14,15] and it is not yet clear which of these equations will ultimately give the best results for nucleon-nucleon scattering.

This question is important because different quasipotential equations do not give the same result in any practical calculation. The reason is that the kernel or potential is inevitably approximated by the exchange of a finite number of particles, and the solution of each equation therefore corresponds to a different approximation of the full sum of all ladder and crossed ladder diagrams on which the dynamics is based. Only in the event that the kernels were summed to all orders could we expect different quasipotential equations to give the same result.

This is already clear in the one-particle-exchange approximation. In this approximation our equation includes off-shell effects not included in the BSLT equation. Specifically, the V^{+-} contributions are not present in BSLT, and as a result they contain no repulsive core. We conjecture that if the BSLT kernel included terms involving the exchange of many pions, that the combined effect of these terms would eventually create the repulsive term which we obtain in the lowest approximation.

In a similar fashion, the very important TPE contribution must be recalculated for use in our equation. Not only do we need to know the contributions to the V^{+-} and V^{--} potentials, but the V^{++} potential is also different because of the different form of the iteration of OPE, which must be subtracted from the nucleon box in calculating the TPE.

In addition to the possible dynamical advantages already discussed, there are other virtues of the approach developed in this paper. In the case of the hydrogen atom, for example, putting the proton on shell leads directly to the Grotch-Yennie equation,[16] which is a Dirac equation with an effective potential which does an optimum job with the e-p system. Another advantage of the approach described here is that it yields directly the vertex function with one particle off shell. When the two-nucleon system is in its bound state, the deuteron, this dnp vertex function, first discussed by Blankenbecler and Cook,[17] is precisely the one needed to discuss in detail the nucleon pole contributions to backward p-d scattering and electro- and photodisintegration of the deuteron.

One could object to the use of the equation developed here on the grounds that it places only one nucleon on shell, and hence treats the two identical particles unsymmetrically. However, as one can see in what follows, this lack of symmetry introduces no inconsistencies, and may be seen as a means of counting the most important effects which come from the fact that the nucleons are off shell. And Fronsdal[18] has argued that only in the unsymmetrical situation where one particle is taken as free can one construct a relativistic classical theory of two interacting particles.

We turn now to a brief discussion of the problem of the TPE potential. As previously indicated, we have introduced the α meson to represent phenomenologically the combined contribution of the TPE and the more massive ϵ meson (950 MeV). We do not wish to suggest that such a simple parametrization does justice to this important contribution, but rather we wished to investigate other aspects of the nuclear force in this paper, and this is impossible without rounding out the dynamics by including some isoscalar attraction which is known to come from the TPE. A calculation of the TPE potential within the framework of this theory is presently in progress.

Two results of this preliminary calculation will be mentioned here. First, we have found that in the static limit our TPE contribution reduces largely to the exchange of a spin-scalar isoscalar meson of distributed mass. We do not believe that the static limit is very accurate quantitatively, but this preliminary result helps provide justification for the replacement of the TPE by the α, even though calculations within the framework of the BSLT theory suggest that the TPE is more complicated.[9,10] Our second remark is that a correct calculation of the TPE contribution must include the role of the $\Delta(1236)$ nucleon resonance, which in a simple pole model gives a large contribution to the A^+ amplitude in π-N scattering and thereby helps to satisfy the Adler self-consistency condition.[19,20] The Δ contribution also cancels most of the contribution to the symmetric π-N scattering length, $a^{(+)}$, which comes from the nucleon poles in the γ^5 theory,[21] so that it has the effect of decreasing the contribution to the TPE over what one would get from the nucleon box and crossed box. Because of these cancellations we believe that a rather precise model of π-N scattering must be developed before the TPE

can be reliably calculated. Another difficulty which we face is that we need to know something about π-N scattering with *off-shell nucleons* before we can evaluate our V^{+-} potentials.

With these preliminary remarks concluded, we turn to our fits to the Reid potentials.

D. Fits to the Reid potentials

In the adiabatic approximation, V^{++} and V^{--} contain central, tensor, and spin-orbit terms, while V^{+-} contain local spin terms (to be described in Sec. III) and velocity-dependent terms involving a single derivative. When the exact quadratic term in (1.6) is calculated, many complicated nonlocal terms are generated. In this paper we neglected some of the smaller of these terms (for details see Sec. III), but our final effective potential still contained significant velocity-dependent terms. These terms were eliminated finally by the effective-mass transformation,[22] giving us an effective potential depending on energy-dependent local central, tensor, and spin-orbit ($\vec{L}\cdot\vec{S}$) terms and a new $\vec{L}\cdot(\vec{\sigma}_1-\vec{\sigma}_2)$ term. This interesting new term is discussed in some detail in Sec. IV, but has not been included in the fits presented in this section. Also, the energy dependence of the potentials (which results from the effective mass transformation) was not investigated, the total energy of the two-nucleon system being fixed at its threshold value of $2M$. Thus, although the potentials presented here are local potentials limited to central, tensor, and $\vec{L}\cdot\vec{S}$ components only, and as such cannot fit all the phase shifts (because of the $^1S_0$, $^1D_2$ splitting for example), this is not a limitation of our approach, and a more exact treatment will include additional nonlocalities. The reader interested in these details should read Sec. IV.

In Figs. 1(a)–1(e) and Figs. 2(a)–2(d) we com-

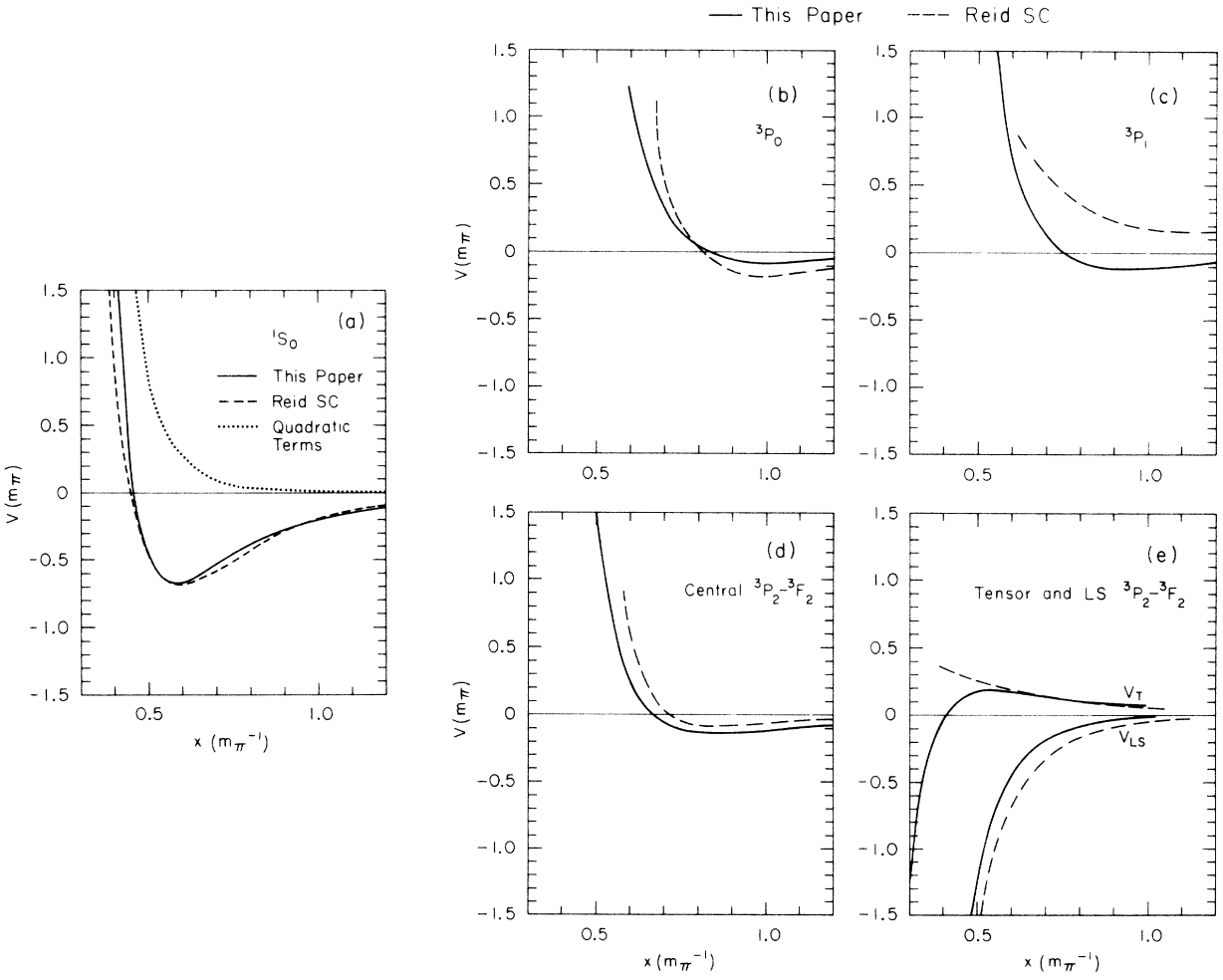

FIG. 1. Isospin triplet states. The solid curves are theoretical potentials presented in this paper. The dashed curves are the soft-core potentials of Reid (Ref. 1).

pare our theoretical potentials with Reid's[1] soft-core potentials. In Figs. 3(a)–3(d) and 4(a)–4(d) we display the spin-independent central, spin-spin, tensor, and spin-orbit potentials separately, and show the contributions to each potential from separate particle exchanges. The Hamada-Johnston[23] potentials are also shown in Figs. 3 and 4 for comparison, but since they are hard-core potentials the comparison is not too relevant.

We will discuss the fits to the Reid potentials first. In Fig. 1 we have displayed all of the isospin-triplet potentials fitted by Reid except the interesting $^1D_2$ state. Our fit to this state would be similar to the $^1S_0$, but the exact situation in this case is quite complex, and discussion of this case is reserved for Sec. IV. The same complications occur for the $^3P_1$ state, so we are not inclined to take our rather poor fit to this state [Fig. 1(c)] too seriously. The fits to the other states are all quite good.

The situation for the isospin singlet states is shown in Fig. 2. Here our fit to the deuteron channel is compared with Reid's alternate soft-core (SCA) potential as well as his soft-core potential. Note that the three curves of Figs. 2(a) and 2(b) almost interpolate between these two cases in a consistent way. The fits shown in Fig. 2 cover all of the isosinglet potentials determined by Reid, although the complex situation discussed in Sec. IV also applies to the $^1P_1$ case [Fig. 2(c)] and the $^3D_2$ case [Fig. 2(d)], so the poorer quality of these fits must again be taken less seriously. For the $^3D_2$ case we also compared our fit with the Reid hard-core (HC) potential.

These fits were not determined by a systematic search in the parameter space. To obtain the final curves presented here, we simply varied four parameters (see Table I) over a lattice and took the curves which gave the best fits to the important ^1S and ^3S- ^3D states, with a little atten-

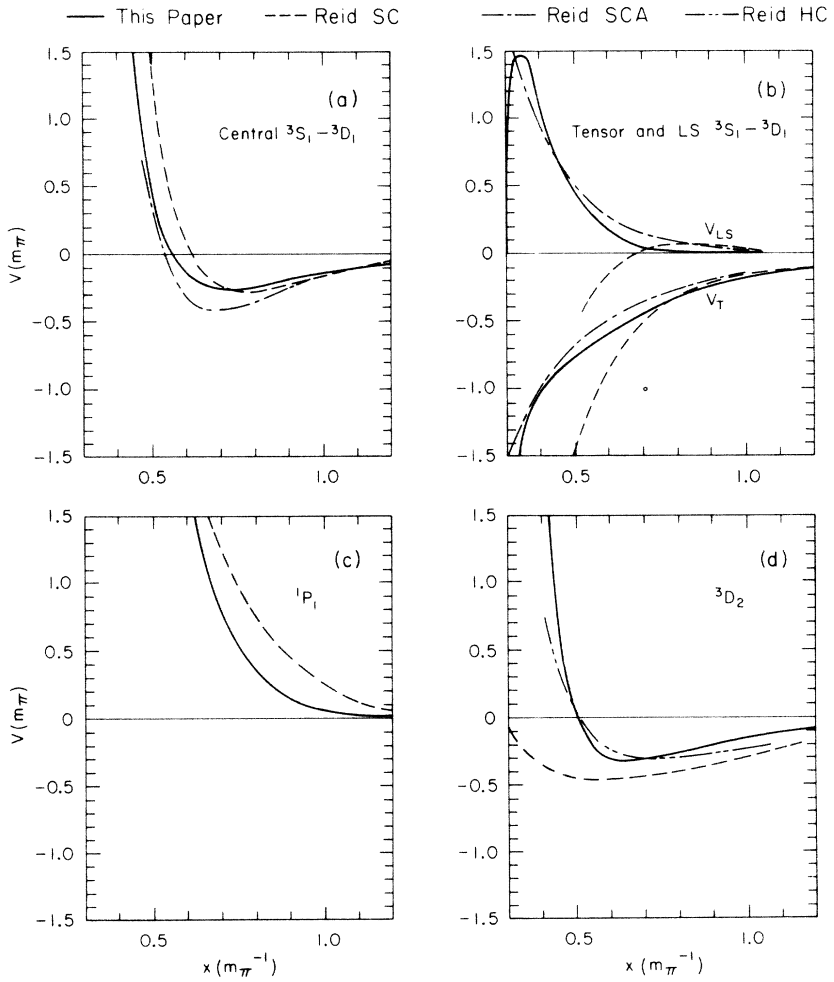

FIG. 2. Isospin singlet states. We also show the Reid alternate soft-core potential (SCA) in (a) and (b) and the hard-core potential (HC) in (d).

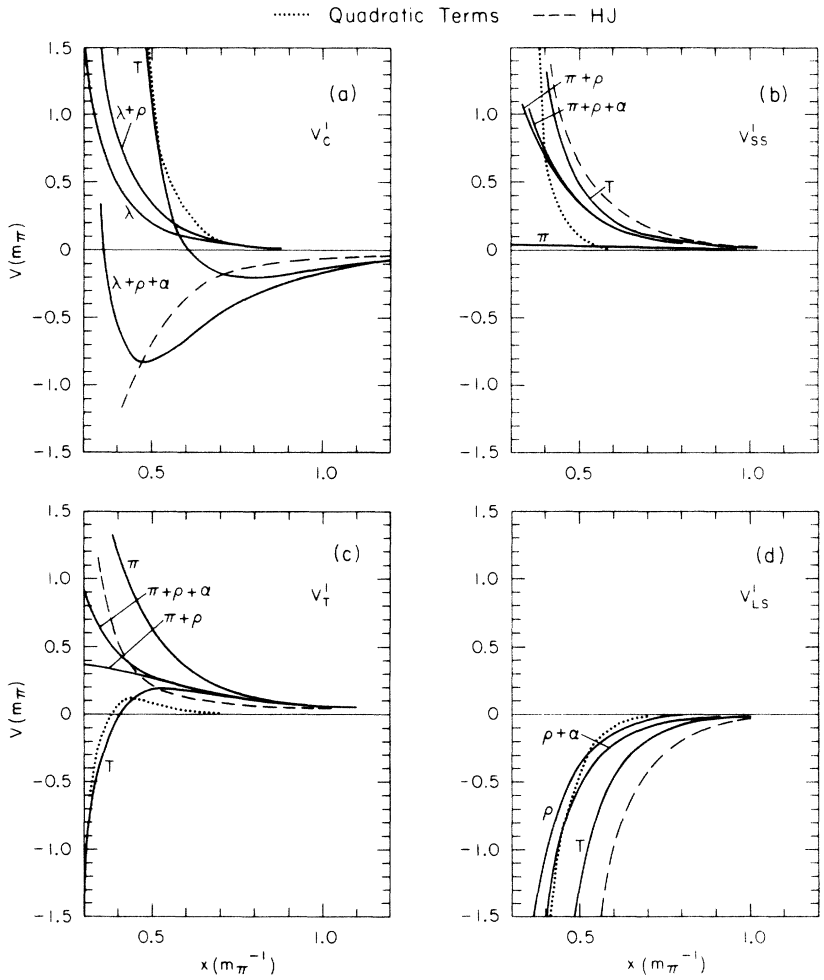

FIG. 3. Isospin triplet potentials defined in Eq. (3.42). The solid lines are the various partial contributions discussed in the text. The lines labeled T are the complete result. The quadratic potentials are defined in Eq. (3.42) and (3.43). The dashed lines are the Hamada-Johnston hard-core potentials given in Ref. 23.

tion paid to the fact that the P waves should be very repulsive. Hence our fits to the P and D waves are presented more to show that these channels are satisfactory, and that the good agreement in the ^1S and ^3S-^3D channels is not accidental.

The values of our parameters are presented in Table I, together with values used in three other recent one-particle-exchange models.[11,12,24] Note that we need fewer particles, a fact which we feel is due to the helpful role of our quadratic potentials. Furthermore, our ρ and ω coupling constants are quite consistent with experimental values,[25,26] and we feel that they are more realistic than the values obtained in Refs. 11 and 24. The ratio $R=(g_\omega/g_\rho)^2=9.0$ is in agreement with the SU(3) nonet scheme,[25] and although this was varied in some earlier fits, it was later fixed at this value. The coupling of the η is not known but a reasonable upper limit is $g_\eta^2/4\pi\cong 1.3$ for a F/D ratio of 0.6. If the F/D ratio is 0.75, then the coupling is zero. In any case, the η coupling used in Refs. 11 and 12 may be too large, and if the value 1.3 or less is used, the η makes little contribution to the potentials.[10] Furthermore, we require only one scalar meson, and no δ meson. Of course we have not yet fitted phase shifts, as have the other authors referred to in Table I, and an attempt to fit phase shifts with high accuracy might very well require more parameters.

We now turn to the dynamical role of the different particle contributions. These are indicated in Figs. 3 and 4, where an exploded view of the isospin-one and -zero potentials are shown. These potentials are defined by

$$V_{\rm eff}=V_C^I+V_{SS}^I\vec{\sigma}_1\cdot\vec{\sigma}_2+V_T^I S_{12}+V_{LS}^I\vec{L}\cdot\vec{S}, \quad (1.10)$$

where $I=0$ or 1 is the isospin of the two-nucleon

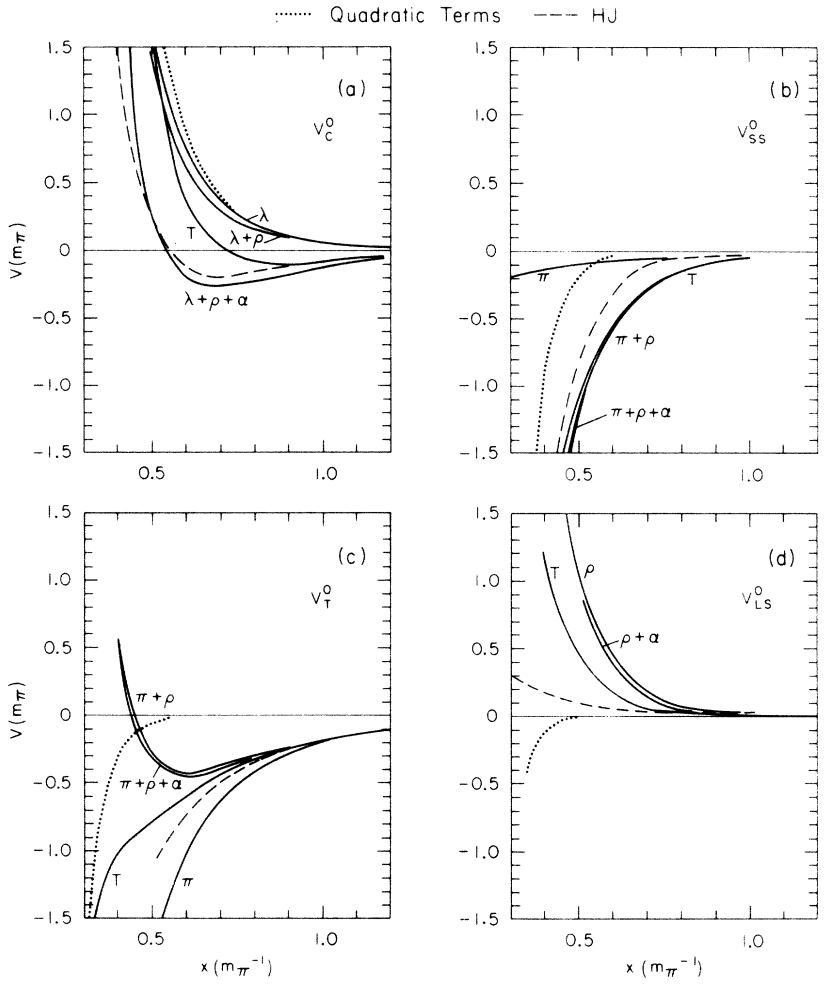

FIG. 4. Isospin singlet potentials. See the caption to Fig. 3.

system, and S_{12} is the tensor operator (see Sec. III).

The solid curves shown in these figures give contributions from different partial combinations of particles. First, we show the π contributions alone, then the $\pi+\rho$ contributions, then the $\pi+\rho+\alpha$ contributions, and finally the total curves which include $\pi+\rho+\alpha+\omega$ and are labeled T.

It turns out that in some cases (V_C) the OPE contribution comes entirely from the quadratic term, while in other cases (V_{SS} and V_T) the quadratic OPE term makes no contribution and the entire OPE contribution is from the long-range part. In still other cases (V_{LS}) neither the quadratic nor the long-range part of the OPE contributes. In the first case, the OPE contribution is labeled by λ (instead of π) to remind the reader that it comes entirely from the quadratic OPE (and is therefore proportional to λ^2—see Sec. III). In the latter case no pion curve is shown at all.

Because the quadratic terms are nonlinear functions of the potentials, there are interference terms and the contribution of the pion and the ρ is not the sum of a pion contribution and a ρ contribution alone. Hence the ρ contribution cannot be determined by subtracting the π curve from the $\rho+\pi$ curve. At intermediate distances, where the quadratic terms are negligible, such a subtraction is valid, but at short range it may be very misleading.

We discuss the role of the four particle exchanges individually.

π: As in any theory, the pion contributes the asymptotic long-range potential and the major share of the tensor potential. A new feature of this theory is that the pion also contributes much repulsion to the spin-independent central potential through its quadratic term (labeled λ). This repulsion gives a major contribution to the soft core, explaining practically all of the repulsion

TABLE I. One-particle-exchange parameters used in this paper are compared with those used in Refs. 11, 12, and 24. The quantities are defined in Sec. II. Those parameters labeled with an asterisk were varied in the final fits to the potentials. The other parameters and $R = (g_\omega/g_\rho)^2 = 9.0$ were fixed. The left column gives the quantum numbers of the exchanged particle using the notation $I(J^P)$, where P is the parity. Masses are in units of the pion mass. The couplings given for Refs. 11 and 24 are adjusted to go with form factors normalized to unity at the boson mass.

		This paper	Ueda and Green, Ref. 24	Gersten, Thompson, and Green, Ref. 11	Schierholz, Ref. 12
	$g_\pi{}^2/4\pi$	14.00	14.95	14.47	14.4
$1(0^-)$	λ*	0.41	•••	•••	•••
$0(0^+)$	$g_\alpha{}^2/4\pi$*	2.41	1.89	•••	1.4
	m_α*	2.6	3.0	•••	2.88
	$g_\epsilon{}^2/4\pi$	•••	18.05	9.92	6.8
	m_ϵ	•••	5.64	4.1	5.03
$0(0^-)$	$g_\eta{}^2/4\pi$	•••	•••	4.28	8.05
	m_η	•••	•••	3.96	3.96
$1(1^-)$	$g_\rho{}^2/4\pi$*	1.0	1.51	0.86	0.605
	K_ρ	3.70	5.06	6.38	4.78
	m_ρ	5.64	5.5	5.5	5.11
$0(1^-)$	$g_\omega{}^2/4\pi$	9.0	23.87	20.63	9.05
	K_ω	−0.12	0.0	0.0	−0.1
	m_ω	5.64	5.64	5.64	5.62
$1(0^+)$	$g_\delta{}^2/4\pi$	•••	2.35	1.14	•••
	m_δ	•••	5.5	6.9	•••
Cutoffs		no	yes	yes	yes

in the isosinglet case [Fig. 4(a)] and much of the repulsion in the isotriplet case [Fig. 3(a)].

The quadratic OPE potential also explains another very important feature of the spin-independent central potentials: The isotriplet V_C is less repulsive (or more attractive) than the isosinglet V_C. Of the particles listed in Table I only the ρ and δ will give a splitting between the two V_C potentials, and the ρ splitting has the wrong sign and the δ probably does not exist. The quadratic OPE potential introduces a splitting in the right direction, and by making $\lambda < 1$ this splitting can be reduced to the point where, when combined with the ρ, it is just the correct size.

But the OPE contributions could never give a satisfactory theory by themselves, for they fail in just about as many ways as they succeed. In particular, the OPE gives negligible contributions to the spin-spin part of the central potential, V_{SS}, and these terms must be large if there is to be attraction in the S states together with strong repulsion in the P states. Hence the OPE provides none of the attraction in the central potential necessary to bind the deuteron and explain the strong threshold ${}^1S_0$ scattering. For these we need other contributions.

ρ: The ρ makes very important contributions to the V_{SS} terms, and in this way helps provide some needed attraction in both the ${}^3S$- ${}^3D$ and ${}^1S$ states and repulsion in the P states. In order to fit all these states simultaneously a large term of the form $(\vec{\tau}_1\cdot\vec{\tau}_2)(\vec{\sigma}_1\cdot\vec{\sigma}_2)$ is needed. The pion contributes just such a term, but it is much too small. The ρ contributes the term

$$(\vec{\tau}_1\cdot\vec{\tau}_2)(\vec{\sigma}_1\cdot\vec{\sigma}_2)\,\frac{m_\rho{}^2}{6M^2}(1+K_\rho)^2\,\frac{g_\rho{}^2}{4\pi}\,\frac{e^{-m_\rho x}}{x}\,. \quad (1.11)$$

This is important only because of the large value of K_ρ, which is enough to overcompensate for the suppression introduced by the coefficient $m_\rho{}^2/6M^2$.

The ρ also makes a major contribution to the $\vec{L}\cdot\vec{S}$ potential. Its contribution to the tensor potential is unfortunately not very helpful, and this deficiency is only rectified after other contributions have been considered.

When the ρ and π have been included, the picture is reasonably satisfactory in all cases but the spin-independent central potentials, V_C, which are much too repulsive. To rectify this situation we need some central attraction.

α: The central attraction is provided by the α meson. As we have repeatedly emphasized, the

α is meant to be a phenomenological representation of the TPE. Note how it makes a decisive impact on the V_C potentials, but that its contributions to the other potentials are very minor. The fact that it contributes to the tensor and spin-spin potentials at all is due to existence of the quadratic potentials.

ω: Finally, the ω meson is added because it is well known to exist, and it would be inconsistent to include the ρ and omit the ω. As one can see, the ω does help the fits in a number of ways. It makes some helpful contributions to the V_{SS} and V_{LS} potentials, and its contribution to the tensor potential is quite decisive in giving a good V_T^0. However, its repulsive contributions to V_C are not needed in this theory, and they serve primarily to force us to increase g_α^2 to compensate for the repulsion. The reason why the ω is a good deal less important than the ρ is due primarily to its small anomalous moment, K_ω. This means that even though its coupling is nine times as strong as the ρ (which is important in V_C) the factor $g_\omega^2(1+K_\omega)^2$ is only about $\frac{1}{3}$ of the corresponding factor $g_\rho^2(1+K_\rho)^2$.

We conclude this section with the following comments:

(i) The fits would be improved by a longer-range ρ type of contribution from the TPE. This would give us larger V_{SS} potentials, which would increase the P-wave repulsion and at the same time increase the S-wave attraction. With such terms a smaller ρ and ω coupling would be acceptable and the range of the α could be made shorter and its coupling smaller. We do expect such contributions to be present in the TPE.

(ii) Examination of all of the partial contributions to V_C [Figs. 3(a) and 4(a)] show that every curve is repulsive at short distances. This is an example of the model independence of our repulsive core, which we emphasized previously in this section. The size and shape of the core does indeed depend on the dynamics, but its existence does not. In all of the cases we looked at while we were searching for a good fit, cores were present.

(iii) Although the over-all effect of the quadratic terms is repulsive, these terms do not make repulsive contributions to every potential. The most striking example of this is the tensor potential, where the quadratic terms are attractive and make important helpful contributions to V_T^0 [Fig. 4(c)] at short distances.

(iv) As we discussed earlier, the potentials presented here do have singularities at short distances. The singularity in the isotriplet potential is at $x_C \cong 0.26 m_\pi^{-1}$, while in the isosinglet potentials it is at $x_C \cong 0.23 m_\pi^{-1}$. These singularities are well inside the distances usually taken for hard cores, and in any case the adiabatic approximation breaks down at this distance, so that the short-range behavior of the potentials and wave functions must be determined by solving the relativistic equations numerically.

The remainder of the paper includes a detailed discussion of the relativistic equations and potentials (Sec. II), the transition to the nonrelativistic limit and the reduction of the effective potential (Sec. III), and a discussion of some of the more usual nonlocalities contained in this theory (Sec. IV).

II. THE RELATIVISTIC THEORY

In this section we develop the explicit form of the relativistic wave equations and potentials discussed qualitatively in the previous section. Our attention is focused on the bound-state equation, but the scattering equation differs only by the addition of the inhomogeneous term describing two free particles.

A. The wave equations

In momentum space our quasipotential equation is[2]

$$(\tilde{\Gamma}C)_{\mu\nu}(\hat{p}) = -\int \frac{d^3k}{(2\pi)^3}\, \upsilon_{\mu\mu',\nu\nu'}(\hat{p},\hat{k},W) \times G_{\mu'\mu'',\nu'\nu''}(\hat{k},W)(\tilde{\Gamma}C)_{\mu''\nu''}(\hat{k}), \tag{2.1}$$

where μ and ν are spinor indices, $P=(W,\vec{0})$ is the total energy-momentum 4-vector, p and k are relative 4-momenta ($\hat{p}$ and $\hat{k}$ are defined below), υ is the interaction kernel with particle 1 on the mass shell, C is the charge conjugation matrix, and $\tilde{\Gamma}_{\mu\nu}$ is the covariant deuteron-nucleon vertex function.[17,27] We shall see how $\tilde{\Gamma}C$ is related to the relativistic wave functions in what follows. The equation (2.1), together with our notation, is illustrated in Fig. 5. In (2.1) summation over repeated indices is implied.

The two-body Green's function, G, is

$$G_{\mu'\mu'',\nu'\nu''}(\hat{k},W) = \frac{[M+\gamma\cdot(\frac{1}{2}P+\hat{k})]_{\mu'\mu''}[M+\gamma\cdot(\frac{1}{2}P-\hat{k})]_{\nu'\nu''}}{2E_k W(2E_k-W)}, \tag{2.2}$$

where, since particle 1 is on the mass shell,

$$\hat{k}=(\hat{k}_0,\vec{\mathrm{k}}),\quad \hat{p}=(\hat{p}_0,\vec{\mathrm{p}}),$$
$$\hat{k}_0=E_k-\tfrac{1}{2}W,\quad \hat{p}_0=E_p-\tfrac{1}{2}W, \tag{2.3}$$
$$E_k=(M^2+\vec{\mathrm{k}}^2)^{1/2}.$$

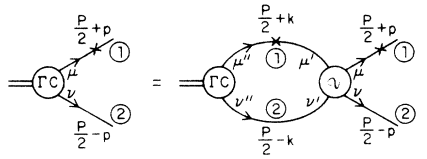

FIG. 5. Diagrammatic representation of Eq. (2.1). The x on line (1) indicates that particle 1 is on its mass shell.

Note that the energy of particle 2 is $W-E_k$, so that particle 2 is also on its mass shell whenever $W=2E_k$.

Our first step is to reduce (2.1) to two coupled integral equations by using the following identity for the projection operator on particle 2 [see also Eq. (1.2)]:

$$[M+\gamma\cdot(\tfrac{1}{2}P-\hat{k})]_{\nu\nu'}\equiv 2M\left(\frac{W}{2E_k}\right)u_\nu^{(s)}(-\vec{k})\bar{u}_{\nu'}^{(s)}(-\vec{k}) - 2M\left(\frac{2E_k-W}{2E_k}\right)v_\nu^{(s)}(\vec{k})\bar{v}_{\nu'}^{(s)}(\vec{k}), \tag{2.4}$$

where $u^{(s)}$ and $v^{(s)}$ are the standard Dirac spinors and a sum over the repeated 2-component spin indices is implied. The identity (2.4) enables us to introduce two wave functions (the over-all normalization of these wave functions is fixed by the normalization condition discussed below)

$$\psi_{rs}^{+}(\vec{p})\equiv\frac{M}{[2W(2\pi)^3]^{1/2}}\frac{\bar{u}_\mu^{(r)}(\vec{p})\bar{u}_\nu^{(s)}(-\vec{p})(\tilde{\Gamma}C)_{\mu\nu}(\hat{p})}{E_p(2E_p-W)}, \tag{2.5}$$

$$\psi_{rs}^{-}(\vec{p})\equiv\frac{-M}{[2W(2\pi)^3]^{1/2}}\frac{\bar{u}_\mu^{(r)}(\vec{p})\bar{v}_\nu^{(-s)}(\vec{p})(\tilde{\Gamma}C)_{\mu\nu}(\hat{p})}{E_p W},$$

and write the two coupled matrix integral equations

$$(2E_p-W)\psi_{rs}^{+}(\vec{p})=-\int\frac{d^3k}{(2\pi)^3}[V_{rr',ss'}^{++}(\vec{p},\vec{k},W)\psi_{r's'}^{+}(\vec{k}) + V_{rr',ss'}^{+-}(\vec{p},\vec{k},W)\psi_{r's'}^{-}(\vec{k})], \tag{2.6}$$

$$-W\psi_{rs}^{-}(\vec{p})=-\int\frac{d^3k}{(2\pi)^3}[V_{rr',ss'}^{-+}(\vec{p},\vec{k},W)\psi_{r's'}^{+}(\vec{k}) + V_{rr',ss'}^{--}(\vec{p},\vec{k},W)\psi_{r's'}^{-}(\vec{k})].$$

Hence the potentials are defined:

$$V_{12}^{++}(\vec{p},\vec{k},W)=\left(\frac{M^2}{E_pE_k}\right)\bar{u}_\mu^{(r)}(\vec{p})\bar{u}_\nu^{(s)}(-\vec{p}) \times\mathfrak{V}_{\mu\mu',\nu\nu'}(\hat{p},\hat{k},W)u_{\mu'}^{(r')}(\vec{k})u_{\nu'}^{(s')}(-\vec{k}),$$

$$V_{12}^{+-}(\vec{p},\vec{k},W)=\left(\frac{M^2}{E_pE_k}\right)\bar{u}_\mu^{(r)}(\vec{p})\bar{u}_\nu^{(s)}(-\vec{p}) \times\mathfrak{V}_{\mu\mu',\nu\nu'}(\hat{p},\hat{k},W)u_{\mu'}^{(r')}(\vec{k})v_{\nu'}^{(-s')}(\vec{k}), \tag{2.7}$$

$$V_{12}^{-+}(\vec{p},\vec{k},W)=\left(\frac{M^2}{E_pE_k}\right)\bar{u}_\mu^{(r)}(\vec{p})\bar{v}_\nu^{(-s)}(\vec{p}) \times\mathfrak{V}_{\mu\mu',\nu\nu'}(\hat{p},\hat{k},W)u_{\mu'}^{(r')}(\vec{k})u_{\nu'}^{(s')}(-\vec{k}),$$

$$V_{12}^{--}(\vec{p},\vec{k},W)=\left(\frac{M^2}{E_pE_k}\right)\bar{u}_\mu^{(r)}(\vec{p})\bar{v}_\nu^{(-s)}(\vec{p}) \times\mathfrak{V}_{\mu\mu',\nu\nu'}(\hat{p},\hat{k},W)u_{\mu'}^{(r')}(\vec{k})v_{\nu'}^{(-s')}(\vec{k}).$$

On the LHS of (2.7) we have used the indices 1 and 2 as a shorthand notation for the 2-component indices $\{r,r'\}$ and $\{s,s'\}$, respectively. This will also be employed in Sec. III.

So far all of our expressions are exact insofar as we have made no nonrelativistic approximation. The individual matrix potentials V and the wave functions ψ^+ and ψ^- are not themselves covariant, but the entire system (2.6) is.

B. The normalization condition

The next task of this section is to write down the relativistic normalization conditions for $\tilde{\Gamma}C$. Correcting the errors in Ref. 2, we have

$$1=\int\frac{d^3p}{(2\pi)^3}(\tilde{\Gamma}C)_{\mu\nu}^{+}(\hat{p})\frac{\partial}{\partial W^2}[G_{\mu\mu',\nu\nu'}(\hat{p},W)] \times(\tilde{\Gamma}C)_{\mu'\nu'}(\hat{p})-R, \tag{2.8}$$

where

$$R=\int\int\frac{d^3pd^3p'}{(2\pi)^6}(\tilde{\Gamma}C)_{\mu\nu}^{+}(\hat{p})G_{\mu\alpha,\nu\beta}(\hat{p},W)\frac{\partial}{\partial W^2} \times[\mathfrak{V}_{\alpha\alpha',\beta\beta'}(\hat{p},\hat{p}',W)] \times G_{\alpha'\mu',\beta'\nu'}(\hat{p}',W)(\tilde{\Gamma}C)_{\mu'\nu'}(\hat{p}'). \tag{2.9}$$

In what follows, we will assume that $\mathfrak{V}$ is independent of W, and therefore $R=0$. Using (2.4) we obtain

$$G_{\mu\mu',\nu\nu'}(\hat{p},W)=\left(\frac{M}{E_p}\right)^2u_\mu(\vec{p})\bar{u}_{\mu'}(\vec{p}) \times\left[\frac{u_\nu^{(s)}(-\vec{p})\bar{u}_{\nu'}^{(s)}(-\vec{p})}{(2E_p-W)}-\frac{v_\nu^{(s)}(\vec{p})\bar{v}_{\nu'}^{(s)}(\vec{p})}{W}\right], \tag{2.10}$$

which is easily differentiated with respect to W^2. The normalization condition reduces to

$$1 = \int d^3p [\,|\psi^+(\vec{p})|^2 + |\psi^-(\vec{p})|^2\,], \tag{2.11}$$

where sums over the two-component spin indices have been suppressed. In one-particle-exchange theories where $\mathcal{V}$ is not a function of W, (2.11) is an exact result. Note that it is a positive-definite condition.

C. The one-particle-exchange potentials

Once the form of the interaction matrix $\mathcal{V}$ is given, the theory is completely specified. In this paper we restrict the dynamics to the exchange of π, α, ρ, and ω mesons.

π: For the OPE potential we take the linear combination of pseudoscalar and pseudovector interactions given in Eq. (1.8). Replacing the Dirac indices of particle 1 by the subscript 1, and similarly for particle 2, gives

$$\mathcal{V}^{\pi}_{12}(\hat{p},\hat{k},W) = \frac{g_\pi{}^2(\vec{\tau}_1\cdot\vec{\tau}_2)\Lambda_1(\hat{q})\Lambda_2(-\hat{q})}{m_\pi{}^2-\hat{t}}\,, \tag{2.12}$$

where $g_\pi{}^2/4\pi = 14.0$ as discussed in the Introduction, $\hat{q} = \hat{p} - \hat{k}$, and from Eq. (1.8)

$$\Lambda_1(\hat{q}) = \gamma_1^5\lambda + (\hat{q}\cdot\gamma_1/2M)\gamma_1^5(1-\lambda)\,, \tag{2.13}$$

where λ is an adjustable parameter which governs the mixture of pseudoscalar to pseudovector coupling. In our final fits $\lambda = 0.41$.

The squared momentum transfer, $q^2 = t$, takes a special form when particle 1 is on shell. We have

$$\begin{aligned}\hat{t} &= (\hat{k}-\hat{p})^2\\ &= (E_k - E_p)^2 - (\vec{k}-\vec{p})^2\\ &= 2M^2 - 2E_kE_p + \vec{k}\cdot\vec{p}\,.\end{aligned} \tag{2.14}$$

In the adiabatic limit where

$$|\vec{p}| \text{ and } |\vec{k}| \ll M\,,$$

then

$$t \simeq -(\vec{k}-\vec{p})^2 \tag{2.15}$$

and we obtain the usual nonrelativistic form for the OPE potential.

Finally, note that the exact form of the OPE potential, Eq. (2.12), is energy-independent, so that the assumption used to obtain the normalization condition (2.11) holds.

α: For the α exchange potential we use the simplest form for the coupling of an isospin-zero, spin-zero meson to nucleons:

$$\mathcal{V}^{\alpha}_{12}(\hat{p},\hat{k}) = \frac{-g_\alpha{}^2 1_1 1_2}{m_\alpha{}^2-\hat{t}}\,, \tag{2.16}$$

where g_α is the αNN coupling constant. Note that this potential is again independent of energy.

As we emphasized in the Introduction, the α used here is not necessarily to be identified with the physical ϵ meson with a mass of about 950 MeV and a width of about 400 MeV. Rather, it is thought of as a simple approximate form for that part of the intermediate attraction which is isoscalar in nature. Hence m_α and g_α are treated as parameters, and the final values are listed in Table I.

ρ and ω: For the vector-meson exchange potentials we use a vector-meson–nucleon coupling of the form

$$g_\rho\tau_1^\alpha\left(\gamma_1^\mu + \frac{iK_\rho}{2M}\sigma_1^{\mu\nu}\hat{q}_\nu\right) = g_\rho\tau_1^\alpha\Sigma_1^\rho(\hat{q})\,, \tag{2.17}$$

where g_ρ is the ρNN coupling and K_ρ is the "anomalous moment" coupling—i.e., it is the F_2/F_1 ratio of the ρ coupling constants. The same form without the τ_1^α is used for the ω coupling.

Using these forms the ρ potential becomes

$$\mathcal{V}^{\rho}_{12}(\hat{p},\hat{k}) = \frac{g_\rho{}^2(\vec{\tau}_1\cdot\vec{\tau}_2)\Sigma_1^\rho(\hat{q})\Sigma_2^\rho(-\hat{q})}{m_\rho{}^2-\hat{t}}\,, \tag{2.18}$$

and the ω potential is the same except for the $\tau_1\cdot\tau_2$ factor, which is missing.

The ρ and ω introduce six parameters. In our final fits, five of these were fixed: The masses we set equal to the ω mass, the K_ρ and K_ω factors set equal to the anomalous moments of the isovector and isoscalar nucleon form factors, respectively, and the ratio $R = g_\omega{}^2/g_\rho{}^2 = 9.0$ as suggested by the nonet scheme. The only coupling we varied was g_ρ, and its value is given in Table I.

The theory is now completely specified. In the next section we treat the difficult problem of taking its nonrelativistic limit.

III. THE NONRELATIVISTIC THEORY

To obtain a simple picture of the behavior of these equations and potentials in the nonrelativistic domain, we go to the adiabatic limit. This is the limit in which the external 3-momentum $\vec{p}$, internal 3-momentum $\vec{k}$, and $\epsilon \equiv W - 2M$ can all be regarded as small compared to M. Of course one can always restrict $\vec{p}$ and W to the nonrelativistic domain, but the assumption that $\vec{k}$ is small compared to M requires that the integrals in (2.6) will be dominated by small values of $\vec{k}$, which in turn will be true only if the range of the force is large compared to M^{-1}. This latter assumption is not really very good, but should suffice at least to give much physical insight into the nuclear force. Ultimately, the results must be checked by integrating Eqs. (2.6) numerically, as discussed in Sec. I.

A. OPE potential

Using Eqs. (2.7) and (2.12) we calculate the OPE potentials to leading order M^{-1}. We obtain

$$(V_\pi)^{++}_{12} = (V_\pi)^{--}_{12} = \frac{-g_\pi^2}{4M^2}(\vec\tau_1\cdot\vec\tau_2)\frac{\vec\sigma_1\cdot\vec q\,\vec\sigma_2\cdot\vec q}{D_\pi(\vec q)}\,,$$
$$(V_\pi)^{+-}_{12} = -(V_\pi)^{-+}_{12} = \frac{-g_\pi^2}{2M}(\vec\tau_1\cdot\vec\tau_2)\frac{\lambda\vec\sigma_1\cdot\vec q}{D_\pi(\vec q)}\,, \tag{3.1}$$

where $\vec q = \vec p - \vec k$ and

$$D_\pi(\vec q) = m_\pi^2 + \vec q^{\,2}\,. \tag{3.2}$$

Note that the potential is local, since it depends on $\vec q$ only. Also, the off-diagonal potentials V^{+-} are large unless λ is small.

Since the potentials are local, we may define position-space potentials in the usual way:

$$V(\vec r) = \frac{1}{(2\pi)^3}\int d^3q\, e^{+i\vec q\cdot\vec r}V(\vec q)\,. \tag{3.3}$$

Fourier-transforming (3.1) we obtain

$$V_\pi^{++}(\vec r) = V_\pi^{--}(\vec r)$$
$$= (\vec\tau_1\cdot\vec\tau_2)[\,V_0^\pi(r)\vec\sigma_1\cdot\vec\sigma_2 + V_2^\pi(r)S_{12}(\vec r)]\,,$$
$$V_\pi^{+-}(\vec r) = -V_\pi^{-+}(\vec r) \tag{3.4}$$
$$= -i(\vec\tau_1\circ\vec\tau_2)\frac{\vec\sigma_1\cdot\vec r}{r}V_1^\pi(r),$$

where if we let $x = m_\pi r$

$$S_{12}(\vec r) = \frac{3\vec\sigma_1\circ\vec r\,\vec\sigma_2\cdot\vec r}{r^2} - \vec\sigma_1\cdot\vec\sigma_2\,, \tag{3.5}$$

$$V_0^\pi\left(\frac{x}{m_\pi}\right) = \frac{g_\pi^2}{4\pi}\,\frac{m_\pi^3}{12M^2}\,\frac{e^{-x}}{x}\,,$$
$$V_1^\pi\left(\frac{x}{m_\pi}\right) = \frac{\lambda g_\pi^2}{4\pi}\,\frac{m_\pi^2}{2M}\left(1+\frac{1}{x}\right)\frac{e^{-x}}{x}\,, \tag{3.6}$$
$$V_2^\pi\left(\frac{x}{m_\pi}\right) = \frac{g_\pi^2}{4\pi}\,\frac{m_\pi^3}{12M^2}\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\frac{e^{-x}}{x}\,.$$

Note that our V^{++} OPE potentials are identical with those obtained by previous workers, while our V^{+-} and V^{--} potentials are totally new.

B. The α exchange potentials

A calculation of the α potential to leading order in M^{-1} gives

$$(V_\alpha^{++})_{12} = -(V_\alpha)^{--}_{12} = \frac{-g_\alpha^2}{D_\alpha(\vec q)}\left[1 - \frac{i}{2M^2}\vec S\cdot(\vec q\times\vec k)\right],$$
$$(V_\alpha^{+-})_{12} = \frac{-g_\alpha^2}{2M}\,\frac{\vec\sigma_2\cdot(2\vec p - \vec q)}{D_\alpha(\vec q)}\,, \tag{3.7}$$

and

$$(V_\alpha^{-+})_{12} = \frac{-g_\alpha^2}{2M}\,\frac{\vec\sigma_2\cdot(\vec q + 2\vec k)}{D_\alpha(\vec q)}\,,$$

where $\vec S$ is the total spin operator, $\vec S = \frac{1}{2}(\vec\sigma_1 + \vec\sigma_2)$. To obtain the result (3.7) we have followed the usual practice of keeping the leading terms of each type of potential even if they are small. Thus we neglected the M^{-2} terms in the central spin-independent part of the potential because there were larger terms of order M^0 present, while M^{-2} terms were kept in the $\vec L\cdot\vec S$ potential because they were the leading terms. The rationale for this somewhat inconsistent procedure is that subsequent adjustment of potential strengths and ranges might be expected to compensate for smaller terms omitted from a potential, but cannot be expected to reproduce types of potentials which have been completely neglected.

The off-diagonal pieces of this potential are not local, so that in position space they will introduce gradient operators. If the position-space wave functions are defined according to

$$\psi(\vec r) = \frac{1}{(2\pi)^{3/2}}\int d^3k\, e^{i\vec k\cdot\vec r}\psi(\vec k) \tag{3.8}$$

(and similarly for $\vec p$), then we have

$$\vec k \to -i\vec\nabla\,, \tag{3.9}$$

and similarly for $\vec p$. Since $\vec p$ is the final momentum, gradient operators which arise from $\vec p$ will always operate on both the potential and the wave function, while those from $\vec k$ operate only on the wave function. In the V^{-+} potentials we will always express the nonlocality through k, and in the V^{+-} potentials it will be expressed through p. In this way, the general relationship $V^{+-} = (V^{-+})^\dagger$ is most conveniently recorded.

In position space the α potentials become

$$V_\alpha^{++}(\vec r) = -V_\alpha^{--}(\vec r)$$
$$= -V_0^\alpha(r) - \frac{1}{Mr}V_1^\alpha(r)\vec L\cdot\vec S\,,$$
$$V_\alpha^{+-}(\vec r, i\vec\nabla) = +\frac{i\vec\sigma_2\cdot\vec r}{r}V_1^\alpha(r) + i\vec\sigma_2\cdot\vec\nabla\frac{1}{M}V_0^\alpha(r)\,, \tag{3.10}$$

and

$$V_\alpha^{-+}(\vec r, i\vec\nabla) = -\frac{i\vec\sigma_2\cdot\vec r}{r}V_1^\alpha(r) + \frac{1}{M}V_0^\alpha(r)i\vec\sigma_2\cdot\vec\nabla\,.$$

The $\vec\nabla$ operates on everything to its right. If we define $\alpha = m_\alpha/m_\pi$, then

$$V_0^\alpha\left(\frac{x}{m_\pi}\right) = \frac{g_\alpha^2}{4\pi}\,m_\pi\,\frac{e^{-\alpha x}}{x}\,,$$
$$V_1^\alpha\left(\frac{x}{m_\pi}\right) = -\frac{m_\pi}{2M}\,\frac{d}{dx}\,V_0^\alpha \tag{3.11}$$
$$= \frac{g_\alpha^2}{4\pi}\,\frac{m_\pi^2}{2M}\,\frac{e^{-\alpha x}}{x}\left(\alpha + \frac{1}{x}\right).$$

C. ρ and ω exchange potentials

When making the M^{-1} expansion for the ρ and ω potentials we again follow the somewhat inconsistent policy of keeping the largest contributions to each different type of spin term, even though in some cases comparable contributions to the central potential are being neglected. The ρ potential gives us

$$(V_\rho^{++})_{12} = (V_\rho^{--})_{12} = \frac{g_\rho^2(\vec\tau_1\cdot\vec\tau_2)}{D_\rho(\vec q)}\left[1+\frac{i}{M^2}\left(\tfrac{3}{2}+2K_\rho\right)\vec S(\vec q\times\vec k)+\frac{\vec q^{\,2}}{12M^2}(1+K_\rho)^2(S_{12}-2\vec\sigma_1\cdot\vec\sigma_2)\right],$$

$$(V_\rho^{+-})_{12} = \frac{g_\rho^2(\vec\tau_1\cdot\vec\tau_2)}{2MD_\rho(\vec q)}\left[-2\vec\sigma_2\cdot\vec p-K_\rho\vec\sigma_2\cdot\vec q+i(1+K_\rho)\vec q\cdot\vec\sigma_1\times\vec\sigma_2\right]. \tag{3.12}$$

$$(V_\rho^{-+})_{12} = \frac{g_\rho^2(\vec\tau_1\cdot\vec\tau_2)}{2MD_\rho(\vec q)}\left[-2\vec\sigma_2\cdot\vec k+K_\rho\vec\sigma_2\cdot\vec q+i(1+K_\rho)\vec q\cdot\vec\sigma_1\times\vec\sigma_2\right].$$

In position space we obtain

$$V_\rho^{++}(\vec r) = V_\rho^{--}(\vec r) = (\vec\tau_1\cdot\vec\tau_2)\left[V_0^\rho(r)-\frac{1}{Mr}\left(\tfrac{3}{2}+2K_\rho\right)V_1^\rho(r)\vec L\cdot\vec S+\frac{m_\rho^2}{6M^2}(1+K_\rho)^2V_0^\rho(r)\vec\sigma_1\cdot\vec\sigma_2-(1+K_\rho)^2V_2^\rho(r)S_{12}(\hat r)\right],$$

$$V_\rho^{+-}(\vec r) = (\vec\tau_1\cdot\vec\tau_2)\left[i\vec\sigma_2\cdot\vec\nabla\frac{1}{M}V_0^\rho(r)-\frac{K_\rho}{2}\frac{i\vec\sigma_2\cdot\vec r}{r}V_1^\rho(r)-\frac{(1+K_\rho)}{2}\frac{\vec r\cdot\vec\sigma_1\times\vec\sigma_2}{r}V_1^\rho(r)\right], \tag{3.13}$$

$$V_\rho^{-+}(\vec r) = (\vec\tau_1\cdot\vec\tau_2)\left[\frac{1}{M}V_0^\rho(r)i\sigma_2\cdot\vec\nabla+\frac{K_\rho}{2}\frac{i\vec\sigma_2\cdot\vec r}{r}V_1^\rho(r)-\frac{(1+K_\rho)}{2}\frac{\vec r\cdot\vec\sigma_1\times\vec\sigma_2}{r}V_1^\rho(r)\right].$$

If we let $\rho = m_\rho/m_\pi$, then

$$V_0^\rho\left(\frac{x}{m_\pi}\right) = \frac{g_\rho^2}{4\pi}m_\pi\frac{e^{-\rho x}}{x},$$

$$V_1^\rho\left(\frac{x}{m_\pi}\right) = \frac{-m_\pi}{M}\frac{d}{dx}V_0^\rho = \frac{g_\rho^2}{4\pi}\frac{m_\pi^2}{M}\frac{e^{-\rho x}}{x}\left(\rho+\frac{1}{x}\right), \tag{3.14}$$

$$V_2^\rho\left(\frac{x}{m_\pi}\right) = \frac{g_\rho^2}{4\pi}\frac{m_\pi^3}{12M^2}\frac{e^{-\rho x}}{x}\left(\rho^2+\frac{3\rho}{x}+\frac{3}{x^2}\right).$$

We obtain the same equations for the ω contribution except that the factor $(\vec\tau_1\cdot\vec\tau_2)$ is missing from (3.12) and (3.13).

D. The coupled Schrödinger equations

We now return to Eq. (2.6) and take the adiabatic limit of both sides. This means, in particular, that we will implement the assumption that the internal k integration is dominated by momenta which are small compared to M, so that terms involving k/M can be treated as small quantities. We obtain

$$\left(\frac{p^2}{M}-\epsilon\right)\psi^+(\vec p) = -\int\frac{d^3k}{(2\pi)^3}\left[V^{++}(\vec q)\psi^+(\vec k)+V^{+-}(\vec q,\vec k)\psi^-(\vec k)\right],$$

$$-2M\psi^-(\vec p) = -\int\frac{d^3k}{(2\pi)^3}\left[V^{-+}(\vec q,\vec k)\psi^+(\vec k)+V^{--}(\vec q)\psi^-(\vec k)\right], \tag{3.15}$$

where each of the potentials is a sum of the potentials (3.1), (3.7), and (3.12), and we have indicated that in this approximation the diagonal potentials V^{++} and V^{--} are local (except for $\vec L\cdot\vec S$ terms).

These equations can be reduced to more familiar form by casting the equation into position space using (3.3) and (3.8). We obtain

$$-\left(\frac{\nabla^2}{M}+\epsilon\right)\psi^+(\vec r) = -V^{++}(\vec r)\psi^+(\vec r)-V^{+-}(\vec r,i\vec\nabla)\psi^-(\vec r), \tag{3.16a}$$

$$-2M\psi^-(\vec r) = -V^{-+}(\vec r,i\vec\nabla)\psi^+(\vec r)-V^{--}(\vec r)\psi^-(\vec r), \tag{3.16b}$$

where the potentials are

$$
\begin{aligned}
V^{++}(\vec{r}) &= U_C + U_{SS}\vec{\sigma}_1\cdot\vec{\sigma}_2 + U_T S_{12}(\vec{r}) + U_{LS}\vec{L}\cdot\vec{S}, \\
U_C &= -V_0^{\alpha} + V_0^{\omega} + (\vec{\tau}_1\cdot\vec{\tau}_2)V_0^{\rho}, \\
U_{SS} &= (\vec{\tau}_1\cdot\vec{\tau}_2)\left[V_0^{\pi} + \frac{m_\rho^2}{6M^2}(1+K_\rho)^2 V_0^{\rho}\right] \\
&\quad + \frac{m_\omega^2}{6M^2}(1+K_\omega)^2 V_0^{\omega}, \\
U_T &= (\vec{\tau}_1\cdot\vec{\tau}_2)[V_2^{\pi} - (1+K_\rho)^2 V_2^{\rho}] \\
&\quad -(1+K_\omega)^2 V_2^{\omega}, \\
U_{LS} &= -\frac{m_\pi}{Mx}[V_1^{\alpha} + (\vec{\tau}_1\cdot\vec{\tau}_2)(1.5+2K_\rho)V_1^{\rho} \\
&\quad + (1.5+2K_\omega)V_1^{\omega}],
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
V^{+-}(\vec{r}) &= V_1^{-}\frac{i\vec{\sigma}_1\cdot\vec{r}}{r} + V_2^{-}\frac{i\vec{\sigma}_2\cdot\vec{r}}{r} \\
&\quad + V_3^{-}\frac{\vec{r}\cdot\vec{\sigma}_1\times\vec{\sigma}_2}{2r} + i\vec{\sigma}_2\cdot\vec{\nabla}v_l, \\
V_1^{-} &= -(\vec{\tau}_1\cdot\vec{\tau}_2)V_1^{\pi}, \\
V_2^{-} &= V_1^{\alpha} - (\vec{\tau}_1\cdot\vec{\tau}_2)\tfrac{1}{2}K_\rho V_1^{\rho} - \tfrac{1}{2}K_\omega V_1^{\omega}, \\
V_3^{-} &= -(\vec{\tau}_1\cdot\vec{\tau}_2)(1+K_\rho)V_1^{\rho} - (1+K_\omega)V_1^{\omega}, \\
v_l &= \frac{1}{M}[V_0^{\alpha} + (\vec{\tau}_1\cdot\vec{\tau}_2)V_0^{\rho} + V_0^{\omega}],
\end{aligned}
\tag{3.18}
$$

and $V^{-+} = (V^{+-})^{\dagger}$. The expression for V^{--} will undergo further approximations, and the final expression is given below.

The reductions of these equations to a single Schrödinger equation was sketched in Sec. I. If one includes all the terms in V^{--}, this reduction is tedious, but offers no difficulties in principle. The final effective potential one obtains is an Hermitian, velocity-dependent potential which contains many nonlocal terms. These terms are very interesting, but they are small and their complexity tends to obscure the main features of the result. These small terms are probably also more sensitive to the errors in the adiabatic approximation. For these reasons we eliminated most of these terms by approximating V^{--} by the leading terms from the α, ρ, and ω contributions. We took

$$
V^{--} = V_0^{\alpha} + (\vec{\tau}_1\cdot\vec{\tau}_2)V_0^{\rho} + V_0^{\omega}, \tag{3.19}
$$

which makes V^{--} a local, spin-independent potential. The terms we have omitted are the OPE and the $\vec{\sigma}_1\cdot\vec{\sigma}_2$, S_{12}, and $\vec{L}\cdot\vec{S}$ terms from the α, ρ, and ω contributions. Since all these terms are down by M^{-2} from the terms given in (3.19), and this potential contributes very little to the details of the intermediate-range force, the approximation is justified within the framework of the adiabatic approximation.

We now turn to the details of using Eq. (3.16b) to eliminate the ψ^{-} wave function from Eq. (3.16a). As long as V^{--} is local, the formal solution was given in Eq. (1.5), with the effective potential defined in Eq. (1.6a). To obtain a practical form for this potential we must reduce the complicated second term, which we have referred to as the quadratic potentials and will denote by V_Q:

$$
V_Q \equiv V^{+-}(2M - V^{--})^{-1}(V^{+-})^{\dagger}. \tag{3.20}
$$

The algebraic details of this reduction will be given in Sec. III E, and the reader not interested in these details may skip directly to the final answer in Sec. III F.

E. Reduction of the quadratic potential

In order to simplify the algebra we introduce some convenient spin-projection operators, although with the simplification (3.19) this technique is not really necessary. However, these operators will be very useful in the future if we wish to include $\vec{\sigma}_1\cdot\vec{\sigma}_2$ and S_{12} terms in V^{--}, such as would come from the OPE.

We may define the projection operators

$$
\begin{aligned}
\hat{S}_0 &= \tfrac{1}{4}(1 - \vec{\sigma}_1\cdot\vec{\sigma}_2), \\
\hat{T}_1 &= \tfrac{1}{8}(3 + \vec{\sigma}_1\cdot\vec{\sigma}_2 + S_{12}), \\
\hat{T}_2 &= \tfrac{1}{8}(\tfrac{3}{2} + \tfrac{1}{2}\vec{\sigma}_1\cdot\vec{\sigma}_2 - S_{12}),
\end{aligned}
\tag{3.21}
$$

which satisfy the relations

$$
x_i^{\,2} = x_i, \quad x_i x_j = 0, \tag{3.22}
$$

where x_i represents any of the $\hat{S}_0$, $\hat{T}_1$, or $\hat{T}_2$. These are complete in the sense that

$$
\begin{aligned}
1 &= \hat{S}_0 + \hat{T}_1 + \hat{T}_2, \\
\vec{\sigma}_1\cdot\vec{\sigma}_2 &= -3\hat{S}_0 + \hat{T}_1 + \hat{T}_2, \\
S_{12} &= 2\hat{T}_1 - 4\hat{T}_2.
\end{aligned}
\tag{3.23}
$$

With these operators we can easily compute the operator $(2M - V^{--})^{-1}$ in the event that V^{--} depends on the invariants in (3.23).

To simplify the treatment of the spin functions which make up the off-diagonal potentials V^{+-} and V^{-+} we introduce the spin operators

$$
\begin{aligned}
\hat{R}_{+} &= \frac{1}{2r}(\vec{\sigma}_1\cdot\vec{r} + \vec{\sigma}_2\cdot\vec{r}), \\
\hat{R}_a &= \frac{1}{4r}(\vec{\sigma}_1\cdot\vec{r} - \vec{\sigma}_2\cdot\vec{r}) + \frac{i}{4}\frac{\vec{r}\cdot\vec{\sigma}_1\times\vec{\sigma}_2}{r} = \hat{R}_b^{\dagger}, \\
\hat{R}_b &= \frac{1}{4r}(\vec{\sigma}_1\cdot\vec{r} - \vec{\sigma}_2\cdot\vec{r}) - \frac{i}{4}\frac{\vec{r}\cdot\vec{\sigma}_1\times\vec{\sigma}_2}{r} = \hat{R}_a^{\dagger}.
\end{aligned}
\tag{3.24}
$$

These operators can be regarded as odd operators in the sense that products of even numbers of R's always give the even operators (3.21), while odd powers of R's reproduce themselves. Note that

$$\hat{R}_+\hat{R}_+ = \hat{T}_1,$$
$$\hat{R}_a^\dagger\hat{R}_a = \hat{R}_b\hat{R}_a = \hat{S}_0, \tag{3.25}$$
$$\hat{R}_b^\dagger\hat{R}_b = \hat{R}_a\hat{R}_b = \hat{T}_2,$$

while all other products involving the R's are zero. When the even operators multiply from the left we have

$$\hat{T}_1\hat{R}_+ = \hat{R}_+,$$
$$\hat{T}_2\hat{R}_a = \hat{R}_a, \tag{3.26}$$
$$\hat{S}_0\hat{R}_b = \hat{R}_b;$$

all other left products are zero. The right products (the results of multiplying the R's on the right-hand side by the even operators) can be obtained from (3.26) by taking the Hermitian conjugate of both sides, being careful to remember that $R_a^\dagger = R_b$. Note the relation

$$\hat{R}^\dagger\hat{R}\hat{R}^\dagger = \hat{R}^\dagger, \tag{3.27}$$

which holds for any R.

These operators are also complete in that

$$\frac{\vec{\sigma}_1\cdot\vec{r}}{r} = \hat{R}_+ + \hat{R}_a + \hat{R}_b,$$
$$\frac{\vec{\sigma}_2\cdot\vec{r}}{r} = \hat{R}_+ - \hat{R}_a - \hat{R}_b, \tag{3.28}$$
$$\frac{i\vec{r}\cdot\vec{\sigma}_1\times\vec{\sigma}_2}{2r} = \hat{R}_a - \hat{R}_b.$$

In terms of these operators, the off-diagonal potentials are

$$V^{+-} = -i(v_+\hat{R}_+ + v_a\hat{R}_a + v_b\hat{R}_b) + i\vec{\sigma}_2\cdot\vec{\nabla}v_l$$
$$= -iU + i\vec{\sigma}_2\cdot\vec{\nabla}v_l, \tag{3.29}$$
$$V^{-+} = iU^\dagger + iv_l\vec{\sigma}_2\cdot\vec{\nabla},$$

where

$$v_+ = -V_1^\alpha + \tfrac{1}{2}K_\omega V_1^\omega + (\vec{\tau}_1\cdot\vec{\tau}_2)(V_1^\pi + \tfrac{1}{2}K_\rho V_1^\rho),$$
$$v_a = V_1^\alpha - (1+\tfrac{3}{2}K_\omega)V_1^\omega + (\vec{\tau}_1\cdot\vec{\tau}_2)[V_1^\pi - (1+\tfrac{3}{2}K_\rho)V_1^\rho], \tag{3.30}$$
$$v_b = V_1^\alpha + (1+\tfrac{1}{2}K_\omega)V_1^\omega + (\vec{\tau}_1\cdot\vec{\tau}_2)[V_1^\pi + (1+\tfrac{1}{2}K_\rho)V_1^\rho],$$

and v_l was previously defined in Eq. (3.18). Introducing

$$D = 1 - \frac{1}{2M}V^{--}$$
$$= 1 - \frac{1}{2M}[V_0^\alpha + V_0^\omega + (\vec{\tau}_1\cdot\vec{\tau}_2)V_0^\rho], \tag{3.31}$$

we obtain

$$2MV_Q = UU^\dagger\left(\frac{1}{D}\right) - \vec{\sigma}_2\cdot\vec{\nabla}\frac{v_l}{D}U^\dagger$$
$$+ U\frac{v_l}{D}\vec{\sigma}_2\cdot\vec{\nabla} - \vec{\sigma}_2\cdot\vec{\nabla}\frac{v_l^2}{D}\vec{\sigma}_2\cdot\vec{\nabla}. \tag{3.32}$$

The first term of (3.32) is reduced using (3.25), but the second term requires knowledge of how $\vec{\sigma}_2\cdot\vec{\nabla}$ commutes with the R's. We use the relations

$$\vec{\sigma}_2\cdot\vec{\nabla}\hat{R}_+ = (R_a+R_b)\vec{\sigma}_2\cdot\vec{\nabla} + \frac{1}{r}(\vec{r}\cdot\vec{\nabla} + \hat{T}_1 + 2\hat{T}_2),$$
$$\vec{\sigma}_2\cdot\vec{\nabla}R_a = \tfrac{1}{2}(R_+ - R_a + R_b)\vec{\sigma}_2\cdot\vec{\nabla}$$
$$- \frac{1}{2r}(\vec{r}\cdot\vec{\nabla} - \vec{\sigma}_1\cdot\vec{L} + 4\hat{S}_0), \tag{3.33}$$
$$\vec{\sigma}_2\cdot\vec{\nabla}R_b = \tfrac{1}{2}(R_+ + R_a - R_b)\vec{\sigma}_2\cdot\vec{\nabla}$$
$$- \frac{1}{2r}(\vec{r}\cdot\vec{\nabla} + \vec{\sigma}_1\cdot\vec{L} + 2\hat{T}_1),$$
$$\vec{\sigma}_2\cdot\vec{\nabla} = \frac{1}{r^2}\vec{\sigma}_2\cdot\vec{r}(\vec{r}\cdot\vec{\nabla} - \vec{\sigma}_2\cdot\vec{L}).$$

Doing the algebra, we obtain a reduced form for the quadratic potential:

$$V_Q = U_C^Q + U_{SS}^Q\vec{\sigma}_1\cdot\vec{\sigma}_2 + U_T^Q S_{12} + U_{LS}^Q\vec{L}\cdot\vec{S}$$
$$+ U_{LD}^Q\vec{L}\cdot\vec{D} - U_E\frac{\nabla^2}{M} - \frac{1}{r}\frac{d}{dr}(U_E)\frac{\vec{r}\cdot\vec{\nabla}}{M}, \tag{3.34}$$

where we have a new spin invariant

$$\vec{L}\cdot\vec{D} \equiv \tfrac{1}{2}\vec{L}\cdot(\vec{\sigma}_1 - \vec{\sigma}_2), \tag{3.35}$$

which seems to violate isospin conservation. This is not the case, however, and an understanding and discussion of this term will be put off for the next section. The potentials U will suffer one more transformation, and their final form will be given below.

The equation (3.34) exhibits the velocity dependence of the quadratic potential through the terms proportional to U_E, where

$$U_E = v_l^2/(2D). \tag{3.36}$$

To compare our effective potential with static potentials we transform this dependence away using the effective-mass transformation.[22] If we introduce a new wave function ψ_T according to

$$\psi^+(x) = \frac{\psi_T(x)}{(1+U_E)^{1/2}}, \tag{3.37}$$

then ψ_T and ψ^+ have the same asymptotic behavior, so that the phase shifts and binding energies are unaffected by the transformation. The Schrödinger equation for ψ_T will contain no velocity-dependent terms. The new effective potential differs from the old, and we have

$$V_T = \frac{D}{D_T}\left(V_{\text{eff}} + \epsilon U_E + \frac{U_E'}{Mr} - \frac{D(U_E')^2}{4MD_T} + \frac{1}{2M}U_E''\right), \tag{3.38}$$

where the prime on the U_E refers to differentiation with respect to r and

$$D_T = D(1+U_E) = D + \tfrac{1}{2}v_l^{\,2}. \tag{3.39}$$

The new effective potential has an energy dependence introduced by the transformation.

F. Summary of final equations

We collect together the final expressions for the potentials. After the effective-mass transformation we obtain a Schrödinger equation for a transformed wave function

$$-\left(\frac{\nabla^2}{M}+\epsilon\right)\psi_T = -V_T\psi_T, \tag{3.40}$$

where the relation between ψ_T and ψ^+ is given in Eq. (3.37). The transformed potential has the form

$$V_T = V_C + V_{SS}\vec{\sigma}_1\cdot\vec{\sigma}_2 + V_T S_{12} + V_{LS}\vec{L}\cdot\vec{S} + V_{LD}\vec{L}\cdot\vec{D}, \tag{3.41}$$

where the new spin invariant $\vec{L}\cdot\vec{D}$ was defined in Eq. (3.35) and will be discussed in the next section. The potentials are all of the form

$$\begin{aligned} V_C &= \frac{D}{D_T}U_C + V_C^Q, \\ V_{SS} &= \frac{D}{D_T}U_{SS} + V_{SS}^Q, \\ V_T &= \frac{D}{D_T}U_T + V_T^Q, \\ V_{LS} &= \frac{D}{D_T}U_{LS} + V_{LS}^Q, \\ V_{LD} &= V_{LD}^Q, \end{aligned} \tag{3.42}$$

where the U potentials have been given in Eq. (3.17) and come from the long-range V^{++} potential. The factor D/D_T which modifies these contributions was defined in Eqs. (3.39), (3.18), and (3.31). This factor arises from the effective-mass transformation.

The quadratic contributions to each potential are

$$\begin{aligned} 8MD_T V_C^Q &= 2v_+^{\,2} + v_a^{\,2} + v_b^{\,2} - \frac{2v_l}{r}(2v_+ - v_a - v_b) - D\left[\frac{v_l}{D}(2v_+ - v_a - v_b)\right]' \\ &\quad + 4M\epsilon v_l^{\,2} - \frac{2v_l}{D_T}\left(v_l' - \frac{v_l D'}{2D}\right)^2 + \frac{8v_l}{r}\left(v_l' - \frac{D'v_l}{2D}\right) + 4\left(v_l' - \frac{v_l D'}{D}\right)^2 + 4v_l\left(v_l'' - \frac{v_l D''}{2D}\right), \\ 8MD_T V_{SS}^Q &= \tfrac{2}{3}v_+^{\,2} + \tfrac{1}{3}v_a^{\,2} - v_b^{\,2} - \frac{2v_l}{r}(\tfrac{2}{3}v_+ - \tfrac{1}{3}v_a + v_b) - D\left[\frac{v_l}{D}(\tfrac{2}{3}v_+ - \tfrac{1}{3}v_a + v_b)\right]', \\ 12MD_T V_T^Q &= v_+^{\,2} - v_a^{\,2} + \frac{v_l}{r}(v_+ + v_a) - D\left[\frac{v_l}{D}(v_+ + v_a)\right]', \\ 2MD_T r V_{LS}^Q &= -v_l(v_+ - v_a) + D\left(\frac{v_l^{\,2}}{D}\right)', \\ 2MD_T r V_{LD}^Q &= v_l(v_+ - v_b) - D\left(\frac{v_l^{\,2}}{D}\right)', \end{aligned} \tag{3.43}$$

where the v's are defined in Eqs. (3.18) and (3.30). In these equations the prime refers to differentiation with respect to r.

The potentials (3.42) are the ones presented in Figs. 1–4. In those figures and in the discussion we ignored the presence of the potential V_{LD}. This potential has some very interesting properties, and we now turn to a discussion of these properties.

IV. SOME SPECIAL NONLOCAL INTERACTIONS

There are many nonlocal interactions present in the theory presented in this paper. The easiest one to deal with was the velocity dependence, which we eliminated by the effective mass transformation. Other nonlocalities at short distances were neglected when we simplified V^{--} [Eq. (3.19)]. It turns out that many of these which we neglected are of the same type as the new nonlocality which results from the $\vec{L}\cdot\vec{D}$ potential.

The existence of the $\vec{L}\cdot\vec{D}$ term defined in Eq. (3.35) means that our potential is not symmetric under interchange of particles 1 and 2, which might at first glance seem to be either a violation of charge independence or the indistinguishability of the particles. Actually, it is not really a manifestation of either, but is due to the fact that particle 2 is off shell and particle 1 is on shell. There is no symmetry between the two particles, because we are working in a dynamical region where symmetry is not expected. In this language the Pauli principle means simply that the sister equation for particle 1 off shell and particle 2 on shell contains no new information.

Because the off-shell nucleon is close to its

mass shell except at short distances, one expects V_{LD} to be of very short range, and this is indeed the case. As Fig. 6 shows, V_{LD} is comparable to the quadratic contributions to V_{LS}, and as one can see from Figs. 3(d) and 4(d) these are of very short range compared to the full potentials.

To study the behavior of $\vec{L}\cdot\vec{D}$, we first assume that the nucleons are nonidentical particles, but that isospin is still conserved. The states are specified by the total angular momentum J, the orbital angular momentum L, the total spin S, and the isospin I. The fact that the particles are nonidentical means that the states do not have to have antisymmetric wave functions, and hence both isospin states exist for each J, L, S. Now, one can easily show that $\vec{L}\cdot\vec{D}$ commutes with $\vec{J}$, L^2, and $\vec{I}$, but does not commute with S^2. Hence $\vec{L}\cdot\vec{D}$ can be expressed as a matrix in block diagonal form, diagonal with respect to J, L, and I. Since S can only be 0 or 1, $\vec{L}\cdot\vec{D}$ is a 2×2 matrix, and its matrix elements are easily shown to be

$$\vec{L}\cdot\vec{D}=\begin{bmatrix} 0 & [J(J+1)]^{1/2} \\ [J(J+1)]^{1/2} & 0 \end{bmatrix}. \tag{4.1}$$

Hence, $\vec{L}\cdot\vec{D}$ changes triplet states into singlet states and vice versa. In the usual nonrelativistic theory the Pauli principle fixes S once J, L, and I have been chosen, so that only the diagonal elements of (4.1) would occur, and hence terms of the form $\vec{L}\cdot\vec{D}$ are absent from the potential.

In our theory the particles are still identical, but the wave functions are not required to be antisymmetric because only particle 1 is on shell. To see why this is so, we restrict our discussion to the hypothetical case of spin-zero fermions, and return to the vertex function, Γ, for two off-shell particles, which is a function of the relative energy and relative 3-momentum. The Pauli principle would require that this function be antisymmetric in its relative 4-momentum:

$$\Gamma(p_0,\vec{p})=-\Gamma(-p_0,-\vec{p}). \tag{4.2}$$

When we put particle 1 on shell, we fix $p_0=E_p-\frac{1}{2}W$, while if particle 2 is on shell $p_0=-E_p+\frac{1}{2}W$. Hence, there is a different wave function to describe each case:

$$\begin{aligned}&\Gamma(E_p-\tfrac{1}{2}W,\vec{p})\equiv\Gamma_1(\vec{p}),\\ &\Gamma(-E_p+\tfrac{1}{2}W,\vec{p})\equiv\Gamma_2(\vec{p}).\end{aligned} \tag{4.3}$$

The antisymmetry now becomes a relation between *two* different wave functions rather than a condition on one wave function. Equation (4.2) becomes

$$\Gamma_1(\vec{p})=-\Gamma_2(-\vec{p}). \tag{4.4}$$

If we were using the BSLT theory (or the Todorov equation), then $p_0=0$, and there would be only one wave function like the nonrelativistic theory, and we would still have the antisymmetry.

Losing the antisymmetry means that all possible J, L, S, and I states contribute to a partial-wave expansion. However, the states that are totally antisymmetric (as $\vec{p}\to-\vec{p}$) and hence have isospin I satisfying the relation

$$I=\tfrac{1}{2}[1-(-1)^{L+S}] \tag{4.5}$$

are the only ones which can contribute to real physical scattering. This is because when *both* particles are on their mass shell, $E_p=\frac{1}{2}W$, and $\Gamma_1=\Gamma_2$, and only antisymmetric states are allowed. These will be referred to as even states. Hence the symmetric states with isospin given by

$$I=\tfrac{1}{2}[1+(-1)^{L+S}] \tag{4.6}$$

are virtual. In this sense they are like the wave function ψ^-, which affects the dynamics even though it does not contribute asymptotically. These will be referred to as odd states.

A way to write the partial-wave expansion for Γ_1 which includes these restrictions is

$$\Gamma_1(\vec{p})=\sum_{J,L,S}\left[\Gamma^e_{JLS}(\vec{p})+\frac{2E_p-W}{W}\Gamma^o_{JLS}(\vec{p})\right], \tag{4.7}$$

where

$$\Gamma^e(\vec{p})=-\Gamma^e(-\vec{p}),\quad \Gamma^o(\vec{p})=+\Gamma^o(-\vec{p}). \tag{4.8}$$

In a similar way [from (4.4)],

$$\Gamma_2(\vec{p})=\sum_{J,L,S}\left[\Gamma^e_{JLS}(\vec{p})-\frac{(2E_p-W)}{W}\Gamma^o_{JLS}(\vec{p})\right]. \tag{4.9}$$

The factor $2E_p-W$ ensures that the Γ^o contribution will vanish on shell. It is also suggested by the requirement that if Γ is even in $\vec{p}$ it must be odd in $p_0-E_p-\frac{1}{2}W$. Note that the wave function in momentum space is (N is a normalization constant)

$$\begin{aligned}\psi_1(\vec{p})&=\frac{N\Gamma_1(\vec{p})}{2E_p-W}\\ &=N\sum_{J,L,S}\left[\frac{\Gamma^e_{JLS}(\vec{p})}{2E_p-W}+\frac{1}{W}\Gamma^o_{JLS}(\vec{p})\right],\end{aligned} \tag{4.10}$$

so that only the antisymmetric part has the (physical) singularity at $E_p=\frac{1}{2}W$.

It is now clear that the role of the $\vec{L}\cdot\vec{D}$ term in the potential is to couple the odd states to the even states. Without this term the odd states would be present but uncoupled, and as such would have no influence on the dynamics. The only states which are affected are those with $J=L\geq1$, for only in this case do $S=0$ and 1 states both exist. Hence the dynamics of four states are affected: the iso-

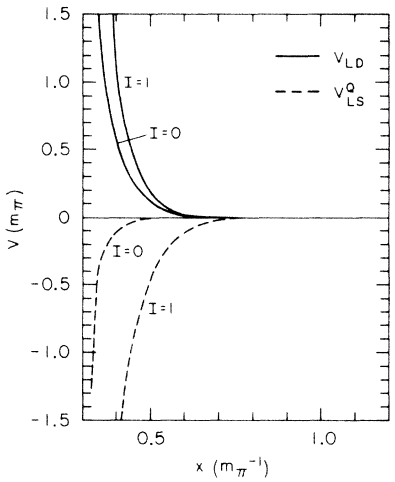

FIG. 6. The potentials V_{LD} are compared with the quadratic contributions to V_{LS}.

triplet $^3P_1$ and $^1D_2$ and the isosinglet $^1P_1$ and $^3D_2$. The fits shown to these potentials in Sec. I will therefore be modified by coupling to unphysical virtual states. This additional coupling can be thought of as an additional nonlocality which enters the theory. As Fig. 6 shows, in the present approximation the coupling only becomes effective inside of $x \cong 0.6 m_\pi^{-1}$.

The equations for these coupled states can be obtained from (4.1) and (4.10). For the odd states it is convenient to introduce

$$\phi_o(\vec{p}) = \frac{-N\Gamma^o(\vec{p})}{2E_p - W} . \tag{4.11}$$

This odd wave function has an asymptotic part just like an even wave function, and is related to the odd part of the wave function defined in (4.10) according to

$$\psi_o(\vec{p}) = -\frac{2E_p - W}{W}\phi_o(\vec{p}) . \tag{4.12}$$

In position space in the nonrelativistic limit we obtain

$$\left(-\frac{\nabla^2}{M} - \epsilon + V^S_{\text{eff}}\right)\psi^S_e = -[J(J+1)]^{1/2}\,\frac{V_{LD}}{2M}\left(\frac{\nabla^2}{M} + \epsilon\right)\phi^{S'}_o ,$$

$$\left(-\frac{\nabla^2}{M} - \epsilon + V^{S'}_{\text{eff}}\right)\frac{1}{2M}\left(\frac{\nabla^2}{M} + \epsilon\right)\phi^{S'}_o = -[J(J+1)]^{1/2} V_{LD}\psi^S_e , \tag{4.13}$$

where S and S' are spin quantum numbers and are either 0 or 1, and $S \neq S'$.

These equations can only be solved numerically. Such a study should not be undertaken until the other small terms of this type have been included. But the best way to handle this problem is to return to the original momentum-space equations which can be solved numerically with less difficulty.

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Isolating the 3-quark component of the proton's wave function

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The 3-quark component of the proton's wave function is extracted from deep-inelastic proton and neutron scattering data. We employ theoretical results based upon a theory of the hadronic wave function which is consistent with the interchange theory of fixed-angle and high-transverse-momentum processes.

INTRODUCTION

In this paper we describe a theoretical extraction of the quark distribution functions of the proton, using the deep-inelastic scattering data for neutrons and protons.[1] The present analysis differs from those previously given[2] in that the Pomeron and Regge contributions are assumed to have specific, theoretically motivated, threshold damping. This enables us to subtract these contributions in a well-defined manner, isolating those portions of the distribution functions most closely related to the simplest three-quark component of the proton's wave function. Among other results we find that for this component the $\mathcal{N}$-quark distribution function is a single power of $(1-x)$ (x is the fraction of the proton's momentum carried by the quark) times that for the $\mathcal{P}$ quark, for all x.

I. THEORETICAL CONSIDERATIONS

We begin by considering the probability $u_i(x)$ for finding a quark i of a given type, carrying a fraction x of the proton's total linear momentum (in a frame in which the proton's linear momentum is large). In general each such probability function may be thought of as having three contributions:

(i) It has a contribution $\hat{u}_i(x)$ arising from the simplest possible quark state consistent with the nucleon's (or meson's) quantum numbers: for instance, $\mathcal{P}\mathcal{P}\mathcal{N}$ for the proton, $\mathcal{P}\bar{\mathcal{N}}$ for a π^+, etc. The wave function (which when integrated over transverse momentum gives the probability distribution function) for this simplest constituent state might, for instance, obey a relatively simple[3] Bethe-Salpeter type of integral equation. Such a component will not exhibit either Regge behavior or Pomeron behavior [$u_i(x) \underset{x\to 0}{\sim} 1/x^\alpha$, with $\alpha = \frac{1}{2}$ or 1, respectively]. It should exhibit a maximum when the quarks present have approximately equal shares of the hadron's momentum ($x \sim \frac{1}{3}$ for a nucleon) and should, of course, be absent for quarks not required to be a part of the *simplest* quark state of the given hadron (e.g., $\bar{\mathcal{P}}\bar{\mathcal{N}}\lambda\bar{\lambda}$ for a proton).

(ii) It also has a contribution which exhibits non-Pomeron Regge behavior. In general, both this contribution and the Pomeron contribution (iii) will be present provided the quark-proton scattering amplitude exhibits these respective types of high-energy behavior (this connection is discussed in Ref. 3, for instance), as expected of a strong-interaction amplitude. In general, however, the