Some rare decay modes of the K meson in a current-current quark model. I. $K \rightarrow \pi l^+ l^-$ decay *

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$K^+ \rightarrow \pi^+ l^+ l^-$ decay is calculated in a zero-parameter modified baryon-loop model which has proven successful in describing the weak radiative kaon decays. The weak Hamiltonian is phenomenologically constructed from one-baryon octet matrix elements. The predicted branching ratio of the loop model $r_+(e^+e^-) = \Gamma(K^+ \rightarrow \pi^+ e^+ e^-)/\Gamma(K^+ \rightarrow \text{all}) = 1.6 \times 10^{-6}$ [with $r_+(\mu^+\mu^-) = 0.3 \times 10^{-6}$] compares reasonably with the prediction of a recent gauge-theory calculation of Lee and Gaillard, $r_+^{\text{G.T.}}(e^+e^-) \sim 10^{-6}$ (with acceptable values ranging between 3×10^{-6} and 0.5×10^{-6}) and is to be compared with the presently available experimental upper bound $r_+^{\text{exp}}(e^+e^-) < 0.4 \times 10^{-6}$. We also find the ratio $\Gamma(K_S^0 \rightarrow \pi^0 e^+ e^-)/\Gamma(K^+ \rightarrow \pi^+ e^+ e^-)$ to be essentially zero in the loop model, as opposed to its value of unity in the gauge-model calculation.

The baryon-loop model,[1] suitably modified for weak interactions,[2-6] continues to prove successful in its description of the weak radiative kaon decays. Attention must be paid to (i) the very recent agreement of the model's prediction[7] of $\Gamma_{\text{theo}}(K_L^0 \rightarrow \pi^+\pi^-\gamma)/\Gamma(K_L^0 \rightarrow \text{all}) = 7.5 \times 10^{-5}$ *without renormalization to* $K_L^0 \rightarrow \gamma\gamma$ with the experimentally measured branching ratio[8] $\Gamma(K_L^0 \rightarrow \pi^+\pi^-\gamma)/\Gamma(K_L^0 \rightarrow \text{all}) = (6.3 \pm 1.9) \times 10^{-5}$, as well as (ii) the excellent agreement of the predicted branching ratio[4]

$$r_\pm = \frac{\Gamma_{\text{theo}}(K^\pm \rightarrow \pi^\pm \pi^0 \gamma;\ 55 \leq T_{\pi^\pm} \leq 90 \text{ MeV})}{\Gamma(K^\pm \rightarrow \text{all})} = 1.6 \times 10^{-5}$$

with the experimental result of Abrams *et al.*,[9] (iii) the model's qualitative explanation for $K_L^0 \rightarrow \gamma\gamma$ decay,[2] and (iv) the consistency of the predicted branching ratio[6] $r = \Gamma_{\text{theo}}(K^+ \rightarrow \pi^+\gamma\gamma)/\Gamma(K^+ \rightarrow \text{all}) = 0.64 \times 10^{-6}$ with the present experimental upper limit $r_{\text{exp}} < 3.5 \times 10^{-5}$, which have been cited in earlier communications.[2-6] In this paper, as part of the further exploration of the predictive power of this *zero-parameter* model, we take up the decays $K^+ \rightarrow \pi^+ l^+ l^-$ and $K_S^0 \rightarrow \pi^0 l^+ l^-$.

In the modified baryon-loop approach, one replaces the (parity-conserving) weak nonleptonic Hamiltonian density relevant for hyperon decays,[2,10]

$$\mathcal{H}_G = \sqrt{2}\, G \cos\theta \sin\theta\, \tfrac{1}{2}\{J_\mu^{(1-i2)}, J^{\mu(4+i5)}\}, \tag{1}$$

by an equivalent weak Hamiltonian,

$$\mathcal{H}_W = -\sqrt{2}\, F\, \text{Tr}([\bar{B}, B]\lambda_6) + \sqrt{2}\, D\, \text{Tr}(\{\bar{B}, B\}\lambda_6), \tag{2}$$

expressed in terms of physical baryon fields.[11] $\mathcal{H}_W$ is constructed from the parity-conserving one-baryon-octet matrix elements of $\mathcal{H}_G$ so that

$$\langle B_j | \mathcal{H}_W | B_i \rangle = \langle B_j | \mathcal{H}_G | B_i \rangle = 2\sqrt{2}\, \bar{u}_j (-i f_{6ji} F + d_{6ji} D) u_i, \tag{3}$$

in a parametrization due to Gronau.[10] He finds in a (semiphenomenological) current-algebraic treatment of nonleptonic hyperon decays based on the symmetric quark model[10] a remarkable fit to the experimental amplitudes for the values $F = 4.7 \times 10^{-5}$ MeV, $D/F = -0.85$. Thus, the total interaction Hamiltonian for virtual photon decay is[12]

$$\begin{aligned}\mathcal{H}_{\text{int}} &= \sqrt{2}\, gf\, \text{Tr}([\bar{B} i\gamma_5, B] M) - \sqrt{2}\, gd\, \text{Tr}(\{\bar{B} i\gamma_5, B\} M) \\ &\quad + \tfrac{1}{2} e A_\mu \text{Tr}([\bar{B}\gamma_\mu, B] Q) \\ &\quad + ie A_\mu \text{Tr}([M, \partial^\mu M] Q) + \mathcal{H}_W,\end{aligned} \tag{4}$$

where, as in our earlier work, we follow Gronau in taking $d/f = 1.8$ ($d + f = 1$) with $g^2/4\pi = 14.6$. Although in principle one has SU(3) conserved at vertices but broken in hadron masses,[10] in practice one neglects such breaking in the baryon octet, replacing the differing intermediate baryon masses by a "mean" baryon mass (m) of 1 GeV.[2-6]

For $K^+ \rightarrow \pi^+ l^+ l^-$ decay there occur two types of loop graphs, "virtual bremsstrahlung" graphs and structure graphs, examples of which are shown in Fig. 1. [A loop-model description of this decay was first suggested by Cabibbo and Ferrari[13] a long time ago and before the introduction of SU(3) symmetry, and it is useful to note that those features of their treatment which pertain to gauge invariance are preserved in the present calculation.] One finds (as in the calculation of Ref. 13) that the "dominant"[14] part of the contribution of the "virtual bremsstrahlung" graphs,

$$A_{\text{brem}}(K^+ \to \pi^+ l^+ l^-) = \frac{i4\sqrt{2}\, g^2 e^2}{m_\pi^2 - m_K^2}[6dfF + 3(f^2 - d^2)D] \times (k+p)^\mu a_\mu(k-p) J, \quad (5)$$

where

$$J = -\frac{im}{2\pi^2}\left\{\left(\frac{m_\pi^2/4m^2}{1-m_\pi^2/4m^2}\right)^{1/2} \tan^{-1}\left[\left(\frac{m_\pi^2/4m^2}{1-m_\pi^2/4m^2}\right)^{1/2}\right] - \left(\frac{m_K^2/4m^2}{1-m_K^2/4m^2}\right)^{1/2} \tan^{-1}\left[\left(\frac{m_K^2/4m^2}{1-m_K^2/4m^2}\right)^{1/2}\right]\right\} \quad (6)$$

and[15]

$$a^\mu(k-p) = -\frac{1}{(k-p)^2}\, \bar{u}(p_-)\gamma^\mu v(p_+), \quad (7)$$

cancels against the corresponding leading contribution coming from the structure graphs.[16] Thus the calculation must be carried to $O(k_i \cdot k_j/m^2)$ in the external invariants $k_i \cdot k_j$ $(k_i, k_j = p, k)$ to produce the gauge-invariant amplitude,

$$A(K^+ \to \pi^+ l^+ l^-) = A_+(0)[(p\cdot k - p^2)k^\mu + (p\cdot k - k^2)p^\mu]a_\mu, \quad (8)$$

where $p^2 = m_\pi^2$, $k^2 = m_K^2$. The contribution of the (four) "virtual bremsstrahlung" graphs to the gauge-invariant amplitude [Eq. (8)] is straightforwardly obtained from the closed form given in Eq. (5) by expansion. However, the contribution from the (six) structure graphs to the expression (8) represents a more formidable calculational effort. One first reduces the six structure contributions to three, making use of (a) the symmetry of the trace of (Dirac) matrices under transposition, (b) the invariance of the Dirac trace under the similarity transformation such that $C^{-1}(\gamma^\mu)^T C = -\gamma^\mu$, (c) the reflection $t \to -t$ of the loop momentum, and (d) the result that

$$\begin{aligned}\mathrm{Tr}(f_a d_b \cdots) &= \mathrm{Tr}[(f_a d_b \cdots)^T] \\ &= \mathrm{Tr}(\cdots d_b^T f_a^T) \\ &= (-1)^N \mathrm{Tr}(\cdots d_b f_a),\end{aligned}$$

where N is the number of f-type matrices in the unitary-spin trace. The unitary-spin traces are themselves accomplished with the aid of the identities[17]

$$\mathrm{Tr}(d_i f_j f_k f_l) = \frac{i3}{4}\sum_n (d_{ijn} f_{nkl} + d_{iln} f_{njk} + d_{ikn} f_{njl}), \quad (9)$$

$$\mathrm{Tr}(f_i d_j d_k d_l) = -\frac{i2}{3}\sum_n (f_{ijn} d_{nkl} + f_{lin} d_{njk}) + \mathrm{Tr}(d_i f_j f_k f_l), \quad (10)$$

so that one obtains for the aforementioned three structure contributions

$$A^{(\mathrm{i})}_{\text{structure}}(K^+ \to \pi^+ l^+ l^-) = i16\sqrt{2}\, g^2 e^2[\tfrac{1}{2}dfF + (\tfrac{1}{4}f^2 - \tfrac{1}{12}d^2)D] \int \frac{d^4t}{(2\pi)^4} \mathrm{Tr}[\gamma_5(\not{t}-m)^{-1}\not{a}(\not{t}+\not{q}-m)^{-2}\gamma_5(\not{t}+\not{q}-\not{k}-m)^{-1}], \quad (11\text{a})$$

$$A^{(\mathrm{ii})}_{\text{structure}}(K^+ \to \pi^+ l^+ l^-) = i16\sqrt{2}\, g^2 e^2[dfF + (\tfrac{1}{2}f^2 - \tfrac{1}{3}d^2)D] \int \frac{d^4t}{(2\pi)^4} \mathrm{Tr}[\gamma_5(\not{t}-m)^{-1}\not{a}(\not{t}+\not{q}-m)^{-1}\gamma_5(\not{t}+\not{q}-\not{k}-m)^{-2}], \quad (11\text{b})$$

$$A^{(\mathrm{iii})}_{\text{structure}}(K^+ \to \pi^+ l^+ l^-) = i16\sqrt{2}\, g^2 e^2[\tfrac{1}{2}dfF + (\tfrac{1}{4}f^2 - \tfrac{1}{12}d^2)D] \int \frac{d^4t}{(2\pi)^4} \mathrm{Tr}[\gamma_5(\not{t}-m)^{-2}\not{a}(\not{t}+\not{q}-m)^{-1}\gamma_5(\not{t}+\not{q}-\not{k}-m)^{-1}]; \quad (11\text{c})$$

note that we have eliminated the pion momentum p in favor of that of the virtual photon, q. In fact, rewriting $A(K^+ \to \pi^+ l^+ l^-)$ in terms of q and k, one finds alternatively

$$A(K^+ \to \pi^+ l^+ l^-) = A_+(0)(-q^2 k^\mu + q\cdot k q^\mu)a_\mu, \quad (12)$$

so that by *choosing* $k\cdot a = 0$, $q^2 = 0$ in the Feynman integrals of (11a)–(11c) we may "read off" $A_+(0)$ as the coefficient of $q\cdot k q\cdot a$; note also that the contribution to $A_+(0) \propto k^2 q\cdot a$ should vanish in this case. One finds the confirmatory results

$A_{\text{structure}}(K^+ \to \pi^+ l^+ l^-; k\cdot a = q^2 = 0)$

$$= i16\sqrt{2}\, g^2 e^2 [\tfrac{1}{2} dfF + (\tfrac{1}{4} f^2 - \tfrac{1}{12} d^2) D] \frac{i}{96\pi^2 m^3}\, q\cdot a[(-2q\cdot k - 2k^2) + 2(2q\cdot k - 4k^2) + (-2k^2)] \quad (13a)$$

and

$$A_{\text{brem}}(K^+ \to \pi^+ l^+ l^-; k\cdot a = q^2 = 0) = 48\sqrt{2}\, g^2 e^2 [dfF + (\tfrac{1}{2} f^2 - \tfrac{1}{6} d^2) D] \frac{1}{96\pi^2 m^3}(-q\cdot a)(2k^2 - 2q\cdot k), \quad (13b)$$

with[18]

$$A_+(0) = \frac{5}{3}\, \frac{\sqrt{2}\, g^2 e^2}{\pi^2 m^3}[\tfrac{1}{2} dfF + (\tfrac{1}{4} f^2 - \tfrac{1}{12} d^2) D]. \quad (14)$$

The decay rate is given by

$$(K^+ \to \pi^+ l^+ l^-) = \alpha^2 \left(\frac{g^2}{4\pi}\right)^2 \frac{200 m_K}{(3\pi)^3 m^6}[dfF + (\tfrac{1}{2} f^2 - \tfrac{1}{6} d^2) D]^2$$
$$\times \int_{m_\pi}^{(m_K{}^2 + m_\pi{}^2 - 4m_l{}^2)/2m_K} d\omega (\omega^2 - m_\pi{}^2)^{3/2} \left(1 - \frac{4m_l{}^2}{m_K + m_\pi{}^2 - 2m_K\omega}\right)^{1/2} \left(1 + \frac{2m_l{}^2}{m_K{}^2 + m_\pi{}^2 - 2m_K\omega}\right), \quad (15)$$

where m_l is the lepton mass ($m_l = m_e, m_\mu$). Expression (15) yields a branching ratio,[19]

$$r_+(e^+e^-) = \frac{\Gamma(K^+ \to \pi^+ e^+ e^-)}{\Gamma(K^+ \to \text{all})} = 1.6\times 10^{-6} \quad (16)$$

[with $r_+(\mu^+\mu^-) = 0.3\times 10^{-6}$], which in view of its sensitive dependence ($\Gamma \propto m^{-6}$) on the "mean" baryonic mass (taken to be 1 GeV)[20] is still compatible with the present experimental upper bound,[21] $r_+^{\text{exp}}(e^+e^-) \lesssim 0.4\times 10^{-6}$. We note furthermore that a recent calculation of this branching ratio by Lee and Gaillard[22] in the gauge model of Salam[23] and Weinberg[23] predicts a branching ratio, $r_+^{\text{G.T.}}(e^+e^-) \sim 10^{-6}$ (with acceptable values ranging from 3×10^{-6} to 0.5×10^{-6}). We hope that experimentalists will meet the challenge these predictions pose with a measurement of this decay rate in the near future.

The decay $K_S^0 \to \pi^0 l^+ l^-$ can also be calculated in the modified baryon-loop model; however, in contrast with the gauge theory calculation of Ref. 22 [where it is found that[25] $A(K^+ \to \pi^+ e^+ e^-) = A(K_S^0 \to \pi^0 e^+ e^-)$], one finds in the loop model no simple relationship between $K_S^0 \to \pi^0 l^+ l^-$ and $K^+ \to \pi^+ l^+ l^-$.[24] Only the structure graphs [Fig. 1(b)] are present in the case of $K_S^0 \to \pi^0 l^+ l^-$, so that their contribution to $A(K_S^0 \to \pi^0 l^+ l^-)$ is gauge-invariant by itself. We find in a calculation completely analogous to that given for $K^+ \to \pi^+ l^+ l^-$ above that

$A(K_S^0 \to \pi^0 l^+ l^-; k\cdot a = q^2 = 0)$

$$= -4\sqrt{2}\, g^2 e^2 [2dfF + (f^2 + d^2) D] \frac{iq\cdot a}{96\pi^2 m^3}$$
$$\times [(-2q\cdot k - 2k^2) - (2q\cdot k - 4k^2) + (-2k^2)], \quad (17)$$

with

$$A_0(0) = \frac{i\sqrt{2}\, g^2 e^2}{6\pi^2 m^3}[2dfF + (f^2 + d^2) D]. \quad (18)$$

However, since $|A_0(0)/A_+(0)| \sim 10^{-4}$, $r_0(e^+e^-)$ *is*

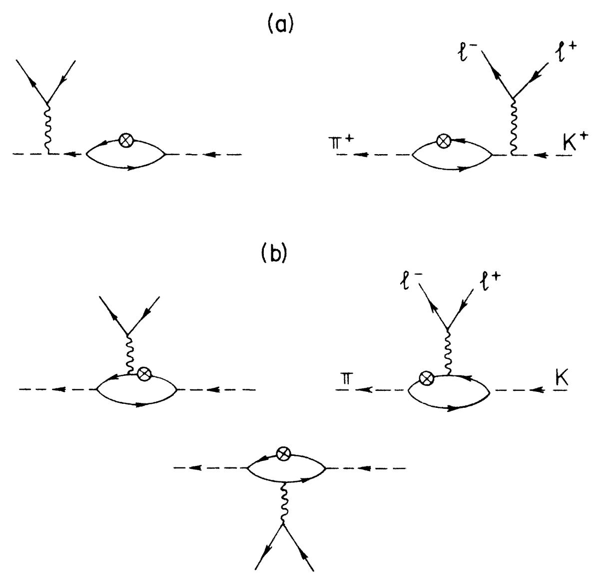

FIG. 1. Graphs contributing to $K \to \pi l^+ l^-$ decay in the baryon-loop model. (a) Graphs for $K^+ \to \pi^+ l^+ l^-$ decay via the emission of a virtual photon from either charged leg ("bremsstrahlung graphs"). There are two additional graphs (not shown) obtained by inserting $\mathcal{H}_W$ in the lower segment of the fermion loop. (b) Structure graphs for $K \to \pi l^+ l^-$ decay. (Only this class of diagrams contributes to $K_S^0 \to \pi^0 l^+ l^-$ decay.) There are three additional graphs (not shown) obtained by inserting $\mathcal{H}_W$ in the lower segment of the fermion loop.

essentially zero in the loop model.

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[12]$M = \lambda_i \phi_i/\sqrt{2}$ is the traceless Hermitian meson matrix; $Q = \lambda_3 + \lambda_8/\sqrt{3}$.

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[14]This is the leading contribution as $m \to \infty$.

[15]$p_\pm^\nu$ is the 4-momentum of the lepton $l^\pm$.

[16]We have verified this explicitly.

[17]R. Rockmore (unpublished). We shall discuss the derivation of such trace identities together with some applications of them elsewhere. One of these results [Eq. (10)] is given in slightly altered form by P. Dittner, Commun. Math. Phys. 22, 238 (1971). We thank P. Singer for calling this work to our attention.

[18]This result and the gauge invariance of the decay amplitude have been verified in detail.

[19]Note our result is one order of magnitude larger than that of Ref. 13.

[20]In the calculation of $K_L^0 \to \gamma\gamma$ in the loop model (Ref. 2), one found a *reduction* from the value of Γ for a *mean* baryonic mass of 1 GeV of approximately 40% using *exact* baryon masses.

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[24]Cf. the model of S. Eliezer and P. Singer [Phys. Rev. 165, 1843 (1968)] for these decays.

[25]There is a numerical inconsistency (and a typographical error) in Eq. (3.30) of Gaillard and Lee (Ref. 22). From their amplitude equality, $i\,T[K^+(p) \to \pi^+(q)e\bar{e}] = i\,T[K_S(p) \to \pi^0(q)e\bar{e}]$, and their Eq. (3.28), one finds using Ref. 21 the result $\Gamma(K_S \to \pi^0 e\bar{e})/\Gamma(K_S \to \text{all}) = 2\times 10^{-8}$ (0.4×10^{-8}). This correction has been confirmed by the authors of Ref. 22.