Lagrangian and (ii) that the coupling is minimal, i.e., local curvature interactions, such as (2), are absent.] A clue as to what these restrictions might be can be obtained by examining the known theorems[4,6,7] that derive the Newton principle from Einstein's theory or from some generalizations of Einstein's theory. In these theorems the crucial assumptions appear to be (i) the presence of only one long-range gravitational field ($g_{\alpha\beta}$) in the Lagrangian and (ii) invariance of the Lagrangian under general coordinate transformations ("general invariance"). This suggests the following reformulation of Schiff's conjecture: *Any theory of gravitation that obeys the weak equivalence principle must be generally invariant and involve only one long-range gravitational field.*[8] Although this statement is in some ways weaker than that given in Ref. 1, a general proof will still be hard to come by.

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[8]In Ref. 1 it is argued that the *matter* part of the Lagrangian can only contain one gravitational field. We are now generalizing this so that only one long-range gravitational field appears in the *entire* Lagrangian. This generalization is necessary if (and only if) one insists that the Newton principle hold *exactly*, even for systems containing gravitational self-energy. Thus, in the Brans-Dicke theory the breakdown of the equality of inertial and gravitational mass (Nordtvedt effect) can be traced to the presence of the long-range scalar field.

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Hartree approximation in relativistic field theory*

Howard J. Schnitzer
Department of Physics, Brandeis University, Waltham, Massachusetts 02154
(Received 25 March 1974)

The large-N limit of $\lambda\phi^4$ theory with O(N) symmetry is derived by functional methods. Since the method does not require the cumbersome, detailed analysis of Feynman graphs, it should make possible the study of more complicated theories in the many-field limit.

Many of the basic questions of relativistic field theory involving its application to hadron physics remain open because of the inability to go beyond the usual perturbation expansion in the coupling constant. A case to the contrary is $\lambda\phi^4$ theory with O(N) symmetry, where one may expand in powers of $1/N$,[1] with each term in the series having a nonperturbative dependence on λ. In particular, the first term of the series in $1/N$ coincides with the Hartree approximation of the theory. It has been advocated[2,3] that other field theories, notably gauge theories, be studied in a similar limit. To date, technical progress along these lines has been restricted to the identification of the dominant Feynman graphs in the limit.[3]

The heavy dependence on a detailed analysis of Feynman graphs is a major drawback of existing studies,[1-3] since this can be quite cumbersome in theories as rich as gauge theories. It would appear that a completely *analytical* technique might be more suitable for an efficient investigation of field theories with many internal (symmetry) degrees of freedom. In this paper we present such an analytical derivation of the large-N limit of $\lambda\phi^4$ theory with O(N) symmetry by a method which is obviously suited to the problems of other relativistic field theories, and gauge theories in particular. Here we limit ourselves to the presentation of the method, and leave specific applications and extensions to future communications.

Consider a $\lambda\phi^4$ theory with O(N) symmetry described by the Lagrangian

$$\mathcal{L}(\tilde{\phi})=\tfrac{1}{2}[(\partial_\mu\tilde{\phi})^2-\mu^2\tilde{\phi}^2]-\frac{\lambda}{4!}(\tilde{\phi}^2)^2, \tag{1}$$

where $\tilde{\phi}^2=\tilde{\phi}_a(x)\tilde{\phi}_a(x)$, and repeated indices are summed from 1 to N. The generating functional for the Green's functions is given by the vacuum amplitude in the presence of an external source,

$$W(j)=\int[d\tilde{\phi}_a]\exp\left\{i\int d^4x[\mathcal{L}(\tilde{\phi})+j_a(x)\tilde{\phi}_a(x)]\right\}. \tag{2}$$

The one-particle irreducible (1PI) graphs are generated by[4]

$$\Gamma(\phi)=-i\ln W(j)-\int d^4x\, j_a(x)\phi_a(x), \tag{3}$$

where

$$\phi_a(x)=-i\,\frac{\delta \ln W(j)}{\delta j_a(x)}.$$

Define

$$\overline{\phi}_a(x)=\tilde{\phi}_a(x)-\phi_a(x) \tag{4}$$

so that

$$\Gamma(\phi)=-i\ln\int[d\overline{\phi}_a]\exp\left\{i\int d^4x[\mathcal{L}(\overline{\phi}+\phi)+j_a(x)\overline{\phi}_a(x)]\right\}, \tag{5}$$

with

$$\delta\Gamma(\phi)/\delta\phi_a(x)=-j_a(x). \tag{6}$$

We may write

$$\Gamma(\phi)=\Gamma_0(\phi)+\Gamma_1(\phi), \tag{7}$$

where

$$\Gamma_0(\phi)=I(\phi)=\int d^4x\,\mathcal{L}(\phi) \tag{8}$$

is the classical action. Combining Eqs. (5)–(8) one obtains the integro-differential equation[5]

$$\Gamma(\phi)=\Gamma_0(\phi)-i\ln\int[d\overline{\phi}_a]\exp\left\{i\left[\hat{I}(\overline{\phi},\phi)-\int d^4x\frac{\delta\Gamma_1(\phi)}{\delta\phi_a(x)}\overline{\phi}_a(x)\right]\right\} \tag{9}$$

where all explicit reference to $j(x)$ has been eliminated, and

$$\hat{I}(\overline{\phi},\phi)=I(\overline{\phi}+\phi)-I(\phi)-\int d^4x\frac{\delta I(\phi)}{\delta\phi_a(x)}\overline{\phi}_a(x) \tag{10a}$$

$$=\int d^4x\left\{\tfrac{1}{2}[(\partial_\mu\overline{\phi})^2-\mu^2\overline{\phi}^2-\tfrac{1}{6}\phi^2\overline{\phi}^2-\tfrac{1}{3}(\phi\cdot\overline{\phi})^2]-\frac{\lambda}{4!}[4(\phi\cdot\overline{\phi})\overline{\phi}^2+(\overline{\phi}^2)^2]\right\}. \tag{10b}$$

Now consider a sequence of steps leading to the $1/N$ expansion of the functional integral (9). First consider the orthogonal transformation

$$\overline{\phi}_a(x)\to R^{-1}{}_{ab}\overline{\phi}_b(x), \tag{11}$$

which is a gauge transformation of the first kind. Such a transformation does not alter the value of the functional integral (9), as it only involves the change of integration variables. Under this transformation

$$\hat{I}(\overline{\phi},\phi)\to\hat{I}(R^{-1}\overline{\phi},\phi)=\hat{I}(\overline{\phi},R\phi) \tag{12a}$$

$$=I_s(\overline{\phi},\phi^2)+I_n(\overline{\phi}^2,\overline{\phi}R\phi), \tag{12b}$$

where the action $\hat{I}(\overline{\phi},\phi)$ has been decomposed into spherical and nonspherical functions with respect to the external field $\phi_a(x)$, as denoted by the subscripts. The spherical action is

$$I_s(\overline{\phi},\phi^2)=\int d^4x\,\mathcal{L}_s(\overline{\phi},\phi^2)$$

$$=\int d^4x\left\{\tfrac{1}{2}[(\partial_\mu\overline{\phi})^2-\mu^2\overline{\phi}^2-\tfrac{1}{6}\phi^2\overline{\phi}^2]-\frac{\lambda}{4!}(\overline{\phi}^2)^2\right\}, \tag{13}$$

while the nonspherical term is

$$I_n(\overline{\phi}^2,\overline{\phi}\cdot\phi)=\int d^4x\,\mathcal{L}_n(\overline{\phi}^2,\overline{\phi}\cdot\phi)$$

$$=-\tfrac{1}{6}\lambda\int d^4x[\overline{\phi}^2(\overline{\phi}\cdot\phi)+(\overline{\phi}\cdot\phi)^2]. \tag{14}$$

Rewrite (9), using (11)–(14), as

$$\Gamma(\phi)-\Gamma_0(\phi)=\Gamma_1(\phi)$$

$$=-i\ln\int[d\overline{\phi}_a]\exp\left\{i\int d^4x\left[\mathcal{L}_s(\overline{\phi},\phi^2)+\mathcal{L}_n(\overline{\phi}^2,\overline{\phi}\cdot R\phi)-\overline{\phi}_a(x)R_{ab}\frac{\delta\Gamma_1(\phi)}{\delta\phi_b(x)}\right]\right\}, \tag{15}$$

and make an expansion of the functional integral in terms of the Green's functions defined by the spherical Lagrangian $\mathcal{L}_s(\overline{\phi},\phi^2)$. This is accomplished by expanding the exponential to obtain the formal power series

$$\Gamma_1(\phi) = -i \ln \int [d\overline{\phi}_a] \left\{ 1 + i \int d^4y \left[\mathcal{L}_n(\overline{\phi}^2, \overline{\phi}\cdot R\phi) - \overline{\phi}_a(y) R_{ab} \frac{\delta\Gamma_1(\phi)}{\delta\phi_b(y)} \right] + \cdots \right\} \exp\left[i \int d^4x\, \mathcal{L}_s(\overline{\phi}, \phi^2) \right]. \tag{16}$$

Since the left side of (16) is independent of any particular choice of the transformation R, we may average the functional integral

$$\int [d\overline{\phi}_a] \left\{ 1 + i \int d^4y \left[\mathcal{L}_n(\overline{\phi}^2, \overline{\phi}\cdot R\phi) - \overline{\phi}_a(y) R_{ab} \frac{\delta\Gamma_1(\phi)}{\delta\phi_b(y)} \right] + \cdots \right\} \exp\left[i \int d^4x\, \mathcal{L}_s(\overline{\phi}, \phi^2) \right] \tag{17}$$

over all possible R_{ab} without changing its value. It is convenient, but not necessary, to make the ansatz

$$\Gamma_1(\phi) = \Gamma_1(\phi^2) + O(1/N), \tag{18}$$

which is justified in the light of our final result. We then perform the average over all R_{ab}, term by term *inside* the functional integration. Since the average is carried out before the functional integration, $\overline{\phi}_a(x)$ is regarded as fixed in the averaging process. Define

$$\phi_a'(x) = R_{ab}\phi_b(x), \tag{19}$$

and the angle between $\phi'(x)$ and $\phi(x)$ by

$$\phi_a(x)\phi_a'(x) = \cos\theta\ \phi^2(x). \tag{20}$$

Note that $\cos\theta$, as defined by (19) and (20), is independent of x since R is a constant transformation, and thus the average to be taken is over the surface of the N sphere defined by $\phi^2(x)$ fixed. Since the N-dimensional angular average of $\cos^{2l}\theta$ is

$$\int d\Omega \cos^{2l}\theta \Big/ \int d\Omega = \frac{1}{\sqrt{\pi}} \frac{\Gamma(l+\frac{1}{2})\Gamma(\frac{1}{2}N)}{\Gamma(l+\frac{1}{2}N)} = O(l!N^{-l}), \tag{21}$$

every term in the angular average of (17), except the first, is suppressed by at least one factor of N^{-1}, so that the spherical term is the leading one in the $1/N$ expansion of (17) and hence (9).

To exemplify the argument, it is instructive to consider the angular average of a prototype term for the special case of constant external field ϕ_a. [If $\phi_a(x)$ is not constant, but bounded, the calculation is similar, although slightly more complicated in detail.] Consider the angular average of

$$\int [d\overline{\phi}_a] \int d^4y_1 \cdots d^4y_{2l} \{\overline{\phi}_{a_1}(y_1)\cdots\overline{\phi}_{a_{2l}}(y_{2l})\} \times \{\phi_{a_1}'\cdots\phi_{a_{2l}}'\} \times \exp\left[i\int d^4x\, \mathcal{L}_s(\overline{\phi}, \phi^2)\right]. \tag{22}$$

It is simple to show that

$$\frac{\int d\Omega\, \phi_{a_1}'\cdots\phi_{a_{2l}}'}{\int d\Omega} = \frac{[\int d\Omega \cos^{2l}\theta/\int d\Omega](\phi^2)^l}{P(a)} \times \{[\delta_{a_1a_2}\cdots\delta_{a_{2l-1},a_{2l}}] + \text{distinct permutations}\}, \tag{23}$$

where $P(a)$ is the number of permutations in (23), a combinatoric factor which cancels in the end. Combining (23) and (22) one finds, on permuting the dummy integration variables y_i, that the angular average of (22) is

$$\left[\int d\Omega \cos^{2l}\theta \Big/ \int d\Omega\right] (\phi^2)^l \int d^4y_1 \cdots d^4y_{2l} \times \int [d\overline{\phi}_a] \{(\overline{\phi}(y_1)\cdot\overline{\phi}(y_2))\cdots(\overline{\phi}(y_{2l-1})\cdot\overline{\phi}(y_{2l}))\} \times \exp\left[i\int d^4x\, \mathcal{L}_s(\overline{\phi}, \phi^2)\right]. \tag{24}$$

From (21) and (24) it is evident that the average of the prototype term (22) is given by an explicit factor of N^{-l} multiplying a spherical Green's function. All other angular averages encountered are carried out in a similar way. It is clear from (24) that the $1/N$ expansion is expected to break down when $\lambda\phi^2(x)$ is $O(N)$.

In summary, for any $\phi_a(x)$ not too large, we have in the large-N limit

$$\begin{aligned}\Gamma(\phi) &= \Gamma_0(\phi) - i \ln \int [d\overline{\phi}_a] \exp\left[i \int d^4x \left\{ \tfrac{1}{2}[(\partial_\mu\overline{\phi})^2 - (\mu^2 + \tfrac{1}{6}\lambda\phi^2)\overline{\phi}^2] - \frac{\lambda}{4!}(\overline{\phi}^2)^2 \right\} \right] + O(N^{-1}) \\ &= \Gamma_0(\phi) - i\ln\int[d\overline{\phi}_a]\exp\left[i\int d^4x\, \mathcal{L}_s(\overline{\phi},\phi^2) + O(N^{-1})\right].\end{aligned} \tag{25}$$

It is clear that $\Gamma(\phi)$ given by (25) generates *only* the 1PI bubble graphs of $\lambda\overline{\phi}^4$ theory, constructed from free propagators with mass term $\mu^2 + \frac{1}{6}\lambda\phi^2(x)$, in agreement with graphical analysis. In order to construct the 1PI amplitudes from (25), it is convenient to define the functional

$$Z(j) = -i \ln \int [d\overline{\phi}_a] \exp\left\{ i \int d^4x [\mathcal{L}_s(\overline{\phi}, \phi^2) + \overline{\phi}_a(x) j_a(x)] \right\} \tag{26}$$

which generates the connected Green's functions of the Lagrangian defined by (13) or (25). Observe that

$$\frac{\delta\Gamma(\phi)}{\delta\phi_a(x)} = \frac{\delta\Gamma_0(\phi)}{\delta\phi_a(x)} - \frac{\lambda}{6}\phi_a(x) \frac{\int [d\overline{\phi}_a] \overline{\phi}^2(x) \exp[i \int d^4y\, \mathcal{L}_s(\overline{\phi}, \phi^2)]}{\int [d\overline{\phi}_a] \exp[i\int d^4y\, \mathcal{L}_s(\overline{\phi}, \phi^2)]} + O(N^{-1}). \tag{27}$$

Making use of (26) one obtains

$$\frac{\delta\Gamma(\phi)}{\delta\phi_a(x)} = \frac{\delta\Gamma_0(\phi)}{\delta\phi_a(x)} + \frac{i\lambda}{6}\phi_a(x)\left[\frac{\delta^2 Z(j)}{\delta j_b(x)\delta j_b(x)}\right]_{j=0} + O(N^{-1}). \tag{28}$$

Similarly

$$\frac{\delta^2\Gamma(\phi)}{\delta\phi_a(x)\delta\phi_b(y)} = \frac{\delta^2\Gamma_0(\phi)}{\delta\phi_a(x)\delta\phi_b(y)} + \frac{i\lambda}{6}\delta_{ab}\delta^4(x-y)\left[\frac{\delta^2 Z(j)}{\delta j_c(x)\delta j_c(x)}\right]_{j=0}$$
$$+\frac{i\lambda^2}{36}\phi_a(x)\phi_b(y)\left\{\left[\frac{\delta^4 Z(j)}{\delta j_c(x)\delta j_c(x)\delta j_d(y)\delta j_d(y)}\right]_{j=0} - 2\left[\frac{\delta^2 Z(j)}{\delta j_c(x)\delta j_d(y)}\right]_{j=0}\left[\frac{\delta^2 Z(j)}{\delta j_c(x)\delta j_d(y)}\right]_{j=0}\right\}$$
$$+O(N^{-1}). \tag{29}$$

Other Green's functions are obtained in the same way by repeated functional differentiation of (25) together with substitutions from functional derivatives of (26).

Equations (28) and (29) are the cornerstones of the theory in the large-N limit.[1] To exemplify their application we formulate the equations[1] which determine the effective potential[6] in this limit by restricting $\phi_a(x)$ to a constant field. We have $\Gamma(\phi) = -V(\phi)\int d^4x$, and write for the two-point function

$$G_{ab}(x,y) = \left[\frac{\delta^2 Z(j)}{\delta j_a(x)\delta j_b(y)}\right]_{j=0} = -\left[\frac{\delta^2\Gamma(\phi)}{\delta\phi_a(x)\delta\phi_b(y)}\right]^{-1}. \tag{30}$$

For constant ϕ, (28) reduces to[1]

$$\frac{\partial V(\phi)}{\partial\phi_a} - \frac{\partial V_0(\phi)}{\partial\phi_a} = -\tfrac{1}{6}i\lambda\,\phi_a G_{bb}(x,x). \tag{31}$$

Observe that (31) requires the trace of G_{ab}, as determined by (29), and that the terms in (29) proportional to δ_{ab} dominate the trace for N large. Thus for the purposes of computing the effective potential, (29) reduces to

$$G^{-1}{}_{ab}(x,y) = G_0{}^{-1}{}_{ab}(x,y) - \tfrac{1}{6}i\lambda\delta_{ab}\delta^4(x-y)\,G_{cc}(x,x), \tag{32}$$

where G_0 is the free propagator with mass $\mu^2 + \frac{1}{6}\lambda\phi^2$. When (32) is transformed to momentum space, it is easily solved for G_{bb}. The result combined with (31) when renormalized, leads to the "gap equation" of the model,[1]

$$\mathfrak{M}^2 = \overline{\mu}^2 + \tfrac{1}{6}\overline{\lambda}\left(1 + \frac{N\overline{\lambda}}{96\pi^2}\right)^{-1}\phi^2 + N\overline{\lambda}(96\pi^2 + N\overline{\lambda})^{-1}\mathfrak{M}^2 \ln\left(\frac{\mathfrak{M}^2}{\overline{\mu}^2}\right), \tag{33}$$

where

$$\mathfrak{M}^2 = 2\partial V(\phi^2)/\partial\phi^2,$$

$$\overline{\mu}^2 = \frac{\partial V}{\partial\phi^2}\bigg|_{\phi^2=0} = \text{finite},$$

and

$$\overline{\lambda} = 12\frac{\partial^2 V}{\partial(\phi^2)^2}\bigg|_{\phi^2=0} = \text{finite}.$$

The consequences of Eq. (33) have already been studied in detail,[1] so that we proceed no further. Finally we remark that (32) is not sufficiently accurate to evaluate the propagator in the case of spontaneous symmetry breakdown[1,7]; one must return to (29), coupled with a consideration of the four-point Green's function in the large-N limit.

The method outlined above makes possible a study of more complicated theories in the many-field limit without the necessity of a detailed analysis of graphs. As such, our machinery seems well suited to the analysis of gauge theories in this limit.

*Research supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)3230.

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Errata

Erratum: Mathematical structure of the Bethe-Salpeter equation for massless exchange reinvestigated [Phys. Rev. D 9, 2411 (1974)]

Marian Günther

(1) The right-hand side of Eq. (9), p. 2413, instead of

$$\frac{-c\cos\varphi}{\cos\varphi-\cos\psi},$$

should read

$$-\frac{c\sin\varphi}{\cos\varphi-\cos\psi}.$$

(2) The right-hand side of Eq. (187), p. 2438, instead of

$$-\frac{2\pi i}{t''}\left(-\frac{1}{t''v'_+v'_-}\right)^l,$$

should read

$$-\frac{2\pi i}{t''}\left(-\frac{u_-^2}{t''v''_+v''_-}\right)^l.$$

(3) In the abstract (p. 2411, lines 17 and 18) and again in the Introduction (p. 2412, second column, lines 18 and 19) the "relativistic Coulomb" propagator was referred to as half the difference between the advanced and retarded propagators, while—inconsistently with it—the same relativistic Coulomb propagator was referred to on p. 2436 (second column, lines 22–24) and in the sequel as half of the sum of the retarded and advanced propagators. Obviously, only the latter statement is correct if the definitions of both the advanced and retarded propagators involve the same direction of the integration with respect to p_0, i.e., from left to right. The author was momentarily misled by thinking of D (satisfying the homogeneous Klein-Gordon equation and equal to the well-known commutators of the free fields) as being defined (incorrectly, according to the now generally adopted conventions) as half of the sum of the retarded and advanced propagators.

Erratum: Addendum to Wilson's theory of critical phenomena and Callan-Symanzik equations in 4 − ϵ dimensions [Phys. Rev. D 9, 1121 (1974)]

E. Brezin, J. C. Le Guillou, and J. Zinn-Justin

There is a misprint in the numerical value of the integral J after formula (4), which should read $J = 0.7494...$ instead of $J = 1.7494....$