Eikonal model with covariant recoil in relativistic quantum field theory in terms of a correlation expansion

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A technique to derive covariant recoil to the eikonal model in relativistic quantum field theory is developed. Recoil is defined relative to the rectilinear-path approximation which gives the eikonal model. It appears in terms of a partially factorizable correlation expansion, where the first term is noncorrelated and is the usual eikonal model. The second term displays pair correlations, etc. By correlation here we mean dynamical correlation. Kinematical correlations due to four-momentum conservation are of course superimposed everywhere. We first consider soft recoil, and then generalize to the hard recoil, without introducing any cutoff. The technique is developed here for a scalar ϕ^3-type theory but it is general enough to be extended to, for example, quantum electrodynamics and scalar electrodynamics.

I. INTRODUCTION

The eikonal model in nonrelativistic quantum theory is a good approximation for elastic two-particle processes at high energy and small momentum transfer.[1] Recently eikonalization has also been studied in the framework of relativistic quantum field theory.[2-5] This might be viewed as a continuation of the question of Reggeization of field theories pursued during the last decade.[6] The primary motivation is to understand diffractive aspects of elastic high-energy scattering in strong interactions, as well as in weak-coupling theories like quantum electrodynamics (QED) and scalar electrodynamics, massive as well as massless. The strong interactions, because of the large coupling constant, must be totally summed up from a formal perturbation expansion. For weaker-coupling theories it shows that the significant high- or intermediate-energy behavior is not just the simple story of a single Born diagram, as (with minor corrections) was the case at low energies.

For example, in QED, if there were no other interaction but exchanges of photons (Fig. 1), the scattering amplitude would take the form

$$A(s;t) = sf(t), \tag{1.1}$$

which does not exceed the Froissart bound,[7] but has elastic-unitarity ambiguities. The form (1.1) is easily obtained from the eikonal approximation of generalized s ladders (Fig. 1), which then exponentiate after summation to infinite order. Other virtual self-energy corrections (Fig. 2) could be treated in the same simple manner,[8] but the addition of these corrections would not affect the form (1.1). At high energies production and annihilation of particles, beautifully accounted for in field theory by second quantization, contribute to the total cross section. Through unitarity, this is reflected in the absorptive part of the elastic amplitude, and therefore all significant high-s influence due to vacuum polarization must be evaluated. To restore unitarity one therefore considers multiexchanges of single chains (towers) (Fig. 3) giving $\sigma_{\text{tot}} = C\ln^2 s$. This "saturation" of the Froissart bound made many people take this scheme very seriously. However, as remarked in Ref. 9, interaction between individual chains can also give a significant contribution.

In the scalar ϕ^3 theory with a reasonable strong coupling constant the situation is even more delicate,[10] due to the problem of defining a rectilinear path for the throughgoing particles. Reasonable arguments have been given for the dominance of nested ladders (Fig. 4), giving rise to Mandelstam-type cuts,[11] over large classes of other diagrams.[12] However, here also we notice the crucial dependence on one single throughgoing large-momentum path. This implies that practically all production should occur in the pionization region,[13] which is hardly the physical situation. To include the possibility of fragmentation[14] of beam and target, factorization properties of the two latter effects from the pionization amplitude in a single ϕ^3 ladder have been studied.[10] To remove the noncovariant cutoffs, which must be introduced to separate "fragments" from "pions," it is also required that the fragmentation amplitudes be calculated to infinite order in the coupling constant. If correlation effects are totally neglected, an exponentiable tail of the latter effects is obtained, thus removing the cutoffs. If all significant physical information is retained in the fragmentation parts, they can only be worked out to some finite order, leaving a noncovariant cutoff dependence.

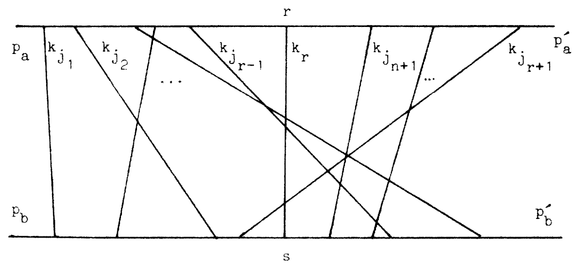

FIG. 1. A typical s-ladder diagram.

(A simple tentative reasoning tells that this is probably more realistic, since out of all dissociation products only a finite number of secondaries could be true fragments. Because of four-momentum conservation infinitely many must be soft "pions".) Nevertheless, we find this approach appealing since we believe that the above factorization properties should survive a better treatment. From a technical point of view the problem is essentially of a kinematical nature, as in infrared-divergent theories. Our previous experience from that problem was that a cutoff could be chosen such that the committed error was minimized, for limited maximal available energy in the process.[15] If, on the other hand, we minimize the error by some s dependence in the cutoff or in a typical mass involved in the factorization (this is necessary in high-energy processes) we get trapped by an undesired s dependence in the fragmentation amplitude, violating the Froissart bound. To sum up the shortcomings, the Regge-eikonal scheme in field theory suffers from lack of recoil, alternatively absence of correlation effects, or noncovariant cutoffs, strongly ordered graphs, and lack of crisscrosses. Besides, there is a general question of arbitrariness in summing relevant classes of diagrams. Some of these deficiencies might very well be symptoms of one and the same disease.

A positive feature is that Regge asymptotic behavior can be generated by the eikonal amplitude derived from field theory,

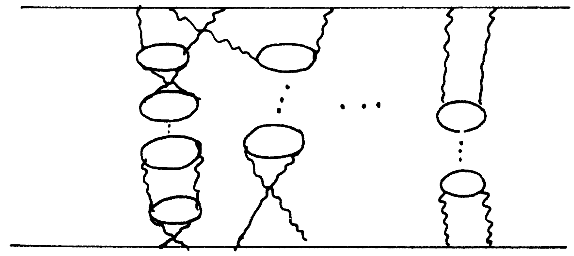

FIG. 3. A diagram with iterated exchanges of single chains (towers).

$$M(s;t)=4\pi i s\int_0^\infty b\,db\,J_0(b\sqrt{-t})(1-e^{i\chi(b;s)}), \qquad (1.2)$$

and that it provides a crude picture of the physical origin of a Pomeron. As indicated above, through the requirement of unitarity, eikonal models of the elastic amplitude provide an excellent tool for studies of multiparticle production processes, or in reverse, the increasing knowledge of multiparticle cross sections gives formidable information about elastic amplitudes. In fact, it has been possible to construct eikonal models for which the scattering operator satisfies full multiparticle unitarity in the direct channel,[16] for a large variety of production mechanisms. They admit a generalization including diffractive excitation of beam and target.[17] The competition of these fragmentation effects with the former multiperipheral effects under the restriction of unitarity is known from more direct phenomenological models.[18] Thus the field-theoretic approach to eikonal models could provide a discriminating language for model building in general. However, from the above-mentioned shortcomings we conclude that a more consistent treatment of the semiclassical limit necessarily goes via one solution or another to the recoil problem.

We here solve the problems for a generalized infinite s ladder within the scalar model $\mathcal{H}_I = g\phi_a^\dagger\phi_b\phi$ − H.c. This ladder, including all types of crisscrosses in contrast to a Bethe-Salpeter ladder, will be the basic ingredient in a

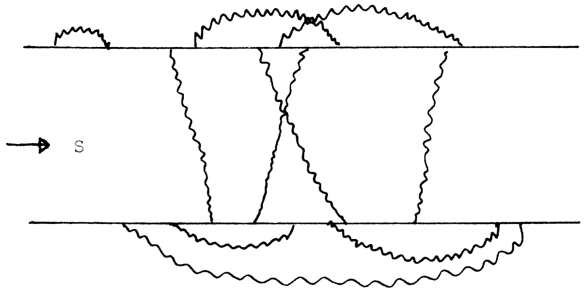

FIG. 2. A typical s-ladder diagram with virtual corrections.

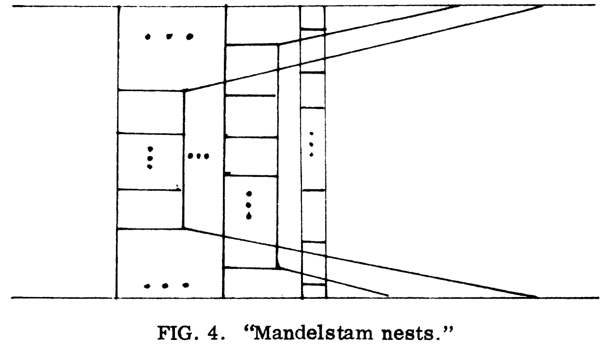

FIG. 4. "Mandelstam nests."

more thorough investigation of the full high-energy amplitude. Recoil is here defined relative to a straight-line path of the throughgoing particles and appears in terms of a partially factorizable dynamical correlation expansion. The first term in this expansion is the usual eikonal model, the second displays pair correlations, etc. A direct application to a generalized t ladder is obvious and the technique applies as well to, e.g., QED and scalar electrodynamics, with or without a mass gap. The solution is first derived by means of a soft-recoil theorem,[19] which is then generalized to the corresponding hard-recoil theorem. However, first we make a slight reformulation of the standard eikonal approach to a generalized s ladder without recoil.

II. THE GENERALIZED INFINITE LADDER WITHOUT RECOIL

The generalized infinite ladder of simple particle exchanges in the s channel (Fig. 1), gives the simplest form of a relativistic eikonal model. The technical clue to eikonalization in this diagrammatic approach is the factorization by an infinite-momentum approximation of the scattered throughgoing particles, or alternatively a soft approximation of the exchanged momenta in the ladders. In a naive manner, part of the amplitude could then be written

$$\sum_{\text{perm}} g^n \prod_{r=1}^{n} (p_i^2 - m_i^2 + 2p_i\cdot K_r + K_r^2)^{-1} \to g^n \prod_{r=1}^{n} (2p_i\cdot k_r)^{-1}, \quad (2.1)$$

where p_i is the external momentum of the ith particle and K_r is a sum $K_r = \sum_{l=1}^{r} k_l$ of exchanged momenta. The eikonal model is then obtained by summation to infinite order in the coupling constant. It is easily seen that k_i^2 terms can be kept in (2.1), whereas $k_i\cdot k_j$ terms with $i\neq j$ spoil the factorization. By use of a technique similar to that which we used in the generalized recoil theorem,[19] we here derive the recoil due to such terms. In spite of the fact that the recoil theorem[19] is derived for soft real quanta, we here demonstrate that it can be generalized for hard virtual quanta (and soft as well, of course). The generalization to the hard spectrum will become clear once we demonstrate the partial-factorization property of the recoil terms, which induce correlation effects. Recoil is here defined relative to the straight-line-path approximation of the throughgoing particles. The same approximation also gives rise to an amplitude totally factorizable in virtual momenta, which tells that the correlation effects due to the above defined recoil are essentially of dynamical nature. The kinematical correlation due to over-all four-momentum conservation is, of course, still present. By elimination of this, the characteristic subtraction (or addition) by one in (1.2) appears. Notations are chosen as close as possible to Lévy and Sucher.[3] Throughgoing masses are denoted m and exchanged ones μ. Since the recoil theorem[19] is valid for any strength of the coupling constant, we just require that g is finite and real. Our working hypothesis is that the correlation expansion decreases with increasing order of correlation, and that therefore some finite order will be sufficient in most physical situations.

Consider a ladder diagram, where $n+1$ quanta with momenta $k_{j_1}, k_{j_2}, \ldots, k_{j_{n+1}}$ are exchanged (Fig. 1). The corresponding amplitude is given by

$$-iM_{n+1}(s;t) = \frac{(-ig)^{2n+2}}{(n+1)!} \int \prod_{l=1}^{n+1} \frac{d^4k_{j_l}}{(2\pi)^4} \Delta_F(k_{j_l}) I (2\pi)^4 \times \delta^4\left(q - \sum_{l=1}^{n+1} k_{j_l}\right), \quad (2.2)$$

where $\tilde{\Delta}_F(k) = i(k^2-\mu^2+i\epsilon)^{-1}$ is the meson propagator, and I is a sum of a- and b-particle propagator products $\Delta_F(p) = i(p^2-m^2+i\epsilon)^{-1}$;

$$q = p_a - p_{a'} = p_{b'} - p_b, \quad (2.3)$$

$$s = (p_a+p_b)^2, \quad (2.4)$$

$$t = q^2. \quad (2.5)$$

The δ function is used to eliminate the rth-momentum dependence and I is then given by

$$I_r = I_r^a \sum_{\Pi} I_r^b(\Pi), \quad (2.6)$$

where

$$I_r^a = \Delta_F(p_a - k_{j_1}) \cdots \Delta_F(p_a - k_{j_1} - \cdots - k_{j_{r-1}}) \times \Delta_F(p_a' + k_{j_{r+1}}) \cdots \Delta_F(p_a' + k_{j_{r+1}} + \cdots + k_{j_{n+1}}). \quad (2.7)$$

Similarly, for a given permutation Π, $I_r^b(\Pi)$ is a product of b-particle propagators. In order to restore certain symmetry properties, which are lost with the special choice of r, we must sum over r:

$$I = \sum_{r=1}^{n+1} I_r. \quad (2.8)$$

Further we must sum over the set D_a of the $(r-1)!$ permutations within the $(r-1)$ quanta emitted before r and the set $D_{a'}$ of the $(n-r+1)!$ permutations within the $(n-r+1)$ quanta emitted after r. We should then also sum over the factor set

$$D_{\mathrm{I}}^{A}=\frac{D_n}{D_a \otimes D_{a'}}\ , \tag{2.9}$$

where D_n is the set of all $n!$ permutations within all $n+1$ quanta except the rth. Thus I in (2.2) should be replaced by the symmetrized expression

$$\begin{aligned} I_{\mathrm{sym}} &= \sum_{r=1}^{n+1} I_{r\ \mathrm{sym}} \\ &= \sum_{r=1}^{n+1} I^{a}_{r\ \mathrm{sym}} \sum_{\Pi} I^{b}_{r}(\Pi)\ , \end{aligned} \tag{2.10}$$

where $I^{a}_{r\ \mathrm{sym}}$ is given by

$$I^{a}_{r\ \mathrm{sym}} = \sum_{D_{\mathrm{I}}^{A} D_a D_{a'}} I^{a}_{r}\ . \tag{2.11}$$

As is seen in Fig. 1 the rth quantum is absorbed at the sth vertex by the b particle. Accordingly, there are $(s-1)$ quanta being absorbed before s, within which we have a set D_b of $(s-1)!$ permutations. Then there are $(n-s+1)$ quanta absorbed after s with a set $D_{b'}$ of $(n-s+1)!$ permutations. Let s_1 be the number of quanta going from a to b without crossing the rth line, and s_2 the number of quanta emitted from a', crossing the rth line and then being absorbed by b. The corresponding permutation sets are denoted respectively D_{ab}, $D_{a'b}$ (Fig. 5). Clearly then we must sum over s_1 and s_2 such that s defined by $s_1+s_2=s-1$ runs from 1 to $n+1$, or, alternatively, first over s_1 and then over s with the given restriction.

In order to include all possible permutations we should then also sum over the factor set $D_{\mathrm{I}}^{B}=D_n/(D_b \otimes D_{b'})$. However, because the quanta absorbed on the bb' side are the same objects as those emitted from the aa' side, D_{I}^{B} is summed over by summation over D_{I}^{A} except for the factor sets of permutations D_{II} and D_{III} defined by (notice that D_a and $D_{a'}$ are not in D_{I}^{A})

$$\begin{aligned} D_{\mathrm{II}} &= D_a/(D_{ab}\otimes D_{ab'})\ , \\ D_{\mathrm{III}} &= D_{a'}/(D_{a'b}\otimes D_{a'b'})\ , \end{aligned} \tag{2.12}$$

where the meaning of $D_{ab'}$ and $D_{a'b'}$ is obvious from the above definitions of D_{ab} and $D_{a'b}$ and Fig. 5. This is because the summation over D_a, $D_{a'}$, D_b, and $D_{b'}$ was performed without exchanges over the rth line. Some of these exchanges are taken care of by D_{II} and D_{III}, given in (2.12), and the rest by D_{I}^{A} given by (2.9). This could also be seen as follows: D_{II} corresponds to the

$$\binom{r-1}{s_1}$$

various ways the s_1 quanta could be connected with the $(r-1)$ quanta before r, and D_{III} corresponds to the

$$\binom{n-r+1}{s_2}$$

ways to connect the s_2 quanta with the $(n-r+1)$ quanta emitted after r. From now on, the coupling constant will be included in the propagators:

$$\begin{aligned} I_{r\ \mathrm{sym}} = &\sum_{\substack{s_1 s_2 \\ D_{\mathrm{I}}^{A} D_{\mathrm{II}} D_{\mathrm{III}}}} \ \sum_{D_a D_{a'}} \prod_{l=1}^{r-1} A_{j_l} \prod_{l=r+1}^{n+1} A'_{j_l} \\ &\times \sum_{D_b D_{b'}} \prod_{l=1}^{s-1} B_{j_l} \prod_{l=s+1}^{n+1} B'_{j_l}\ , \end{aligned} \tag{2.13}$$

where

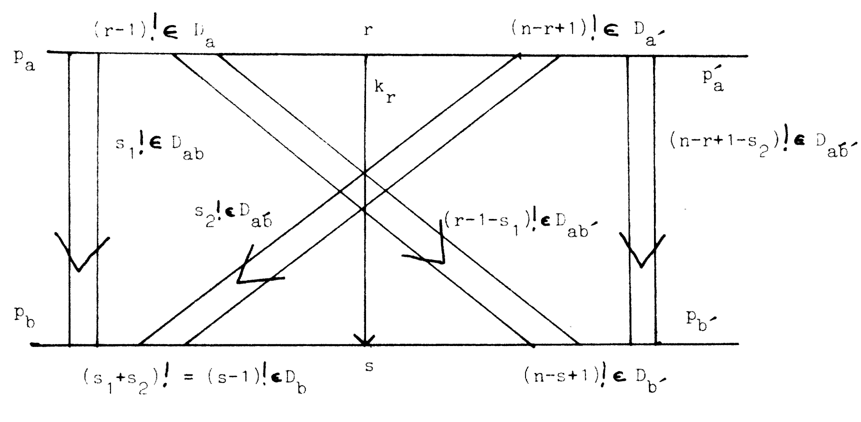

FIG. 5. Schematic picture of the permutation sets relative to the rth quantum.

$$A_{j_l}=\frac{ig}{-2p_a\cdot K_{j_l}+K_{j_l}{}^2+i\epsilon}\;,\quad K_{j_l}=\sum_{t=j_1}^{j_l}k_t\,,\qquad l=1,\ldots,r-1 \tag{2.14}$$

$$A'_{j_l}=\frac{ig}{2p'_a\cdot K_{j_l}+K_{j_l}{}^2+i\epsilon}\;,\quad K_{j_l}=\sum_{t=j_{r+1}}^{j_l}k_t\,,\qquad l=r+1,\ldots,n+1 \tag{2.15}$$

$$B_{j_l}=\frac{ig}{2p_b\cdot K_{j_l}+K_{j_l}{}^2+i\epsilon}\;,\quad K_{j_l}=\sum_{t=j_1}^{j_l}k_t\,,\qquad l=1,\ldots,s-1 \tag{2.16}$$

$$B'_{j_l}=\frac{ig}{-2p'_b\cdot K_{j_l}+K_{j_l}{}^2+i\epsilon}\;,\quad K_{j_l}=\sum_{t=j_{s+1}}^{j_l}k_t\,,\qquad l=s+1,\ldots,n+1\,. \tag{2.17}$$

As a trivial check we now apply the eikonal approximation, which is essentially to collect all terms of order $g^{2n}k^{-2n}$. The derivation will be carried out in some detail in order to see how things generalize when recoil corrections are included. Summing over D_a, $D_{a'}$, D_b, and $D_{b'}$ (2.10) easily factorizes to

$$I_{r\,\mathrm{sym}}=\sum_{\substack{s_1s_2\\ D_{\mathrm{I}}^AD_{\mathrm{II}}D_{\mathrm{III}}}}\prod_{l=1}^{r-1}f_{j_l}^{0a}\prod_{l=r+1}^{n+1}f_{j_l}^{0a'}\prod_{l=1}^{s-1}f_{j_l}^{0b*}\prod_{l=s+1}^{n+1}f_{j_l}^{0b'*}\,, \tag{2.18}$$

with the currents f^0 defined as in Ref. 19,

$$f^0(p;k)=\frac{ig}{2\epsilon_ip_i\cdot k+i\epsilon}\,. \tag{2.19}$$

The ϵ_i is +1 for outgoing particles and −1 for ingoing particles. If the quanta are absorbed, then clearly $k\to-k$ and

$$f^0(p;\,-k)=f^{0*}(p;\,k). \tag{2.20}$$

According to the above scheme of connecting the different quanta (see also Fig. 5), and after regrouping the elements we obtain

$$I_{r\,\mathrm{sym}}=\sum_{D_{\mathrm{I}}^A}\sum_{\substack{s_1\\D_{\mathrm{II}}}}\prod_{l=1}^{s_1}f_{j_l}^{0a}f_{j_l}^{0b*}\prod_{l=s_1+1}^{r-1}f_{j_l}^{0a}f_{j_l}^{0b'*}\times\sum_{\substack{s_2\\D_{\mathrm{III}}}}\prod_{l=r+1}^{r+s_2}f_{j_l}^{0a'}f_{j_l}^{0b*}\prod_{l=r+s_2+1}^{n+1}f_{j_l}^{0a'}f_{j_l}^{0b'*}\,. \tag{2.21}$$

For fixed r, s_1, and s_2 we then sum over D_{I}^A, D_{II}, and D_{III}:

$$I_{r\,\mathrm{sym}}=\binom{n}{r-1}\sum_{s_1s_2}\binom{r-1}{s_1}\prod_{l=1}^{s_1}f_l^{0a}f_l^{0b*}\times\prod_{l=s_1+1}^{r-1}f_l^{0a}f_l^{0b'*}\binom{n-r+1}{s_2}\times\prod_{l=r+1}^{r+s_2}f_l^{0a'}f_l^{0b*}\prod_{l=r+s_2+1}^{n+1}f_l^{0a'}f_l^{0b'*}\,. \tag{2.22}$$

After summation over s_1 and s_2 and by use of the binomial theorem we then find

$$I_{r\,\mathrm{sym}}=\binom{n}{r-1}\prod_{l=1}^{r-1}f_l^{0a}(f^{0b}+f^{0b'})_l^*\times\prod_{l=r+1}^{n+1}f_l^{0a'}(f^{0b}+f^{0b'})_l^*\,, \tag{2.23}$$

which, after relabeling of the last $n-r+1$ currents by letting $l\to l-1$, repeated use of the binomial theorem, and summation over r, factorizes to

$$I_{\mathrm{sym}}=\sum_{r=0}^{n}I_{r\,\mathrm{sym}}=\prod_{l=1}^{n}J_l^{0A}J_l^{0B*}\,. \tag{2.24}$$

The currents J^0 are defined as in Ref. 19,

$$J_l^0=f_l^0+f_l^{0'}\,. \tag{2.25}$$

We then define a scalar product for the uncorrelated currents

$$U(x;\,s;\,t)=i\int\frac{d^4k}{(2\pi)^4}\,\tilde{\Delta}_F(k)e^{-ik\cdot x}J^{0A}(k)\cdot J^{0B*}(k)\,. \tag{2.26}$$

Summing over n to infinite order in the coupling constant we get the usual eikonal result

$$M(s;\,t)=ig^2\int d^4x\,e^{iq\cdot x}\Delta_F(x)\sum_{n=0}^{\infty}\frac{(iU)^n}{(n+1)!}=g^2\int d^4x\,e^{iq\cdot x}\Delta_F(x)\frac{e^{iU}-1}{iU}\,. \tag{2.27}$$

III. SOFT RECOIL

The totally factorizable and exponentiable form (2.27) thus obtains if all $k_i\cdot k_j$ terms in (2.2) are neglected. We here approach this problem by first making a soft-recoil expansion around $p_i{}^2=m_i{}^2$ (no internal mass excitations). The clue to this problem appears already by a first-order recoil expansion, where all terms of essential order $g^{2n}k^{-2n+1}$ must be carried out. They prove to be partially factorizable from the rest of the expansion, which is completely factorizable and still summable to infinite order in the coupling constant. It is clear that a first-order recoil expansion is only valid in a certain ϵ domain in k space around $p_i^2 = m_i^2$, and we therefore must include all orders in the recoil expansion. As mentioned in Sec. II, k_1^2 terms can be kept without spoiling the factorization in (2.18). This means, as we show later, that noncorrelated recoil can be kept to all orders in the recoil expansion, still giving a closed form (2.27), but now with slightly modified currents. A consequence of this is that the scalar product (2.26) becomes finite also for $t \neq 0$.

We then demonstrate how pair-correlation effects to all orders in the recoil expansion factorize partially, and we demonstrate this property in a nonexpanded form. This enlarges the above ϵ domain for both uncorrelated and pair-correlated currents to the whole of k space. The complete recoil then of course includes triple and higher correlations, but for the moment we shall discard them, in line with our working hypothesis that some few orders of correlations will be sufficient in a large variety of physical situations. We first concentrate on the soft-pair correlation, meanwhile discarding k_i^2 terms since they are easily included later.

Straightforward application of the generalized recoil theorem[19] to (2.13) and summation over the permutation sets D_a, $D_{a'}$, D_b, and $D_{b'}$ gives the first-order recoil

$$
\begin{aligned}
I^R_{r\,\mathrm{sym}} = \sum_{\substack{s_1 s_2\\ D_{\mathrm{I}}^A D_{\mathrm{II}} D_{\mathrm{III}}}} \Bigg[& \Bigg(\sum_{\substack{p<q\\1}}^{r-1} \chi^{0a}_{j_p j_q} \prod_{l \neq pq}^{r-1} f^{0a}_{j_l} \Bigg) \prod_{l=r+1}^{n+1} f^{0a'}_{j_l} \prod_{l=1}^{s-1} f^{0b*}_{j_l} \prod_{l=s+1}^{n+1} f^{0b'*}_{j_l} + \prod_{l=1}^{r-1} f^{0a}_{j_l} \Bigg(\sum_{\substack{p<q\\r+1}}^{n+1} \chi^{0a'}_{j_p j_q} \prod_{\substack{l \neq pq\\r+1}}^{n+1} f^{0a'}_{j_l} \Bigg) \prod_{l=1}^{s-1} f^{0b*}_{j_l} \prod_{l=s+1}^{n+1} f^{0b'*}_{j_l} \\
& + \prod_{l=1}^{r-1} f^{0a}_{j_l} \prod_{l=r+1}^{n+1} f^{0a'}_{j_l} \Bigg(\sum_{\substack{p<q\\1}}^{s-1} \chi^{0b}_{j_p j_q} \prod_{\substack{l \neq pq\\1}}^{s-1} f^{0b*}_{j_l} \Bigg) \prod_{l=s+1}^{n+1} f^{0b'*}_{j_l} + \prod_{l=1}^{r-1} f^{0a}_{j_l} \prod_{l=r+1}^{n+1} f^{0a'}_{j_l} \prod_{l=1}^{s-1} f^{0b*}_{j_l} \Bigg(\sum_{\substack{p<q\\s+1}}^{n+1} \chi^{0b'}_{j_p j_q} \prod_{\substack{l \neq pq\\s+1}}^{n+1} f^{0b'*}_{j_l} \Bigg) \Bigg].
\end{aligned}
\tag{3.1}
$$

The pair currents are given by

$$
\chi^{0i}_{pq} = -(ig)^2 \epsilon_i \frac{k_p \cdot k_q}{p_i \cdot (k_p + k_q)} \frac{1}{2p_i \cdot k_p} \frac{1}{2p_i \cdot k_q}, \qquad i = a, a' \tag{3.2}
$$

and

$$
\chi^{0i}_{pq} = +(ig)^2 \epsilon_i \frac{k_p \cdot k_q}{p_i \cdot (k_p + k_q)} \frac{1}{2p_i \cdot k_p} \frac{1}{2p_i \cdot k_q}, \qquad i = b, b'. \tag{3.3}
$$

In Appendix A we regroup the elements, exactly as in the uncorrelated case in Sec. II, with respect to s_1, s_2, and the rth line (Fig. 5) giving (A1). Under the integral we can relabel $r \to r+1$, and for the last $n - r + 1$ currents we let $l \to l - 1$. Summation over s_1, s_2, D_{II}, and D_{III} then gives

$$
I^R_{\mathrm{sym}} = \sum_{r=0}^{n} \sum_{\substack{p<q\\1}}^{n} (\chi^{0A}_{pq} J^{0B*}_p J^{0B*}_q + \chi^{0B}_{pq} J^{0A}_p J^{0A}_q) \times \sum_{D_{\mathrm{I}}^{A\,pq}} \Bigg(\prod_{l=1}^{r} f^{0a}_{j_l} J^{0B*}_{j_l} \prod_{l=r+1}^{n} f^{0a'}_{j_l} J^{0B*}_{j_l} \Bigg)_{l \neq pq}. \tag{3.4}
$$

The permutation set $D_{\mathrm{I}}^{A\,pq}$ is the factor set $D_{\mathrm{I}}^{A}/\mathrm{perm}(pq)$. The total pair-correlation currents are defined by

$$
\chi^{0Q}_{pq} = \sum_{i \in Q} \chi^{0i}_{pq}, \quad Q = A, B. \tag{3.5}
$$

Summation on r then gives I^R_{sym}, which we must add to (2.24),

$$
I^R_{\mathrm{sym}} = \sum_{\substack{p<q\\1}}^{n} (\chi^{0A}_{pq} J^{0B*}_p J^{0B*}_q + \chi^{0B}_{pq} J^{0A}_p J^{0A}_q) \prod_{\substack{l \neq pq\\1}}^{n} J^{0A}_l J^{0B*}_l. \tag{3.6}
$$

When inserted in the $(n+1)$th-order generalized ladder (2.10) we can utilize this partial decoupling of coordinates and extract a form-invariant pair-correlation functional

$$
\begin{aligned}
P(x; s; t) = i^2 \int \frac{d^4 k_p}{(2\pi)^4} \frac{d^4 k_q}{(2\pi)^4} \Delta_F(k_p) \Delta_F(k_q) e^{-i(k_p + k_q)\cdot x} & \times [\chi^{0A}(k_p k_q) J^{0B*}(k_p) J^{0B*}(k_q) \\
& + \chi^{0B}(k_p k_q) J^{0A}(k_p) J^{0A}(k_q)],
\end{aligned}
\tag{3.7}
$$

which together with the scalar product (2.26) inserted in (2.2) gives

$$
-iM_{n+1}(s; t) = \frac{g^2}{(n+1)!} \int d^4 x\, e^{iq\cdot x} \Delta_F(x) \times \left[(iU)^n + \binom{n}{2} i^2 P (iU)^{n-2} \right]. \tag{3.8}
$$

The number of possible pair correlations $\binom{n}{2}$ does not spoil the explicit summation. Reversing summation and integration we obtain the closed form

$$M(s;t)=ig^2\int d^4x\,e^{iq\cdot x}\Delta_F(x)\times\left\{\frac{e^{iU}-1}{iU}+\frac{i^2P}{2(iU)^2}\left[(iU-2)e^{iU}+2\frac{e^{iU}-1}{iU}\right]\right\}. \tag{3.9}$$

Before generalizing to the corresponding hard formula we notice that if the rth quantum is not identical to the others, we should divide by $n!$ instead of $(n+1)!$ in (3.8) and would then obtain

$$M(s;t)=\int d^4x\,e^{iq\cdot x}\mathcal{T}(x)e^{iU}\left(1+\frac{i^2}{2!}P\right). \tag{3.10}$$

This form requires the presence of at least one totally different quantum or other different mechanism T. Therefore T could be regarded as the initiator of the whole process, whereas in (3.9) the initiator could be any one of the exchanged quanta. At least one quantum must be exchanged in order for something at all to happen, therefore the subtraction by one in (2.27). If $\mathcal{T}$ displays the exchange of a scalar particle, then it is given by

$$\mathcal{T}(x)=ig_s{}^2\Delta_F(x). \tag{3.11}$$

IV. HARD RECOIL

Integrating (2.26) and (3.7) over the whole k space we notice that the former is logarithmically divergent and the latter is linearly divergent, at least for $x=0$. This is due to the fact that we have taken only the first term in the recoil expansion, e.g., the soft approximation, which requires some type of noncovariant ϵ cutoff in the spatial k space. To enlarge the integrations in (2.26) and (3.7) to the whole k space we must find the full uncorrelated and pair-correlated currents compatible with (2.13), to all orders in the recoil expansion.

We first demonstrate this for the uncorrelated currents. For a prong i emitting n quanta the amplitude is proportional to

$$A_n^i=\sum_{\Pi}\prod_{l=1}^{n}A_{j_l}^i(\Pi), \tag{4.1}$$

where A_j is given by (2.14) and (2.15). For $n=1$ this trivially gives the uncorrelated current to all orders in the recoil expansion:

$$A_1^i=\frac{ig}{2\epsilon_i p_i\cdot k_1+k_1{}^2+i\epsilon}=f_1^i,\qquad J_l^Q=\sum_{i\in Q}f_l^i,\quad Q=A,B. \tag{4.2}$$

This is just a modification of the soft currents (2.19) and (2.25). For $n=2$ we obtain

$$A_2=\frac{ig}{2\epsilon_1 p_i\cdot k_1+k_1{}^2}\,\frac{ig}{2\epsilon_i p_i\cdot k_2+k_2{}^2}\times\left[1-\frac{2k_1\cdot k_2}{2\epsilon_i p_i\cdot(k_1+k_2)+k_1{}^2+k_2{}^2}\right], \tag{4.3}$$

from which factorization of the currents (4.2) is evident and where the pair-correlation current can be read off directly:

$$\chi_{pq}'^{\,i}=-x_{pq}^i f_p^i f_q^i, \tag{4.4}$$

with

$$x_{pq}^i=\frac{2k_p\cdot k_q}{2\epsilon_i p_i\cdot(k_p+k_q)+k_p{}^2+k_q{}^2}. \tag{4.5}$$

The nth-order formula follows from (B6):

$$A_n^i=\prod_{l=1}^{n}f_l^i+\sum_{\substack{p<q\\1}}^{n}\chi_{pq}'^{\,i}\prod_{\substack{l\neq pq\\1}}^{n}f_l^i. \tag{4.6}$$

Thus partial factorization of soft pair effects survives when uncorrelated recoil to all orders in the recoil expansion is included, although the form of (4.5) is somewhat modified compared to (3.2). The currents (4.2) are then the total uncorrelated currents compatible with (2.13), since all other terms in the recoil expansion consist of higher-order correlations.

We then extract all pair-correlating effects from the all-orders recoil expansion of (2.13). Neglecting all higher-order correlations than pair correlations for n emitted quanta from the ith prong, the mth-order recoil can be written in the form

$$R_n^{i\,(m)}=\prod_{l=1}^{n}f_l^i\left[(-1)^m\sum_{\substack{p<q\\1}}^{n}(x_{pq}^i)^m+\text{higher correlations}\right]. \tag{4.7}$$

As is shown in Appendix C, Eq. (C12), this holds for $|x_{pq}^i|<1$, but in (C13)–(C15) we demonstrate that the summed form

$$M_n^i=\prod_{l=1}^{n}f_l^i\sum_{\substack{p<q\\1}}^{n}\frac{1}{1+x_{pq}^i} \tag{4.8}$$

is valid for all x's. The full pair-correlation currents are thus given by

$$\chi_{pq}^i=\chi_{pq}'^{\,i}\frac{1}{1+x_{pq}^i},\quad \chi_{pq}^Q=\sum_{i\in Q}\chi_{pq}^i,\quad Q=A,B \tag{4.9}$$

since all other terms in the recoil expansion consist of higher-order correlations.

Inserting (4.2) and (4.9), we are again led to (3.8) and (3.9) or (3.10) now with modified U and P given by

$$U(x;s;t)=i\int\frac{d^4k}{(2\pi)^4}\tilde{\Delta}_F(k)e^{-ik\cdot x}J^A(k)J^B(k) \tag{4.10}$$

and

$$P(x;s;t)=i^2\int\frac{d^4k_p}{(2\pi)^4}\frac{d^4k_q}{(2\pi)^4}\tilde{\Delta}_F(k_p)\tilde{\Delta}_F(k_q)e^{-i(k_p+k_q)\cdot x}\times(\chi^A_{pq}J^B_pJ^B_q+\chi^B_{pq}J^A_pJ^A_q+\chi^A_{pq}\chi^B_{pq}),\tag{4.11}$$

where the logarithmic and the linear divergences, respectively, are now removed. The integration is now extended from the former ϵ domain to the whole k space, thus circumventing the noncovariant cutoff problem. Simple power counting in (4.10) and (4.11) shows that they are well behaved in the ultraviolet limit with divergency indices $D_U=-2$ and $D_P=-4$. The third term in (4.11) is of course not present in (3.7), since (3.7) is derived in a soft first-order recoil expansion.

In order to get an idea of how higher-order correlation effects appear in this expansion, we notice that in a second-order soft-recoil expansion triple and double pair correlations also occur. From (B15) we get

$$R_n^{i(2)}=\prod_{t=1}^{n}f_t^i\left[(-1)^2\sum_{\substack{p<q\\1}}^{n}(x^i_{pq})^2+\sum_{\substack{p<q<l\\1}}^{n}x'^i_{pql}+\frac{1}{2}\sum_{(p<q)\neq(k<l)}^{n}x^i_{pq}x^i_{kl}\right],\tag{4.12}$$

where n is again the number of emitted quanta from the ith prong, e.g., the order of the coupling constant to be summed over at the end. Concentrating on triple-correlation effects we notice that the exact triple-correlation current to all orders in the recoil expansion should already appear for $n=3$, as the difference between (C13) and the two first terms in (C15). Inserted in the infinite generalized ladder (2.13), it induces a triple-correlation function of the type

$$T^i(x;s;t)=i^3\int\frac{d^4k_p\,d^4k_q\,d^4k_l}{(2\pi)^4(2\pi)^4(2\pi)^4}\tilde{\Delta}_F(k_p)\tilde{\Delta}_F(k_q)\times\tilde{\Delta}_F(k_l)e^{-i(k_p+k_q+k_l)\cdot x}\times J^B_pJ^B_qJ^B_lx^i_{pql}J^A_pJ^A_qJ^A_l.\tag{4.13}$$

To get the total triple-correlation effect for $i\in A(B)$ we must consider all combinations of uncorrelated, pair-, and triple-correlation effects in the noncorrelated B (A) currents, similar to the third term in (4.11). This does not spoil the partial factorization, which means total factorization except for arbitrary three quanta. Similar to (3.8) we get

$$-iM_{n+1}(s;t)=\frac{g^2}{(n+1)!}\times\int d^4x\,e^{iq\cdot x}\Delta_F(x)\times\left[(iU)^n+\binom{n}{2}i^2P(iU)^{n-2}+\binom{n}{3}i^3T(iU)^{n-3}\right],\tag{4.14}$$

where $k!$ in the $\binom{n}{k}$ factors mirrors the independence of intrinsic order among correlating quanta. Summing up (4.14) we get

$$M(s;t)=ig^2\int d^4x\,e^{iq\cdot x}\Delta_F(x)\left\{\frac{e^{iU}-1}{iU}+\frac{i^2P}{2!(iU)^2}\left[(iU-2)e^{iU}+2\frac{e^{iU}-1}{iU}\right]+\frac{i^3T}{3!(iU)^3}\left([6-3iU+(iU)^2]e^{iU}-6\frac{e^{iU}-1}{iU}\right)\right\}.\tag{4.15}$$

If $|U|<1$, then some few terms in the kernel expansion are significant,

$$I(x;s;t)=[1+\tfrac{1}{2}iU+\tfrac{1}{6}(iU)^2+\tfrac{1}{24}(iU)^3+\tfrac{1}{120}(iU)^4+\tfrac{1}{720}(iU)^5+\cdots]+i^2P[\tfrac{1}{6}+\tfrac{1}{8}iU+\tfrac{1}{20}(iU)^2+\tfrac{1}{36}(iU)^3+\cdots]+i^3T[\tfrac{1}{24}+\tfrac{1}{30}iU+\tfrac{1}{72}(iU)^2+\cdots],\tag{4.16}$$

whereas if $|U|\geq1$, we must stick to the closed form (4.15). In the latter case we notice that the strength of the coupling constant alone is not a relevant measure of the strength of the correlation, but rather the summed factor which multiplies P in (4.15).

If one of the quanta is different, the δ function in (2.2) could be eliminated via this quantum giving a third term in the expansion (3.10):

$$M(s;t)=\int d^4x\,e^{iq\cdot x}J(x)e^{iU}\left(1+\frac{i^2}{2!}P+\frac{i^3}{3!}T+\cdots\right).\tag{4.17}$$

In this form the strength of correlations are partially determined by the strength of the coupling constant to a power twice the order of correlation, irrespective of whether it is a weak- or strong-coupling theory. In the form (4.15) this is only true for the weak-coupling case. In the strong-coupling case (4.15) and (4.17) then require a separate thorough investigation to see if our working hypothesis is correct, or rather to check which models have the property of a rapidly decreasing correlation expansion. It should then be remembered that any other possible link between the two correlating quanta in the above ladder expansion will contribute to the dynamical pair correlations. For example, we could make a t-ladder insertion in this special pair and apply the same technique to this.

This will be discussed further in the summary. We here content ourselves with the first "correction" term to the eikonal result and therefore just drop higher than pair-correlated terms. However, as we saw in Appendix C, we could also correct for triple correlations without too much complication, and in principle to an arbitrary high order, thereby defining what we would like to call the correlation expansion. Its correspondence to the ordinary perturbation expansion will be discussed in a subsequent paper on scalar electrodynamics.[20]

V. SUMMARY

A technique to derive covariant recoil to the eikonal model has been developed. The obtained result is no longer crucially dependent on a rectilinear path approximation or a strict infinite momentum limit. Although derived here for the technically simple ϕ^3 theory it is general enough to be applied to, for example, QED and scalar electrodynamics, massive as well as massless.

Recoil appears in terms of a dynamical correlation expansion in which the pure eikonal model is the first term. Thus recoil is here defined relative to the straight-line-path approximation for the throughgoing particles, since this gives the eikonal model. The second term in the above expansion displays pair-correlation effects, etc. By kinematical correlations we mean here those due to over-all four-momentum conservation, which is superimposed everywhere.

Let us for a moment rotate the s ladder into an infinite generalized t ladder, which is the simplest multiperipheral production mechanism.[21] In principle, it should then be possible to relate our correlations to the usual correlations of emitted particles in multiparticle processes. (Usually, in a multiperipheral model one discards not only the dynamical, but also the kinematical correlations.) It therefore provides a method to construct more realistic "gas" models[22–24] guided by relativistic quantum field theory, without appealing to the partition-function technique[25] in deriving dynamical correlations (other than those due to energy-momentum conservation). The dynamical correlations could be thought of as some "collision" mechanism in the "gas." It is clear that the above scalar ϕ^3 theory is probably too simple a structure for high-energy hadron physics. However, it will no doubt provide interesting information on correlations between clusters in more realistic models. Going back to the s ladder, it should be mentioned here that a t-ladder insertion between the two correlating quanta would give a more complete pair-correlation effect. However, here we restrict to the simple ladder structure itself, since such a second ladder insertion is managed with the same technique.

The above collision idea is not new, but from the point of view of field theory we were earlier trapped by the fact that we could just derive some few orders in the hard spectrum. To derive a result to infinite order in the coupling constant, we previously had to apply some semiclassical or eikonal type of approximation, with a corresponding large loss of information in the original model. In a Bethe-Salpeter approach we must neglect crisscrosses and transverse momenta. To speak in terms of information theory, the above collision mechanism renders the gas a certain mean free path, e.g., periodicity or short-range order. Accordingly, we could then also expect a corresponding reduction of coordinates. In fact something similar takes place here. The various correlation effects partially factorize out from the full amplitude and the correlation functionals are form-invariant, irrespective to which individual quanta are correlating. Because of these two properties we can now sum to infinite order in the coupling constant.

As we will see in Ref. 21 this closed expression for the elastic amplitude, because of its explicit form, can be used to derive directly measurable correlations. It therefore provides an effective tool with which to discriminate among various models. In the case of a rapidly decreasing correlation expansion, this provides an alternative to the Bethe-Salpeter equation where we can now retain all crisscrosses and still obtain a closed explicit result (no recursion formula). Another good feature is that in this method we can clearly see the separation between dynamical and kinematical correlations.

It should further be noticed that the noncorrelated currents, physically significant in the infrared (IR) part of the spectrum, and the nontrivial correlations, displaying residual hard effects, are factorized and separated without any use of cutoff. In our approach there is no difference between soft and hard quanta in an individual meaning, since all quanta are both soft and hard in that they are present in both parts of spectrum. All quanta have a totally factorizable tail (exponentiable) and their hard residual effects appear in terms of the above correlation expansion, a relativistic invariant series representing two-, three-particle interactions, etc. (As mentioned above, then we must consider all possible links between the pairs, triplets, etc.)

Nevertheless, from a practical point of view we could still speak of soft and hard quanta, not as individuals but as effective interacting quanta. This is because in the uncorrelated currents (4.2) there is a covariant cutoff automatically built in through the k^2 term in the denominator. In the corresponding QED-variant form, this has been known since 1961 as Yennie's form.[26,27] This form can be exploited at low and intermediate energies in weak-coupling theories where some few orders or even the first-order Born approximation of the hard spectrum give a good result. At extremely high energies, however, from tower insertions, etc.,[2] we get increasing terms which are powers of $\ln s$, which necessarily require infinite-order considerations. (In passing, we notice that these calculations also include ordering, etc.) Then if we increase the strength of the coupling slowly, we realize that these terms become important already at lower energies, and we therefore must stick to the summed form both in the hard and soft spectrum. In the subsequent paper on scalar electrodynamics,[20] we discuss what must be included in this correlation expansion in order to get an exactly identical result to some finite order in the ordinary perturbation expansion, since of course we want to reproduce the nice low-energy results. For a more compressed discussion of the result of this paper, on scalar electrodynamics[20] and on the new multiperipheral model, where because of identical clusters all crisscross graphs are coherently summed over,[21] see Ref. 28.

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APPENDIX A

In Eq. (3.1) pair effects from each particle leg have been partially factorized. The four particle legs in this process are somewhat arbitrary until we have fixed the vertices r and s, which determine the endpoints and the starting points, respectively, of such legs. We then regroup the factors in (3.1) with respect to the indices s_1 and s_2. After summation over r we find

$$
\begin{aligned}
I^R_{\text{sym}} = \sum_{\substack{r\\ D_{\text{I}}^A}} \Bigg[\sum_{\substack{s_1\\ D_{\text{II}}}} \Bigg(& \sum_{\substack{p<q\\ 1}}^{s_1} \chi^{0a}_{j_p j_q} f^{0b*}_{j_p} f^{0b*}_{j_q} \prod_{\substack{l\neq pq\\ 1}}^{s_1} f^{0a}_{j_l} f^{0b*}_{j_l} \prod_{l=s_1+1}^{r-1} f^{0a}_{j_l} f^{0b'*}_{j_l} + \sum_{p=1}^{s_1} \sum_{q=s_1+1}^{r-1} \chi^{0a}_{j_p j_q} f^{0b*}_{j_p} f^{0b'*}_{j_q} \prod_{\substack{l\neq p\\ 1}}^{s_1} f^{0a}_{j_l} f^{0b*}_{j_l} \prod_{\substack{l\neq q\\ s_1+1}}^{r-1} f^{0a}_{j_l} f^{0b'*}_{j_l} \\
& + \sum_{q=1}^{s_1} \sum_{p=s_1+1}^{r-1} \chi^{0a}_{j_p j_q} f^{0b*}_{j_q} f^{0b'*}_{j_p} \prod_{\substack{l\neq q\\ 1}}^{s_1} f^{0a}_{j_l} f^{0b*}_{j_l} \prod_{\substack{l\neq p\\ s_1+1}}^{r-1} f^{0a}_{j_l} f^{0b'*}_{j_l} \\
& + \sum_{\substack{p<q\\ s_1+1}}^{r-1} \chi^{0a}_{j_p j_q} f^{0b'*}_{j_p} f^{0b'*}_{j_q} \prod_{l=1}^{s_1} f^{0a}_{j_l} f^{0b*}_{j_l} \prod_{\substack{l\neq pq\\ s_1+1}}^{r-1} f^{0a}_{j_l} f^{0b'*}_{j_l} \Bigg) \sum_{\substack{s_2\\ D_{\text{III}}}} [\text{see (2.21)}] \\
& + \text{all terms including } \chi^{0a'} \text{ and } \chi^{0b} \\
& + \sum_{\substack{s_1\\ D_{\text{II}}}} [\text{see (2.21)}] \sum_{\substack{s_2\\ D_{\text{III}}}} \Bigg(\sum_{\substack{p<q\\ r-s_2+1}}^{n+1} \chi^{0b'}_{j_p j_q} f^{0a'}_{j_p} f^{0a'}_{j_q} \prod_{l=p+1}^{r+s_2} f^{0a'}_{j_l} f^{0b*}_{j_l} \prod_{\substack{l\neq pq\\ r+s_2+1}}^{n+1} f^{0a'}_{j_l} f^{0b'*}_{j_l} \Bigg) \\
& + \sum_{\substack{s_1 s_2\\ D_{\text{II}} D_{\text{III}}}} \Bigg(\sum_{p=s_1+1}^{r-1} \sum_{q=r+s_2+1}^{n+1} \chi^{0b'}_{j_p j_q} f^{0a}_{j_p} f^{0a'}_{j_q} \Bigg) \prod_{l=1}^{s_1} f^{0a}_{j_l} f^{0b*}_{j_l} \prod_{\substack{l\neq p\\ s_1+1}}^{r-1} f^{0a}_{j_l} f^{0b'*}_{j_l} \prod_{l=r+1}^{r+s_2} f^{0a'}_{j_l} f^{0b*}_{j_l} \prod_{\substack{l\neq q\\ r+s_2+1}}^{n+1} f^{0a'}_{j_l} f^{0b'*}_{j_l} + (p \leftrightarrow q) \\
& + \sum_{\substack{s_1\\ D_{\text{II}}}} \Bigg(\sum_{\substack{p<q\\ s_1+1}}^{r-1} \chi^{0b'}_{j_p j_q} f^{0a}_{j_p} f^{0a}_{j_q} \Bigg) \prod_{l=1}^{s_1} f^{0a}_{j_l} f^{0b*}_{j_l} \prod_{\substack{l\neq pq\\ s_1+1}}^{r-1} f^{0a}_{j_l} f^{0b'*}_{j_l} \sum_{\substack{s_2\\ D_{\text{III}}}} [\text{see (2.21)}] \Bigg] \,.
\end{aligned} \tag{A1}
$$

Under the integral we can relabel $r \to r+1$ and for the last $n-r+1$ currents we put $l \to l-1$. By means of

(2.25) and (3.5), summation over $D_{\rm II}$, $D_{\rm III}$, s_1, and s_2, and repeated use of the binomial theorem, we obtain

$$I^R_{\rm sym}=\sum_{\substack{r\\ D^A_{I}}}\left(\sum_{\substack{p<q\\1}}^{r}\chi^{0a}_{j_p j_q}J^{0B}_{j_p}J^{0B*}_{j_q}\prod_{\substack{l\neq pq\\1}}^{r}f^{0a}_{j_l}J^{0B}_{j_l}\prod_{l=r+1}^{n}f^{0a'}_{j_l}J^{0B*}_{j_l}+\prod_{l=1}^{r}f^{0a}_{j_l}J^{0B*}_{j_l}\sum_{\substack{p<q\\r+1}}^{n}\chi^{0a'}_{j_p j_q}J^{0B*}_{j_p}J^{0B*}_{j_q}\prod_{\substack{l\neq pq\\r+1}}^{n}f^{0a'}_{j_l}J^{0B*}_{j_l}\right.$$
$$+\sum_{\substack{p<q\\1}}^{r}\chi^{0B}_{j_p j_q}f^{0a}_{j_p}f^{0a}_{j_q}\prod_{\substack{l\neq pq\\1}}^{r}f^{0a}_{j_l}J^{0B*}_{j_l}\prod_{l=r+1}^{n}f^{0a'}_{j_l}J^{0B*}_{j_l}+\sum_{p=1}^{r}\sum_{q=r+1}^{n}\chi^{0B}_{j_p j_q}f^{0a}_{j_p}f^{0a'}_{j_q}\prod_{\substack{l\neq p\\1}}^{r}f^{0a}_{j_l}J^{0B*}_{j_l}\prod_{\substack{l\neq q\\r+1}}^{n}f^{0a'}_{j_l}J^{0B*}_{j_l}$$
$$\left.+\sum_{q=1}^{r}\sum_{p=r+1}^{n}\chi^{0B}_{j_p j_q}f^{0a'}_{j_p}f^{0a}_{j_q}\prod_{\substack{l\neq q\\1}}^{r}f^{0a}_{j_l}J^{0B*}_{j_l}\prod_{\substack{l\neq p\\r+1}}^{n}f^{0a'}_{j_l}J^{0B*}_{j_l}+\sum_{\substack{p<q\\r+1}}^{n}\chi^{0B}_{j_p j_q}f^{0a'}_{j_p}f^{0a'}_{j_q}\prod_{l=1}^{r}f^{0a}_{j_l}J^{0B*}_{j_l}\prod_{\substack{l\neq pq\\r+1}}^{n}f^{0a'}_{j_l}J^{0B*}_{j_l}\right). \qquad \text{(A2)}$$

This is verified as follows:

$$\sum_{\substack{p<q\\1}}^{r}\chi^{0a}_{j_p j_q}J^{0B*}_{j_p}J^{0B*}_{j_q}\prod_{\substack{l\neq pq\\1}}^{r}f^{0a}_{j_l}J^{0B*}_{j_l}=\sum_{\substack{p<q\\1}}^{r}\chi^{0a}_{j_p j_q}J^{0B*}_{j_p}J^{0B*}_{j_q}\sum_{\substack{s_1\\ D^{pq}_{\rm II}}}\prod_{\substack{l\neq pq\\1}}^{s_1}f^{0a}_{j_l}f^{0b*}_{j_l}\prod_{\substack{l\neq pq\\s_1+1}}^{r}f^{0a}_{j_l}f^{0b'*}_{j_l}$$
$$=\sum_{\substack{s_1\\D_{\rm II}}}\left(\sum_{\substack{p<q\\1}}^{s_1}\chi^{0a}_{j_p j_q}f^{0b*}_{j_p}f^{0b*}_{j_q}\prod_{\substack{l\neq pq\\1}}^{s_1}f^{0a}_{j_p}f^{0b*}_{j_l}\prod_{l=s_1+1}^{r}f^{0a}_{j_l}f^{0b'*}_{j_l}\right.$$
$$+\sum_{p=1}^{s_1}\sum_{q=s_1+1}^{r}\chi^{0a}_{j_p j_q}f^{0b*}_{j_p}f^{0b'*}_{j_q}\prod_{\substack{l\neq p\\1}}^{s_1}f^{0a}_{j_l}f^{0b*}_{j_l}\prod_{\substack{l\neq q\\s_1+1}}^{r}f^{0a}_{j_l}f^{0b'*}_{j_l}$$
$$+\sum_{q=1}^{s_1}\sum_{p=s_1+1}^{r}\chi^{0a}_{j_p j_q}f^{0b*}_{j_q}f^{0b'*}_{j_p}\prod_{\substack{l\neq q\\1}}^{s_1}f^{0a}_{j_l}f^{0b*}_{j_l}\prod_{\substack{l\neq p\\s_1+1}}^{r}f^{0a}_{j_l}f^{0b'*}_{j_l}$$
$$\left.+\sum_{\substack{p<q\\s_1+1}}^{r}\chi^{0a}_{j_p j_q}f^{0b'*}_{j_p}f^{0b'*}_{j_q}\prod_{l=1}^{s_1}f^{0a}_{j_l}f^{0b*}_{j_l}\prod_{\substack{l\neq pq\\s_1+1}}^{r}f^{0a}_{j_l}f^{0b'*}_{j_l}\right).$$

Relabeling $l\to l+1$ and $r\to r-1$, this gives the four first sums in (A1). Similar relations give the rest of (A1). By straightforward calculations we further find

$$I^R_{\rm sym}=\sum_{r=0}^{n}\sum_{\substack{p<q\\1}}^{n}(\chi^{0A}_{pq}J^{0B*}_{p}J^{0B*}_{q}+\chi^{0B}_{pq}J^{0A}_{p}J^{0A}_{q})$$
$$\times\sum_{D^{A\,pq}_{I}}\left(\prod_{l=1}^{r}f^{0a}_{j_l}J^{0B*}_{j_l}\prod_{l=r+1}^{n}f^{0a'}_{j_l}J^{0B*}_{j_l}\right)_{l\neq pq}, \qquad \text{(A3)}$$

which follows from relations like

$$\sum_{r=0}^{n}\sum_{\substack{p<q\\1}}^{n}\chi^{0A}_{pq}\sum_{D^{A\,pq}_{I}}\left(\prod_{l=1}^{r}f^{0a}_{j_l}\prod_{l=r+1}^{n}f^{0a'}_{j_l}\right)_{l\neq pq}$$
$$=\sum_{\substack{r\\D^A_{I}}}\left(\sum_{\substack{p<q\\1}}^{r}\chi^{0a}_{j_p j_q}\prod_{\substack{l\neq pq\\1}}^{r}f^{0a}_{j_l}\prod_{l=r+1}^{n}f^{0a'}_{j_l}\right.$$
$$\left.+\prod_{l=1}^{r}f^{0a}_{j_l}\sum_{\substack{p<q\\r+1}}^{n}\chi^{0a'}_{j_p j_q}\prod_{\substack{l\neq pq\\r+1}}^{n}f^{0a'}_{j_l}\right), \qquad \text{(A4)}$$

while $D^A_{I}=[D(1;n)/D(1;r)]/D(r+1;n)$ as in Sec. I. As an exercise we here verify (A4) for the case $n=3$. For the left-hand side we get

$$\chi^{0A}_{12}\sum_{D^{A_{12}}_{I}}\left(\prod_{l=1}^{r}f^{0a}_{j_l}\prod_{l=r+1}^{3}f^{0a'}_{j_l}\right)_{l\neq1,2}$$
$$+\chi^{0A}_{13}\sum_{D^{A_{13}}_{I}}\left(\prod_{l=1}^{r}f^{0a}_{j_l}\prod_{l=r+1}^{3}f^{0a'}_{j_l}\right)_{l\neq1,3}$$
$$+\chi^{0A}_{23}\sum_{D^{A_{23}}_{I}}\left(\prod_{l=1}^{r}f^{0a}_{j_l}\prod_{l=r+1}^{3}f^{0a'}_{j_l}\right)_{l\neq2,3}$$
$$=(\chi^{0A}_{12}\underset{(r=3)}{f^{0a}_{3}}+\chi^{0A}_{12}\underset{(r=0)}{f^{0a'}_{3}})_{r\neq1,2}$$
$$+(\chi^{0A}_{13}\underset{(r=2)}{f^{0a}_{2}}+\chi^{0A}_{13}\underset{(r=0)}{f^{0a'}_{2}})_{r\neq1,3}$$
$$+(\chi^{0A}_{23}\underset{(r=1)}{f^{0a}_{1}}+\chi^{0A}_{23}\underset{(r=0)}{f^{0a'}_{1}})_{r\neq2,3}.$$

Similarly we get for the right-hand side

$$\sum_{D_I^A}^{r}\left(\sum_{p<q\atop 1}^{r}\chi^{0a}_{j_p j_q}\prod_{l\neq pq\atop 1}^{r} f^{0a}_{j_l}\prod_{l=r+1}^{3} f^{0a'}_{j_l}+\prod_{l=1}^{r} f^{0a}_{j_l}\sum_{p<q\atop r+1}^{3}\chi^{0a'}_{j_p j_q}\prod_{l\neq pq\atop r+1}^{3} f^{0a'}_{j_l}\right)$$

$$=\sum_{D_I^A r=2}\sum_{p<q\atop 1}^{2}\chi^{0a}_{j_p j_q}\prod_{l=3}^{3} f^{0a'}_{j_l}+\sum_{D_I^A r=3}\sum_{p<q\atop 1}^{3}\chi^{0a}_{j_p j_q}\prod_{l\neq pq\atop 1}^{3} f^{0a}_{j_l}+\sum_{D_I^A r=0}\sum_{p<q\atop 1}^{3}\chi^{0a'}_{j_p j_q}\prod_{l\neq pq\atop 1}^{3} f^{0a'}_{j_l}+\sum_{D_I^A r=1}\prod_{l=1}^{1} f^{0a}_{j_l}\sum_{p<q\atop 2}^{3}\chi^{0a'}_{j_p j_q}$$

$$=\chi^{0a}_{12}(f^{0a}+f^{0a'})_3+\chi^{0a}_{13}(f^{0a}+f^{0a'})_2+\chi^{0a}_{23}(f^{0a}+f^{0a'})_1+\chi^{0a'}_{12}(f^{0a}+f^{0a'})_3+\chi^{0a'}_{13}(f^{0a}+f^{0a'})_2+\chi^{0a'}_{23}(f^{0a}+f^{0a'})_1$$

= left-hand side.

Finally summation over r in (A3) gives

$$I^R_{\rm sym}=\sum_{p<q\atop 1}^{n}(\chi^{0A}_{pq}J^{0B*}_p J^{0B*}_q+\chi^{0B}_{pq}J^{0A}_p J^{0A}_q)\prod_{l\neq pq\atop 1}^{n} J^{0A}_l J^{0B}_l\,. \tag{A5}$$

We have thus performed the proof for the A side, but could as well have done it for the B side by summation over r instead of s in (A1). The A-B symmetry proves (A5).

APPENDIX B: PARTIAL FACTORIZATION OF THE SECOND-ORDER RECOIL

Notations and conventions. For simplicity we shall consider emitted quanta from one outgoing particle leg. Permutation within a set of n emitted quanta are denoted Π_n. If m of these are already summed over we shall use the notation $\Pi_{n/m}$, the factor set of permutations. We will further frequently make use of the symbols

$$f_l=\frac{1}{2p\cdot k_l+k_l^{\,2}}\;, \tag{B1}$$

$$y_{12\cdots l}=2p\cdot(k_1+\cdots+k_l)+k_1^{\,2}+\cdots+k_l^{\,2}, \tag{B2}$$

$$x_{12\cdots l}=(y_{12\cdots l})^{-1}2\sum_{i<j\atop 1}^{l}k_i\cdot k_j\;. \tag{B3}$$

If all x's are discarded ($x\to 0$), then the well-known eikonal approximation is obtained. We therefore first expand around $x=0$ for the case $|x|<1$ and then derive the obtained summed forms for arbitrary $|x|$ values from the original form without expanding in a series. In passing it should be noticed that $|x|<1$ also yields extremely hard quanta. In the following we shall use the notation M for the total amplitudes.

$n=2$: *all x values.*

$$M_2=\sum_{\Pi_2}\frac{1}{2p\cdot k_1+k_1^{\,2}}\,\frac{1}{\epsilon p\cdot(k_1+k_2)+k_1^{\,2}+2k_1\cdot k_2+k_2^{\,2}}$$

$$=\frac{1}{y_{12}(1+x_{12})}\sum_{\Pi_2}f_1$$

$$=f_1f_2\frac{1}{1+x_{12}}\;. \tag{B4}$$

The corresponding pair-correlation current is defined by

$$\chi_{12}=M_2-f_1f_2$$

$$=f_1f_2\frac{-x_{12}}{1+x_{12}}\,. \tag{B5}$$

For higher n values we will first concentrate on the case $|x|<1$. In Appendix C we will extend these results to arbitrary x values.

$n=3$: $|x|<1$.

$$M_3=\frac{1}{y_{123}(1+x_{123})}\sum_{\Pi_{3/2}}M_2$$

and by the use of (B1)–(B4)

$$M_3=\frac{1}{y_{123}}\,(1-x_{123}+x_{123}^{\,2})\sum_{\Pi_{3/2}}f_1f_2(1-x_{12}+x_{12}^{\,2})$$

$$=R_3^{(0)}+R_3^{(1)}+R_3^{(2)}, \tag{B6}$$

where the upper index is the "order" of the recoil. Clearly then $R_3^{(0)}=f_1f_2f_3$ and

$$R_3^{(1)}=-\frac{x_{123}}{y_{123}}\sum_{\Pi_{3/2}}f_1f_2-\frac{1}{y_{123}}\sum_{\Pi_{3/2}}f_1f_2x_{12}$$

$$=-x_{123}f_1f_2f_3-f_1f_2f_3\sum_{\Pi_{3/2}}x_{12}+\frac{f_1f_2f_3}{y_{123}}\sum_{\Pi_{3/2}}y_{12}x_{12}$$

$$=-f_1f_2f_3\sum_{\Pi_{3/2}}x_{12}$$

$$=-f_1f_2f_3(x_{12}+x_{13}+x_{23}),$$

where $y_{12}x_{12}=2k_1\cdot k_2$ and $y_{123}^{\;-1}\sum_{\Pi_{3/2}}y_{12}x_{12}=x_{123}$ have been used.

In the general case with n emitted quanta, the first-order recoil is given by

$$R_n^{(1)}=-\prod_{l=1}^{n}f_l\sum_{i<j\atop 1}^{n}x_{ij}\,,$$

which is easily proved by induction.

Second-order recoil in M_3 is given by

$$R_3^{(2)} = y_{123}^{-1}\left(\sum_{\Pi_{3/2}} f_1 f_2 x_{12}^2 + x_{123}^2 \sum_{\Pi_{3/2}} f_1 f_2 + x_{123}\sum_{\Pi_{3/2}} f_1 f_2 x_{12}\right),$$

where we decompose the three terms according to

(i) $$\sum_{\Pi_{3/2}} f_1 f_2 x_{12}^2 = f_1 f_2 f_3 y_{123}\sum_{\Pi_{3/2}} x_{12}^2 - f_1 f_2 f_3 \sum_{\Pi_{3/2}} 2k_1\cdot k_2 x_{12},$$

(ii) $$x_{123}\sum_{\Pi_{3/2}} f_1 f_2 = x_{123} y_{123} f_1 f_2 f_3 = f_1 f_2 f_3 \sum_{\Pi_{3/2}} 2k_1\cdot k_2 = f_1 f_2 f_3 \sum_{\Pi_{3/2}} y_{12} x_{12},$$

(iii) $$\sum_{\Pi_{3/2}} f_1 f_2 x_{12} = f_1 f_2 f_3 \sum_{\Pi_{3/2}} y_3 x_{12}.$$

Insertion of (i)–(iii) gives

$$R_3^{(2)} = f_1 f_2 f_3 \sum_{\Pi_{3/2}} x_{12}^2 + \frac{f_1 f_2 f_3}{y_{123}}\left[2(k_1\cdot k_2 + k_1\cdot k_3 + k_2\cdot k_3)\sum_{\Pi_{3/2}} x_{12} - \sum_{\Pi_{3/2}} 2k_1\cdot k_2 x_{12}\right]$$
$$= f_1 f_2 f_3 \sum_{\Pi_{3/2}} (x_{12}^2 + x'_{123}), \tag{B7}$$

where the notation

$$x'_{123} = x_{12}\frac{2(k_1+k_2)\cdot k_3}{y_{123}} \tag{B8}$$

is used for the triple correlation. Then M_3 is given by

$$M_3 = f_1 f_2 f_3\left[1 - \sum_{\Pi_{3/2}} x_{12} + \sum_{\Pi_{3/2}} (x_{12}^2 + x'_{123})\right]. \tag{B9}$$

$n=4$: $|x|<1$.

$$M_4 = y_{1234}^{-1}(1 - x_{1234} + x_{1234}^2)\sum_{\Pi_{4/3}} M_3 .$$

Since the general forms of $R_n^{(0)}$ and $R_n^{(1)}$ are known, only the second-order recoil term will be studied:

$$R_4^{(2)} = y_{1234}^{-1}\left[\sum_{\Pi_{4/3}} f_1 f_2 f_3 \sum_{\Pi_{3/2}} (x_{12}^2 + x'_{123}) + x_{1234}^2 \sum_{\Pi_{4/3}} f_1 f_2 f_3 + x_{1234}\sum_{\Pi_{4/3}} f_1 f_2 f_3 \sum_{\Pi_{3/2}} x_{12}\right]. \tag{B10}$$

Insertion of

$$x_{1234}^2 \sum_{\Pi_{4/3}} f_1 f_2 f_3 = x_{1234}^2 y_{1234} f_1 f_2 f_3 f_4$$

and

$$x_{1234}\sum_{\Pi_{4/3}} f_1 f_2 f_3 \sum_{\Pi_{3/2}} x_{12} = x_{1234} f_1 f_2 f_3 f_4 \sum_{\Pi_{4/3}} y_4 \sum_{\Pi_{3/2}} x_{12} = x_{1234} f_1 f_2 f_3 f_4\left(y_{1234}\sum_{\substack{i<j\\1}}^{4} x_{ij} - y_{1234}x_{1234}\right),$$

which is found by means of

$$\sum_{\Pi_{4/3}} y_4 \sum_{\Pi_{3/2}} x_{12} = \sum_{\substack{(i<j)\neq(k<l)\\1}}^{4} x_{ij} y_{kl} = y_{1234}\sum_{\substack{i<j\\1}}^{4} x_{ij} - \sum_{\substack{i<j\\1}}^{4} x_{ij} y_{ij} = y_{1234}\sum_{\substack{i<j\\1}}^{4} x_{ij} - x_{1234} y_{1234}, \tag{B11}$$

gives

$$x_{123}^2 \sum_{\Pi_{4/3}} f_1 f_2 f_3 + x_{1234}\sum_{\Pi_{4/3}} f_1 f_2 f_3 \sum_{\Pi_{3/2}} x_{12} = x_{1234} f_1 f_2 f_3 f_4 y_{1234}\sum_{\substack{i<j\\1}}^{4} x_{ij}. \tag{B12}$$

The first term in (B10) is decomposed according to

$$\sum_{\Pi_{4/3}} f_1 f_2 f_3 \sum_{\Pi_{3/2}} (x_{12}^2 + x'_{123}) = f_1 f_2 f_3 f_4\left(\sum_{\Pi_{4/3}} y_4 \sum_{\Pi_{3/2}} x_{12}^2 + \sum_{\Pi_{4/3}} y_4 \sum_{\Pi_{3/2}} x'_{123}\right)$$
$$= f_1 f_2 f_3 f_4\left(y_{1234}\sum_{i<j}^{4} x_{ij}^2 - \sum_{\substack{i<j\\1}}^{4} 2k_i\cdot k_j x_{ij} + y_{1234}\sum_{\substack{i<j<l\\1}}^{4} x'_{ijl}\right) - \sum_{\substack{l\neq(i<j)\\1}}^{4} x_{ij}\, 2(k_i+k_j)\cdot k_l . \tag{B13}$$

Equation (B12) contains $2k_i k_j x_{kl}$ for all combinations $(ijkl)$. However, all terms in which all four indices are not different cancel by the second and the fourth terms in (B13). By insertion of (B12) and (B13) and use of

$$y_{1234}^{-1}\, 2(k_1\cdot k_2 x_{34} + k_1\cdot k_3 x_{24} + k_1\cdot k_4 x_{23} + k_2\cdot k_3 x_{14} + k_2\cdot k_4 x_{13} + k_3\cdot k_4 x_{12}) = x_{12}x_{34} + x_{13}x_{24} + x_{14}x_{23} = \tfrac{1}{2}\sum_{(i<j)\neq(k<l)\atop 1}^{4} x_{ij}x_{kl}\,,$$

(B10) takes on the form

$$R_4^{(2)} = f_1 f_2 f_3 f_4\left(\sum_{i<j\atop 1}^{4} x_{ij}^2 + \sum_{i<j<k\atop 1}^{4} x'_{ijk} + \tfrac{1}{2}\sum_{(i<j)\neq(k<l)\atop 1}^{4} x_{ij}x_{kl}\right). \qquad \text{(B14)}$$

Repeated use of this technique gives the general formula for arbitrary n:

$$R_n^{(2)} = \prod_{l=1}^{n} f_l\left(\sum_{i<j\atop 1}^{n} x_{ij}^2 + \sum_{i<j<k\atop 1}^{n} x'_{ijk} + \tfrac{1}{2}\sum_{(i<j)\neq(k<l)\atop 1}^{n} x_{ij}x_{kl}\right). \qquad \text{(B15)}$$

APPENDIX C: PARTIAL FACTORIZATION OF PAIR EFFECTS IN THIRD- AND HIGHER-ORDER RECOIL EXPANSION

Again we let n denote the number of emitted quanta.

$n=2$: $|x|<1$. The third-order recoil is easily identified in the expansion of (B4)

$$M_2 = f_1 f_2 (1 - x_{12} + x_{12}^2 - x_{12}^3 + \cdots)\,. \qquad \text{(C1)}$$

$n=3$: $|x|<1$.

$$M_3 = y_{123}^{-1}(1 - x_{123} + x_{123}^2 - x_{123}^3) \times \sum_{\Pi_{3/2}} f_1 f_2 (1 - x_{12} + x_{12}^2 - x_{12}^3) = R_3^{(0)} + R_3^{(1)} + R_3^{(2)} + R_3^{(3)}\,.$$

The third-order recoil is thus given by

$$R_3^{(3)} = (-1)^3 y_{123}^{-1}\left(\sum_{\Pi_{3/2}} f_1 f_2 x_{12}^3 + x_{123}\sum_{\Pi_{3/2}} f_1 f_2 x_{12}^2 + x_{123}^2\sum_{\Pi_{3/2}} f_1 f_2 x_{12} + x_{123}^3\sum_{\Pi_{3/2}} f_1 f_2\right). \qquad \text{(C2)}$$

The first and the second terms decompose according to

$$\begin{aligned}\sum_{\Pi_{3/2}} f_1 f_2 x_{12}^3 &= f_1 f_2 f_3 \sum_{\Pi_{3/2}} y_3 x_{12}^3 \\ &= f_1 f_2 f_3 y_{123}\sum_{\Pi_{3/2}} x_{12}^3 - f_1 f_2 f_3\sum_{\Pi_{3/2}} y_{12}x_{12}^3 \\ &= f_1 f_2 f_3 y_{123}\sum_{\Pi_{3/2}} x_{12}^3 - f_1 f_2 f_3\sum_{\Pi_{3/2}} 2k_1\cdot k_2 x_{12}^2\,,\end{aligned} \qquad \text{(C3)}$$

$$\begin{aligned}x_{123}\sum_{\Pi_{3/2}} f_1 f_2 x_{12}^2 &= f_1 f_2 f_3 x_{123}\sum_{\Pi_{3/2}} y_3 x_{12}^2 \\ &= f_1 f_2 f_3 x_{123} y_{123}\sum_{\Pi_{3/2}} x_{12}^2 - f_1 f_2 f_3 x_{123}\sum_{\Pi_{3/2}} 2k_1\cdot k_2 x_{12}\,.\end{aligned} \qquad \text{(C4)}$$

The two last terms in (C3) after decomposition,

$$x_{123}^2\sum_{\Pi_{3/2}} f_1 f_2 x_{12} = x_{123}^2 f_1 f_2 f_3\sum_{\Pi_{3/2}} y_3 x_{12}\,, \qquad \text{(C5)}$$

$$x_{123}^3\sum_{\Pi_{3/2}} f_1 f_2 = x_{123}^3 y_{123} f_1 f_2 f_3 = x_{123}^2 f_1 f_2 f_3\sum_{\Pi_{3/2}} x_{12} y_{12}\,, \qquad \text{(C6)}$$

combine to one single term

$$f_1 f_2 f_3 x_{123}^2 y_{123}\sum_{\Pi_{3/2}} x_{12}\,. \qquad \text{(C7)}$$

Insertion of (C3), (C4), and (C7) in (C2) then gives

$$R_3^{(3)} = (-1)^3\frac{f_1 f_2 f_3}{y_{123}}\left(y_{123}\sum_{\Pi_{3/2}} x_{12}^3 - \sum_{\Pi_{3/2}} 2k_1\cdot k_2 x_{12}^2 + x_{123}y_{123}\sum_{\Pi_{3/2}} x_{12}^2 - x_{123}\sum_{\Pi_{3/2}} 2k_1\cdot k_2 x_{12} + x_{123}^2 y_{123}\sum_{\Pi_{3/2}} x_{12}\right). \qquad \text{(C8)}$$

Using

$$x_{123}y_{123}=2(k_1\cdot k_2+k_1\cdot k_3+k_2\cdot k_3)$$

we then find that all pair effects in the third and fifth terms are effectively canceled by the second and the fourth terms, respectively. Thus, the four last terms contain only higher correlations than pair correlations and therefore the first term correctly describes the pair effects in a third-order recoil expansion.

Since we are here mainly interested in the pair effects, for n emitted quanta the mth-order recoil can be written in the form

$$R_n^{(m)}=\left[\frac{(-1)^m}{y_{1\cdots n}}\sum_{\nu=0}^{m}x_{1\cdots n}{}^{\nu}\times\sum_{\Pi_{n/(n-1)}}f_1\cdots f_{n-1}\sum_{\substack{i<j\\1}}^{n-1}x_{ij}{}^{m-\nu}+\text{higher correlations}\right]. \tag{C9}$$

As in (B11) the first term decomposes into

$$\begin{aligned}\sum_{\Pi_{n/(n-1)}}f_1\cdots f_{n-1}\sum_{\substack{i<j\\1}}^{n-1}x_{ij}{}^{m}&=f_1\cdots f_n\sum_{\Pi_{n/(n-1)}}y_n\sum_{\substack{i<j\\1}}^{n-1}x_{ij}\\&=f_1\cdots f_n\sum_{\substack{i<j\\1}}^{n}x_{ij}{}^{m}(y_{1\cdots n}-y_{ij})\\&=f_1\cdots f_n\,y_{1\cdots n}\sum_{\substack{i<j\\1}}^{n}x_{ij}{}^{m}-f_1\cdots f_n\sum_{\substack{i<j\\1}}^{n}x_{ij}{}^{m}\,y_{ij}\,.\end{aligned} \tag{C10}$$

The second term in (C9) gives

$$\begin{aligned}x_{1\cdots n}\sum_{\Pi_{n/(n-1)}}f_1\cdots f_{n-1}\sum_{\substack{i<j\\1}}^{n-1}x_{ij}{}^{m-1}&=x_{1\cdots n}f_1\cdots f_n\sum_{\Pi_{n/(n-1)}}y_n\sum_{\substack{i<j\\1}}^{n-1}x_{ij}{}^{m-1}\\&=x_{1\cdots n}f_1\cdots f_n\sum_{\substack{i<j\\1}}^{n}x_{ij}{}^{m-1}(y_{1\cdots n}-y_{ij})\,.\end{aligned} \tag{C11}$$

As is seen, the first term in (C11) causes pair effects since

$$x_{1\cdots n}\,y_{1\cdots n}=2(k_1\cdot k_2+\cdots+k_{n-1}\cdot k_n).$$

However, these are effectively canceled by the second term in (C10). Successive decomposition and pairwise recombination proceeds exactly as in third-order recoil (C2)–(C8), except for the first term, which is thus the only pair-correlating term in (C10). The termination of this procedure occurs also in this case since the last terms,

$$\begin{aligned}x_{1\cdots n}{}^{m}\sum_{\Pi_{n/(n-1)}}f_1\cdots f_{n-1}&=x_{1\cdots n}{}^{m}f_1\cdots f_n\,y_{1\cdots n}\\&=x_{1\cdots n}{}^{m-1}f_1\cdots f_n\sum_{\substack{i<j\\1}}^{n}x_{ij}y_{ij}\,,\end{aligned}$$

$$\begin{aligned}x_{1\cdots n}{}^{m-1}\sum_{\Pi_{n/(n-1)}}f_1\cdots f_{n-1}\sum_{\substack{i<j\\1}}^{n-1}x_{ij}&=x_{1\cdots n}{}^{m-1}f_1\cdots f_n\sum_{\Pi_{n/(n-1)}}y_n\sum_{\substack{i<j\\1}}^{n-1}x_{ij}\\&=x_{1\cdots n}{}^{m-1}f_1\cdots f_n\sum_{\substack{i<j\\1}}^{n}x_{ij}(y_{1\cdots n}-y_{ij})\,,\end{aligned}$$

recombine into one single term [compare (C7)],

$$x_{1\cdots n}{}^{m-1}f_1\cdots f_n\,y_{1\cdots n}\sum_{\substack{i<j\\1}}^{n}x_{ij}\,.$$

It should here be noticed that only in those terms where all triple-correlating mechanisms are removable is it important to investigate the survival or canceling of pairs, since the other terms are of higher-order correlation anyway, and can therefore be neglected in deriving the pair effects. Thus, the mth-order recoil for n emitted quanta can be put in the form [compare (C9)]

$$R_n^{(m)}=\prod_{l=1}^{n}f_l\left[(-1)^m\sum_{\substack{i<j\\1}}^{n}x_{ij}{}^{m}+\text{higher correlations}\right]. \tag{C12}$$

The case of arbitrary x values. In Appendix B we saw that in the case $n=2$ we could derive (B4) and (B5) for any x value. Starting from $n=2$ we extend this derivation to the case of 3 "emitted" quanta.

$n=3$. *The exact M_3 amplitude reads*

$$\begin{aligned}M_3&=\frac{1}{y_{123}(1+x_{123})}\sum_{\Pi_{3/2}}f_1f_2\frac{1}{1+x_{12}}\\&=y_{123}{}^{-1}\left(1-\frac{x_{123}}{1+x_{123}}\right)\sum_{\Pi_{3/2}}f_1f_2\frac{1}{1+x_{12}}\,,\end{aligned} \tag{C13}$$

where the second term for nonvanishing x_{123} is triple correlated. The rest decomposes as follows:

$$\begin{aligned}M_3'&=y_{123}{}^{-1}\sum_{\Pi_{3/2}}f_1f_2\frac{1}{1+x_{12}}\\&=y_{123}{}^{-1}\sum_{\Pi_{3/2}}f_1f_2\left(1-\frac{x_{12}}{1+x_{12}}\right),\end{aligned}$$

for all x_{12}. We then write

$$M_3'=f_1f_2f_3y_{123}^{-1}\sum_{\Pi_{3/2}}y_3\left(1-\frac{x_{12}}{1+x_{12}}\right)$$

$$=f_1f_2f_3\left(1-\sum_{\Pi_{3/2}}\frac{x_{12}}{1+x_{12}}+y_{123}^{-1}\sum_{\Pi_{3/2}}\frac{y_{12}x_{12}}{1+x_{12}}\right), \tag{C14}$$

where the last term is triple correlated through y_{123}^{-1}.

This can be iterated to an arbitrary n, and the general formula reads

$$M_n=\prod_{l=1}^{n}f_l\left(1-\sum_{\substack{i<j\\1}}^{n}\frac{x_{ij}}{1+x_{ij}}+\text{higher correlations}\right), \tag{C15}$$

where the first term is again the well-known eikonal amplitude, which is completely factorizable. The second term in (C15) is, in fact, the same pair correlation as in (C12), a fact which becomes clear after summation over m in (C12), with no other restriction on x but $x\neq -1$, a case which will be examined once the final summed-up result has been obtained.

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