Angular momentum analysis of the four-nucleon Green's function

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A formalism for a complete partial-wave expansion of the four-nucleon Green's function is given by generalizing a standard method. All of the constraints imposed by the symmetry properties (parity, time reversal, and exchange symmetry) are worked out, and an extended unitarity condition is shown to be satisfied. The use of Padé approximations to the Green's function is shown to provide a physical amplitude which is unitary, has correct threshold behavior in all waves, and has good analyticity properties. Therefore, such a scheme is well suited to explore beyond the Born term the dynamical content of various Lagrangian models proposed for the nucleon-nucleon interaction.

I. INTRODUCTION

The Bethe-Salpeter equation (BSE) has been considered as a possible scheme to treat the low-energy nucleon-nucleon scattering in a relativistic framework.[1] It is only recently that a reliable numerical solution for this equation has been obtained[2] in the ladder approximation.

Two different methods were used and compared with each other: matrix inversion after kernel regularization and Padé approximations (PAs) constructed from iterated series. The use of PA was suggested by the rigorous convergence results, well established for the BSE in the spinless case.[3] However, a new idea was introduced recently[4] which consists of approximating through the Padé method the Green's function rather than its restriction to the physical transition matrix T. The reasons will become clear in the following.

In the case of the ladder series (computed by iterating the BSE), it was found[5] that while a [4/4] PA was necessary to reach numerically good results when working with the T matrix, a [2/2] PA to the Green's function G was enough to reach the same accuracy. Such a result is certainly not surprising in itself because the Green's function G contains much more information than the T matrix, and the PAs make explicit use of this extra information. Indeed it is a nonlinear approximation, which mixes up physical and unphysical elements at any order. Furthermore, it is far easier to compute unphysical terms of G in low perturbation order than physical elements of T in higher orders.

We propose in this work a systematic investigation of the PAs to the four-nucleon Green's function, as has already been done for πN.[6] This requires a detailed analysis of the general four-nucleon Green's function.

The physical T matrix is a 4×4 matrix which, due to the symmetrical properties (parity, time reversal, Pauli principle), has only 5 independent amplitudes (corresponding to the well-known Fermi invariants), while the complete Green's function G is a 16×16 matrix with 41 independent amplitudes.

We are mainly interested in the low-energy region (from zero to a few hundred MeV), and therefore the unitarity condition plays an important role. It is well known[7] that the $[N/N]$ PAs to the T matrix are rigorously unitary. On the other hand, the Green's function enjoys an extended unitarity condition which reduces to the usual one for the physical elements; therefore, the PAs on the Green's function also automatically fulfill the extended unitarity condition. The elastic unitarity equation becomes a purely algebraic condition when we diagonalize the Green's function under the rotation group (namely when we choose a set of basis vectors with definite total angular momentum J).

As a consequence we shall investigate the general structure of G for a given angular momentum J, parity π, and isospin I. It will be shown that for $G^{J,I,\pi}$ the extended unitarity condition becomes

$$2\,\mathrm{Im}G^{J,I,\pi}(s)=\rho G^{J,I,\pi}(s)PG^{J,I,\pi*}(s)$$
$$=\rho G^{J,I,\pi*}(s)PG^{J,I,\pi}(s),$$

where P is a projector on the physical states, $\sqrt{s}$ is the energy in the center-of-mass system, and $\rho=m^2/2\pi\sqrt{s}$. We shall also prove that $G^{J,I,\pi}$ is a 4×4 quasisymmetrical matrix if the physical element is a singlet or an uncoupled triplet $\pi=(-1)^J$, or a 6×6 quasisymmetric matrix when the physical amplitude is a coupled triplet $\pi=(-1)^{J+1}$. This produces $10+10+21=41$ independent amplitudes.

Finally, one should remark that historically the PAs to the Green's function were not introduced in order to improve convergence, but rather to cure a disease of the [1/1] PA to the T matrix in the case of pseudoscalar interactions. In such a case the Born term gives rise to wrong threshold behavior for the $^1S_0$ and $^3(J-1)_J$ waves (this is

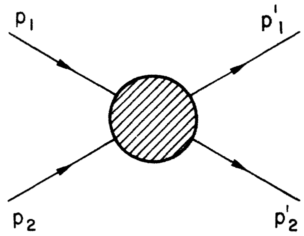

FIG. 1. Diagrammatic representation of the Green's function.

unavoidable since it is related to the intrinsic parity of the pion). This wrong threshold behavior which is still present in the [1/1] PA to T is completely eliminated by the use of the [1/1] PA to the Green's function.

In this first paper we give the complete angular momentum expansion of the Green's function in a form suited for actual field-theoretical calculations. The partial-wave analysis performed in the case of BSE[8] is not suitable to such purposes. We actually extend the method of Goldberger, Grisaru, McDowell, and Wong[9] (GGMW) to the Green's function with a choice of a set of invariant amplitudes which are free from kinematical singularities, easy to compute for any Feynman diagram, and for which the extended Pauli principle is a trivial algebraic relation and the parity and time-reversal constraints are extremely simple. The final partial-wave elements of G are linearly related to the Legendre projection of such amplitudes. In a future paper we shall apply this method to specific models such as the Yukawa model, the nonlinear σ model, and the gauge field models.

The plan of the work is the following. In Sec. II the general structure of the Green's function is examined together with its symmetry properties under parity and time-reversal operations and the exchange operation (Pauli principle). Using the angular momentum basis, selection rules are derived and the Green's function is shown to be reducible to a direct sum of two 4×4 matrices, a 6×6 matrix, and an irrelevant 2×2 matrix. In Sec. III the extended unitarity equation, the properties of the PAs to the Green's function, and the threshold behavior are analyzed. In Sec. IV we quote the basis of invariants we have chosen and their Fierz transformation properties. The final formulas relate the transitions in the spectroscopic frame to the Legendre projections of the invariant amplitudes. Appendixes A, B, C, D, and E contain the most relevant technical details.

II. THE GREEN'S FUNCTION

A. Definitions

Let $\psi(x)$ be the Heisenberg field for the nucleon. Then the four-point Green's function in configuration space reads

$$\mathcal{G}(x_1', x_2', x_1, x_2) = \langle 0 | T(\psi(x_1') \otimes \psi(x_2')\, \bar{\psi}(x_1) \otimes \bar{\psi}(x_2)) | 0 \rangle , \tag{2.1}$$

where T is the time-ordering operator. We split the Green's function into a disconnected part (sum of products of two propagators) and a connected one, which reads in momentum space (see Fig. 1)

$$\mathcal{G}^{\text{conn}}(p_1', p_2', p_1, p_2) = (2\pi)^{-16} \int e^{i(p_1 x_1 + p_2 x_2 - p_1' x_1' - p_2' x_2')} \times \mathcal{G}^{\text{conn}}(x_1', x_2', x_1, x_2)\, dx_1 dx_2 dx_1' dx_2' . \tag{2.2}$$

Amputation of the external legs yields

$$\mathcal{G}^{\text{amp}}(p_1', p_2', p_1, p_2) = S_F^{-1}(p_1') \otimes S_F^{-1}(p_2')\, \mathcal{G}^{\text{conn}}(p_1', p_2', p_1, p_2) \times S_F^{-1}(p_1) \otimes S_F^{-1}(p_2) , \tag{2.3}$$

where $S_F(p)$ is the exact fermion propagator. The connection with the S matrix is fixed by the normalization of the physical nucleon wave function

$$N(p) = (2\pi)^{-3/2} \left(\frac{m}{p_0} \right)^{1/2} , \tag{2.4}$$

where $m^2 = p_1^{\,2} = p_2^{\,2} = p_1'^{\,2} = p_2'^{\,2}$ is the square of the nucleon mass. The S matrix is given by

$$S(p_1', p_2', p_1, p_2) = \delta(p_1 - p_1')\, \delta(p_2 - p_2') + N(p_1) N(p_2) N(p_1') N(p_2')\, \bar{u}(p_1') \otimes \bar{u}(p_2')\, \mathcal{G}^{\text{amp}}(p_1', p_2', p_1, p_2)\, u(p_1) \otimes u(p_2) , \tag{2.5}$$

where u is a Dirac spinor with positive energy (see Appendix A) and where all the momenta are on the mass shell.

The relativistically invariant transition amplitude T is related to S by the standard relation

$$S(p_1', p_2', p_1, p_2) = \delta(p_1 - p_1')\, \delta(p_2 - p_2') + i(2\pi)^4 N(p_1) N(p_2) N(p_1') N(p_2') \times \delta(p_1 + p_2 - p_1' - p_2') T(p_1', p_2', p_1, p_2). \tag{2.6}$$

In order to exploit translational invariance and to have a simpler relation with the transition amplitude we define the Green's function from now on as $G(p_1', p_2', p_1, p_2)$, where we drop the superscript amp for convenience:

$$\mathcal{G}^{\text{amp}}(p_1', p_2', p_1, p_2) = i(2\pi)^4\, \delta(p_1 + p_2 - p_1' - p_2') \times G(p_1', p_2', p_1, p_2) , \tag{2.7}$$

so that from the previous relations G is related to the transition amplitude by

$$T(p_1', p_2', p_1, p_2) = \bar{u}(p_1') \otimes \bar{u}(p_2') G(p_1', p_2', p_1, p_2) \times u(p_1) \otimes u(p_2) \ . \quad (2.8)$$

The Green's function, whose perturbation expansion is given by the Gell-Mann–Low series, is, from its definition, a 16×16 matrix and has the expansion

$$G = \sum_{i,j} G^{(i,j)}(p_1', p_2', p_1, p_2)\, \Gamma_i \otimes \Gamma_j \ , \quad (2.9)$$

with

$$\Gamma_i = (1, \gamma_\mu, i\gamma_5\gamma_\mu, 2^{-1/2}\sigma_{\mu\nu}, \gamma_5). \quad (2.10)$$

Our convention for the γ matrices is given in Appendix A. A covariant version of (2.9) is obtained if, from the external momenta, we construct a set of orthogonal four-vectors $W_\mu^{(0)}, W_\mu^{(1)}, W_\mu^{(2)}, W_\mu^{(3)}$ such that in a given frame $W_\mu^{(\nu)} = g_{\mu\nu}$. With this choice

$$\Gamma_i = (1, \not{W}^{(\mu)}, i\gamma_5 \not{W}^{(\mu)}, 2^{-3/2}[\not{W}^{(\mu)}, \not{W}^{(\nu)}], \gamma_5) , \quad (2.11)$$

and the $G^{(i,j)}$'s become invariant functions of the momenta.

For deeper physical insight we shall not use for G the representation (2.9) or its covariant form, but rather choose a new and better suited basis (in the 16-dimensional space in which G operates for fixed momenta), for which all of the symmetry properties (parity, time reversal, rotational invariance, exchange symmetry) become obvious. This will be the object of Sec. II B.

B. Choice of basis vectors

Let us introduce four independent spinors

$$|\eta, \lambda, p\rangle = \gamma_5^{(1-\eta)/2} u(\lambda, p), \quad \eta = \pm 1, \quad \lambda = \pm\tfrac{1}{2} \quad (2.12)$$

where u satisfies the Dirac equation

$$(\not{p} - m) u(\lambda, p) = 0, \quad p_0 > 0, \quad m^2 = p^2 \quad (2.13)$$

normalized by $\bar{u}u = 1$. λ is the helicity and η can be thought of as an intrinsic nucleon parity. We remind the reader that the relativistic nucleon field [which belongs to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz group] has to be a spinor with four independent components, while the physical wave function (which has a definite parity and belongs to the $\frac{1}{2}$ representation of the rotation group) is a spinor with two independent components, as in the nonrelativistic case.

A suitable basis for G is provided by the spinors of (2.12). By defining the dual of (2.12) not as the adjoint but rather as

$$\langle \eta, \lambda, p | = \bar{u}(\lambda, p) \gamma_5^{(1-\eta)/2} \ , \quad (2.14)$$

orthonormality and closure relations are satisfied, and moreover this choice is clearly Lorentz-invariant (Appendix A).

The space-reflection transformation properties of the above states are quite evident; however, since we deal with identical fermions we must take into account the symmetry properties arising from the interchange of two of them. In order to make this exchange symmetry transparent, we shall change our basis $|\eta_1, \eta_2, \lambda_1, \lambda_2; p_1, p_2\rangle$ by a linear transformation into a new basis

$$|\beta, \lambda_1, \lambda_2; p_1, p_2\rangle = \Gamma_\beta |\lambda_1, \lambda_2; p_1, p_2\rangle \ , \quad (2.15)$$

where $\beta = +, -, e, o$ and

$$\begin{aligned} &\Gamma_+ = 1 \otimes 1, \quad \Gamma_- = \gamma_5 \otimes \gamma_5, \\ &\Gamma_e = 2^{-1/2}(1 \otimes \gamma_5 + \gamma_5 \otimes 1) \ , \\ &\Gamma_o = 2^{-1/2}(1 \otimes \gamma_5 - \gamma_5 \otimes 1) \ . \end{aligned} \quad (2.16)$$

The states with $\beta = +, -$ have positive intrinsic parity, while the ones with $\beta = e, o$ have negative intrinsic parity; moreover, $\Gamma_+, \Gamma_-, \Gamma_e$, are exchange-invariant, while Γ_o goes into $-\Gamma_o$ (the indices e, o, which stand for even and odd, explicitly show this property). Letting P and Ω be the parity and exchange operators in Fock space and $\mathcal{V}_P, \mathcal{V}_\Omega$ be their restriction to the 16-dimensional space spanned by the set (2.15), we have (see Appendix B)

$$\begin{aligned} &\mathcal{V}_P \Gamma_\beta = \eta_\beta \Gamma_\beta \mathcal{V}_P \ , \\ &\mathcal{V}_\Omega \Gamma_\beta = \omega_\beta \Gamma_\beta \mathcal{V}_\Omega \ , \end{aligned} \quad (2.17)$$

with

$$\begin{aligned} &\eta_\beta = \begin{cases} +1, & \text{for } \beta = +, - \\ -1, & \text{for } \beta = e, o \end{cases} \\ &\omega_\beta = \begin{cases} +1, & \text{for } \beta = +, -, e \\ -1, & \text{for } \beta = o \ . \end{cases} \end{aligned} \quad (2.18)$$

The time-reversal operator acts on Γ_β in the same way as parity (see Appendix B for the proof). The states $|\lambda_1, \lambda_2; p_1, p_2\rangle$ have no definite space and exchange parities. In any low-energy theory the elastic unitarity condition plays a fundamental role; therefore, it is crucial to introduce a basis in which the unitarity equation becomes algebraic and in which, at the same time, the symmetry properties are evident. Such a basis is provided by the eigenstates of total angular momentum J, orbital momentum L, and total spin S. For the same reason as used previously, define $|\beta, J, L, S\rangle$ to be the states corresponding to (2.15) in the angular momentum basis. One checks immediately, using (2.18), that their space and exchange parities are given by

$$\begin{aligned} &P|\beta, J, L, S\rangle = \eta_\beta (-1)^L |\beta, J, L, S\rangle \ , \\ &\Omega|\beta, J, L, S\rangle = \omega_\beta (-1)^{L+S+I} |\beta, J, L, S\rangle \ , \end{aligned} \quad (2.19)$$

where I is the total isospin.

The partial-wave expansion of G can be obtained by solving a simpler equivalent problem. With any couple β', β we associate the Green's function $\Gamma_{\beta'} G \Gamma_\beta$ and we expand the corresponding transition amplitude,

$$T_{\beta'\beta} = \bar{u}(p_1') \otimes \bar{u}(p_2') \Gamma_{\beta'} G \Gamma_\beta u(p_1) \otimes u(p_2) , \quad (2.20)$$

in partial waves by using the Jacob and Wick procedure.[10] A pictorial view of this procedure is to think of the Green's function as a coupled-channel system of nucleons of both parities. The Jacob and Wick expansion $T_{\beta'\beta}$ gives us all the transition amplitudes of G corresponding to a given "channel" $\beta'\beta$, namely,

$$\langle J, L', S' \mid T_{\beta'\beta} | J, L, S \rangle = \langle \beta', J, L', S' \mid G \mid \beta, J, L, S \rangle . \quad (2.21)$$

If we fix J, the Green's function is a 16×16 matrix connecting a set of states specified by β, L, S (we shall often designate such states by using the generalized spectroscopic notation $\{^{2S+1}L_J\}_\beta$). In this frame, time-reversal invariance implies

$$\langle \beta', J, L', S' | G | \beta, J, L, S \rangle = \eta_\beta \eta_{\beta'} \langle \beta, J, L, S | G | \beta', J, L', S' \rangle . \quad (2.22)$$

C. Selection rules

The conservation of space parity in the nucleon-nucleon interaction implies

$$\eta_\beta (-1)^L = \eta_{\beta'} (-1)^{L'} . \quad (2.23)$$

The exchange parity is also conserved and since only the eigenvalues ± 1 are allowed (from $\Omega^2 = 1$), we may split the Hilbert space into the direct sum of two eigenspaces $\mathcal{H}_\pm$. The physical two-nucleon states belong to $\mathcal{H}_-$; therefore since we want to approximate G by the Padé method, which preserves the direct-sum decomposition, we shall be concerned only with the restriction of G to $\mathcal{H}_-$. Taking into account (2.18) we get

$$\omega_\beta (-1)^{L+S+I} = \omega_{\beta'} (-1)^{L'+S'+I} = -1 . \quad (2.24)$$

Tables I and II summarize the values of parity and exchange parity for the singlet ^1J_J, uncoupled-triplet ^3J_J, and coupled-triplet $^3(J \pm 1)_J$ states. In Table III, for fixed J and isospin I, which are good quantum numbers, we classify into four sets A, B, C, D the states with the same space and exchange parities. In Table IV we quote the value of isospin I for which these sets of states belong to $\mathcal{H}_-$ or to $\mathcal{H}_+$.

TABLE I. Values of $\bar{\eta} = (-)^J \times$ (parity) for the spectroscopic states.

$^{2S+1}L_J$ \ β	$+,-$	e,o
^1J_J	+	−
^3J_J	+	−
$^3(J \pm 1)_J$	−	+

TABLE II. Values of $\bar{\omega} = (-)^{J+1} \times$ (exchange parity) for the spectroscopic states.

$^{2S+1}L_J$ \ β	$+,-,e$	o
^1J_J	+	−
^3J_J	−	+
$^3(J \pm 1)_J$	+	−

Finally, as a consequence of the previous statements, we see that the Green's function G splits into the direct sum of two 4×4 matrices, one 6×6 matrix, and one 2×2 matrix corresponding respectively to the transition between the states of the sets A, B, C, D of Table III. The first three matrices will be referred to as singlet, uncoupled-triplet, and coupled-triplet matrices, in analogy with the name of the physical amplitude ($\beta = \beta' = +$) associated with them. The fourth matrix is totally irrelevant for the Padé approximation because it never contains any physical amplitude.

D. Symmetry properties of $T_{\beta'\beta}$

In Sec. II B we stated that the partial-wave expansion of G is obtained by expanding the transition amplitudes $T_{\beta'\beta}$ of the associated Green's functions $\Gamma_{\beta'} G \Gamma_\beta$. We first notice that the transformation properties of $\Gamma_{\beta'} G \Gamma_\beta$ are given by

space inversion:

$$\Gamma_{\beta'} G \Gamma_\beta \to \eta_\beta \eta_{\beta'} \Gamma_{\beta'} G \Gamma_\beta ,$$

exchange operation:

$$\Gamma_{\beta'} G \Gamma_\beta \to \omega_\beta \omega_{\beta'} \Gamma_{\beta'} G \Gamma_\beta , \quad (2.25)$$

time reversal:

$$\Gamma_{\beta'} G \Gamma_\beta \to \eta_\beta \eta_{\beta'} \Gamma_\beta G^T \Gamma_{\beta'} .$$

Therefore, the invariance properties of the corresponding transition amplitude $T_{\beta'\beta}$ will be characterized by $\eta_\beta \eta_{\beta'}$ and $\omega_\beta \omega_{\beta'}$.

From (2.9), (2.19), and (2.20) it is evident that in the basis (2.15) G has the form

TABLE III. Classification of spectroscopic states according to $\bar{\eta}$ and $\bar{\omega}$ for fixed J and I.

Set	Label	$\bar{\eta}$	$\bar{\omega}$	States in spectroscopic notation $\{^{2S+1}L_J\}_\beta$
A	Singlet	+1	+1	$\{^1J_J\}_+$, $\{^1J_J\}_-$, $\{^3(J-1)_J\}_e$, $\{^3(J+1)_J\}_e$
B	Uncoupled triplet	+1	−1	$\{^3J_J\}_+$, $\{^3J_J\}_-$, $\{^3(J-1)_J\}_o$, $\{^3(J+1)_J\}_o$
C	Coupled triplet	−1	+1	$\{^3(J-1)_J\}_+$, $\{^3(J+1)_J\}_+$, $\{^3(J-1)_J\}_-$, $\{^3(J+1)_J\}_-$, $\{^1J_J\}_e$, $\{^3J_J\}_o$
D	Irrelevant	−1	−1	$\{^3J_J\}_e$, $\{^1J_J\}_o$

$$G=\begin{bmatrix} T_{++} & T_{+-} & T_{+e} & T_{+o} \\ T_{-+} & T_{--} & T_{-e} & T_{-o} \\ T_{e+} & T_{e-} & T_{ee} & T_{eo} \\ T_{o+} & T_{o-} & T_{oe} & T_{oo} \end{bmatrix}, \qquad (2.26)$$

where each $T_{\beta'\beta}$ is itself a 4×4 matrix.

According to (2.25) we distinguish four sets of transition amplitudes $T_{\beta'\beta}$ (see Table V).

For fixed angular momentum the initial and final states for given J can be specified by L,S and L',S'; the corresponding transition matrix $T_{\beta'\beta}$ goes into $T^J_{\beta'\beta}$. Only a few elements in $T^J_{\beta'\beta}$ are nonzero because (2.24) implies that

$$(-1)^{(L'-L)}=\eta_\beta\eta_{\beta'},$$
$$(-1)^{(S'-S)}=\eta_\beta\eta_{\beta'}\omega_\beta\omega_{\beta'}, \qquad (2.27)$$

and the number of allowed transitions varies from two to six according to the class to which $T_{\beta'\beta}$ belongs (see Table VI).

For fixed L the Green's function G goes into G^J, which is obtained from (2.26) by replacing $T_{\beta'\beta}$ by $T^J_{\beta\beta'}$, and can be decomposed, though a suitable transformation, into the direct sum of the two 4×4 matrices, one 6×6 matrix, and one 2×2 matrix, as mentioned in Sec. IIC.

III. PROPERTIES OF THE GREEN'S FUNCTION AND ITS PADÉ APPROXIMATION

A. Unitarity

The exact Green's function G fulfills an extended unitarity equation, which can be written formally $G-G^\dagger=iG\mathcal{P}G^\dagger$ and which explicitly reads in the elastic region

$$G(p_1',p_2',p_1,p_2)-G^\dagger(p_1',p_2',p_1,p_2)=i\left(\frac{m}{\pi}\right)^2\int d\Omega_2(k_1,k_2)\,G(p_1',p_2',k_1,k_2)\,\mathcal{P}G^\dagger(k_1,k_2,p_1,p_2)\,, \qquad (3.1)$$

where $\mathcal{P}$ is the projector over the physical states, G is given by (2.9), $G^\dagger$ is given by

$$G^\dagger=\sum_{i,j}G^{(i,j)*}\Gamma_i\otimes\Gamma_j, \qquad (3.2)$$

and $d\Omega_2$ is the invariant two-body phase space

$$d\Omega_2(k_1,k_2)=d^4k_1\,d^4k_2\,\delta^{(+)}(k_1^2-m^2)\,\delta^{(+)}(k_2^2-m^2)\times\delta(k_1+k_2-p_1-p_2)\,. \qquad (3.3)$$

This equation holds order by order in perturbation theory if one assumes the Cutkosky rules for the Feynman diagrams contributing to G (see Appendix C for an explicit proof at second order).

From (3.1) we see that the physical transition amplitude $T=T_{++}$ fulfills the usual elastic unitarity condition. In the angular momentum basis the elastic unitarity equation becomes purely algebraic and reads

TABLE IV. Splitting of the spectroscopic states classified in Table III into positive and negative exchange parity sets $\mathcal{K}_+$ and $\mathcal{K}_-$ according to J and I.

T \ J	$\mathcal{K}_-$ Even	$\mathcal{K}_-$ Odd	$\mathcal{K}_+$ Even	$\mathcal{K}_+$ Odd
0	B,D	A,C	A,C	B,D
1	A,C	B,D	B,D	A,C

TABLE V. Classification of the transition amplitudes $T_{\beta'\beta}$ according to space and exchange parity.

Class	$\eta_\beta\eta_{\beta'}$	$\omega_\beta\omega_{\beta'}$	Amplitudes which belong to the class
I	+1	+1	T_{++}, T_{+-}, T_{-+}, T_{--}, T_{ee}, T_{oo}
II	+1	−1	T_{eo}, T_{oe}
III	−1	+1	T_{+e}, T_{-e}, T_{e+}, T_{e-}
IV	−1	−1	T_{+o}, T_{-o}, T_{o+}, T_{o-}

TABLE VI. Selection rules for transition amplitudes $T_{\beta'\beta}$ according to orbital momentum and total spin.

Class	$\|\Delta L\|$	$\|\Delta S\|$	Transitions allowed
I	0, 2	0	${}^1J_J \leftrightarrow {}^1J_J$, ${}^3J_J \leftrightarrow {}^3J_J$, ${}^3(J\pm1) \leftrightarrow {}^3(J\pm1)_J$
II	0, 2	1	${}^1J_J \leftrightarrow {}^3J_J$
III	1	1	${}^1J_J \leftrightarrow {}^3(J\pm1)_J$
IV	1	0	${}^3J_J \leftrightarrow {}^3(J\pm1)_J$

$$G^J - G^{J\dagger} = i\rho G^J P G^{J\dagger} , \qquad (3.4)$$

where P projects on the physical states and ρ is a kinematical factor $\rho = m^2/2\pi\sqrt{s}$; we shall come back to this point at the end of Sec. IV.

We remind the reader that the matrix elements of G^J are specified by the orbital momentum L, total spin S, and "intrinsic parity" index β and that they are analytic functions of the energy s and of the square of the external four-momenta:

$$G^J = G^J(s, p_1'^2, p_2'^2, p_1^2, p_2^2) .$$

The off-shell structure of the NN amplitude is relevant for some applications in nuclear physics such as the three-body problems. However, in the following we shall always restrict ourself to the case where $p_1^2 = p_2^2 = p_1'^2 = p_2'^2 = m^2$ since we want first to develop and test an approximation to the physical amplitude.

B. The approximation method

Interest in the Green's function, beyond its applications in the N-body problem, is motivated by the use of a nonlinear approximation scheme (Padé approximation) in which the physical and unphysical perturbative amplitudes are coupled to each other. This situation is very usual, for instance, in computing the resolvent of a given operator. It can be shown that the $[N/N]$ PA on G^J has the remarkable and unexpected feature of fulfilling identically the extended unitarity relation (3.4). Actually, only the first two terms in the perturbative expansion of G^J can be easily computed for a renormalizable Lagrangian,

$$G^J = \alpha G^J_{\text{Born}} + \alpha^2 G^J_{2-\text{Born}} + \cdots , \qquad (3.5)$$

where α is an expansion parameter. The corresponding $[1/1]$ PA reads

$$[1/1]^J = \alpha G^J_{\text{Born}} (G^J_{\text{Born}} - \alpha G^J_{2-\text{Born}})^{-1} G^J_{\text{Born}} . \qquad (3.6)$$

It can be shown that[7] the PAs have the same analytic structure in s and in $p_i^2, p_i'^2$ as the series (3.5), and, in addition, have extra poles given by

$$\det[G^J_{\text{Born}}(s) - \alpha G^J_{2-\text{Born}}(s)] = 0 , \qquad (3.7)$$

some of which are good candidates for the spectrum. Two remarks can be made to close this subsection:

(i) The $[N/N]$ PA can be derived from the Lippmann-Schwinger variational principle by using the perturbative ansatz.[7]

(ii) Some of the anomalies in the threshold behavior of the Born term remain in the $[1/1]$ PA to the physical amplitude T, but they all disappear when the Green's function approximation is used (this point is developed in Sec. III C).

C. Threshold behavior

It can be shown[11] using only Lorentz invariance that the one-pion-exchange contribution produces an anomalous threshold behavior for the two s waves as well as for all the ${}^3(J-1)_J$ waves (this is due to the pion's negative parity). At zero energy the pseudoscalar pion cannot be coupled to the physical nucleon (P-wave coupling), and therefore the amplitude vanishes. This is not true for the other elements contributing to the Green's function. For instance, a physical nucleon and an unphysical one of opposite "intrinsic parity" can be coupled by the pion even at zero energy (s-wave coupling). Therefore, near threshold, these unphysical transitions are very large compared to the physical one and will contribute through the Padé approximation to the physical phase shifts in a crucial way. The one-pion exchange is actually a very bad low-energy approximation, because it gives zero scattering lengths, while experimentally they are quite large (+5 F for ${}^3S_1$ and −20 F for ${}^1S_0$). The $[1/1]$ PA to the Green's function has already proved to give the good scattering lengths.[4,12]

For a given isospin I and angular momentum J, the Green's function has matrix elements which are functions of k (the center-of-mass momentum) and of α (the expansion parameter). For k small we can write

$$G^J_{f,i}(k, \alpha) = k^{L_f + L_i + 1} A^J_{f,i}(k, \alpha) , \qquad (3.8)$$

where L_f and L_i are the angular momenta of the final and initial states $|L_f, S_f\rangle$ and $|L_i, S_i\rangle$.

Another way to express the content of (3.8) is to write

$$G^J(k, \alpha) = kI(k) A^J(k, \alpha) I(k) , \qquad (3.9)$$

where $I(k)$ is a diagonal matrix independent of α whose elements are k^{L_m}. When $k \to 0$ we shall say that the threshold behavior is regular if

$$\lim_{k\to 0} A^J_{f,i}(k) = A^J_{f,i}(0) \neq 0 . \qquad (3.10)$$

From the well-known properties[7] of Padé approximation we can write

$$[N/N]_G\,\mathcal{J}(k,\alpha) = kI(k)\,[N/N]_A\,\mathcal{J}(k,\alpha)\,I(k)\ . \qquad (3.11)$$

It now becomes clear that if some of the matrix elements of the Born approximation to $A^J(k,\alpha)$ vanish for $k=0$, the PA to $A^J(k,\alpha)$ will in general not reflect this pathology. There is however an exception to this general rule: the [1/1] PA to a 1×1 matrix. However, we shall never encounter such a case as long as we deal with Green's functions which are at least 4×4 irreducible matrices (or 3×3 in the exceptional case $J=0$). The Green's function PA restores completely the correct threshold behavior in all partial waves.

More physical intuition of these phenomena can be obtained by considering, for instance, the $^1S_0$ wave. The Born term for the physical amplitude gives a zero contribution to the scattering length. For other elements of the Green's function the contribution of the Born term is large; this occurs for transitions to negative-parity states. Therefore, these matrix elements can never be neglected. But, in a linear perturbation theory such as ordinary perturbation theory, they will not affect the physical value because they cannot recouple with the physical element, in any order. In a Padé approximation which is nonlinear the unphysical matrix element recouples strongly with the physical one.

IV. PARTIAL-WAVE ANALYSIS OF THE NUCLEON-NUCLEON GREEN'S FUNCTION

A. Classification of the invariant amplitudes

Let us consider a transition between two given "intrinsic parity states" β', β. From (2.5) and according to the study of Ref. 13 we can decompose the transition amplitude into 16 Lorentz invariants $\langle O_i\rangle$ and $\langle O_i^5\rangle$, $i=1,\ldots,8$:

$$T_{\beta'\beta} = \sum_{\substack{i=1,\ldots,8\\ I=0,1}} [(A_i)^I_{\beta'\beta}\langle O_i\rangle + (A_i^5)^I_{\beta'\beta}\langle O_i^5\rangle]\,\mathcal{P}_I\ , \qquad (4.1)$$

where $\mathcal{P}_I$ is the projection operator on the state of isospin I, and where

$$\langle O_i\rangle = \bar{u}(p_1')\otimes\bar{u}(p_2')\,O_i\,u(p_1)\otimes u(p_2)\ ,$$
$$\langle O_i^5\rangle = \bar{u}(p_1')\otimes\bar{u}(p_2')\,O_i^5\,u(p_1)\otimes u(p_2)\ , \qquad (4.2)$$

with

$$\begin{aligned}
&O_1 = 1\otimes 1,\quad O_2=\tfrac{1}{2}\sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\ ,\\
&O_3 = i\gamma_5\gamma_\mu\otimes i\gamma_5\gamma^\mu\ ,\\
&O_4=\gamma_\mu\otimes\gamma^\mu,\quad O_5=\gamma_5\otimes\gamma_5\ ,\\
&O_6 = m^{-1}[\gamma_5(\not p_2+\not p_2')\otimes\gamma_5+\gamma_5\otimes\gamma_5(\not p_1+\not p_1')]\ ,\\
&O_7 = m^{-1}[\gamma_5(\not p_2+\not p_2')\otimes\gamma_5-\gamma_5\otimes\gamma_5(\not p_1+\not p_1')]\ ,\\
&O_8 = m^{-1}[(\not p_2+\not p_2')\otimes 1-1\otimes(\not p_1+\not p_1')]\ ,
\end{aligned} \qquad (4.3)$$

and

$$\begin{aligned}
&O_1^5 = 1\otimes\gamma_5+\gamma_5\otimes 1\ ,\\
&O_2^5 = m^{-1}[(\not p_2+\not p_2')\otimes\gamma_5+\gamma_5\otimes(\not p_1+\not p_1')]\ ,\\
&O_3^5 = m^{-1}[\gamma_5(\not p_2+\not p_2')\otimes 1+1\otimes\gamma_5(\not p_1+\not p_1')],\\
&O_4^5 = \gamma_\mu\otimes\gamma_5\gamma^\mu+\gamma_5\gamma_\mu\otimes\gamma^\mu,\\
&O_5^5 = 1\otimes\gamma_5-\gamma_5\otimes 1\ ,\\
&O_6^5 = m^{-1}[(\not p_2+\not p_2')\otimes\gamma_5-\gamma_5\otimes(\not p_1+\not p_1')]\ ,\\
&O_7^5 = m^{-1}[\gamma_5(\not p_2+\not p_2')\otimes 1-1\otimes\gamma_5(\not p_1+\not p_1')]\ ,\\
&O_8^5 = \gamma_\mu\otimes\gamma_5\gamma^\mu-\gamma_5\gamma_\mu\otimes\gamma^\mu\ .
\end{aligned} \qquad (4.4)$$

We note that the five first invariants $\langle O_i\rangle$, $i=1,\ldots,5$, are just the Fermi invariants S, T, A, V, P, and that the A_i's and the A_i^5's are functions of the kinematical invariants. The study of the transformation laws under parity, time reversal, and exchange operations, which we have performed in Sec. II for the Green's function, can be applied in a straightforward way to the invariant amplitudes (cf. Appendix B). The results are summarized in Table VII.

The parity, time-reversal, and particle-exchange transformation properties make it possible to decompose $T_{\beta'\beta}$, using that subset of the

TABLE VII. Transformation properties of the invariants O_i and O_i^5 according to space inversion, exchange operation, and time reversal.

Operation	Action
Space inversion	$O_i\to O_i$
	$O_i^5\to -O_i^5$
Exchange operation	$O_i\to O_i$, $i=1,\ldots,6$
	$O_i\to -O_i$, $i=7,8$
	$O_i^5\to O_i^5$, $i=1,\ldots,4$
	$O_i^5\to -O_i^5$, $i=5,\ldots,8$
Time reversal	$O_i\to O_i$, $i=1,2,3,4,5,8$
	$O_i\to -O_i$, $i=6,7$
	$O_i^5\to O_i^5$, $i=3,4,7,8$
	$O_i^5\to -O_i^5$, $i=1,2,5,6$

TABLE VIII. Classification of the invariants into which the $T_{\beta'\beta}$ are decomposed according to symmetry properties.

Class	Allowed invariants
I	$O_1, \ldots, O_6$ ($\beta=\beta'$ eliminates O_6 by time reversal)
II	O_7, O_8
III	$O_1^5, \ldots, O_4^5$
IV	$O_5^5, \ldots, O_8^5$

16 invariants which transform in the same way (Tables V and VIII).

B. Fierz transformation

Let us consider the invariants $\langle O_i\rangle$ and $\langle O_i^5\rangle$ [cf. (4.2)] and define their Fierz transform $\langle O_i\rangle_{\text{FT}}$ and $\langle O_i^5\rangle_{\text{FT}}$ by

$$\langle O_i\rangle_{\text{FT}} = \bar{u}(p_2')\otimes\bar{u}(p_1')\,O_i(p_1,p_2,p_2',p_1')\,u(p_1)\otimes u(p_2)\,,$$
$$\langle O_i^5\rangle_{\text{FT}} = \bar{u}(p_2')\otimes\bar{u}(p_1')\,O_i^5(p_1,p_2,p_2',p_1')\,u(p_1)\otimes u(p_2)\,.$$

The calculation of the Fierz transformation gives (Appendix D)

$$\begin{bmatrix}\langle O_1\rangle_{\text{FT}}\\ \langle O_2\rangle_{\text{FT}}\\ \langle O_3\rangle_{\text{FT}}\\ \langle O_4\rangle_{\text{FT}}\\ \langle O_5\rangle_{\text{FT}}\end{bmatrix} = \tfrac{1}{4}\begin{bmatrix}1&1&1&1&1\\ 6&-2&0&0&6\\ 4&0&-2&2&-4\\ 4&0&2&-2&-4\\ 1&1&-1&-1&1\end{bmatrix}\begin{bmatrix}\langle O_1\rangle\\ \langle O_2\rangle\\ \langle O_3\rangle\\ \langle O_4\rangle\\ \langle O_5\rangle\end{bmatrix}, \quad (4.5)$$

$$\langle O_6\rangle_{\text{FT}} = \langle O_6\rangle\,,$$

$$\begin{pmatrix}\langle O_7\rangle_{\text{FT}}\\ \langle O_8\rangle_{\text{FT}}\end{pmatrix} = \begin{pmatrix}0&1\\ 1&0\end{pmatrix}\begin{pmatrix}\langle O_7\rangle\\ \langle O_8\rangle\end{pmatrix}, \quad (4.6)$$

$$\begin{bmatrix}\langle O_1^5\rangle_{\text{FT}}\\ \langle O_2^5\rangle_{\text{FT}}\end{bmatrix} = \begin{bmatrix}\dfrac{t+s-u}{2t} & -\dfrac{m^2}{t}\\ \dfrac{s-u-t}{t} & \dfrac{t-2m^2}{t}\end{bmatrix}\begin{bmatrix}\langle O_1^5\rangle\\ \langle O_2^5\rangle\end{bmatrix},$$
$$\begin{pmatrix}\langle O_3^5\rangle_{\text{FT}}\\ \langle O_4^5\rangle_{\text{FT}}\end{pmatrix} = \begin{pmatrix}1&-2\\ 0&-1\end{pmatrix}\begin{pmatrix}\langle O_3^5\rangle\\ \langle O_4^5\rangle\end{pmatrix}, \quad (4.7)$$

$$\begin{bmatrix}\langle O_5^5\rangle_{\text{FT}}\\ \langle O_6^5\rangle_{\text{FT}}\\ \langle O_7^5\rangle_{\text{FT}}\\ \langle O_8^5\rangle_{\text{FT}}\end{bmatrix} = \begin{bmatrix}0&0&0&-\frac{1}{2}\\ 0&0&1&1\\ 2&1&0&0\\ -2&0&0&0\end{bmatrix}\begin{bmatrix}\langle O_5^5\rangle\\ \langle O_6^5\rangle\\ \langle O_7^5\rangle\\ \langle O_8^5\rangle\end{bmatrix}. \quad (4.8)$$

Since the transition amplitude must have a definite exchange parity, it is convenient to parametrize $T_{\beta'\beta}$ with a new set of invariant amplitudes which are linear combinations of the previous ones and of their Fierz transforms:

$$T_{\beta'\beta} = \sum_{i,I}\{(F_i)^I_{\beta'\beta}[\langle O_i\rangle + (-)^i\langle O_i\rangle_{\text{FT}}] + (F_i^5)^I_{\beta'\beta}[\langle O_i^5\rangle - (-)^i\langle O_i^5\rangle_{\text{FT}}]\}\mathcal{P}_I\,. \quad (4.9)$$

The linear relations between the F_i's (or F_i^5's) and the A_i's (or A_i^5's) of (4.1) are given explicitly in Appendix D. One can easily check that the amplitudes F_i and F_i^5 are free from kinematical singularities. Furthermore, if one splits $F_i(s,t,u)$ and $F_i^5(s,t,u)$ into even and odd components for the exchange $t\leftrightarrow u$, they correspond to transitions in $\mathcal{K}_+$ and $\mathcal{K}_-$, respectively. Precisely, one has

$$\left.\begin{matrix}(F_i(s,t,u))^I_{\beta'\beta} \pm \omega_{\beta'}(-)^{I+i+1}(F_i(s,u,t))^I_{\beta'\beta}\\ (F_i^5(s,t,u))^I_{\beta'\beta} \pm \omega_{\beta'}(-)^{I+i}(F_i^5(s,u,t))^I_{\beta'\beta}\end{matrix}\right\} \to \mathcal{K}_\pm\,. \quad (4.10)$$

C. Partial-wave expansion in the helicity formalism

In order to project out the partial waves we shall use the Jacob and Wick formalism.[10] To use it, we must change the normalization by setting

$$\Phi_{\beta'\beta} = \frac{m^2}{2\pi\sqrt{s}}\,T_{\beta'\beta}\,. \quad (4.11)$$

With this new normalization, the differential cross section reads

$$\frac{d\sigma}{d\Omega} = |\Phi_{++}|^2\,,$$

where Ω is the solid angle.

Following Jacob and Wick[10] we have

$$\langle\lambda_1'\lambda_2'|\Phi_{\beta'\beta}|\lambda_1\lambda_2\rangle = \frac{1}{k}\sum_J(2J+1)\,d^J_{\lambda\lambda'}(\cos\theta)\times\langle\lambda_1'\lambda_2'|\Phi^J_{\beta'\beta}|\lambda_1\lambda_2\rangle\,, \quad (4.12)$$

where k is the center-of-mass three-momentum

$$k=(\tfrac{1}{4}s-m^2)^{1/2}$$

and where

$$\lambda=\lambda_1-\lambda_2\ ,$$
$$\lambda'=\lambda_1'-\lambda_2'\ .$$

Equation (4.12) differs by a factor of 2 from Jacob and Wick since we want, accounting for the identity of particles, to have the same unitarity equation for $\Phi^J_{\beta'\beta}$. Using the orthogonality relation of the d^J functions,

$$\int_0^\pi d^J_{\lambda\lambda'}(\cos\theta)\,d^{J'}_{\mu\mu'}(\cos\theta)\sin\theta\,d\theta=\frac{2}{2J+1}\,\delta_{JJ'}\times\delta_{\lambda\mu}\delta_{\lambda'\mu'}\ , \tag{4.13}$$

we invert (4.12) through

$$\langle\lambda_1'\lambda_2'|\Phi^J_{\beta'\beta}|\lambda_1\lambda_2\rangle=\tfrac{1}{2}k\int_{-1}^{+1}\langle\lambda_1'\lambda_2'|\Phi_{\beta'\beta}|\lambda_1\lambda_2\rangle\times d^J_{\lambda\lambda'}(\cos\theta)\,d(\cos\theta)\ . \tag{4.14}$$

The relevant functions $d^J_{\lambda\lambda'}$ are related to the Legendre polynomials by

$$d^J_{00}(\cos\theta)=P_J(\cos\theta)\ ,$$

$$d^J_{11}(\cos\theta)=\frac{1}{1+\cos\theta}\times\left[P_J(\cos\theta)+\frac{JP_{J+1}(\cos\theta)+(J+1)P_{J-1}(\cos\theta)}{2J+1}\right]\ ,$$

$$d^J_{-11}(\cos\theta)=\frac{1}{1-\cos\theta}\times\left[-P_J(\cos\theta)+\frac{JP_{J+1}(\cos\theta)+(J+1)P_{J-1}(\cos\theta)}{2J+1}\right]\ ,$$

$$d^J_{10}(\cos\theta)=-d^J_{01}(\cos\theta)=\frac{[J(J+1)]^{1/2}}{2J+1}\frac{1}{\sin\theta}[P_{J+1}(\cos\theta)-P_{J-1}(\cos\theta)]\ . \tag{4.15}$$

Following GGMW[9] notations, we shall call $(\phi_i)_{\beta'\beta}$ the matrix elements of $\Phi_{\beta'\beta}$ in the standard helicity basis $|+\tfrac{1}{2},+\tfrac{1}{2}\rangle$, $|+\tfrac{1}{2},-\tfrac{1}{2}\rangle$, $|-\tfrac{1}{2},+\tfrac{1}{2}\rangle$, $|-\tfrac{1}{2},-\tfrac{1}{2}\rangle$. In this basis the $\Phi_{\beta'\beta}$ takes the form given in Table IX(a).

Using the explicit representation of the helicity spinors (Appendix A), we get the linear relations which give the ϕ_i's in terms of the invariant amplitudes F_i and F_i^5. For amplitudes belonging to classes I and II (as defined in Table V), the results are

$$\begin{aligned}
\phi_1&=\frac{1}{4\pi k_0}[m^2F_1+m^2(F_2+F_4)\cos\theta-(k^2+3k_0^2)F_3]\ ,\\
\phi_2&=\frac{1}{4\pi k_0}\{-k_0^2F_1+[(k_0^2+k^2)F_2+m^2F_4]\cos\theta+3m^2F_3-p^2F_5\}\ ,\\
\phi_3&=(1+\cos\theta)\frac{1}{4\pi k_0}\times[m^2F_2+k_0^2F_4+\tfrac{1}{2}k^2(-F_1+2F_3+F_5)]\ ,\\
\phi_4&=(1-\cos\theta)\frac{1}{4\pi k_0}\times[m^2F_2+k_0^2F_4-\tfrac{1}{2}k^2(-F_1+2F_3+F_5)]\ ,\\
\phi_5&=\sin\theta\frac{-1}{4\pi m}[m^2(F_2+F_4)+4k^2(F_6+F_8)]\ ,\\
\phi_6&=\sin\theta\frac{1}{4\pi m}[m^2(F_2+F_4)-4k^2(F_6+F_7)]\ ,\\
\phi_7&=\sin\theta\frac{1}{4\pi m}[m^2(F_2+F_4)+4k^2(F_6-F_8)]\ ,\\
\phi_8&=\sin\theta\frac{1}{4\pi m}[m^2(F_2+F_4)-4k^2(F_6-F_7)]\ .
\end{aligned} \tag{4.16}$$

We recall that k and k_0 are respectively the center-of-mass momentum and energy.

TABLE IX. (a) Structure of the transition matrix $\Phi_{\beta'\beta}$ in the standard helicity basis with $\eta=\eta_\beta\eta_{\beta'}$, i.e., the product of intrinsic parities. The indices $\beta'\beta$ have been omitted for convenience. (b) Structure of the partial-wave transition matrix $\phi^J_{\beta'\beta}$ in the standard helicity basis. The indices $\beta'\beta$ have been omitted for convenience.

(a)

	$f\backslash i$	$++$	$+-$	$-+$	$--$
$\Phi=$	$++$	ϕ_1	ϕ_5	ϕ_7	ϕ_2
	$+-$	ϕ_6	ϕ_3	ϕ_4	ϕ_8
	$-+$	$-\eta\phi_8$	$+\eta\phi_4$	$+\eta\phi_3$	$-\eta\phi_6$
	$--$	$\eta\phi_2$	$-\eta\phi_7$	$-\eta\phi_5$	$+\eta\phi_1$

(b)

	$f\backslash i$	$++$	$+-$	$-+$	$--$
$\Phi^J=$	$++$	ϕ_1^J	ϕ_5^J	ϕ_7^J	ϕ_2^J
	$+-$	ϕ_6^J	ϕ_3^J	ϕ_4^J	ϕ_8^J
	$-+$	$\eta\phi_8^J$	$\eta\phi_4^J$	$\eta\phi_3^J$	$\eta\phi_6^J$
	$--$	$\eta\phi_2^J$	$\eta\phi_7^J$	$\eta\phi_5^J$	$\eta\phi_1^J$

For amplitudes belonging to classes III and IV the results are

$$\phi_1 = \frac{k}{\pi}(-2F_3^5\cos\theta - 2F_4^5)\,,$$

$$\phi_2 = \frac{k}{\pi}(-\tfrac{1}{2}F_1^5\cos\theta + F_2^5)\,,$$

$$\phi_3 = (1+\cos\theta)\frac{k}{2\pi}(-\tfrac{1}{2}F_5^5 + F_6^5 - 2F_7^5 + F_8^5)\,,$$

$$\phi_4 = (1-\cos\theta)\frac{k}{2\pi}(\tfrac{1}{2}F_5^5 + F_6^5 + 2F_7^5 + F_8^5)\,, \tag{4.17}$$

$$\phi_5 = \sin\theta\frac{k}{4\pi}\frac{m}{k_0}\left(F_1^5 + \frac{4k_0^2}{m^2}F_3^5 + F_5^5 + \frac{4k_0^2}{m^2}F_7^5\right),$$

$$\phi_6 = \sin\theta\frac{k}{4\pi}\frac{m}{k_0}\left[F_1^5 - \frac{4k_0^2}{m^2}F_3^5 + \frac{2(k^2+k_0^2)}{m^2}F_6^5 + 2F_8^5\right],$$

$$\phi_7 = \sin\theta\frac{k}{4\pi}\frac{m}{k_0}\left(-F_1^5 - \frac{4k_0^2}{m^2}F_3^5 + F_5^5 + \frac{4k_0^2}{m^2}F_7^5\right),$$

$$\phi_8 = \sin\theta\frac{k}{4\pi}\frac{m}{k_0}\left(-F_1^5 + \frac{4k_0^2}{m^2}F_3^5 + 2\frac{k^2+k_0^2}{m^2}F_6^5 + 2F_8^5\right).$$

We notice from (4.18) and (4.19) that the amplitudes free from kinematical singularities and suitable for the partial-wave projections are

$$\phi_1,\ \phi_2,\ \frac{\phi_3}{1+\cos\theta},\ \frac{\phi_4}{1-\cos\theta},$$

$$\frac{\phi_i}{\sin\theta}\ (i=5,\ldots,8)\,.$$

Let us call $(\phi_i^J)_{\beta'\beta}$ the elements of the partial-wave transition matrix $\Phi^J_{\beta'\beta}$ which takes the form given in Table IX(b).

Combining (4.14) and (4.16), we get

$$\phi_1^J = \tfrac{1}{2}k\int_{-1}^{+1}\phi_1 P_J(\cos\theta)\,d(\cos\theta)\,,$$

$$\phi_2^J = \tfrac{1}{2}k\int_{-1}^{+1}\phi_2 P_J(\cos\theta)\,d(\cos\theta)\,,$$

$$\phi_3^J = \tfrac{1}{2}k\int_{-1}^{+1}\frac{\phi_3}{1+\cos\theta}\times\left[P_J(\cos\theta) + \frac{JP_{J+1}(\cos\theta)+(J+1)P_{J-1}(\cos\theta)}{2J+1}\right]d(\cos\theta),$$

$$\phi_4^J = \tfrac{1}{2}k\int_{-1}^{+1}\frac{\phi_4}{1-\cos\theta}\times\left[-P_J(\cos\theta) + \frac{JP_{J+1}(\cos\theta)+(J+1)P_{J-1}(\cos\theta)}{2J+1}\right]d(\cos\theta), \tag{4.18}$$

$$\phi_5^J = \tfrac{1}{2}k\frac{[J(J+1)]^{1/2}}{2J+1}\times\int_{-1}^{+1}\frac{\phi_5}{\sin\theta}[P_{J+1}(\cos\theta) - P_{J-1}(\cos\theta)]\,d(\cos\theta)\,,$$

$$\phi_6^J = -\tfrac{1}{2}k\frac{[J(J+1)]^{1/2}}{2J+1}\times\int_{-1}^{+1}\frac{\phi_6}{\sin\theta}[P_{J+1}(\cos\theta) - P_{J-1}(\cos\theta)]\,d(\cos\theta)\,,$$

$$\phi_7^J = -\tfrac{1}{2}k\frac{[J(J+1)]^{1/2}}{2J+1}\times\int_{-1}^{+1}\frac{\phi_7}{\sin\theta}[P_{J+1}(\cos\theta) - P_{J-1}(\cos\theta)]\,d(\cos\theta)\,,$$

$$\phi_8^J = -\tfrac{1}{2}k\frac{[J(J+1)]^{1/2}}{2J+1}\times\int_{-1}^{+1}\frac{\phi_8}{\sin\theta}[P_{J+1}(\cos\theta) - P_{J-1}(\cos\theta)]\,d(\cos\theta)\,.$$

D. Partial waves in the spectroscopic basis

In order to achieve our program we must now go from the angular momentum helicity basis to the spectroscopic one (SB), where the states are chosen in the following order: $({}^1J_J, {}^3J_J, {}^3(J-1)_J, {}^3(J+1)_J)$. An easy calculation shows that this transformation is given by

$$(\Phi^J_{\beta',\beta})_{SB} = M(\Phi^J_{\beta',\beta})M^\dagger\,, \tag{4.19}$$

where the transformation matrix reads

$$M = \frac{1}{[2(2J+1)]^{1/2}}\begin{bmatrix} (2J+1)^{1/2} & 0 & 0 & -(2J+1)^{1/2} \\ 0 & (2J+1)^{1/2} & -(2J+1)^{1/2} & 0 \\ (J)^{1/2} & (J+1)^{1/2} & (J+1)^{1/2} & (J)^{1/2} \\ -(J+1)^{1/2} & (J)^{1/2} & (J)^{1/2} & -(J+1)^{1/2} \end{bmatrix}. \tag{4.20}$$

We quote in Appendix E the formulas which give the spectroscopic final transition directly in terms of the invariant amplitudes A_i and A_i^5 defined in (4.1). Through (4.9), (4.16), (4.17), (4.18), and (4.19) one finally checks that in $(\Phi^J_{B',B})_{SB}$ all transitions, except the ones classified in Table VI, vanish identically.

Since the matrices $T^J_{B'B}$ are related to the $T_{B'B}$ through the same expansion as (4.12), the unitarity equation reads

$$T^J_{B'B} - (T^J_{B'B})^* = i\,\frac{m^2}{2\pi\sqrt{s}}\,T^J_{B'+}(T^J_{+B})^*\ , \qquad (4.21)$$

while for the matrices $\Phi_{B'B}$ it becomes simply

$$\Phi^J_{B'B} - (\Phi^J_{B'B})^* = i\,\Phi^J_{B'+}(\Phi^J_{+B})^*\ . \qquad (4.22)$$

V. CONCLUSION

In this work we have given a simple formalism to obtain the angular momentum expansion of the four-nucleon Green's function by generalizing the method used by GGMW for the physical amplitude.

We proved that the Green's function fulfills an extended unitarity condition, which, in the elastic region, is completely diagonalized by choosing the angular momentum basis, just as for the physical amplitude.

Our aim is to apply this formalism to the relevant field-theoretical models such as the Yukawa model, the nonlinear σ model, and the Yang-Mills models with gauge fields associated with the baryonic number and the isospin number (ω and ρ particles).

The [1/1] Padé approximant to the Green's function, which involves the computation up to second order of the above-mentioned models, will produce a physical amplitude with all the nice features one expects in a low-energy theory: correct threshold behavior in all waves, elastic unitarity, and correct analyticity in the energy and bound states. The PA is defined in such a way that it reproduces, for small coupling, the perturbation series up to second order and consequently has the same symmetry properties as the Lagrangian from which it is constructed.

Furthermore, such an approximation to the Green's function allows a *unique* field-theoretical off-shell extension and provides an off-shell T matrix which can be used for three-body calculations in nuclear physics.

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APPENDIX A

1. Notation and definitions

Our metric will be defined by the metric tensor

$$g_{00} = 1,\quad g_{\mu\nu} = -1, \quad \text{for } \mu=\nu=1,2,3 \qquad (A1)$$
$$g_{\mu\nu}=0, \quad \text{for } \mu\neq\nu .$$

The scalar product is then

$$a\cdot b = a_0 b_0 - \vec{a}\cdot\vec{b}\ . \qquad (A2)$$

The Dirac equation for the spinors $u(p,\lambda)$ is

$$(\gamma\cdot p - m)\,u(p,\lambda) = 0\ , \qquad (A3)$$

where we have defined

$$\gamma\cdot p = p_0\gamma_0 - \vec{p}\cdot\vec{\gamma}\ .$$

We normalize the spinors in such a way that

$$\bar{u}(p,\lambda)\,u(p,\lambda') = \delta_{\lambda\lambda'}, \qquad \bar{u} = u^\dagger \gamma_0$$

$$\sum_\lambda u(p,\lambda)\,\bar{u}(p,\lambda) = \frac{\not{p}+m}{2m}\ .$$

For the Dirac matrices we use the representation in which γ_0 is diagonal and γ_5 is antidiagonal, i.e.,

$$\gamma_0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \vec{\gamma} = \begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix},\quad \gamma_5 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad (A4)$$

where

$$\gamma_5 = i\,\gamma_0\gamma_1\gamma_2\gamma_3\ . \qquad (A5)$$

The σ_i are the 2×2 Pauli matrices. In this representation one has

$$(\gamma_5)^2 = (\gamma_0)^2 = -(\vec{\gamma})^2 = 1\ . \qquad (A6)$$

For completeness, we also quote the well-known anticommutation relations of the γ matrices:

$$[\gamma_\mu, \gamma_\nu]_+ = 2 g_{\mu\nu}\ . \qquad (A7)$$

The tensor operator is defined in the usual way

$$\sigma_{\mu\nu} = \tfrac{1}{2} i\,[\gamma_\mu, \gamma_\nu]\,.$$

2. Kinematics

We denote by p_1, p_2 and p_1', p_2' the four-momenta of the ingoing and outgoing nucleons, respectively. We define the Mandelstam variables s, t, and u, as usual, by

$$s = (p_1+p_2)^2 = (p_1'+p_2')^2\ ,$$
$$t = (p_1-p_1')^2 = (p_2-p_2')^2\ , \qquad (A8)$$
$$u = (p_2-p_1')^2 = (p_1-p_2')^2\ ,$$

so that we have

$$s+t+u=4m^2\ ,$$

where m is the nucleon mass. Since nucleon-nucleon kinematics is well known we will not be discussing it here.

We quote also the well-known kinematical relations

$$s=4(k^2+m^2)=4k_0^{\ 2}\ , \tag{A9}$$

$$t=-2k^2(1-\cos\theta)\ , \tag{A10}$$

$$u=-2k^2(1+\cos\theta)\ , \tag{A11}$$

where k is the c.m. momentum of each nucleon, k_0 is the c.m. energy of each nucleon, and θ is the c.m. scattering angle in the s channel. Let us recall also that in this picture the s channel describes a nucleon-nucleon process, whereas the t and u channels describe nucleon-antinucleon processes.

In order to make the partial-wave expansion through the Jacob and Wick procedure we shall have to calculate the invariant amplitudes in the helicity basis. In the center-of-mass helicity basis, we shall write the Dirac spinors as

$$u(\lambda_1)=\left(\frac{k_0+m}{2m}\right)^{1/2}\begin{pmatrix}\chi(\lambda_1)\\ \dfrac{2k\lambda_1}{k_0+m}\,\chi(\lambda_1)\end{pmatrix},$$

$$u(\lambda_2)=\left(\frac{k_0+m}{2m}\right)^{1/2}\begin{pmatrix}\chi(-\lambda_2)\\ \dfrac{2k\lambda_2}{k_0+m}\,\chi(-\lambda_2)\end{pmatrix},$$

$$u(\lambda_1')=\left(\frac{k_0+m}{2m}\right)^{1/2}\times\begin{pmatrix}e^{-i(\theta/2)\sigma_y}\chi(\lambda_1')\\ \dfrac{2k\lambda_1'}{k_0+m}\,e^{-i(\theta/2)\sigma_y}\chi(\lambda_1')\end{pmatrix}, \tag{A12}$$

$$u(\lambda_2')=\left(\frac{k_0+m}{2m}\right)^{1/2}\times\begin{pmatrix}e^{-i(\theta/2)\sigma_y}\chi(-\lambda_2')\\ \dfrac{2k\lambda_2'}{k_0+m}\,e^{-i(\theta/2)\sigma_y}\chi(-\lambda_2')\end{pmatrix},$$

where $\chi(\lambda)$ is a two-dimensional spinor which is an eigenstate of $\frac{1}{2}\sigma_z$.

APPENDIX B: SYMMETRY TRANSFORMATIONS

In order to analyze the symmetry properties of the Green's function G, we quote explicitly the transformations that it undergoes for space inversion, particle exchange, and time reversal. We shall only quote the results since the derivation can be found in any standard book on quantum field theory.

The action of parity is given by

$$G(p_1',p_2',p_1,p_2)\to\mathcal{V}_P\,G(\underline{p}_1',\underline{p}_2',\underline{p}_1,\underline{p}_2)\,\mathcal{V}_P\ ,$$

where $p=(p_0,\vec{p})$ and $\underline{p}=(p_0,-\vec{p})$, and $\mathcal{V}_P=\gamma_0\otimes\gamma_0$

The exchange operation is defined by

$$G(p_1',p_2',p_1,p_2)\to\mathcal{V}_\Omega\,G(p_2',p_1',p_2,p_1)\,\mathcal{V}_\Omega\ ,$$

where $\mathcal{V}_\Omega$ is a matrix defined in such a way that

$$\mathcal{V}_\Omega A\otimes B\,\mathcal{V}_\Omega=B\otimes A\ .$$

Finally, time reversal acts as follows:

$$G(p_1',p_2',p_1,p_2)\to\mathcal{V}_\Theta\,G^T(\underline{p}_1,\underline{p}_2,\underline{p}_1',\underline{p}_2')\,\mathcal{V}_\Theta^{\ -1}\ ,$$

where $\mathcal{V}_\Theta=i\gamma_5\gamma_2\otimes i\gamma_5\gamma_2$.

If one assumes G to be invariant under space inversion, exchange of particles, and time reversal, it follows that the transformation properties of the associated Green's functions $\Gamma_{B'}G\Gamma_B$, and consequently of the associated transition matrices $T_{B'B}$, defined by (2.20), are obtained by looking at the action of $\mathcal{V}_P$, $\mathcal{V}_\Omega$, and $\mathcal{V}_\Theta$ on Γ_B. We find

$$\mathcal{V}_P\Gamma_B\mathcal{V}_P=\eta_B\Gamma_B\ ,$$

$$\mathcal{V}_\Omega\Gamma_B\mathcal{V}_\Omega=\omega_B\Gamma_B\ ,$$

$$\mathcal{V}_\Theta\Gamma_B\mathcal{V}_\Theta^{\ -1}=\eta_B\Gamma_B\ ,$$

and finally,

parity:

$$\Gamma_{B'}\,G(p_1',p_2',p_1,p_2)\,\Gamma_B\to\eta_B\eta_{B'}\,\Gamma_{B'}\times G(p_1',p_2',p_1,p_2)\,\Gamma_B\ ,$$

exchange:

$$\Gamma_{B'}\,G(p_1',p_2',p_1,p_2)\,\Gamma_B\to\omega_B\omega_{B'}\,\Gamma_{B'}\times G(p_1',p_2',p_1,p_2)\,\Gamma_B\ ,$$

time reversal:

$$\Gamma_{B'}\,G(p_1',p_2',p_1,p_2)\,\Gamma_B\to\eta_B\eta_{B'}\,\Gamma_B\times G(p_1',p_2',p_1,p_2)\,\Gamma_{B'}\ .$$

The transformation properties of the invariants O_i given in Table VII can be obtained in a similar way.

APPENDIX C: PERTURBATIVE PROOF OF THE EXTENDED UNITARY RELATION

By expanding Eq. (3.1) in the πN coupling constant g, we find at order g^2

$$G_{\rm Born}=G_{\rm Born}^\dagger$$

and at order g^4

$$G_{2\,\rm Born}-G_{2\,\rm Born}^\dagger=i\left(\frac{m}{\pi}\right)^2\int d\Omega_2\,G_{\rm Born}\,\mathcal{P}\,G_{\rm Born}^\dagger\ , \tag{C1}$$

where $d\Omega_2$ is the invariant two-body phase space given by

$$d\Omega_2 = d^4k_1\, d^4k_2\, \delta^{(+)}(k_2{}^2 - m^2) \times \delta(k_1 + k_2 - p_1 - p_2)\ .$$

At order g^4 the only Feynman diagram for which $G_{2\,\text{Born}} - G^\dagger_{2\,\text{Born}}$ is nonzero is the direct box diagram (see Fig. 2). Its contribution in isospin 1 reads

$$G_{2\,\text{Born}} = -i(2\pi)^{-4} \int \frac{d^4k_1 d^4k_2}{D_1D_2D_3D_4}\, \delta(k_1 + k_2 - p_1 - p_2) \times \gamma_5(\not{k}_1 + m)\, \gamma_5 \otimes \gamma_5 (\not{k}_2 + m)\gamma_5\ , \tag{C2}$$

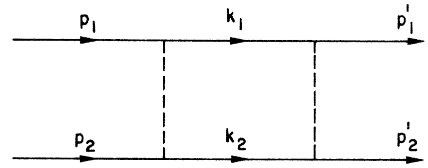

FIG. 2. The direct box diagram.

where $D_1 = k_1{}^2 - m^2$, $D_2 = (p_1 - k_1)^2 - \mu^2$, $D_3 = k_3{}^2 - m^2$, $D_4 = (p_1' - k_1)^2 - \mu^2$.

Cutkosky rules applied to $G_{2\,\text{Born}}$ give us the absorptive parts $G^{(ij)}_{2\,\text{Born}} - G^{*(ij)}_{2\,\text{Born}}$ of the invariant amplitudes $G^{(i,j)}_{2\,\text{Born}}$, into which $G_{2\,\text{Born}}$ can be decomposed [cf. (2.9), (2.11)]. In fact

$$G_{2\,\text{Born}} - G^\dagger_{2\,\text{Born}} = \text{disc}\, G_{2\,\text{Born}} = (2\pi i)^2(-i)(2\pi)^{-4} \int d^4k_1 d^4k_2\, \delta^{(+)}(k_1{}^2 - m^2)\, \delta^{(+)}(k_2{}^2 - m^2)\, \delta(k_1 + k_2 - p_1 - p_2) \times \frac{1}{D_2D_4}\, \gamma_5 \otimes \gamma_5 (\not{k}_1 + m) \otimes (\not{k}_2 + m)\, \gamma_5 \otimes \gamma_5\ . \tag{C3}$$

We notice that

$$(\not{k}_1 + m) \otimes (\not{k}_2 + m) = 4m^2 \sum_{\lambda_1, \lambda_2} u(k_1, \lambda_1)\, \bar{u}(k_1, \lambda_1) \otimes u(k_2, \lambda_2)\, \bar{u}(k_2, \lambda_2) = 4m^2\, \mathcal{P}\ , \tag{C4}$$

where $\mathcal{P}$ is the projector over the physical states, and that

$$G_{\text{Born}}(k_1, k_2, p_1, p_2) = \frac{\gamma_5 \otimes \gamma_5}{(p_1 - k_1)^2 - \mu^2}\ . \tag{C5}$$

Consequently, when we replace (C4), (C5), and (3.3) in (C3), we get (C1).

Finally, we remark that the definition (3.2) implies that

$$\langle \beta | G^\dagger | \alpha \rangle = \langle \alpha | G | \beta \rangle^*$$

[where $|\beta\rangle = \Gamma_\beta u(p_1) \otimes u(p_2)$, $|\beta'\rangle = \Gamma_{\beta'} u(p_1') \otimes u(p_2')$, and the adjoints are defined according to (2.14), (2.15) by $\langle \beta | = \bar{u}(p_1) \otimes \bar{u}(p_2)\Gamma_\beta$ and $\langle \beta' | = \bar{u}(p_1') \otimes \bar{u}(p_2')\Gamma_{\beta'}$] only if the Dirac matrices Γ_i are defined in such a way that

$$\gamma_0 \Gamma_i^\dagger \gamma_0 = \Gamma_i \tag{C7}$$

so that

$$(\gamma_0 \otimes \gamma_0)\, \Gamma_\beta \Gamma_i \otimes \Gamma_j \Gamma_\alpha (\gamma_0 \otimes \gamma_0) = (\Gamma_\alpha \Gamma_i \otimes \Gamma_j \Gamma_\beta)^\dagger . \tag{C8}$$

In order to fulfill condition (C7) the Γ_i defined in (2.9) should be defined in the simple change

$$\gamma_5 \to i\gamma_5\ .$$

The same obviously holds for the Γ_β defined in (2.16). We want to stress that such an exchange would not affect any of our results or conclusions.

APPENDIX D: FIERZ TRANSFORMATIONS

Let us consider an invariant amplitude

$$\langle F \otimes G \rangle = \bar{u}(p_1') \otimes \bar{u}(p_2') \times [F(p_1, p_2, p_1', p_2') \otimes G(p_1, p_2, p_1', p_2')] \times u(p_1) \otimes u(p_2)\ . \tag{D1}$$

We define its Fierz transform by

$$\langle F \otimes G \rangle_{\text{FT}} = \bar{u}(p_2') \otimes \bar{u}(p_1') \times [F(p_1, p_2, p_2', p_1') \otimes G(p_1, p_2, p_2', p_1')] \times u(p_1) \otimes u(p_2)\ . \tag{D2}$$

The computation goes through the basic identity

$$\langle F \otimes G \rangle_{\text{FT}} = \tfrac{1}{4} \sum_A \langle \Gamma_A \otimes F(p_1, p_2, p_2', p_1')\, \Gamma^A G(p_1, p_2, p_2', p_1') \rangle = \tfrac{1}{4} \sum_A \langle G(p_1, p_2, p_2', p_1')\, \Gamma_A F(p_1, p_2, p_2', p_1') \otimes \Gamma^A \rangle\ , \tag{D3}$$

where Γ_A is the complete set of 16 matrices

$$\Gamma_A=\{1,\gamma_5,\gamma_\mu,i\gamma_5\gamma_\mu,\tfrac{1}{2}\sigma_{\mu\nu}\}\ , \tag{D4}$$

such that

$$\mathrm{tr}[\Gamma_i,\Gamma_j]_+=4\delta_{ij}\ ,\quad \text{when } \Gamma_i\in\Gamma_A,\ \Gamma_j\in\Gamma_A . \tag{D5}$$

It is obvious to check from the definition that

$$\langle\langle F\otimes G\rangle_{\mathrm{FT}}\rangle_{\mathrm{FT}}=\langle F\otimes G\rangle . \tag{D6}$$

For a 4 spin-$\frac{1}{2}$ interaction the number of amplitudes in the transition matrix (2.9) is *a priori* equal to the number of amplitudes in the Green's function, but using the Dirac equation and eliminating the Levi-Civita ϵ symbols arising from the choice of the basis vectors $W^{(\mu)}$ used in (2.7), one can show[13] that the number of independent amplitudes reduces to 16 and that for equal masses the following equalities hold:

$$\tfrac{1}{2}t\langle\sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\rangle=\tfrac{1}{2}t\langle\gamma_5\sigma_{\mu\nu}\otimes\gamma_5\sigma^{\mu\nu}\rangle$$
$$=4m^2\langle\gamma_\mu\otimes\gamma^\mu\rangle+(s-u)\langle\gamma_5\otimes\gamma_5\rangle$$
$$-4m\langle 1\otimes\not\pi_1+\not\pi_2\otimes 1\rangle+(s-u)\langle 1\otimes 1\rangle\ ,$$

$$\tfrac{1}{2}t\langle\gamma_5\sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\rangle=\tfrac{1}{2}t\langle\sigma_{\mu\nu}\otimes\gamma_5\sigma^{\mu\nu}\rangle$$
$$=-4m\langle\not\pi_2\otimes\gamma_5+\gamma_5\otimes\not\pi_1\rangle$$
$$+(s-u)\langle 1\otimes\gamma_5+\gamma_5\otimes 1\rangle,$$

$$\langle\gamma_\mu\otimes\gamma_5\gamma^\mu\not\pi_1\rangle=m\langle\gamma_5\gamma_\mu\otimes\gamma^\mu\rangle+m\langle 1\otimes\gamma_5\rangle-\langle\gamma_5\not\pi_2\otimes 1\rangle\ ,$$

$$\langle\gamma_5\gamma_\mu\otimes\gamma_5\gamma^\mu\not\pi_1\rangle+\tfrac{1}{2}m\langle\sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\rangle=m\langle\gamma_\mu\otimes\gamma^\mu\rangle-\langle\not\pi_2\otimes 1\rangle ,$$

$$\langle\gamma_\mu\not\pi_2\otimes\gamma^\mu\not\pi_1\rangle+\tfrac{1}{8}(s-u)\langle\sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\rangle$$
$$=\tfrac{1}{4}(s-u)\langle 1\otimes 1\rangle-(2m^2-\tfrac{1}{4}t)\langle\gamma_5\otimes\gamma_5\rangle$$
$$+m^2\langle i\gamma_5\gamma_\mu\otimes i\gamma_5\gamma^\mu\rangle\ ,$$

$$\langle\gamma_5\gamma_\mu\not\pi_2\otimes\gamma^\mu\not\pi_1\rangle+\tfrac{1}{8}(s-u)\langle\gamma_5\sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\rangle$$
$$=\tfrac{1}{4}(s-u)\langle\gamma_5\otimes 1\rangle-(m^2-\tfrac{1}{4}t)\langle 1\otimes\gamma_5\rangle-m\langle\gamma_5\not\pi_2\otimes 1\rangle ,$$

$$\langle\gamma_5\gamma_\mu\not\pi_2\otimes\gamma_5\gamma^\mu\not\pi_1\rangle+\tfrac{1}{8}(s-u)\langle\sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\rangle \tag{D7}$$
$$=\tfrac{1}{4}(s-u)\langle\gamma_5\otimes\gamma_5\rangle+\tfrac{1}{4}t\langle 1\otimes 1\rangle$$
$$-m\langle\gamma_5\not\pi_2\otimes\gamma_5+\gamma_5\otimes\gamma_5\not\pi_1\rangle\ ,$$

$$-\langle\gamma_5\not\pi_2\otimes\gamma_5\not\pi_1\rangle+\tfrac{1}{2}m^2\langle\sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\rangle$$
$$=+\tfrac{1}{4}(s-u)\langle i\gamma_5\gamma_\mu\otimes i\gamma_5\gamma^\mu\rangle+(2m^2-\tfrac{1}{4}t)\langle\gamma_\mu\otimes\gamma^\mu\rangle$$
$$-m\langle 1\otimes\not\pi_1+\not\pi_2\otimes 1\rangle ,$$

$$\langle\not\pi_2\otimes\gamma_5\not\pi_1\rangle=\tfrac{1}{4}(s-u)\langle\gamma_\mu\otimes\gamma_5\gamma^\mu\rangle$$
$$-(m^2-\tfrac{1}{4}t)\langle\gamma_5\gamma_\mu\otimes\gamma^\mu\rangle+m\langle\gamma_5\not\pi_2\otimes 1\rangle\ ,$$

$$\langle\not\pi_2\otimes\not\pi_1\rangle=\tfrac{1}{4}(s-u)\langle\gamma_\mu\otimes\gamma^\mu\rangle-\tfrac{1}{4}t\langle i\gamma_5\gamma_\mu\otimes i\gamma_5\gamma^\mu\rangle$$
$$+m^2\langle\gamma_5\otimes\gamma_5\rangle\ ,$$

$$\langle\gamma_\mu\otimes\gamma^\mu\not\pi_1\rangle=m\langle 1\otimes 1\rangle-\langle\gamma_5\not\pi_2\otimes\gamma_5\rangle\ ,$$

$$\langle\gamma_5\gamma_\mu\otimes\gamma^\mu\not\pi_1\rangle+\tfrac{1}{2}m\langle\gamma_5\sigma_{\mu\nu}\otimes\sigma^{\mu\nu}\rangle=-\langle\not\pi_2\otimes\gamma_5\rangle\ ,$$

where we define

$$\pi_1=\tfrac{1}{2}(p_1+p_1')\ ,\qquad \pi_2=\tfrac{1}{2}(p_2+p_2')\ . \tag{D8}$$

The above relations are taken from Ref. 13.

From this set of identities we can derive a new one using the following remark: If we consider a relation between invariants of the type

$$\langle A(p_1,p_2,p_1',p_2')\otimes B(p_1,p_2,p_1',p_2')\rangle=\sum_k C_k(p_1,p_2,p_1',p_2')\langle A_k(p_1,p_2,p_1',p_2')\otimes B_k(p_1,p_2,p_1',p_2')\rangle ,$$

we can also prove that the following relation holds:

$$\langle B(p_2,p_1,p_2',p_1')\otimes A(p_2,p_1,p_2',p_1')\rangle=\sum_k C_k(p_2,p_1,p_2',p_1')\langle B_k(p_2,p_1,p_2',p_1')\otimes A_k(p_2,p_1,p_2',p_1')\rangle ;$$

i.e., we interchange the order in the tensor products and we interchange $\pi_1\leftrightarrow\pi_2$.

Taking into account the Fierz transforms (4.7)–(4.10), which have been calculated using (D3), we can relate the amplitudes F_i and F_i^5 defined by (4.11) to the amplitudes A_i and A_i^5 given by (4.1) and get

$$F_1=\tfrac{1}{2}(A_1-3A_2-2A_4),\quad F_1^5=\tfrac{1}{2}\left(A_1^5+\frac{s-2m^2}{m^2}A_2^5\right),$$

$$F_2=\tfrac{1}{2}(\tfrac{1}{2}A_1+A_2+\tfrac{1}{2}A_5),\quad F_2^5=\tfrac{1}{2}\left(\tfrac{1}{2}A_1^5+\frac{t-u+2m^2}{2m^2}A_2^5\right),$$

$$F_3=\tfrac{1}{2}(A_3-A_4),\quad F_3^5=\tfrac{1}{2}A_3^5\ ,$$

$$F_4=\tfrac{1}{2}(\tfrac{1}{2}A_1+A_3+A_4-\tfrac{1}{2}A_5),\quad F_4^5=\tfrac{1}{2}(A_3^5+A_4^5)\ , \tag{D9}$$

$$F_5=\tfrac{1}{2}(A_5-3A_2+2A_4),\quad F_5^5=\tfrac{1}{2}(A_5^5-2A_6^5-2A_8^5)\ ,$$

$$F_6=\tfrac{1}{2}A_6,\quad F_6^5=\tfrac{1}{2}(A_6^5-A_7^5),$$

$$F_7=\tfrac{1}{2}(A_7-A_8),\quad F_7^5=\tfrac{1}{2}(A_6^5+A_7^5)\ ,$$

$$F_8=\tfrac{1}{2}(A_7+A_8),\quad F_8^5=\tfrac{1}{2}(A_8^5+\tfrac{1}{2}A_5^5-A_7^5)\ .$$

APPENDIX E: SUMMARY OF THE RESULTS AND FINAL PROJECTION FORMULAS

In the spectroscopic basis (SB) the transition amplitudes $(T^J_{\beta',\beta})_{\mathrm{SB}}$ are 4×4 matrices with the structure given in Table X.

The only nonzero elements of any amplitudes $(T^J_{\beta',\beta})_{\rm SB}$ are the ones associated to the class to which this amplitude belongs (cf. Table V). The Green's function for definite angular momentum $(G^J)_{\rm SB}$ is built up from the 16 transition matrices $(T^J_{\beta',\beta})_{\rm SB}$ following the same scheme as (2.26); it is given in Table XI.

This structure allows us to verify the announced decomposition of $(G^J)_{\rm SB}$ into a direct sum of four matrices we have labeled in Table III as the singlet, uncoupled-triplet, coupled-triplet, and irrelevant one. The structure of these matrices is given in Tables XII(a)–XII(d).

In Table IV we have shown for fixed I and J which of these matrices correspond to the exchange parity -1 and $+1$, respectively.

In order to give the final results for the transition amplitudes $\langle L'S'|\Phi^J_{\beta',\beta}|LS\rangle$ we introduce a set of auxiliary amplitudes G_k, G^5_k with simple transformation properties for $t \leftrightarrow u$ exchange and for time reversal:

$$4\pi G_1 = F_1 - 4F_3 + F_5\,, \quad 4\pi G_2 = 2F_2\,,$$
$$4\pi G_3 = F_1 - 2F_3 - F_5\,, \quad 4\pi G_4 = 2F_4\,,$$
$$4\pi G_5 = F_1 + 4F_3 + F_5\,, \quad 4\pi G_6 = 2F_6\,,$$
$$4\pi G_7 = F_7 + F_8\,, \quad 4\pi G_8 = F_7 - F_8\,,$$

TABLE X. Structure of the transition amplitude in the spectroscopic basis. We associate $*$, $\triangle$, $+$, and $\square$ with classes I, II, III, and IV, respectively.

$(T^J_{\beta',\beta})_{\rm SB} =$ $f \backslash i$	${}^1J_J$	${}^3J_J$	${}^3(J-1)_J$	${}^3(J+1)_J$
${}^1J_J$	$*$	$\triangle$	$+$	$+$
${}^3J_J$	$\triangle$	$*$	$\square$	$\square$
${}^3(J-1)_J$	$+$	$\square$	$*$	$*$
${}^3(J+1)_J$	$+$	$\square$	$*$	$*$

and

$$4\pi G^5_1 = 2F^5_1\,, \quad 4\pi G^5_2 = 2\,\frac{F^5_1 - 2F^5_2}{1-\cos\theta}\,, \tag{E1}$$
$$4\pi G^5_3 = 8F^5_3\,, \quad 4\pi G^5_4 = 8F^5_4\,,$$
$$4\pi G^5_5 = F^5_5 + 2F^5_6 + 4F^5_7 + 2F^5_8\,,$$
$$4\pi G^5_6 = F^5_5 + 2\,\frac{k_0{}^2 + k^2}{m^2}\,F^5_6 + 4\,\frac{k_0{}^2}{m^2}\,F^5_7 + 2F^5_8\,,$$
$$4\pi G^5_7 = -F^5_5 + 2F^5_6 - 4F^5_7 + 2F^5_8\,,$$
$$4\pi G^5_8 = -F^5_5 + 2\,\frac{k_0{}^2 + k^2}{m^2}\,F^5_6 - 4\,\frac{k_0{}^2}{m^2}\,F^5_7 + 2F^5_8\,.$$

We can check that $F^5_1 - 2F^5_2|_{t=0} = 0$ and $F^5_1 + 2F^5_2|_{u=0}$

TABLE XI. Structure of the partial-wave Green's function in the spectroscopic basis. $*$, $\triangle$, $+$, and $\square$ are associated with classes I, II, III, and IV, respectively, and the dots are understood as zeros.

$(G^J)_{\rm SB} =$	$+$	$-$	e	o
$+$	$\begin{pmatrix} * & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot \\ \cdot & \cdot & * & * \\ \cdot & \cdot & * & * \end{pmatrix}$	$\begin{pmatrix} * & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot \\ \cdot & \cdot & * & * \\ \cdot & \cdot & * & * \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & + & + \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \square & \square \\ \cdot & \square & \cdot & \cdot \\ \cdot & \square & \cdot & \cdot \end{pmatrix}$
$-$	$\begin{pmatrix} * & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot \\ \cdot & \cdot & * & * \\ \cdot & \cdot & * & * \end{pmatrix}$	$\begin{pmatrix} * & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot \\ \cdot & \cdot & * & * \\ \cdot & \cdot & * & * \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & + & + \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \square & \square \\ \cdot & \square & \cdot & \cdot \\ \cdot & \square & \cdot & \cdot \end{pmatrix}$
e	$\begin{pmatrix} \cdot & \cdot & + & + \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & + & + \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} * & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot \\ \cdot & \cdot & * & * \\ \cdot & \cdot & * & * \end{pmatrix}$	$\begin{pmatrix} \cdot & \triangle & \cdot & \cdot \\ \triangle & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
o	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \square & \square \\ \cdot & \square & \cdot & \cdot \\ \cdot & \square & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \square & \square \\ \cdot & \square & \cdot & \cdot \\ \cdot & \square & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \triangle & \cdot & \cdot \\ \triangle & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} * & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot \\ \cdot & \cdot & * & * \\ \cdot & \cdot & * & * \end{pmatrix}$

TABLE XII. (a) Structure of the singlet matrix in the spectroscopic basis. * and + are associated with classes I and III, respectively. (b) Structure of the uncoupled-triplet matrix in the spectroscopic basis. * and □ are associated with classes I and IV, respectively. (c) Structure of the coupled-triplet matrix in the spectroscopic basis. *, Δ, +, and □ are associated with classes I, II, III, and IV, respectively. (d) Structure of the irrelevant matrix in the spectroscopic basis. * and Δ are associated with classes I and II, respectively.

(a)

	$f \backslash i$	$({}^1J_J)_+$	$({}^1J_J)_-$	$[{}^3(J-1)_J]_e$	$[{}^3(J+1)_J]_e$
(singlet) →	$({}^1J_J)_+$	*	*	+	+
	$({}^1J_J)_-$	*	*	+	+
	$[{}^3(J-1)_J]_e$	+	+	*	*
	$[{}^3(J+1)_J]_e$	+	+	*	*

(b)

	$f \backslash i$	$({}^3J_J)_+$	$({}^3J_J)_-$	$[{}^3(J-1)_J]_o$	$[{}^3(J+1)_J]_o$
(uncoupled triplet) →	$({}^3J_J)_+$	*	*	□	□
	$({}^3J_J)_-$	*	*	□	□
	$[{}^3(J-1)_J]_o$	□	□	*	*
	$[{}^3(J+1)_J]_o$	□	□	*	*

(c)

	$f \backslash i$	$[{}^3(J-1)_J]_+$	$[{}^3(J+1)_J]_+$	$[{}^3(J-1)_J]_-$	$[{}^3(J+1)_J]_-$	$({}^1J_J)_e$	$({}^3J_J)_o$
(coupled triplet) →	$[{}^3(J-1)_J]_+$	*	*	*	*	+	□
	$[{}^3(J+1)_J]_+$	*	*	*	*	+	□
	$[{}^3(J-1)_J]_-$	*	*	*	*	+	□
	$[{}^3(J+1)_J]_-$	*	*	*	*	+	□
	$({}^1J_J)_e$	+	+	+	+	*	Δ
	$({}^1J_J)_o$	□	□	□	□	Δ	*

(d)

	$f \backslash i$	$({}^3J_J)_e$	$({}^1J_J)_o$
(irrelevant) →	$({}^3J_J)_e$	*	Δ
	$({}^1J_J)_o$	Δ	*

$=0$ so that G_2^5 has no kinematical singularity.

If in the $F_k(F_k^5)$ we take the components of given $t \leftrightarrow u$ parity, which correspond to $\mathcal{K}_+$ or $\mathcal{K}_-$ [see (4.10)], then the G_k for $k=1,\ldots,6$ have the same $t \leftrightarrow u$ parity as F_k for the same k, while for $k=7, 8$ the combinations $G_7 \pm G_8$ have definite $t \leftrightarrow u$ parity.

For the other set, one may notice that G_1^5, G_3^5, $G_5^5 - G_7^5$, $G_6^5 - G_8^5$ have the same parity as F_k^5 for k odd, while $t(G_1^5 - G_2^5)$, G_4^5, $G_5^5 + G_7^5$, $G_6^5 + G_8^5$ have the same parity as F_k^5 for k even.

We finally quote the expressions for G_i (G_i^5) in terms of A_i (A_i^5) which make the time-reversal transformation properties transparent. Using (D9) we get

$$4\pi G_1 = \tfrac{1}{2}(A_1 - 6A_2 - 4A_3 + 4A_4 + A_5),$$
$$4\pi G_2 = \tfrac{1}{2}(A_1 + 2A_2 + A_5),$$
$$4\pi G_3 = \tfrac{1}{2}(A_1 - 2A_3 - 2A_4 - A_5),$$
$$4\pi G_4 = \tfrac{1}{2}(A_1 + 2A_3 + 2A_4 - A_5),$$
$$4\pi G_5 = \tfrac{1}{2}(A_1 - 6A_2 + 4A_3 - 4A_4 + A_5),$$
$$4\pi G_6 = A_6, \quad 4\pi G_7 = A_7, \quad 4\pi G_8 = A_8,$$

and

$$4\pi G_1^5 = A_1^5 + 2\frac{k^2 + k_0^2}{m^2} A_2^5, \quad 4\pi G_2^5 = 4\frac{k^2}{m^2} A_2^5,$$
$$4\pi G_3^5 = 4A_3^5, \quad 4\pi G_4^5 = 4(A_3^5 + A_4^5),$$
$$4\pi G_5^5 = A_5^5 + 2A_6^5, \quad 4\pi G_6^5 = A_5^5 + 2\frac{k^2 + k_0^2}{m^2} A_6^5,$$
$$4\pi G_7^5 = -4A_7^5 + 2A_8^5, \quad 4\pi G_8^5 = -4\frac{k_0^2}{m^2} A_7^5 + 2A_8^5. \tag{E2}$$

The amplitudes G_k, G_k^5 are linear combinations of the amplitudes A_i, A_i^5, having the same transformations properties under time reversal. From the results quoted in Table VII for the time-reversal transformation properties of the invariants O_i, O_i^5 we easily derive the following result. If

$$\langle L'S|\Phi^J_{\beta'\beta}|LS\rangle = \mathcal{G}(G_k, G_k^5), \tag{E3}$$

then

$$\langle LS|\Phi^J_{\beta'\beta}|L'S'\rangle = \mathcal{G}(\epsilon_k G_k, \epsilon_k^5 G_k^5), \tag{E4}$$

where

$$\epsilon_k = \begin{cases} +1, & \text{for } k = 1, 2, 3, 4, 5, 8 \\ -1, & \text{for } k = 6, 7; \end{cases}$$

$$\epsilon_k^5 = \begin{cases} +1, & \text{for } k = 3, 4, 7, 8 \\ -1, & \text{for } k = 1, 2, 5, 6. \end{cases}$$

Accounting for (4.22), (4.20), (4.21), (4.18), (4.19), (D9), and (E2) and defining

$$(\mathcal{Q}(G_k, G_k^5))^J = \int_{-1}^{+1} \mathcal{Q}(G_k, G_k^5) P_J(\cos\theta)\, d(\cos\theta),$$

where $\mathcal{Q}(G_k, G_k^5)$ is a linear form in the G_k's and G_k^5's [when only one G_k or G_k^5 will be involved, we shall write simply G_k^J or $(G_k^5)^J$], the final formulas read (we drop the indices $\beta'\beta$ for convenience):

singlet-singlet transition:

$$\langle {}^1J_J|\Phi^J|{}^1J_J\rangle = \frac{1}{2}\frac{k}{k_0}\frac{1}{2J+1}\left[(2J+1)(k_0^2 G_1 + m^2 G_3)^J - (J+1)k^2 G_2^{J+1} - Jk^2 G_2^{J-1}\right]; \tag{E6}$$

uncoupled-triplet–uncoupled-triplet transition:

$$\langle {}^3J_J|\Phi^J|{}^3J_J\rangle = \frac{1}{2}\frac{k}{k_0}\frac{1}{2J+1}\left[(2J+1)(m^2 G_2 + k_0^2 G_4)^J - Jk^2 G_3^{J+1} - (J+1)k^2 G_3^{J-1}\right]; \tag{E7}$$

coupled-triplet–coupled-triplet transitions:

$$\langle {}^3(J-1)_J|\Phi^J|{}^3(J-1)_J\rangle = \frac{1}{2}\frac{k}{k_0}\frac{1}{(2J+1)^2}\{J(J+1)(k_0 - m)^2(G_2 + G_4)^{J+1} + [Jk_0 + (J+1)m]^2 G_2^{J-1}$$
$$+ [Jm + (J+1)k_0]^2 G_4^{J-1} - J(2J+1)k^2 G_5^J - (J+1)(2J+1)k^2 G_3^J\}, \tag{E8}$$

$$\langle {}^3(J+1)_J|\Phi^J|{}^3(J+1)_J\rangle = \frac{1}{2}\frac{k}{k_0}\frac{1}{(2J+1)^2}\{[(J+1)k_0 + Jm]^2 G_2^{J+1} + [(J+1)m + Jk_0]^2 G_4^{J+1}$$
$$+ J(J+1)(k_0 - m)^2(G_2 + G_4)^{J-1} - (J+1)(2J+1)k^2 G_5^J - J(2J+1)k^2 G_3^J\}, \tag{E9}$$

$$\langle {}^3(J-1)_J|\Phi^J|{}^3(J+1)_J\rangle = \frac{1}{2}\frac{k}{k_0}\frac{[J(J+1)]^{1/2}}{(2J+1)^2}\left[k^2 J(G_4 - G_2)^{J+1} + k^2 J(G_4 - G_2)^{J-1} + (k_0 - m)(mG_4 - k_0 G_2)^{J+1}\right.$$
$$+ (k_0 - m)(k_0 G_4 - mG_2)^{J-1} + k^2(2J+1)(G_5 - G_3)^J$$
$$\left.- 4(k_0/m)k^2(2J+1)(G_6^{J+1} - G_6^{J-1})\right]; \tag{E10}$$

singlet–uncoupled-triplet transitions:

$$\langle {}^1J_J|\Phi^J|{}^3J_J\rangle = \frac{1}{2}\frac{k}{m}\frac{[J(J+1)]^{1/2}}{2J+1}(-4k^2)\left[(G_7 + G_8)^{J+1} - (G_7 + G_8)^{J-1}\right]; \tag{E11}$$

singlet–coupled-triplet transitions:

$$\langle {}^1J_J|\Phi^J|{}^3(J-1)_J\rangle = \frac{1}{2}\frac{k^2}{2J+1}\left(\frac{J}{2J+1}\right)^{1/2}\Big[-(J+1)(G_1^5-G_2^5+G_3^5)^{J+1}-J(G_1^5-G_2^5+G_3^5)^{J-1}+(2J+1)(G_1^5-G_2^5-G_4^5)^J$$
$$+\frac{m}{k_0}(J+1)\left(G_1^5+\frac{k_0^2}{m^2}G_3^5\right)^{J+1}-\frac{m}{k_0}(J+1)\left(G_1^5+\frac{k_0^2}{m^2}G_3^5\right)^{J-1}\Big], \tag{E12}$$

$$\langle {}^1J_J|\Phi^J|{}^3(J+1)_J\rangle = \frac{1}{2}\frac{k^2}{2J+1}\left(\frac{J+1}{2J+1}\right)^{1/2}\Big[(J+1)(G_1^5-G_2^5+G_3^5)^{J+1}+J(G_1^5-G_2^5+G_3^5)^{J-1}-(2J+1)(G_1^5-G_2^5-G_4^5)^J$$
$$+\frac{m}{k_0}J\left(G_1^5+\frac{k_0^2}{m^2}G_3^5\right)^{J+1}-\frac{m}{k_0}J\left(G_1^5+\frac{k_0^2}{m^2}G_3^5\right)^{J-1}\Big]; \tag{E13}$$

uncoupled-triplet–coupled-triplet transitions:

$$\langle {}^3J_J|\Phi^J|{}^3(J-1)_J\rangle = \frac{1}{2}\frac{k^2}{2J+1}\left(\frac{J+1}{2J+1}\right)^{1/2}\Big[J(G_5^5+G_7^5)^{J+1}+(J+1)(G_5^5+G_7^5)^{J-1}-(2J+1)(G_5^5-G_7^5)^J$$
$$-\frac{m}{k_0}J(G_6^5+G_8^5)^{J+1}+\frac{m}{k_0}J(G_6^5+G_8^5)^{J-1}\Big], \tag{E14}$$

$$\langle {}^3J_J|\Phi^J|{}^3(J+1)_J\rangle = \frac{1}{2}\frac{k^2}{2J+1}\left(\frac{J}{2J+1}\right)^{1/2}\Big[J(G_5^5+G_7^5)^{J+1}+(J+1)(G_5^5+G_7^5)^{J-1}-(2J+1)(G_5^5-G_7^5)^J$$
$$+\frac{m}{k_0}(J+1)(G_6^5+G_8^5)^{J+1}-\frac{m}{k_0}(J+1)(G_6^5+G_8^5)^{J-1}\Big]. \tag{E15}$$

Using the results of Table VIII, one can verify from the previous formulas that any amplitude $\Phi_{\beta'\beta}$ contributes only to the transitions classified in Table VI.

We should also remark that for any $\langle L'S'|\Phi^J|LS\rangle$, only the G_k or G_k^5 of a given $t \leftrightarrow u$ parity contribute to the $(\cdots)^J$ Legendre projections and those of opposite parity to the $(\cdots)^{J\pm1}$ Legendre projections.

It is only in the singlet–coupled-triplet transitions that G_2^5 appears in which have no definite $t \leftrightarrow u$ parity. However, letting $G_2^{5(\pm)}$ be the components of G_2^5 with $t \leftrightarrow u$ parity equal to ±1, one writes $G_2^5 = G_2^{5(+)} + G_2^{5(-)}$ and remarks that

$$F_2^{5(\epsilon)} = \pi[G_1^{5(\epsilon)} - G_2^{5(\epsilon)}] + \pi\cos\theta\, G_2^{5(-\epsilon)} = 0, \tag{E16}$$

where $\epsilon = \pm\beta'(-1)^{I+1}$ if the amplitude belongs to $\mathcal{H}_\pm$. Therefore

$$(J+1)(G_2^{5(-\epsilon)})^{J+1} + J(G_2^{5(-\epsilon)})^{J-1} + (2J+1)(G_1^{5(\epsilon)} - G_2^{5(\epsilon)})^J = 0.$$

and in (E12) we find that only the appropriate combinations $(G_1^{5(\epsilon)} - G_2^{5(\epsilon)} + G_3^{5(\epsilon)})^{J\pm1}$, $(G_2^{5(-\epsilon)} + G_4^{5(-\epsilon)})^J$, and

$$\left(G_1^{5(\epsilon)} + \frac{k_0^2}{m^2}G_3^{5(\epsilon)}\right)^{J\pm1}$$

contribute to $\langle {}^1J_J|\Phi|{}^3(J\pm1)_J\rangle$.

This remark is useful since in perturbation theory one may compute the G_k for any given graph and obtain the partial-wave contribution for the sum of that graph and the one for which the final lines are interchanged simply by multiplying the previous results for $\langle L'S'|\Phi|LS\rangle$ by a factor 2.

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