Inverse problem of scattering and off-shell continuation of the transition amplitude

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It is shown how dispersion relations for the Jost solution can be used to generate a class of integral equations, among them the Marchenko equation, that all solve the inverse problem for the s-wave Schrödinger equation. Motivated by the fact that scattering data (the phase shifts) are functions of momentum, we develop momentum-space versions of some of these equations. We also construct a Fredholm series solution to the inverse problem; it is formulated entirely in momentum space, and is convergent for a large class of potentials with or without bound states. As a by-product we obtain simple procedures for the continuation of the physical transition amplitude off the energy shell.

I. INTRODUCTION

The conventional way to formulate the inverse problem of scattering is to ask for that local potential which via the Schrödinger equation reproduces the scattering data represented by a phase shift (and certain bound-state parameters). However, when dealing with multiparticle scattering theories like the Faddeev three-body theory, it is not the potential but the two-body off-shell transition amplitude that forms the appropriate input, and one is faced with the closely related problem of reconstructing the off-shell parts of this amplitude from its on-shell, "physical," values. Assuming a local, energy-independent potential, this can of course be done by combining the theory for the conventionally stated inverse problem (finding the potential) with the Schrödinger or Lippmann-Schwinger equation (finding the off-shell amplitude from a potential), but in quite a roundabout fashion. For one thing, the solution of the inverse problem involves a function that directly corresponds to the half-off-shell amplitude, so that the last step is in fact unnecessary.[1]

From the point of view of off-shell continuation and, for that matter, also of finding the potential, it is further unsatisfactory to have the theory of the inverse problem formulated in configuration space, thus demanding at one point or another the Fourier transformation of the initial data into a configuration-space function and, at a later point, the transformation of the solution back to momentum space. The methods devised in Ref. 1 to improve this situation were based on the idea of formally solving the Marchenko equation for the inverse problem in configuration space (iterative series, Fredholm series, etc.), and transforming the solution term by term into momentum space in such a way that all intermediate references to configuration space were eliminated. In the simple case of an s wave without bound states, a momentum-space series based on the iterative series solution to the Marchenko equation was considered in detail, and found to be simple and well suited for numerical evaluation.

In the present paper we are going to extend the results of Ref. 1 in several ways. After some preliminaries in Sec. II we consider in Sec. III the straightforward but less-known approach to the inverse problem that is based on dispersion relations (in the momentum parameter) for solutions to the (s-wave) Schrödinger equation.[2] The outcome is not only a rederivation of the Marchenko equation, but the derivation of a whole class of equivalent equations which all solve the inverse problem. However, while the Marchenko equation might be a good choice for the coordinate-space version of the inverse theory, we prefer an alternative equation for the momentum-space formulation in Sec. IV. The equation chosen generalizes the momentum-space series solution obtained in Ref. 1 in that it reproduces this series upon iteration, but the equation is valid also when the series diverges (when there are bound states).

From the solutions to the momentum-space equation, the half-off-shell transition amplitude as well as the (momentum-space) potential and bound-state wave function can be obtained in a straightforward manner, the half-off-shell amplitude simply as a linear combination of two solutions.

In Sec. VI we present another treatment of the inverse problem in momentum space which is based on the Fredholm series solution to one of the dispersion relations obtained in Sec. III. Without being much more difficult to evaluate, it has the advantage over the iterative series of Ref. 1 of being always convergent.

II. NOTATION

In this section we collect some well-known results from potential scattering theory.[3] We consider solutions to the s-wave Schrödinger equation

$$\left(\frac{d^2}{dr^2}+k^2-\mathcal{V}(r)\right)\psi(k,r)=0 \tag{2.1}$$

for potentials that satisfy the conditions

$$\int_0^\infty dr\, r^\alpha |\mathcal{V}(r)|<\infty, \quad \alpha=1,2 \ . \tag{2.2}$$

The Jost solutions to Eq. (2.1) are defined by boundary conditions at infinity,

$$\lim_{r\to\infty} f_\pm(k,r)e^{\mp ikr}=1 \ , \tag{2.3}$$

and the Jost functions are the Jost solutions at the origin,

$$\mathcal{L}_\pm(k)=f_\pm(k,0) \ . \tag{2.4}$$

For fixed r, the Jost solution $f_+(k,r)$ [and the Jost function $\mathcal{L}_+(k)$] is a regular analytic function of k in $\mathrm{Im}k>0$, and it is continuous with a continuous k derivative in $\mathrm{Im}k\geq 0$. On the real k axis we have $f_+^*(k,r)=f_-(k,r)$, and for real and positive k values in addition we have $f_+(ke^{i\pi},r)=f_-(k,r)$. Similar relations hold for the Jost functions $\mathcal{L}_\pm(k)$. At infinity, i.e., for $|k|\to\infty$ in $\mathrm{Im}k\geq 0$, $f_+(k,r)$ has the behavior

$$f_+(k,r)=e^{ikr}+o(e^{-\nu r}), \quad \nu=\mathrm{Im}k \ . \tag{2.5}$$

The "regular" solution $\phi(k,r)$ to Eq. (2.1) is defined by the boundary condition at the origin

$$\phi(k,0)=0, \quad \phi'(k,0)=1 \ . \tag{2.6}$$

It is an entire function of k for all k with the symmetry property $\phi(-k,r)=\phi(k,r)$, and it has the behavior

$$\phi(k,r)=k^{-1}\sin kr+o(|k|^{-1}e^{|\nu|r}) \tag{2.7}$$

when $|k|\to\infty$. It is related to the Jost solutions through (k real)

$$\phi(k,r)=(2ik)^{-1}[\mathcal{L}_-(k)f_+(k,r)-\mathcal{L}_+(k)f_-(k,r)] \tag{2.8}$$

and to the physical wave function through

$$\psi^+(k,r)=\frac{k\phi(k,r)}{\mathcal{L}_+(k)} \ . \tag{2.9}$$

$\psi^+(k,r)$ is a meromorphic function in $\mathrm{Im}k>0$, with a finite number of (necessarily simple) poles on the imaginary axis. These poles come from zeros of $\mathcal{L}_+(k)$ and correspond to bound states. Let $k=iK$, K real and positive, be such a pole. Then $f_+(iK,0)=\mathcal{L}_+(iK)=0$, and the Jost solution is proportional to the regular solution

$$f_+(iK,r)=c\phi(iK,r) \ . \tag{2.10}$$

The S matrix, finally, is defined in terms of the Jost functions for real, positive k values as

$$S(k)=\frac{\mathcal{L}_-(k)}{\mathcal{L}_+(k)}=e^{2i\delta(k)} \ , \tag{2.11}$$

where $\delta(k)$ is the scattering phase shift, and $\mathcal{L}_\pm(k)=|\mathcal{L}_\pm(k)|e^{\mp i\delta(k)}$. For negative k we take $\delta(k)=-\delta(-k)$, so that $S(-k)=S^*(k)$.

III. DISPERSION RELATIONS

With the knowledge of the analytic properties in k of $f_\pm(k,r)$, $\phi(k,r)$, and $\mathcal{L}_\pm(k)$ summarized in Sec. II, various dispersion relations for these entities can be obtained. Probably the best known is a dispersion relation for $\mathcal{L}_\pm(k)$, or rather $\ln\mathcal{L}_\pm(k)$, which leads to an explicit formula for $|\mathcal{L}_\pm(k)|$ in terms of the phase shift $\delta(k)$.[3] Less known is a rederivation of the equation of Gel'fand and Levitan for the inverse problem of scattering.[2] In this case a dispersion relation for a particular combination of solutions to the Schrödinger equation is turned into an integral equation in k for the regular solution $\phi(k,r)$ in terms of $|\mathcal{L}_\pm(k)|$.

The Marchenko equation for the inverse problem can also be obtained from a dispersion relation. In fact, from the analytic properties of the Jost solutions it follows that $f_+(k,r)e^{-ikr}$ satisfies the Cauchy formula

$$f_+(k,r)e^{-ikr}=\frac{1}{2\pi i}\int_C dk'\,\frac{f_+(k',r)e^{-ik'r}}{k'-k} \ , \tag{3.1}$$

where $\mathrm{Im}k>0$. The contour of integration is the line segment $k'\in[-\Lambda,\Lambda]$, closed by the upper half plane part of the circle $|k'|=\Lambda$. If in Eq. (3.1) the variable of integration along the line segment is changed from k' to $-k'$, the relation (2.8) in the form

$$f_+(-k,r)=f_-(k,r)=S(k)f_+(k,r)-\frac{2ik\phi(k,r)}{\mathcal{L}_+(k)} \tag{3.2}$$

can be used to write (3.1) as

$$f_+(k,r)e^{-ikr}=-\frac{1}{2\pi i}\int_{-\Lambda}^{\Lambda} dk'\,\frac{S(k')f_+(k',r)e^{ik'r}}{k'+k}+\frac{1}{\pi}\int_C dk'\,\frac{k'\phi(k',r)e^{ik'r}}{\mathcal{L}_+(k')(k'+k)}$$
$$+\frac{1}{2\pi i}\int_{C'} dk'\left(\frac{f_+(k',r)e^{-ik'r}}{k'-k}-\frac{2ik'\phi(k',r)e^{ik'r}}{\mathcal{L}_+(k')(k'+k)}\right) \ , \tag{3.3}$$

where C' is the circular part of C. In the first integral, $S(k')\to 1$ when $k'\to\pm\infty$ and, since

$$\lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda} dk' \frac{f_+(k', r)e^{ik'r}}{k'+k} = 0 , \tag{3.4}$$

the integral is well defined when $\Lambda\to\infty$. The contribution to the second integral comes from the bound-state poles at $k=iK_n$ ($K_n>0$), i.e., from the zeros of $\mathfrak{L}_+(k)$, and amounts to

$$2i\sum_n \frac{iK_n\phi(iK_n, r)e^{-K_n r}}{(k+iK_n)\mathfrak{L}_+(iK_n)} . \tag{3.5}$$

Here it is instructive to see what happens for more restrictive classes of potentials then considered so far. Let us assume that the potential is such that $f_+(k, r)$ is analytic in the domain $\mathrm{Im}k>-\mu$, $\mu>0$, so that $f_-(k, r)$ is analytic in $\mathrm{Im}k<\mu$. Then the S matrix and the relation (2.8) are well defined in the strip $|\mathrm{Im}k|<\mu$. If, in addition, all the bound-state poles fall in this strip, we can use Eq. (2.8) to determine the constant c in Eq. (2.10), and rewrite the expression (3.5) as

$$\sum_n \frac{f_+(iK_n, r)}{k+iK_n}\frac{\mathfrak{L}_-(iK_n)}{\mathfrak{L}_+(iK_n)} e^{-K_n r} . \tag{3.6}$$

But this is exactly the contribution to the first integral in Eq. (3.3) from the bound-state pole singularities of $S(k)$, with opposite sign.

With the less stringent conditions (2.2) on the potential, the constant c in Eq. (2.10) cannot be determined from Eq. (2.10), but it is nevertheless convenient to combine the contribution from the second integral in Eq. (3.3) with the first one. For this purpose, let us introduce a function $S^{BS}(k)$ which is analytic in the upper half plane except for simple poles at the bound-state points $k=iK_n$, which has suitably chosen residues at these points, and which behaves reasonably at infinity. Then, from (3.3),

$$f_+(k, r) = e^{ikr} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{S(k')-S^{BS}(k')}{k'+k} \times e^{i(k+k')r} f_+(k', r) , \tag{3.7}$$

where we have used the fact that the last integral in (3.3) tends to 1 when $\Lambda\to\infty$. In order for the integrand in Eq. (3.7) to decrease more rapidly for large k', we assume that $S^{BS}(k')\to 1$, $k'\to\pm\infty$, and that $S^{BS}(k')\equiv 1$ when there are no bound states. This we can do in view of the relation Eq. (3.4).

For $\mathrm{Im}k\to 0^+$, Eq. (3.7) is an integral equation for $f_+(k, r)$ in terms of $S(k)-S^{BS}(k)$.

With the present approach it is evident that Eq. (3.7) is not a unique equation that solves the inverse problem starting from $S(k)-S^{BS}(k)$. For instance, if on the right-hand side of (3.7) f_+ is represented by the expression

$$f_+(q, r) = e^{iqr} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{f_+(k', r)-e^{ik'r}}{k'-q-i\epsilon} e^{i(q-k')r} , \tag{3.8}$$

we obtain a new integral equation with a Hermitian kernel [we always take $S^{BS}(-q)=S^{BS*}(q)$]

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dq \frac{S(q)-S^{BS}(q)}{(k+q+i\epsilon)(k'-q-i\epsilon)} e^{i(k+2q-k')r} . \tag{3.9}$$

When written for the Fourier transform,

$$A(r, r') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq[f_+(q, r)-e^{iqr}]e^{-iqr'} , \tag{3.10}$$

this equation is recognized as the familiar Marchenko equation for the inverse problem. As another example, a whole class of equations is obtained if to the right-hand side of Eq. (3.7) we add functions of k times the expression (3.4).[4] In Sec. IV we shall consider the equation

$$f_+(k, r) = e^{ikr} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{S(k')-S^{BS}(k')-[S(k)-S^{BS}(k)]^*}{k+k'} e^{i(k+k')r} f_+(k', r) , \tag{3.11}$$

which has been obtained in precisely this way. Unlike Eq. (3.7), Eq. (3.11) has a nonsingular kernel.

Let us finally comment on the well-known bound-state normalization ambiguity of the inverse problem. For the more restricted class of potentials considered above, the poles of $S(k')$ due to zeros of $\mathfrak{L}_+(k)$ cancel the poles of $S^{BS}(k)$, and $S(k)-S^{BS}(k)$ has the singularity structure of $\mathfrak{L}_-(k)$ in the upper half-plane. In this case there is no ambiguity. However, in the more general case, we do not know to what extent the singularity of $S(k)$ at the bound-state point $k=iK_n$ should be attributed to a zero of $\mathfrak{L}_+(k)$ or to the singularity structure of $\mathfrak{L}_-(k)$. Consequently, the choice of residue of $S^{BS}(k)$ at $k=iK_n$ is ambiguous, and we are left with an unspecified parameter in the solution to the inverse problem.

IV. INVERSE PROBLEM IN MOMENTUM SPACE

Since we aim at a momentum-space formulation of the inverse problem we want to transform Eqs. (3.7) and (3.9) into equations for the momentum-space transform of $f_+(k, r)$. It turns out to be convenient to define a function

$$\mathcal{F}(k, q) = \frac{2}{\pi} \int_0^\infty dr\, e^{i(2q-k)r} f_+(k, r)\,, \tag{4.1}$$

which is related to the normalized momentum-space physical wave function through [compare Eq. (2.9)]

$$\Psi^+(k,p) \equiv \frac{2}{\pi} \int_0^\infty dr \sin pr\, \psi^+(k, r) = \frac{2}{\pi}\, e^{i\delta(k)} \int_0^\infty dr \sin pr\; \mathrm{Im}[f_+(k, r)^{i\delta(k)}]$$

$$= e^{i\delta(k)} \mathrm{Im}\left(\frac{1}{2i} e^{i\delta(k)} [\,\mathcal{F}(k, \tfrac{1}{2}(k+p)) - \mathcal{F}(k, \tfrac{1}{2}(k-p))]\right)\;. \tag{4.2}$$

Equations (3.7) and (3.11) take a particularly simple form when expressed in terms of $\mathcal{F}$:

$$\mathcal{F}(k, q) = \frac{1}{\pi}\, \frac{i}{q+i\epsilon} - \frac{1}{2\pi i} \int_{-\infty}^\infty dk'\, \frac{S(k') - S^{\mathrm{BS}}(k')}{k+k'+i\epsilon}\, \mathcal{F}(k', k'+q) \tag{4.3}$$

and

$$\mathcal{F}(k, q) = \frac{1}{\pi}\, \frac{i}{q+i\epsilon} - \frac{1}{2\pi i} \int_{-\infty}^\infty dk'\, \frac{S(k') - S^{\mathrm{BS}}(k') - [S(k) - S^{\mathrm{BS}}(k)]^*}{k+k'}\, \mathcal{F}(k', k'+q)\;. \tag{4.4}$$

In addition to having a nonsingular kernel, the latter equation has the advantage of being equivalent to a real equation in the absence of bound states. Indeed, for $\mathrm{Im}[(1/2i)e^{i\delta(k)}\mathcal{F}(k, q)]$, which is the entity of interest for Eq. (4.2), one has

$$\mathrm{Im}\left(\frac{1}{2i}\, e^{i\delta(k)} \mathcal{F}(k, q)\right) = \frac{1}{2\pi} \mathrm{Im}\, \frac{e^{i\delta(k)}}{q+i\epsilon} - \frac{1}{\pi} \int_{-\infty}^\infty dk'\, \frac{\sin[\delta(k)+\delta(k')]}{k+k'}\; \mathrm{Im}\left(\frac{1}{2i}\, e^{i\delta(k')} \mathcal{F}(k', k'+q)\right). \tag{4.5}$$

When iterated, this last equation reproduces the series solution to the inverse problem in momentum space obtained in a previous paper.[1] From this paper we also recall the result that for a simple Bargmann potential without bound states the iterative series corresponding to Eq. (4.5) converges, while with one bound state the iterative series based on Eq. (4.3) or (4.4) diverges.

In order to complete the solution of the inverse problem, we also give the expression for the potential in terms of $\mathcal{F}(k, q)$. The momentum-space potential is a function of two variables p and q,

$$\mathcal{V}(p, q) = \frac{2}{\pi} \int_0^\infty dr \sin pr\; \mathcal{V}(r) \sin qr\;. \tag{4.6}$$

However, it is easy to show that

$$\mathcal{V}(p, q) = \mathcal{V}(\tfrac{1}{2}(p+q), \tfrac{1}{2}(p+q)) - \mathcal{V}(\tfrac{1}{2}(p-q), \tfrac{1}{2}(p-q)) \tag{4.7}$$

and that

$$\frac{1}{p}\mathcal{V}(p,p) = \frac{2}{\pi} \int_0^\infty dr \sin 2pr \int_r^\infty dr'\, \mathcal{V}(r')\;. \tag{4.8}$$

From the conventional Marchenko-type discussion of the inverse problem it is known that[5]

$$\tfrac{1}{2} \int_r^\infty dr'\, \mathcal{V}(r') = \lim_{\substack{r' \to r \\ r' > r}} A(r, r')\,, \tag{4.9}$$

where $A(r, r')$ is the function defined in Eq. (3.10). Combining Eqs. (3.7), (3.10), (4.8), and (4.9), we get

$$\frac{1}{p}\mathcal{V}(p,p) = \frac{1}{2\pi i} \int_{-\infty}^\infty dk' [S(k') - S^{\mathrm{BS}}(k')] \times [\mathcal{F}(k', k'+q) - \mathcal{F}(k', k'-q)]\;, \tag{4.10}$$

where on the right-hand side the contribution to the imaginary part vanishes. In particular, if there are no bound states,

$$\frac{1}{p}\mathcal{V}(p,p) = -\frac{2}{\pi} \int_{-\infty}^\infty dk' \sin\delta(k') \times \mathrm{Im}\left(\frac{1}{2i} e^{i\delta(k')} [\mathcal{F}(k', k'+q) - \mathcal{F}(k', k'-q)]\right). \tag{4.11}$$

With these relations, we have completed the momentum-space formulation of the solution to the inverse problem of scattering.

V. THE HALF-OFF-SHELL AMPLITUDE AND THE BOUND-STATE WAVE FUNCTION

It was seen in Sec. IV that the physical wave function $\Psi^+(k,p)$ is a by-product of the solution to the inverse problem, Eq. (4.2). So is the half-off-shell transition amplitude t, as is obvious from the relation[6]

$$\Psi^{+}(k,p)=\delta(p-k)-2\mu\frac{pk}{p^{2}-k^{2}-i\epsilon}\times t(p,k;(2\mu)^{-1}k^{2}+i\epsilon)\ . \quad (5.1)$$

Also the bound-state wave functions can be obtained in a straightforward way. Assume there is a bound state at $k=iK$. Then the bound-state wave function $f_+(iK,r)$ can be obtained by taking $k=iK$ in Eq. (3.7) or (3.11), or from the Cauchy-type formula

$$f_{+}(iK,r)=\frac{K}{\pi}\int_{-\infty}^{\infty}dk'\,\frac{1}{k'^{2}+K^{2}}\,f_{+}(k',r)\ . \quad (5.2)$$

The momentum-space bound-state wave function is now (apart from normalization)

$$f_{+}(iK,p)=\frac{2}{\pi}\int_{0}^{\infty}dr\sin pr\, f_{+}(iK,r)=\frac{K}{\pi}\int_{-\infty}^{\infty}dk'\,\frac{1}{k'^{2}+K^{2}}\,\frac{1}{2i}[\mathfrak{F}(k',\tfrac{1}{2}(k'+p))-\mathfrak{F}(k',\tfrac{1}{2}(k'-p))]\ . \quad (5.3)$$

Since $\mathfrak{F}(-K,-q)=\mathfrak{F}^{*}(K,q)$, the imaginary part of the right-hand side vanishes identically.

VI. THE FREDHOLM SOLUTION

Since the iterative series solutions to the momentum-space equations of Sec. IV do not always converge, we shall in this section develop another, always convergent, series solution which is based on the Fredholm theory for integral equations. This approach was proposed already in Ref. 1, but at that time it was far from evident that it could be given a convenient momentum-space formulation. However, this is the case, and from the point of view of actual evaluation, the series solution presented in this section is almost as convenient as the iterative series.

The Fredholm theory is not directly applicable to the equations of Sec. IV, and our starting point will again be Eq. (3.7). The reason for choosing this equation rather than Eq. (3.11) is that we have not found a convenient momentum-space representation of the recursion relation for the Fredholm denominator for this later equation. With the notation

$$K(k,k';r)=-\frac{1}{2\pi i}\,\frac{S(k')-S^{\rm BS}(k')}{k+k'+i\epsilon}\,e^{i(k+k')r}\ , \quad (6.1)$$

Eq. (3.7) can be written (with a parameter η added)

$$f_{+}(k,r)=e^{ikr}+\eta\int_{-\infty}^{\infty}dk'K(k,k';r)f_{+}(k',r)\ . \quad (6.2)$$

Its Fredholm solution is[7]

$$f_{+}(k,r)=e^{ikr}+\eta\frac{1}{\Delta(r)}\int_{-\infty}^{\infty}dk'Y(k,k';r)e^{ik'r}\ , \quad (6.3)$$

where $\Delta(r)$ is defined by

$$\Delta(r)=\exp[\mathrm{Tr}\ln(1-\eta K)]\ . \quad (6.4)$$

We note in passing that $\Delta(r)$ is real; this follows from the fact that $K(-k,-k';r)=K(k,k';r)$, so that all the traces of powers of K implicit in (6.4) are real.

Equation (6.2) can now be written as an equation for Y:

$$\int_{-\infty}^{\infty}dk'Y(k,k';r)e^{ik'r}=\Delta(r)\int_{-\infty}^{\infty}dk'K(k,k';r)e^{ik'r}+\eta\int_{-\infty}^{\infty}dk''K(k,k'';r)\times\int_{-\infty}^{\infty}dk'Y(k'',k';r)e^{ik'r}. \quad (6.5)$$

For a large class of phase shifts, $\Delta(r)$ and $Y(k,k';r)$ can be written as everywhere-convergent power series in η (see the Appendix):

$$\Delta(r)=\sum_{n}\eta^{n}\Delta_{n}(r),\qquad \Delta_0=1$$

$$Y(k,k';r)=\sum_{n}\eta^{n}Y_{n}(k,k';r)\ , \quad (6.6)$$

$$Y_0(k,k';r)=K(k,k';r)\ .$$

If these expansions are combined with Eq. (6.5) and a differentiated (with respect to η) form of Eq. (6.4), the coefficients for each power of η give the standard recursion relations for Y_n and Δ_n, $n=1,2,3,\ldots$. In particular,

$$\int_{-\infty}^{\infty}dk'Y_{n}(k,k';r)e^{ik'r}=\Delta_{n}(r)\int_{-\infty}^{\infty}dk'K(k,k';r)e^{ik'r}+\int_{-\infty}^{\infty}dk''K(k,k'';r)\int_{-\infty}^{\infty}dk'Y_{n-1}(k'',k';r)e^{ik'r}\ . \quad (6.7)$$

While this relation can easily be transformed into a momentum-space relation, as will be seen below, this is not the case for the corresponding relation for Δ_n. For this reason we consider

$$\frac{d}{dr}\Delta(r) = -\eta\Delta(r)\mathrm{Tr}\left((1-\eta K)^{-1}\frac{d}{dr}K\right) = \eta\Delta(r)\frac{1}{2\pi}\int_{-\infty}^{\infty}dk'[S(k')-S^{\mathrm{BS}}(k')]e^{ik'r}f_+(k',r)\,, \tag{6.8}$$

which follows from Eqs. (6.4) and (6.1). When this expression is combined with Eqs. (6.3) and (6.6), we obtain as coefficient for each power of η

$$\frac{d}{dr}\Delta_n(r) = \Delta_{n-1}(r)\frac{1}{2\pi}\int_{-\infty}^{\infty}dk'[S(k')-S^{\mathrm{BS}}(k')]e^{2ik'r} + \frac{1}{2\pi}\int_{-\infty}^{\infty}dk[S(k)-S^{\mathrm{BS}}(k)]e^{ikr} \times\int_{-\infty}^{\infty}dk'Y_{n-2}(k,k';r)e^{ik'r}\,. \tag{6.9}$$

This recursion relation for $\Delta_n(r)$ is of course equivalent to the standard relation, but more convenient for our purposes. It is valid for $n=2,3,4,\ldots$, while for $n=1$ the last term is absent.

Let us now make the transition to momentum space. For this purpose we define [in analogy with Eq. (4.1)]

$$\hat{Y}_n(k,q) = \frac{2}{\pi}\int_0^{\infty}dr\,e^{i(2q-k)r}\int_{-\infty}^{\infty}dk'Y_n(k,k';r)e^{ik'r} \tag{6.10}$$

and

$$\hat{\Delta}_n(q) = \frac{2}{\pi}\int_0^{\infty}dr\,e^{2iqr}\Delta_n(r) = -\frac{2}{\pi}\int_0^{\infty}dr\,\frac{e^{2iqr}-1}{2iq}\,\frac{d}{dr}\Delta_n(r)\,. \tag{6.11}$$

In terms of these entities, Eqs. (6.7) and (6.9) take the form

$$\hat{Y}_n(k,q) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk'\,\frac{S(k')-S^{\mathrm{BS}}(k')}{k+k'+i\epsilon}[\hat{Y}_{n-1}(k',k'+q)+\Delta_n(k'+q)]\,, \tag{6.12}$$

$$\hat{\Delta}_n(q) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk'[S(k')-S^{\mathrm{BS}}(k')]\frac{1}{2q}[\hat{Y}_{n-2}(k',k'+q)+\hat{\Delta}_{n-1}(k'+q)-\hat{Y}_{n-2}(k',k')-\hat{\Delta}_{n-1}(k')]\,. \tag{6.13}$$

These relations are obviously of the same complexity as the corresponding relations for the iterative solution to Eq. (4.3), but have the advantage of always producing convergent series. Similar relations, but with a nonsingular integrand and corresponding to the iterative solution of Eq. (4.4), can be obtained by replacing $S(k') - S^{\mathrm{BS}}(k')$ with $S(k')-S^{\mathrm{BS}}(k')-[S(k)-S^{\mathrm{BS}}(k)]^*$ in Eq. (6.12) and optionally also in Eq. (6.13). This replacement is justified, since it is justified in Eq. (6.5) and, after multiplication with the factor $e^{2iqr}-1$ of Eq. (6.11), also in Eq. (6.8). If the replacements are carried out in both equations, and there are no bound states ($S^{\mathrm{BS}}\equiv 1$), Eqs. (6.12) and (6.13) are equivalent to the two real, nonsingular relations

$$\mathrm{Im}\left(\frac{1}{2i}e^{i\delta(k)}\hat{Y}_n(k,q)\right) = -\frac{1}{\pi}\int_{-\infty}^{\infty}dk'\,\frac{\sin[\delta(k)+\delta(k')]}{k+k'}\,\mathrm{Im}\left(\frac{1}{2i}e^{i\delta(k')}[Y_{n-1}(k',k'+q)+\hat{\Delta}_n(k'+q)]\right)\,, \tag{6.14}$$

$$\mathrm{Im}\left(\frac{1}{2i}e^{i\delta(k)}\hat{\Delta}_n(q)\right) = -\frac{1}{\pi}\int_{-\infty}^{\infty}dk'\sin[\delta(k)+\delta(k')] \times\mathrm{Im}\left(\frac{1}{2i}e^{i\delta(k')}\frac{1}{2q}[\hat{Y}_{n-2}(k',k'+q)+\hat{\Delta}_{n-1}(k'+q)-\hat{Y}_{n-2}(k',k')-\hat{\Delta}_{n-1}(k')]\right)\,. \tag{6.15}$$

Finally, combining Eq. (6.3) with Eqs. (6.12) and (6.13) we get for $\mathfrak{F}(k,q)$

$$e^{i\delta(k)}\mathfrak{F}(k,q) = \frac{1}{\pi}\,\frac{i}{q+i\epsilon}\,e^{i\delta(k)} + \int_{-\infty}^{\infty}dk'\Delta^{-1}(k,k')e^{i\delta(k')}\hat{Y}(k',q)\,, \tag{6.16}$$

where $\Delta^{-1}(k,k')$ is the inverse of $\Delta(k,k')$, and [recall that $\Delta(r)$ is real]

$$\Delta(k,k') = \frac{2}{\pi}\int_0^{\infty}dr\sin kr\,\Delta(r)\,\sin k'r = \mathrm{Im}\left(\frac{1}{2i}[\hat{\Delta}(\tfrac{1}{2}(k+k'))-\hat{\Delta}(\tfrac{1}{2}(k-k'))]\right)\,. \tag{6.17}$$

In particular, in the absence of bound states,

$$\operatorname{Im}\left(\frac{1}{2i}e^{i\delta(k)}\mathfrak{F}(k,q)\right)=\frac{1}{2\pi}\operatorname{Im}\frac{e^{i\delta(k)}}{q+i\epsilon}+\int_{-\infty}^{\infty}dk'\Delta^{-1}(k,k')\times\operatorname{Im}\left(\frac{1}{2i}e^{i\delta(k')}\mathfrak{F}(k',q)\right). \tag{6.18}$$

The potential, the half-off-shell transition amplitude, etc. can be obtained from $\mathfrak{F}(k,q)$ as in Secs. IV and V.

Owing to the straightforward character of the relations (6.12) and (6.13) the need for the inversion of $\Delta(k,k')$ is the main (but inessential) computational complication of the Fredholm series solution as compared to the iterative series of Ref. 1.

The usefulness of the Fredholm series has been tested in two cases. It turns out that for the simple Bargmann S matrix considered in Sec. VII of Ref. 1, the Fredholm series terminates with the full solution after one term. In a numerical test case the procedure of the present section has been found to work well for a spherical well potential strong enough to support one bound state.

VII. CONCLUSIONS

The first thing we want to emphasize is the flexibility and usefulness of the dispersion-theory formulation of the inverse problem of scattering. In this paper it has only been exploited in quite a rudimentary fashion, and one immediate further application would be to consider weighted dispersion relations for solutions to the Schrödinger equation.

The second point we want to make is that the solution to the inverse problem should preferably be formulated in momentum space, since the input data are given in momentum space. From the present (and a previous[1]) work we conclude that (for S waves with or without bound states) this can be done in a both simple and practically useful manner.

We finally recall that the half-off-shell continuation of the on-shell, "physical" transition amplitude is a by-product of the solution to the inverse problem. Consequently, with the methods developed here, we have established a practically useful momentum-space procedure for constructing the half-off-shell amplitude from its on-shell values.[8]

APPENDIX

The kernel of Eq. (6.2) is not of the Hilbert-Schmidt type, but the equation has, nevertheless, a convergent Fredholm solution. To see this, we assume that the function $A(r,r')$ of (3.10) is well defined, and rewrite (6.2) as an equation for this function:

$$A(r,r')=\eta\,\theta(r'-r)F(r'+r)+\eta\int_{-\infty}^{\infty}dr''\,\theta(r'-r)F(r'+r'')A(r,r'')\,. \tag{A1}$$

Here, $\theta(x)=1$ for $x>0$, $\theta(x)=0$ for $x<0$, $\theta(r'-r)\times F(r'+r'')$ is the Fourier transform of the kernel $K(k,k';r)$ of Eq. (6.1), and

$$F(r)=(2\pi)^{-1}\int_{-\infty}^{\infty}dq[S(q)-S^{BS}(q)]e^{iqr}\,.$$

However, the Fredholm solution to (A1) [i.e., to Eq. (6.2)] is term by term identical with that of the Marchenko equation,

$$A(r,r')=\eta\,\theta(r'-r)F(r'+r)+\eta\int_{-\infty}^{\infty}dr''\,\theta(r'-r)F(r'+r'')\times\theta(r''-r)A(r,r'')\,. \tag{A2}$$

The convergence of the solution now follows from the fact that the kernel of (A2) *is* a Hilbert-Schmidt kernel.[5] More precisely, the Fredholm solution of Eq. (6.2) converges at least for the (quite large) class of potentials for which the kernel of the Marchenko equation (A2) is of the Hilbert-Schmidt type.

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[1]B. R. Karlsson, Phys. Rev. D 6, 1662 (1972).

[2]M. Verde, Nucl. Phys. 9, 255 (1958).

[3]R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966), Chap. 12.1.

[4]The corresponding slower decrease of the kernel for large k' can be avoided with more complicated modifications of Eq. (3.7).

[5]V. De Alfaro and T. Regge, *Potential Scattering* (North-Holland, Amsterdam, 1965), Chap. 12.

[6]$t(k,k;(2\mu)^{-1}k^2+i\epsilon)=-(\pi\mu k)^{-1}\sin\delta(k)e^{i\delta(k)}$. μ is the reduced mass implicit in $\mathcal{V}(r)=2\mu V(r)$.

[7]R. G. Newton, Ref. 3, Chap. 9.

[8]I.e., that off-shell extension that corresponds to a local, energy-independent interaction.