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Meson couplings from conserved vector current and partially conserved axial-vector current

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By requiring only that conserved vector current (CVC) and partially conserved axial-vector current (PCAC) be expressed by the field equations, a matrix formalism is developed for the nonlinear meson Lagrangian density, incorporating both conditions in any group representation. The pseudoscalar-meson "mass term" is given explicitly. The concept of the chiral covariant derivative is employed to treat a general system of vector, axial-vector, and pseudoscalar mesons in an elegant manner. In the context of invariant pseudoscalar-meson coupling constant f ($\approx 0.7 m_\pi^{-1}$?) and vector-meson coupling constant g (≈ 6) there follows immediately a relation between the (unrenormalized) axial-vector and vector-meson masses: $m_A{}^2 = m_V{}^2 + (g/2f)^2$.

I. INTRODUCTION

One of the major problems confronting attempts to describe the phenomena of elementary particles within a field-theoretical framework is the relationship of the internal quantum numbers to the dynamical properties of fields. The aspirations of mathematical esthetes to form a unified group structure providing "higher symmetries" appear to contain insufficient currency to conquer the towering difficulties involved. It would seem, therefore, that we should concentrate our efforts to understand the "internal" interactions of fields on the basis of their properties with respect to space-time with which we can deal effectively.

There are a number of beautiful and powerful theories employing dynamical subsidiary conditions or conservation laws in electrodynamics, gravitation,[1] and strong and weak interactions[2]; we may take as a relevant example the principle of conservation of vector current (CVC), leading to the so-called F-type coupling of the representative vector meson.[2] The purpose of the present paper is to point out that the power of at least one such dynamical condition, partial conservation of axial-vector current (PCAC),[3] has not heretofore been fully exploited. We show explicitly, in a nonlinear system of pseudoscalar mesons, with vector and axial-vector mesons, how PCAC in a form unified with CVC can completely determine the form of the couplings as a generalization of what is usually referred to as chiral dynamics. These results can be considered to be the (chiral) extension of the Yang-Mills theory[4]; they are model-independent, not only with respect to the form of the (unitary) pseudoscalar-meson functional, but also with respect to the representation of the "higher symmetry" (given that a second-rank tensorial, i.e., matrix, representation is valid). Further, a relation among the vector-meson mass, the axial-vector-meson mass, and the vector- and pseudoscalar-meson coupling constants follows directly.

In Sec. II we review the results of previous work[5-8] on nonlinear systems of pseudoscalar mesons, and give a general derivation of PCAC from the chiral dynamical form of the pseudoscalar meson Lagrangian in any representation [SU(2), SU(3), etc.]. Section III gives a brief resume of the formalism for vector mesons, with self-interactions, demonstrating how the supplementary condition follows from the field equations. The vector mesons are then added to the nonlinear pseudoscalar system with the development of the concept of covariant derivatives; it is shown how both CVC and (modified) PCAC are maintained. The axial-vector mesons are introduced into the combined system in Sec. IV; a broadening of the concept of the covariant derivative is evolved, along with a natural basis for the increase in the mass of the axial-vector meson over that of the vector meson. Section V discusses further extensions of the method, including the effects of "symmetry breaking" and the presence of baryons. An outline of the derivation of the nonlinear Lagrangian density from CVC and PCAC for a system of pions, ρ mesons, and A mesons is consigned to the Appendix.

II. PSEUDOSCALAR MESONS AND THE DIVERGENCE CONDITION

Let us consider a general system of pseudoscalar mesons P described by a local, Hermitian Lagrangian density. With an SU(2) representation, it has been shown explicitly in previous work[5,6,9] that in order for the divergence condition

$$-\mu^2 P = \partial_\mu J_{\mu 5} \tag{1}$$

($J_{\mu 5}$ is an initially unspecified axial-vector quantity) to follow from the field equations, the Lagrangian density *must* have the form

$$\mathcal{L}_P = -\frac{1}{8f^2}(\partial_\mu U)(\partial_\mu U)^\dagger - \frac{\mu^2}{2}A\,. \tag{2}$$

U is any unitary functional of the pseudoscalar meson field:

$$U(i\gamma_5 fP) = 1 + 2i\gamma_5 fP + \sum_{n=2}^{\infty} a_n (i\gamma_5 fP)^n\;, \tag{3}$$

with

$$U^\dagger U = UU^\dagger = 1\,, \tag{4}$$

and f is the expansion parameter, equal to the (inverse) pion decay constant, defined by the second term on the right-hand side of Eq. (3). μ is the pseudoscalar meson mass. If the P appearing in Eq. (1) is the same as in the argument of U in Eq. (3) (which is not necessary for the subsequent development), the "mass term" A can be expressed formally as a one-dimensional integral:

$$A = -\frac{1}{f^2}\int_0^{x=i\gamma_5 fP} x\,dx\, U^\dagger(x)\frac{dU(x)}{dx}\,. \tag{5}$$

The γ_5 appears in conjunction with P not only as a reminder that P is a pseudoscalar, but chiefly to facilitate the future addition of baryons to the system, which can be done in a straightforward manner without altering the basic results. Note that γ_5 will not appear explicitly in $\mathcal{L}_P$.

Since P (or, more generally, fP) is assumed to have a possible matrix representation

$$P = \sum_i \epsilon_i P_i\,, \tag{6}$$

where the ϵ_i are the (matrix) elements of the representation, satisfying

$$\mathrm{Tr}\,\epsilon_i\,\epsilon_j = \delta_{ij} \tag{7}$$

and the P_i are simply the component pseudoscalar meson fields, an over-all trace over the matrix expression in Eq. (2) is understood, so that the Lagrangian density is a scalar. Equation (7) implies that the ϵ_i used here correspond to $2^{-1/2}\tau_i$ for an isospin representation and correspond to $2^{-1/2}$ times the Gell-Mann λ matrices for an SU(3) representation. Equations (1) and (3) are matrix equations (no trace taken).

While the derivation of Eq. (2) from Eq. (1) was originally done in detail for the SU(2) representation, and extended to SU(3), with baryons included, we shall show in the remainder of this section how Eq. (1) will follow from the Lagrangian density (2) for *any* matrix representation.

The variation (sometimes called the Euler derivative) of a Lagrangian density $\mathcal{L}$ with respect to some parameter $\mathcal{P}$ (not necessarily the pseudoscalar meson field P), will be denoted by $\delta_{\mathcal{P}}\mathcal{L}$:

$$\delta_{\mathcal{P}}\mathcal{L} \equiv \frac{\partial\mathcal{L}}{\partial\mathcal{P}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathcal{P})}\right). \tag{8}$$

Now, if $\mathcal{P}$ is a pseudoscalar parameter which has been normalized to be equal to P in lowest order, it can be expressed in general as a series in (odd) powers of P:

$$\mathcal{P} = P\left(1 + \sum_{n=2}^{\infty} c_n (i\gamma_5 fP)^n\right), \tag{9}$$

and hence any algebraic function of $\mathcal{P}$ commutes with any algebraic function of P. Further, any functional $U(i\gamma_5 fP)$ can be written as a functional of $(i\gamma_5 f\mathcal{P})$:

$$U(i\gamma_5 f\mathcal{P}) = 1 + 2i\gamma_5 f\mathcal{P} + \sum_{n=2}^{\infty} b_n (i\gamma_5 f\mathcal{P})^n\;, \tag{10}$$

with the b_n in general different from the a_n in Eq. (3), except for $b_2 = a_2 = 2$ as required by Eq. (4). It is not difficult to see that for any component $\mathcal{P}_i$ in a possible matrix representation of $\mathcal{P}$

$$\frac{\partial(\partial_\mu U)}{\partial\mathcal{P}_i} = \partial_\mu\frac{\partial U}{\partial\mathcal{P}_i} \tag{11}$$

and

$$\frac{\partial(\partial_\mu U)}{\partial(\partial_\nu\mathcal{P}_i)} = \delta_{\mu\nu}\frac{\partial U}{\partial\mathcal{P}_i}\;; \tag{12}$$

therefore, taking the variation of $\mathcal{L}_P$, Eq. (2), with respect to $\mathcal{P}_i$ gives

$$\delta_{\mathcal{P}_i}\mathcal{L}_P = -\frac{1}{8f^2}\,\mathrm{Tr}\left(-\frac{\partial U}{\partial \mathcal{P}_i}(\partial_\mu\partial_\mu U)^\dagger - (\partial_\mu\partial_\mu U)\frac{\partial U^\dagger}{\partial \mathcal{P}_i}\right) - \frac{\mu^2}{2}\,\mathrm{Tr}\left(\frac{\partial U}{\partial \mathcal{P}_i}\frac{\partial A}{\partial U}\right). \tag{13}$$

This set of equations (for all i) can be put in matrix form by multiplying throughout by ϵ_i and summing over i:

$$\begin{aligned}\delta_{\mathcal{P}}\mathcal{L}_P &= \sum_i \epsilon_i\,\delta_{\mathcal{P}_i}\mathcal{L}_P \\ &= +\frac{1}{8f^2}\left((i\gamma_5 f)\sum_{n=1}^{\infty} b_n n \sum_{m=0}^{n-1}(i\gamma_5 f\mathcal{P})^m(\partial_\mu\partial_\mu U)^\dagger(i\gamma_5 f\mathcal{P})^{n-m-1}\right.\\ &\qquad\left. -(i\gamma_5 f)\sum_{n=1}^{\infty} b_n n\sum_{m=0}^{n-1}(-i\gamma_5 f\mathcal{P})^m(\partial_\mu\partial_\mu U)(-i\gamma_5 f\mathcal{P})^{n-m-1}\right)\\ &\quad -\frac{\mu^2}{2}(i\gamma_5 f)\sum_{n=1}^{\infty} b_n n\sum_{m=0}^{n-1}(i\gamma_5 f\mathcal{P})^m\frac{\partial A}{\partial U}(i\gamma_5 f\mathcal{P})^{n-m-1}.\end{aligned} \tag{14}$$

From the explicit form of $\delta_{\mathcal{P}}\mathcal{L}_P$ given in Eq. (14) it is clear, with Eq. (13), that for any algebraic function of the matrix $(i\gamma_5 f\mathcal{P})$, $\mathcal{R}(i\gamma_5 f\mathcal{P})=\mathcal{R}$, we have

$$\delta_{\mathcal{P}}\mathcal{L}_P\mathcal{R} = \sum_i \epsilon_i\left[\frac{1}{8f^2}\,\mathrm{Tr}\left(\mathcal{R}\frac{\partial U}{\partial\mathcal{P}_i}(\partial_\mu\partial_\mu U)^\dagger + (\partial_\mu\partial_\mu U)\mathcal{R}\frac{\partial U^\dagger}{\partial\mathcal{P}_i}\right) - \frac{\mu^2}{2}\,\mathrm{Tr}\left(\mathcal{R}\frac{\partial U}{\partial\mathcal{P}_i}\frac{\partial A}{\partial U}\right)\right] \tag{15}$$

and

$$\mathcal{R}\delta_{\mathcal{P}}\mathcal{L}_P = \sum_i \epsilon_i\left[\frac{1}{8f^2}\,\mathrm{Tr}\left(\frac{\partial U}{\partial\mathcal{P}_i}\mathcal{R}(\partial_\mu\partial_\mu U)^\dagger + (\partial_\mu\partial_\mu U)\frac{\partial U^\dagger}{\partial\mathcal{P}_i}\mathcal{R}\right) - \frac{\mu^2}{2}\,\mathrm{Tr}\left(\frac{\partial U}{\partial\mathcal{P}_i}\mathcal{R}\frac{\partial A}{\partial U}\right)\right], \tag{16}$$

or, as anticommutators and commutators:

$$\{\mathcal{R},\delta_{\mathcal{P}}\mathcal{L}_P\} = \sum_i \epsilon_i\left(\frac{1}{8f^2}\,\mathrm{Tr}\left(\left\{\mathcal{R},\frac{\partial U}{\partial\mathcal{P}_i}\right\}(\partial_\mu\partial_\mu U)^\dagger + (\partial_\mu\partial_\mu U)\left\{\mathcal{R},\frac{\partial U^\dagger}{\partial\mathcal{P}_i}\right\}\right) - \frac{\mu^2}{2}\,\mathrm{Tr}\left(\left\{\mathcal{R},\frac{\partial U}{\partial\mathcal{P}_i}\right\}\frac{\partial A}{\partial U}\right)\right) \tag{17}$$

and

$$[\mathcal{R},\delta_{\mathcal{P}}\mathcal{L}_P] = -\sum_i \epsilon_i\left(\frac{1}{8f^2}\,\mathrm{Tr}\left(\left[\mathcal{R},\frac{\partial U}{\partial\mathcal{P}_i}\right](\partial_\mu\partial_\mu U)^\dagger + (\partial_\mu\partial_\mu U)\left[\mathcal{R},\frac{\partial U^\dagger}{\partial\mathcal{P}_i}\right]\right) - \frac{\mu^2}{2}\,\mathrm{Tr}\left(\left[\mathcal{R},\frac{\partial U}{\partial\mathcal{P}_i}\right]\frac{\partial A}{\partial U}\right)\right). \tag{18}$$

We shall now show that the pseudoscalar meson field equation which yields the divergence condition, Eq. (1), is

$$\tfrac{1}{4}\{(U+U^\dagger),\delta_{\mathcal{P}c}\mathcal{L}_P\}=0, \tag{19}$$

with the variational parameter $\mathcal{P}c$ chosen as the imaginary (pseudoscalar) part of U, normalized to the pseudoscalar meson field in lowest order:

$$\mathcal{P}c = -i\gamma_5\frac{1}{4f}(U-U^\dagger). \tag{20}$$

The proof follows simply by using, for the quantity $\mathcal{R}$ in Eq. (17), half the real part of U, $\frac{1}{4}(U+U^\dagger)$ and noting that

$$\left\{(U+U^\dagger),\frac{\partial U}{\partial\mathcal{P}_i}\right\} = \left\{\frac{\partial(U-U^\dagger)}{\partial\mathcal{P}_i},U\right\}, \tag{21}$$

$$\left\{(U+U^\dagger),\frac{\partial U^\dagger}{\partial\mathcal{P}_i}\right\} = -\left\{\frac{\partial(U-U^\dagger)}{\partial\mathcal{P}_i},U^\dagger\right\}, \tag{22}$$

which is a consequence of the unitarity of U:

$$\frac{\partial}{\partial\mathcal{P}_i}(UU^\dagger)=\frac{\partial}{\partial\mathcal{P}_i}(U^\dagger U)=0. \tag{23}$$

Now, for the choice $\mathcal{P}_i=\mathcal{P}c_i$ as given in Eq. (20),

$$\frac{\partial(U-U^\dagger)}{\partial\mathcal{P}c_i}=4i\gamma_5 f\epsilon_i. \tag{24}$$

We can therefore obtain from Eqs. (21) and (22)

$$\tfrac{1}{4}\left\{(U+U^\dagger),\frac{\partial U}{\partial\mathcal{P}c_i}\right\}=i\gamma_5 f\{\epsilon_i,U\}, \tag{25}$$

$$\tfrac{1}{4}\left\{(U+U^\dagger),\frac{\partial U^\dagger}{\partial\mathcal{P}c_i}\right\}=-i\gamma_5 f\{\epsilon_i,U^\dagger\}. \tag{26}$$

The relation (25) and its conjugate (26) are the key to the derivation of the divergence condition; using them in Eq. (17) with $\mathcal{R}=\frac{1}{4}(U+U^\dagger)$, and putting the result in matrix form by noting that, for any matrix M we have

$$\sum_i \epsilon_i \,\mathrm{Tr}(\epsilon_i M)=M\,, \tag{27}$$

we get

$$\tfrac{1}{4}\{(U+U^\dagger),\,\delta_{\mathcal{P}c}\mathcal{L}_P\}=i\gamma_5\frac{1}{8f}(\{U,(\partial_\mu\partial_\mu U)^\dagger\}-\{U^\dagger,(\partial_\mu\partial_\mu U)\})-i\gamma_5\frac{\mu^2}{2}f\left\{U,\frac{\partial A}{\partial U}\right\}. \tag{28}$$

It is evident that, as the "mass term" A, Eq. (5), can be written as

$$A=-\frac{1}{f^2}\int_0^{x=i\gamma_5 fP} x(U)U^\dagger\,dU\,, \tag{29}$$

and hence

$$\left\{U,\frac{\partial A}{\partial U}\right\}=-2i\gamma_5\frac{1}{f}P \tag{30}$$

[x commutes with $U(x)$, and $UU^\dagger=1$]. We have achieved the desired result by proving the identity

$$\tfrac{1}{4}\{(U+U^\dagger),\,\delta_{\mathcal{P}c}\mathcal{L}_P\}=-\partial_\mu J_{\mu 5}-\mu^2 P\,, \tag{31}$$

with

$$J_{\mu 5}=+i\gamma_5\frac{1}{8f}(\{U^\dagger,\partial_\mu U\}-\{U,\partial_\mu U^\dagger\})\,, \tag{32}$$

and $\mathcal{P}c$ given explicitly by Eq. (20). The divergence condition (1) holds as a consequence of the field equation

$$\delta_{\mathcal{P}c}\mathcal{L}_P=0\,. \tag{33}$$

It is hoped that the straightforward mathematical development of this section dispels any mystery about the basis of chiral dynamics[10-14] and its relationship to PCAC.

III. VECTOR MESONS, CVC, AND COVARIANT DERIVATIVES

The novel aspects of the material in the previous section dictated a rather detailed development; the vector-meson formalism, on the other hand, is well known, and we shall in this section content ourselves with a resume to introduce the reader to the background and notation used for the introduction of axial-vector mesons in Sec. IV.

We use the symbol $\mathfrak{M}_\mu$ to denote a matrix representation of vector mesons with nonzero mass m. It can then be shown[7] that, in order for the subsidiary condition

$$\partial_\mu\mathfrak{M}_\mu=0 \tag{34}$$

to follow from the field equation

$$\delta_{\mathfrak{M}_\mu}\mathcal{L}_{\mathfrak{M}}=0\,, \tag{35}$$

the vector-meson Lagrangian density $\mathcal{L}_{\mathfrak{M}}$ must be given by

$$\mathcal{L}_{\mathfrak{M}}=-\tfrac{1}{4}\mathfrak{M}_{\mu\nu}\mathfrak{M}_{\mu\nu}-\tfrac{1}{2}m^2\mathfrak{M}_\mu\mathfrak{M}_\mu\,, \tag{36}$$

with

$$\mathfrak{M}_{\mu\nu}=\partial_\mu\mathfrak{M}_\nu-\partial_\nu\mathfrak{M}_\mu-i\tfrac{1}{2}g[\mathfrak{M}_\mu,\mathfrak{M}_\nu]\,, \tag{37}$$

g being the vector-meson coupling constant. Again, a trace is understood to be taken on the right-hand side of Eq. (36). Starting with the Lagrangian density (36), we can obtain the subsidiary condition (34) from the field equation (35) easily. The variation of Eq. (36) with respect to $\mathfrak{M}_\mu$ is

$$\delta_{\mathfrak{M}_\mu}\mathcal{L}_{\mathfrak{M}}=\partial_\nu\mathfrak{M}_{\nu\mu}-i\tfrac{1}{2}g[\mathfrak{M}_\nu,\mathfrak{M}_{\nu\mu}]-m^2\mathfrak{M}_\mu\,. \tag{38}$$

Let us introduce the "covariant derivative," D_μ, the properties of which will be discussed further, acting on $\mathfrak{M}_\nu$ or $\mathfrak{M}_{\nu\lambda}$:

$$D_\mu\mathfrak{M}=\partial_\mu\mathfrak{M}-i\tfrac{1}{2}g[\mathfrak{M}_\mu,\mathfrak{M}]\,, \tag{39}$$

where $\mathfrak{M}$ stands for either $\mathfrak{M}_\nu$ or $\mathfrak{M}_{\nu\lambda}$. Then the variation (38) can be written as

$$\delta_{\mathfrak{M}_\mu}\mathcal{L}_{\mathfrak{M}}=D_\nu\mathfrak{M}_{\nu\mu}-m^2\mathfrak{M}_\mu\,. \tag{40}$$

We can now write the covariant derivative of the variation (40) as

$$D_\mu\delta_{\mathfrak{M}_\mu}\mathcal{L}_{\mathfrak{M}}=D_\mu D_\nu\mathfrak{M}_{\nu\mu}-m^2D_\mu\mathfrak{M}_\mu=-m^2\partial_\mu\mathfrak{M}_\mu\,. \tag{41}$$

This last step follows from the fact that

$$D_\mu D_\nu\mathfrak{M}_{\nu\mu}=\tfrac{1}{2}[D_\mu,D_\nu]\mathfrak{M}_{\nu\mu}=-i\tfrac{1}{4}g[\mathfrak{M}_{\mu\nu},\mathfrak{M}_{\nu\mu}]=0\,. \tag{42}$$

From Eq. (41) it is clear that the subsidiary condition (34) follows from the field equation (35). More generally, in the context of field-current equivalence, the subsidiary condition represents a manifestation of conservation of the corresponding vector current.

Enlarging our considerations to a system of interacting fields described by a total Lagrangian density $\mathcal{L}$, we note that the subsidiary condition (34) will follow from the field equations in general only if

$$D_\mu S_\mu=0\,, \tag{43}$$

where the "source current" S_μ is defined by

$$\delta_{\mathfrak{M}_\mu}\mathcal{L} = D_\nu \mathfrak{M}_{\nu\mu} - m^2\mathfrak{M}_\mu + S_\mu \qquad (44)$$

or

$$S_\mu = \delta_{\mathfrak{M}_\mu}(\mathcal{L} - \mathcal{L}_{\mathfrak{M}}). \qquad (45)$$

If we define a related vector current J_μ by

$$J_\mu = S_\mu - i\tfrac{1}{2}g[\mathfrak{M}_\nu, \mathfrak{M}_{\nu\mu}] \qquad (46)$$

then Eq. (44) becomes

$$\delta_{\mathfrak{M}_\mu}\mathcal{L} = \partial_\nu \mathfrak{M}_{\nu\mu} - m^2\mathfrak{M}_\mu + J_\mu\ , \qquad (47)$$

and taking the ordinary divergence of Eq. (47) gives

$$\partial_\mu\delta_{\mathfrak{M}_\mu}\mathcal{L} = -m^2\partial_\mu\mathfrak{M}_\mu + \partial_\mu J_\mu \qquad (48)$$

so that the field equation $\delta_{\mathfrak{M}_\mu}\mathcal{L} = 0$ yields

$$m^2\partial_\mu\mathfrak{M}_\mu = \partial_\mu J_\mu \qquad (49)$$

and the subsidiary condition (34) follows from conservation of the vector current J_μ, or vice versa. As has been shown,[7,15] in order for the total Lagrangian density $\mathcal{L}$ to lead to a conserved current J_μ, the interactions with the vector-meson fields must be introduced via a covariant derivative D_μ, which replaces the ordinary derivative in the Lagrangian density without the vector-meson interactions. This covariant derivative is defined in general[4] as

$$D_\mu = \partial_\mu - igT\cdot\mathfrak{M}_\mu\ , \qquad (50)$$

where the components T_i of T are Hermitian matrices (of the same order as the vector-meson representation) representing the generators of the corresponding components of the conserved current, acting upon the field to which D_μ is applied. For our present purposes, it will suffice to assume that the fields of interest (the representations of the pseudoscalar and axial-vector mesons as well as the vector mesons) have the same transformation properties under T, so that, in particular,

$$D_\mu\mathcal{P} = \partial_\mu\mathcal{P} - i\tfrac{1}{2}g[\mathfrak{M}_\mu, \mathcal{P}]\ , \qquad (51)$$

corresponding to Eq. (39), with $\mathcal{P}$ any pseudoscalar parameter defined in Eq. (9). It will then follow that

$$D_\mu\delta_{\mathfrak{M}_\mu}\mathcal{L} - i\tfrac{1}{2}g[\mathcal{P}, \delta_{\mathcal{P}}\mathcal{L}] = -m^2\partial_\mu\mathfrak{M}_\mu\ . \qquad (52)$$

There are three salient features of the covariant derivative:

1. It establishes a relationship between the vector-meson interactions and the generators of the conserved current such that the subsidiary condition (34) and CVC are interdependent, as explained above.

2. It is distributive, i.e., for a product of field representations $P_AP_B\cdots P_Z$ we have

$$D_\mu(P_AP_B\cdots P_Z) = (D_\mu P_A)P_B\cdots P_Z + P_A(D_\mu P_B)\cdots P_Z + \cdots + P_AP_B\cdots(D_\mu P_Z)\ . \qquad (53)$$

Note in particular that the commutator in, say, Eq. (51) is distributive.

3. It leaves invariant the form of the field equations for the various representations upon which it acts. For example, the replacement of the ordinary derivative of $\mathcal{P}$ by the covariant derivative given by Eq. (51) in a Lagrangian density $\mathcal{L}(\partial)$ to give $\mathcal{L}(D)$ results in a variation with respect to $\mathcal{P}$ of

$$\begin{aligned}\delta_{\mathcal{P}}\mathcal{L}(D) &= \frac{\partial\mathcal{L}(D)}{\partial\mathcal{P}} - \partial_\mu\frac{\partial\mathcal{L}(D)}{\partial(\partial_\mu\mathcal{P})}\\ &= \frac{\partial\mathcal{L}(\partial)}{\partial\mathcal{P}} + i\tfrac{1}{2}g\left[\mathfrak{M}_\mu, \frac{\partial\mathcal{L}(\partial)}{\partial(\partial_\mu\mathcal{P})}\right] - \partial_\mu\frac{\partial\mathcal{L}(\partial)}{\partial(\partial_\mu\mathcal{P})}\\ &= \frac{\partial\mathcal{L}(\partial)}{\partial\mathcal{P}} - D_\mu\frac{\partial\mathcal{L}(\partial)}{\partial(\partial_\mu\mathcal{P})}\\ &= \frac{\partial\mathcal{L}(\partial)}{\partial\mathcal{P}} - D_\mu\frac{\partial\mathcal{L}(D)}{\partial(D_\mu\mathcal{P})}\ . \end{aligned} \qquad (54)$$

Thus the covariant derivative neatly replaces the ordinary derivative not only in the Lagrangian densities, but in the consequent field equations.

The addition of vector mesons into the nonlinear system of pseudoscalar mesons treated in the previous section is now simple. Owing to the three properties discussed above, the replacement of ordinary derivatives of P, $\mathcal{P}$, or U by covariant derivatives throughout as demanded by CVC allows the derivation in SU(2) (see Ref. 7) of the Lagrangian density

$$\mathcal{L}_P = -\frac{1}{8f^2}(D_\mu U)(D_\mu U)^\dagger - \tfrac{1}{2}\mu^2 A \qquad (55)$$

from the "covariant divergence condition"

$$-\mu^2 P = D_\mu J_{\mu 5}\ , \qquad (56)$$

which now replaces Eq. (1), with

$$J_{\mu 5} = -i\gamma_5\frac{1}{8f}(\{U, D_\mu U^\dagger\} - \{U^\dagger, D_\mu U\})\ . \qquad (57)$$

The proof of Eq. (56) starting with the field equation from the Lagrangian density (55) likewise proceeds in a straightforward manner.

IV. AXIAL-VECTOR MESONS, PCAC, AND CHIRAL COVARIANT DERIVATIVES

Using the ideas established in the previous two sections, we shall now add axial-vector mesons to the system, and the full power of PCAC will be revealed.

First, in the absence of pseudoscalar mesons both vector and axial-vector current are assumed to be conserved (nonconservation due to "mass symmetry breaking" is discussed in Ref. 8). The requirement that the subsidiary condition (34) be obtainable from the field equations for both the vector meson $\mathcal{V}_\mu$ and the axial-vector meson $\mathcal{A}_\mu$ then yields the result[16-18] that the combined Lagrangian density must be given by Eqs. (36) and (37), with the replacement

$$\mathfrak{M}_\mu = \mathcal{V}_\mu + \gamma_5 \mathcal{A}_\mu \,. \tag{58}$$

Again γ_5 is used not only to separate the axial-vector from the vector parts of $\mathfrak{M}_\mu$, but also with a view toward the inclusion of baryons in the formalism. A trace over Dirac space as well as over the "internal" representation space is then assumed in Lagrangian density (36). The formalism in Sec. III remains unchanged, although one should keep in mind that the (matrix) equations have both Dirac scalar and γ_5 parts, and there are therefore really two equations in each. In particular, notice that $\mathfrak{M}_{\mu\nu}$ defined by Eq. (37) can be separated into "1" and γ_5 parts as

$$\mathfrak{M}_{\mu\nu} = \mathcal{V}_{\mu\nu} + \gamma_5 \mathcal{A}_{\mu\nu} \,, \tag{59}$$

and that now the variation with respect to $\mathfrak{M}_\mu$ is

$$\delta_{\mathfrak{M}_\mu} = \delta_{\mathcal{V}_\mu} + \gamma_5 \delta_{\mathcal{A}_\mu} \,. \tag{60}$$

Similarly, the covariant derivative in Eq. (39) becomes

$$D_\mu \mathfrak{M} = \partial_\mu \mathfrak{M} - i\tfrac{1}{2} g[\mathcal{V}_\mu, \mathfrak{M}] - \gamma_5 i \tfrac{1}{2} g[\mathcal{A}_\mu, \mathfrak{M}] \,. \tag{61}$$

Finally we are ready to consider the full system of vector, axial-vector, and pseudoscalar mesons. The fundamental problem to be addressed is that of finding the most general form of the total Lagrangian yielding field equations which reduce to the subsidiary conditions

$$m^2 \partial_\mu \mathcal{V}_\mu = 0 \tag{62}$$

and

$$m^2 \partial_\mu \mathcal{A}_\mu = -\mathcal{F}\mu^2 P \,, \tag{63}$$

with a constant $\mathcal{F}$, and μ^2 being the pseudoscalar meson mass squared as in Sec. II. Equation (63) is the PCAC condition in our field-equation form, and is as powerful a statement as the corresponding CVC condition, Eq. (62). It can be shown that the required form of the Lagrangian density, assumed to be nonlinear in the pseudoscalar meson field and local (restricted to terms containing not more than two vector indices in the pseudoscalar-meson part and not more than four in the vector-meson part) is

$$\mathcal{L} = \mathcal{L}_{\mathfrak{M}} + \mathcal{L}_U - \tfrac{1}{2}\mu^2 A \,, \tag{64}$$

with the pseudoscalar mass term A the same as in Sec. II, and

$$\mathcal{L}_U = -\frac{1}{8f^2}\left(\partial_\mu U - i\tfrac{1}{2} g[\mathcal{V}_\mu, U] + i\gamma_5 \tfrac{1}{2} g\{\mathcal{A}_\mu, U\}\right) \times \left(\partial_\mu U^\dagger - i\tfrac{1}{2} g[\mathcal{V}_\mu, U^\dagger] - i\gamma_5 \tfrac{1}{2} g\{\mathcal{A}_\mu, U^\dagger\}\right). \tag{65}$$

Again U is the general nonlinear unitary pseudoscalar-meson functional as in Eq. (3). The lengthy derivation of the above form of $\mathcal{L}_U$ for an SU(2) representation of the mesons is relegated to the Appendix to avoid interrupting the flow of thought at this point. We can demonstrate readily, however, that for any matrix representation the Lagrangian density (64) with (65) yields field equations giving the CVC and PCAC conditions (62) and (63). For this purpose, we introduce the "chiral covariant derivative," $\mathfrak{D}_\mu$, defined acting on a pseudoscalar functional U as

$$\begin{aligned} \mathfrak{D}_\mu U &= \partial_\mu U - i\tfrac{1}{2} g[\mathcal{V}_\mu, U] + i\gamma_5 \tfrac{1}{2} g\{\mathcal{A}_\mu, U\} \\ &= D_\mu U + i\gamma_5 g \mathcal{A}_\mu U \,, \end{aligned} \tag{66}$$

with the covariant derivative D_μ given by Eq. (61); the adjoint operation $\mathfrak{D}_\mu^\dagger$ is then expressed by

$$\begin{aligned} (\mathfrak{D}_\mu U)^\dagger &= \mathfrak{D}_\mu^\dagger U^\dagger \\ &= \partial_\mu U^\dagger - i\tfrac{1}{2} g[\mathcal{V}_\mu, U^\dagger] - i\gamma_5 \tfrac{1}{2} g\{\mathcal{A}_\mu, U^\dagger\} \\ &= D_\mu U^\dagger - i\gamma_5 g U^\dagger \mathcal{A}_\mu \,. \end{aligned} \tag{67}$$

The kinematic (derivative) part of the pseudoscalar meson Lagrangian, with interactions, can now be written as

$$\mathcal{L}_U = -\frac{1}{8f^2}(\mathfrak{D}_\mu U)(\mathfrak{D}_\mu U)^\dagger \,. \tag{68}$$

(A trace is assumed to be taken on the right so that $\mathcal{L}_U$ is a scalar.) Let us examine the properties of the chiral covariant derivative $\mathfrak{D}_\mu$, in parallel with the properties of the covariant derivative D_μ discussed in the previous section:

1. $\mathfrak{D}_\mu$ has been defined in such a way that the couplings of the vector and axial-vector mesons to the pseudoscalar meson functional required by CVC and PCAC have been "built in."

2. $\mathfrak{D}_\mu$ is neither a Hermitian nor a distributive operation like D_μ; indeed, mathematical manipulations involving $\mathfrak{D}_\mu$ are such as to make very questionable its association with the usual concept of a derivative. However, there is one relation it has with respect to the covariant derivative that makes the formalism of Sec. II extendible to the full meson system: For any two quantities X and Y having the same order representation as $\mathfrak{M}_\mu$, it is easy to see, using the definitions (66) and (67), that

$$(\mathfrak{D}_\mu X)^\dagger Y + X^\dagger \mathfrak{D}_\mu Y = D_\mu (X^\dagger Y) \,. \tag{69}$$

3. Just as with the covariant derivative D_μ, the form of the pseudoscalar-meson field equations as

given in, say, Eq. (13) or (28) will remain the same, except for the replacement of the ordinary derivative ∂_μ by the chiral covariant derivative $\mathfrak{D}_\mu$ throughout. Thus, starting with the Lagrangian density (64) with (68), which we can write as

$$\mathfrak{L} = \mathfrak{L}_M + \mathfrak{L}(\mathfrak{D}), \tag{70}$$

where $\mathfrak{L}(\mathfrak{D})$ indicates that $\mathfrak{D}_\mu$ has replaced ∂_μ everywhere in the pseudoscalar part of $\mathfrak{L}$, we can proceed as in Eq. (54), to arrive at the conclusion that

$$\delta_{\mathcal{P}} \mathfrak{L}(\mathfrak{D}) = \frac{\partial \mathfrak{L}(\partial)}{\partial \mathcal{P}} - \mathfrak{D}_\mu^\dagger \frac{\partial \mathfrak{L}(\mathfrak{D})}{\partial(\mathfrak{D}_\mu \mathcal{P})} - \mathfrak{D}_\mu \frac{\partial \mathfrak{L}(\mathfrak{D})}{\partial(\mathfrak{D}_\mu \mathcal{P})^\dagger}. \tag{71}$$

In particular, with $\mathfrak{L}$ given in terms of the pseudoscalar functional U, we now have, for a component $\mathcal{P}_i$ of $\mathcal{P}$,

$$\delta_{\mathcal{P}_i} \mathfrak{L} = -\frac{1}{8f^2}\left[-\frac{\partial U}{\partial \mathcal{P}_i}(\mathfrak{D}_\mu \mathfrak{D}_\mu U)^\dagger - (\mathfrak{D}_\mu \mathfrak{D}_\mu U)\frac{\partial U^\dagger}{\partial \mathcal{P}_i}\right] - \frac{\mu^2}{2}\frac{\partial U}{\partial \mathcal{P}_i}\frac{\partial A}{\partial U} \tag{72}$$

rather than Eq. (13) [traces are assumed on the right-hand side of Eq. (72)].

The above features of the chiral covariant derivative make possible an elegant formulation of the field equations for the total meson system which yields a combined vector and axial-vector subsidiary condition expressing both CVC and PCAC. First, notice that the results of Sec. II follow directly with the replacement of the double chiral covariant derivative, $\mathfrak{D}_\mu \mathfrak{D}_\mu U$, for the ordinary derivative $\partial_\mu \partial_\mu U$; starting with Eq. (72) replacing Eq. (13), we obtain the pseudoscalar meson equation as

$$\tfrac{1}{4}\{(U+U^\dagger), \delta_{\mathcal{P}c}\mathfrak{L}\} = i\gamma_5 \frac{1}{8f}(\{U, (\mathfrak{D}_\mu \mathfrak{D}_\mu U)^\dagger\} - \{U^\dagger, \mathfrak{D}_\mu \mathfrak{D}_\mu U\}) - i\gamma_5 \tfrac{1}{2}\mu^2 f\left\{U, \frac{\partial A}{\partial U}\right\}. \tag{73}$$

In addition to the anticommutator (73), we need to compute the commutator of $\delta_{\mathcal{P}c}\mathfrak{L}$ with $\mathcal{P}c$ for use in the CVC subsidiary condition according to Eq. (52). Starting with Eq. (18) with $\partial_\mu \partial_\mu U$ replaced by $\mathfrak{D}_\mu \mathfrak{D}_\mu U$ and $\mathfrak{R} = \frac{1}{4}(U - U^\dagger)$, we use

$$\left[(U-U^\dagger), \frac{\partial U}{\partial \mathcal{P}_i}\right] = \left[U, \frac{\partial(U-U^\dagger)}{\partial \mathcal{P}_i}\right] \tag{74}$$

and

$$\left[(U-U^\dagger), \frac{\partial U^\dagger}{\partial \mathcal{P}_i}\right] = \left[U^\dagger, \frac{\partial(U-U^\dagger)}{\partial \mathcal{P}_i}\right] \tag{75}$$

obtained from the unitarity of U, Eq. (23). Thence, with the variational parameter $\mathcal{P} = \mathcal{P}c$ to get Eq. (24), we put the result in matrix form via Eq. (27) *et seq.*:

$$\tfrac{1}{4}[(U-U^\dagger), \delta_{\mathcal{P}c}\mathfrak{L}] = i\gamma_5 \frac{1}{8f}([U, (\mathfrak{D}_\mu \mathfrak{D}_\mu U)^\dagger] + [U^\dagger, \mathfrak{D}_\mu \mathfrak{D}_\mu U]) \tag{76}$$

(U commutes with $\partial A/\partial U$). To get the combined pseudoscalar-meson field equation contributions to the subsidiary conditions, we take

$$\tfrac{1}{4}(\{(U+U^\dagger), \delta_{\mathcal{P}c}\mathfrak{L}\} - [(U-U^\dagger), \delta_{\mathcal{P}c}\mathfrak{L}]) = \tfrac{1}{2}(U^\dagger \delta_{\mathcal{P}c}\mathfrak{L} + \delta_{\mathcal{P}c}\mathfrak{L} U)$$
$$= -i\gamma_5 \frac{1}{4f}[U^\dagger \mathfrak{D}_\mu \mathfrak{D}_\mu U - (\mathfrak{D}_\mu \mathfrak{D}_\mu U)^\dagger U] - i\gamma_5 \tfrac{1}{2}\mu^2 f\left\{U, \frac{\partial A}{\partial U}\right\}. \tag{77}$$

The contributions to the vector and axial-vector meson field equations from the interactions with the pseudoscalar field given in $\mathfrak{L}_U$, Eq. (65), are

$$\delta_{\mathcal{V}_\mu}\mathfrak{L}_U = i\frac{g}{16f^2}([U^\dagger, \mathfrak{D}_\mu U] + [U, (\mathfrak{D}_\mu U)^\dagger]) \tag{78}$$

and

$$\delta_{\mathcal{A}_\mu}\mathfrak{L}_U = i\gamma_5 \frac{g}{16f^2}(\{U^\dagger, \mathfrak{D}_\mu U\} - \{U, (\mathfrak{D}_\mu U)^\dagger\}). \tag{79}$$

The quantity $(2f/g)\delta_{\mathcal{A}_\mu}\mathfrak{L}_U$ is the generalization of the axial-vector current defined by Eq. (32), although it cannot now be regarded as the source current for the pseudoscalar meson field. Instead, we construct a "mixed parity" source current S_μ:

$$S_\mu = \delta_{\mathcal{V}_\mu}\mathfrak{L}_U + \gamma_5 \delta_{\mathcal{A}_\mu}\mathfrak{L}_U = i\frac{g}{8f^2}[U^\dagger \mathfrak{D}_\mu U - (\mathfrak{D}_\mu U)^\dagger U]. \tag{80}$$

Using the relation (69), we have for the covariant divergence of this current

$$D_\mu S_\mu = i\frac{g}{8f^2}[U^\dagger \mathfrak{D}_\mu \mathfrak{D}_\mu U - (\mathfrak{D}_\mu \mathfrak{D}_\mu U)^\dagger U]. \tag{81}$$

Now comparing with Eq. (77), we see that the altered pseudoscalar-meson variation of the Lagrangian density is

$$\tfrac{1}{2}(U^\dagger \delta_{\mathcal{P}c}\mathfrak{L} + \delta_{\mathcal{P}c}\mathfrak{L} U) = -\gamma_5 \frac{2f}{g} D_\mu S_\mu - i\gamma_5 \tfrac{1}{2}\mu^2 f\left\{U, \frac{\partial A}{\partial U}\right\}. \tag{82}$$

Further, the combined vector and axial-vector field equations can now be written as in Sec. III in the form

$$\delta_{\mathcal{V}_\mu}\mathcal{L}+\gamma_5\,\delta_{\mathcal{a}_\mu}\mathcal{L}=D_\nu\mathfrak{M}_{\nu\mu}-m^2\mathfrak{M}_\mu+S_\mu\ , \qquad (83)$$

with S_μ given by Eq. (80). The combined subsidiary conditions, referring to Eq. (41), are then derived from

$$D_\mu(\delta_{\mathcal{V}_\mu}\mathcal{L}+\gamma_5\delta_{\mathcal{a}_\mu}\mathcal{L})=-m^2\partial_\mu\mathfrak{M}_\mu+D_\mu S_\mu\ . \qquad (84)$$

But the covariant divergence of the source current, $D_\mu S_\mu$, is related to the pseudoscalar-meson field equations through Eq. (82); thus

$$D_\mu(\delta_{\mathcal{V}_\mu}\mathcal{L}+\gamma_5\delta_{\mathcal{a}_\mu}\mathcal{L})+\gamma_5\frac{g}{4f}(U^\dagger\delta_{\mathcal{P}_c}\mathcal{L}+\delta_{\mathcal{P}_c}\mathcal{L}U)$$
$$=-m^2\partial_\mu(\mathcal{V}_\mu+\gamma_5\mathcal{a}_\mu)-i\tfrac{1}{4}g\mu^2\left\{U,\frac{\partial A}{\partial U}\right\}. \qquad (85)$$

If relation (30) is used for the last term in Eq. (85),

$$-i\tfrac{1}{4}g\mu^2\left\{U,\frac{\partial A}{\partial U}\right\}=-\gamma_5\frac{g}{2f}\mu^2P\ , \qquad (86)$$

then the field equations

$$\delta_{\mathcal{V}_\mu}\mathcal{L}=0\ , \qquad (87)$$

$$\delta_{\mathcal{a}_\mu}\mathcal{L}=0\ , \qquad (88)$$

$$\delta_{\mathcal{P}_c}\mathcal{L}=0 \qquad (89)$$

yield the combined subsidiary condition

$$-m^2\partial_\mu\mathfrak{M}_\mu-\gamma_5\frac{g}{2f}\mu^2P=0\ , \qquad (90)$$

whose "1" and γ_5 parts are

$$-m^2\partial_\mu\mathcal{V}_\mu=0\ , \qquad (91)$$

$$-m^2\partial_\mu\mathcal{a}_\mu-\frac{g}{2f}\mu^2P=0\ . \qquad (92)$$

Therefore Eq. (85), which follows from the meson Lagrangian (64) and (36) with (58) and (59) regardless of whether or not the field equations (87)–(89) are satisfied, embodies the essence of the relationship between the field equations and the subsidiary conditions (91) and (92) expressing CVC and PCAC. It is clear that PCAC fixes the axial-vector meson couplings in the same fashion that CVC fixes the vector-meson couplings.

There is an additional noteworthy feature of the Lagrangian density (64). The derivative part of the pseudoscalar meson Lagrangian density, $\mathcal{L}_\mu$, can be written as

$$\mathcal{L}_\mu=-\frac{1}{8f^2}\mathrm{Tr}(D_\mu U\cdot D_\mu U^\dagger$$
$$+2i\gamma_5 g\mathcal{a}_\mu U\cdot D_\mu U^\dagger+g^2\mathcal{a}_\mu\mathcal{a}_\mu)\ . \qquad (93)$$

If we make the *a priori* valid interpretation of the field quantities in $\mathcal{L}$ as corresponding to the physical particles, it is evident that the last term in Eq. (93) contributes to the effective mass of the axial-vector meson. This term cannot be eliminated by a counterterm in $\mathcal{L}_{\mathfrak{M}}$, or else the PCAC condition (92) will not follow; it must be regarded as an intrinsic mass increase due to the existence of the pseudoscalar meson. Indeed, if the PCAC condition is taken at face value, we must conclude that

$$m_{\mathcal{a}}^2=m_{\mathcal{V}}^2+\left(\frac{g}{2f}\right)^2\ , \qquad (94)$$

where $m_{\mathcal{a}}$ is the axial-vector meson mass, $m_{\mathcal{V}}$ is the vector-meson mass, $g\sim 6$, and $2f\sim 1.4\mu^{-1}$. Thus, another way of expressing the PCAC condition, independent of the coupling constants, would be

$$\partial_\mu\mathcal{a}_\mu=-(m_{\mathcal{a}}^2-m_{\mathcal{V}}^2)^{1/2}\frac{\mu^2}{m_{\mathcal{V}}^2}P\ . \qquad (95)$$

Another possible interpretation of Eq. (93) is that the field $\mathcal{a}_\mu$ contains a mixture of the "physical" axial-vector and derivative pseudoscalar fields,[16–18] with open parameters governing the amount of mixing. The latter interpretation necessitates pseudoscalar meson couplings with more than two derivatives, and seems to lead to more disorder than we presently wish to consider.

V. DISCUSSION

The formalism of the preceding three sections has been presented in the most straightforward manner and in the context of the most uncomplicated system which yields the basic results, in order to concentrate on the basic idea: that CVC and PCAC *alone* determine the form of the meson Lagrangian density *independent* of any particular group-theoretical representation, and that the two concepts can be formally combined in an elegant fashion. However, before much contact with reality can be made [in, for example, Eq. (94)], this skeleton must be fleshed out with addition considerations, such as the inclusion of differences among the various meson masses (or coupling constants), loosely referred to as "symmetry breaking." Our approach can easily accommodate mass differences, which leads in a natural manner to precise expressions for the amount of nonconservation of strange vector currents (see, for example, Ref. 8). Another obvious direction of research involves the additions of baryons to the mesons, the framework for which has already been constructed.[8,19] By means of the application of general covariant derivatives one may extract the so-called D and F parts of the vector and axial-vector baryon currents.

Lastly, the combined vector and axial-vector interaction employed here lays the groundwork for a possible unified description of strong and weak interactions which is not tied to any group-theoretical model. In fact, differing representations can be handled in our formalism through the simple expedient of embedding them in a common inclusive (matrix) representation, without altering the fundamental results.

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APPENDIX: DERIVATION OF THE MESON LAGRANGIAN DENSITY FROM CVC AND PCAC IN SU(2)

We shall here construct a general Lagrangian density for a system of vector (ρ), axial-vector ($\mathfrak{a}$), and pseudoscalar (π) isotriplet mesons, the pertinent terms of which have undetermined coefficients that are assumed to be in general functions of the π field via the isoscalar π^2, and we shall show that CVC and PCAC determine all the coefficients completely, up to a unitary function U of π and a constant identifiable as the pion coupling constant f.

The Lagrangian density can be written as

$$\mathfrak{L}=\mathfrak{L}_{\rho+\mathfrak{a}}+\mathfrak{L}_{\pi}\,, \tag{A1}$$

where it has been demonstrated that CVC (and, equally, PCAC in the absence of the pion) requires that the ρ and $\mathfrak{a}$ Lagrangian be given by

$$\mathfrak{L}_{\rho+\mathfrak{a}}=-\tfrac{1}{4}({\rho_{\mu\nu}}^2+{\mathfrak{a}_{\mu\nu}}^2)-\tfrac{1}{2}m^2({\rho_\mu}^2+{\mathfrak{a}_\mu}^2)\,, \tag{A2}$$

with implied summation over the Greek indices, and where

$$\rho_{\mu\nu}=\partial_\mu\rho_\nu-\partial_\nu\rho_\mu+g\rho_\mu\times\rho_\nu+g\mathfrak{a}_\mu\times\mathfrak{a}_\nu \tag{A3}$$

and

$$\mathfrak{a}_{\mu\nu}=\partial_\mu\mathfrak{a}_\nu-\partial_\nu\mathfrak{a}_\mu+g\rho_\mu\times\mathfrak{a}_\nu+g\mathfrak{a}_\mu\times\rho_\nu\,. \tag{A4}$$

Furthermore, the ρ-meson couplings must occur through the vector covariant derivative

$$\partial_\mu\pi+g\rho_\mu\times\pi\,. \tag{A5}$$

We are therefore presented with the general pion Lagrangian, with interactions,

$$\begin{aligned}\mathfrak{L}_\pi=&-\tfrac{1}{2}(1-2b)(\partial_\mu\pi+g\rho_\mu\times\pi)^2\\&+c(\pi\cdot\partial_\mu\pi)^2-\tfrac{1}{2}(\mu^2\pi^2-2a)\\&+\alpha(\partial_\mu\pi+g\rho_\mu\times\pi)\cdot\mathfrak{a}_\mu+A{\mathfrak{a}_\mu}^2\\&+\beta(\pi\cdot\partial_\mu\pi)(\pi\cdot\mathfrak{a}_\mu)+B(\pi\cdot\mathfrak{a}_\mu)^2\,,\end{aligned} \tag{A6}$$

where a, b, c, α, β, A, and B are all assumed to be functions of π^2 and have been chosen to correspond to the notation of previous work[6] involving the pion with the ρ field; in particular, in the absence of interactions $a(0)=b(0)=0$ in order to obtain the free pion Lagrangian. The variations of the Lagrangian density $\mathfrak{L}$ with respect to ρ_μ and $\mathfrak{a}_\mu$ are

$$\begin{aligned}\delta_{\rho_\mu}\mathfrak{L}=&(\partial_\nu+g\rho_\nu\times)\rho_{\nu\mu}+g\mathfrak{a}_\nu\times\mathfrak{a}_{\nu\mu}-m^2\rho_\mu\\&+\alpha g\pi\times\mathfrak{a}_\mu-g(1-2b)\pi\times(\partial_\mu+g\rho_\mu\times)\pi\end{aligned} \tag{A7}$$

and

$$\begin{aligned}\delta_{\mathfrak{a}_\mu}\mathfrak{L}=&(\partial_\nu+g\rho_\nu\times)\mathfrak{a}_{\nu\mu}+g\mathfrak{a}_\nu\times\rho_{\nu\mu}-(m^2-2A)\mathfrak{a}_\mu\\&+\alpha(\partial_\mu+g\rho_\mu\times)\pi+\beta(\pi\cdot\partial_\mu\pi)\pi+2B(\pi\cdot\mathfrak{a}_\mu)\pi\,,\end{aligned} \tag{A8}$$

where the variations mean

$$\delta_{\rho_\mu}\mathfrak{L}=\frac{\partial\mathfrak{L}}{\partial_{\rho_\mu}}-\partial_\nu\frac{\partial\mathfrak{L}}{\partial(\partial_\nu\rho_\mu)}\,, \tag{A9}$$

etc. For the pion variation, we eliminate the non-independent term $\pi\cdot(\partial_\mu+g\rho_\mu\times)^2\pi$ by using the quantity

$$\begin{aligned}\Delta_\pi\mathfrak{L}=&\,\delta_\pi\mathfrak{L}+\frac{2b'}{1-2b-2b'\pi^2}(\pi\cdot\delta_\pi\mathfrak{L})\pi\\
=&\,(1-2b)(\partial_\mu+g\rho_\mu\times)^2\pi-4b'(\pi\cdot\partial_\mu\pi)(\partial_\mu+g\rho_\mu\times)\pi+\frac{1-2b}{1-2b-2b'\pi^2}2(b'-c)\partial_\mu(\pi\cdot\partial_\mu\pi)\pi\\
&-\frac{(1-2b)2c'+8b'^2}{1-2b-2b'\pi^2}(\pi\cdot\partial_\mu\pi)^2\pi-\frac{1-2b}{1-2b-2b'\pi^2}\left(1-\frac{2}{\mu^2}a'\right)\mu^2\pi\\
&-\alpha(\partial_\mu+g\rho_\mu\times)\mathfrak{a}_\mu+[(\beta-2\alpha')\pi\cdot\partial_\mu\pi+2B\pi\cdot\mathfrak{a}_\mu]\mathfrak{a}_\mu\\
&-\frac{(1-2b)\beta+2b'\alpha}{1-2b-2b'\pi^2}\partial_\mu(\pi\cdot\mathfrak{a}_\mu)\pi+\frac{1-2b}{1-2b-2b'\pi^2}2A'{\mathfrak{a}_\mu}^2\pi+\frac{2b'}{1-2b-2b'\pi^2}(\beta-2\alpha')(\pi\cdot\partial_\mu\pi)(\pi\cdot\mathfrak{a}_\mu)\pi\\
&+\frac{(1-2b)2B'+4b'B}{1-2b-2b'\pi^2}(\pi\cdot\mathfrak{a}_\mu)^2\pi+\frac{(1-2b)2\alpha'+2b'\alpha}{1-2b-2b'\pi^2}(\partial_\mu\pi+g\rho_\mu\times\pi)\cdot\mathfrak{a}_\mu\pi\,.\end{aligned} \tag{A10}$$

A prime in a coefficient indicates differentiation with respect to the function variable, π^2. To form the subsidiary condition for the $\mathfrak{A}$ field that we identify with PCAC, we construct the quantity

$$\Delta\mathcal{L} = \delta(\pi^2)\Delta_\pi\mathcal{L} + (\partial_\mu + g\rho_\mu \times)\delta_{\mathfrak{A}_\mu}\mathcal{L} + g\mathfrak{A}_\mu \times \delta_{\rho_\mu}\mathcal{L} , \quad \text{(A11)}$$

with $\delta(\pi^2)$ an as yet undetermined function of the pion field, and the relation of the last two terms established by CVC. If the π, ρ, and $\mathfrak{A}$ field equations are satisfied $\Delta\mathcal{L}$ vanishes; we therefore demand that it be identically

$$\Delta\mathcal{L} = -m^2\partial_\mu\mathfrak{A}_\mu - \mathfrak{F}\mu^2\pi , \quad \text{(A12)}$$

with $\mathfrak{F}$ some constant. Using expressions (A10), (A7), and (A8) to write $\Delta\mathcal{L}$ explicitly, we get

$$\begin{aligned}
\Delta\mathcal{L} = &-m^2\partial_\mu\mathfrak{A}_\mu - \frac{1-2b}{1-2b-2b'\pi^2}\left(1-\frac{2}{\mu^2}a'\right)\delta\mu^2\pi \\
&+[\alpha+(1-2b)\delta](\partial_\mu+g\rho_\mu\times)^2\pi + [(2\alpha'+\beta)-4b'\delta](\pi\cdot\partial_\mu\pi)(\partial_\mu+g\rho_\mu\times)\pi \\
&+\left[\beta+\frac{2(1-2b)}{1-2b-2b'\pi^2}(b'-c)\delta\right]\partial_\mu(\pi\cdot\partial_\mu\pi)\pi + \left[2\beta'-\frac{2(1-2b)c'+8b'^2}{1-2b-2b'\pi^2}\delta\right](\pi\cdot\partial_\mu\pi)^2\pi \\
&+[2A-\alpha\delta](\partial_\mu+g\rho_\mu\times)\mathfrak{A}_\mu + [4A'-(2\alpha'-\beta)\delta](\pi\cdot\partial_\mu\pi)\mathfrak{A}_\mu \\
&+[-g^2\alpha+2B\delta](\pi\cdot\mathfrak{A}_\mu)\mathfrak{A}_\mu + \left[g^2\alpha+\frac{1-2b}{1-2b-2b'\pi^2}2A'\delta\right]\mathfrak{A}_\mu{}^2\pi + [2B+g^2(1-2b)](\pi\cdot\mathfrak{A}_\mu)(\partial_\mu+g\rho_\mu\times)\pi \\
&+\left[2B-\frac{(1-2b)\beta+2b'\alpha}{1-2b-2b'\pi^2}\delta\right]\partial_\mu(\pi\cdot\mathfrak{A}_\mu)\pi + \left[4B'+\frac{2b'}{1-2b-2b'\pi^2}(\beta-2\alpha')\delta\right](\pi\cdot\partial_\mu\pi)(\pi\cdot\mathfrak{A}_\mu)\pi \\
&+\left[-g^2(1-2b)+\frac{(1-2b)2\alpha'+2b'\alpha}{1-2b-2b'\pi^2}\delta\right]\mathfrak{A}_\mu\cdot(\partial_\mu\pi+g\rho_\mu\times\pi)\pi + \left[\frac{(1-2b)2B'+4b'B}{1-2b-2b'\pi^2}\delta\right](\pi\cdot\mathfrak{A}_\mu)^2\pi . \quad \text{(A13)}
\end{aligned}$$

For (A12) to hold, all of the coefficients in square brackets must vanish. The twelve consequent equations yield, with some redundancy, the following solutions for δ, α, β, A, B, c, and a in terms of the function b and the constant $\mathfrak{F}$:

$$(1-2b)\delta^2 = \mathfrak{F}^2 - g^2(1-2b)\pi^2 , \quad \text{(A14)}$$

$$\alpha = -(1-2b)\delta , \quad \text{(A15)}$$

$$\beta = 2(1-2b)\delta' , \quad \text{(A16)}$$

$$A = -\tfrac{1}{2}(1-2b)\delta^2 , \quad \text{(A17)}$$

$$B = -\tfrac{1}{2}g^2(1-2b) , \quad \text{(A18)}$$

$$c = b' + (1-2b-2b'\pi^2)\frac{\delta'}{\delta} , \quad \text{(A19)}$$

and

$$1-\frac{2}{\mu^2}a' = \frac{\mathfrak{F}(1-2b-2b'\pi^2)}{\mathfrak{F}^2-g^2(1-2b)\pi^2}\delta . \quad \text{(A20)}$$

With the relation of the constant $\mathfrak{F}$ to the pion coupling constant f

$$f = \frac{g}{2\mathfrak{F}} , \quad \text{(A21)}$$

$\mathcal{L}_\pi$ can now be written as

$$\begin{aligned}
\mathcal{L}_\pi = &-\tfrac{1}{2}(1-2b)[(\partial_\mu\pi+g\rho_\mu\times\pi)^2 - g^2(\pi\times\mathfrak{A}_\mu)^2] - \frac{g^2}{8f^2}\mathfrak{A}_\mu{}^2 \\
&-\frac{g}{2f}[1-2b-4f^2(1-2b)^2\pi^2]^{1/2}(\partial_\mu\pi+g\rho_\mu\times\pi)\cdot\mathfrak{A}_\mu \\
&+\frac{g}{2f}\frac{2b'-4f^2(1-2b)^2}{[1-2b-4f^2(1-2b)^2\pi^2]^{1/2}}(\pi\cdot\partial_\mu\pi)(\pi\cdot\mathfrak{A}_\mu) + c(\pi\cdot\partial_\mu\pi)^2 - \tfrac{1}{2}(\mu^2\pi^2-2a) , \quad \text{(A22)}
\end{aligned}$$

with c and a the same as for the pure pion case. To put the results (A22) in "unitary" form, we define two other functions of π^2, ρ and σ:

$$\rho = (1-2b)^{1/2} , \quad \text{(A23)}$$

$$\sigma = (1-4f^2\rho^2\pi^2)^{1/2} = [1-4f^2(1-2b)\pi^2]^{1/2} \quad \text{(A24)}$$

so that

$$\delta = \mathfrak{F}\frac{\sigma}{\rho} = \frac{g}{2f}\frac{\sigma}{\rho} , \quad \text{(A25)}$$

and it can be verified that the Lagrangian (A22) can be written as

$$\mathcal{L}_\pi = -\frac{1}{4f^2}[i\gamma_5 f\rho(\partial_\mu + g\rho_\mu \times)\pi + i\gamma_5 \tfrac{1}{2} g\sigma \mathfrak{A}_\mu + \sigma'(\pi\cdot\partial_\mu\pi) + 2i\gamma_5 f\rho'(\pi\cdot\partial_\mu\pi)\pi - gf\rho(\pi\cdot\mathfrak{A}_\mu)]$$
$$\times[-i\gamma_5 f\rho(\partial_\mu + g\rho_\mu \times)\pi - i\gamma_5 \tfrac{1}{2} g\sigma \mathfrak{A}_\mu + \sigma'(\pi\cdot\partial_\mu\pi) - 2i\gamma_5 f\rho'(\pi\cdot\partial_\mu\pi)\pi - gf\rho(\pi\cdot\mathfrak{A}_\mu)] - \tfrac{1}{2}(\mu^2\pi^2 - 2a). \qquad \text{(A26)}$$

The "unitary" form of $\mathcal{L}_\pi$, given in Eqs. (64) and (65) of the text, follows by setting

$$U = \sigma + 2i\gamma_5 f\rho\pi. \qquad \text{(A27)}$$

The matrix form of $\mathcal{L}_P$ using U, as given in the text, is derivable from CVC and PCAC as above for any SU(2) group (not just isospin).

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