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Nonlinear group realization involving 2-dimensional space-time symmetry

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The nonlinear-group-realization methods of Coleman, Wess, and Zumino are extended to include the effects of space-time symmetry in a simple 2-dimensional model. A coset decomposition of the 4-dimensional Lorentz group is made with respect to the subgroup generated by a Lorentz boost and one internal rotation. Nonlinear Goldstone fields are identified with the coset parameters in this decomposition. The physical properties of these fields are determined by the subgroup. Charge pairs $\pm q$ are found which are not scalars under the Lorentz boost. Physical momentum and energy operators are introduced. Covariant derivatives are then calculated and the phenomenological Lagrangian is discussed.

I. INTRODUCTION

Although chiral symmetry is believed to be an approximate symmetry of nature, no parity doublets are observed. Weinberg[1] resolved this difficulty by suggesting that chiral symmetry is not an ordinary symmetry such as isospin, which is realized by a multiplet structure, but rather a dynamical symmetry which is realized by the appearance of a set of massless Goldstone particles. These particles form a nonlinear realization of the dynamical symmetry. The basic difference between an ordinary and a dynamical symmetry is that the ordinary symmetry is characterized by conservation laws and relates processes involving fixed numbers of particles, while the dynamical symmetry does not produce conservation laws but relates processes involving different numbers of these Goldstone fields.[2,3] This is made explicit in the nonlinear formalism. Also the nonlinearity of the Goldstone fields implies that the vacuum must be degenerate in the dynamical-symmetry limit. The physical vacuum is thus not invariant under the higher-symmetry group, but only under some physically exact subgroup of the symmetry, and thus provides the mechanism for spontaneous breaking of the higher symmetry.

Much of the previous work on nonlinear group realizations has concentrated on the internal chiral-symmetry groups $SU(2)\times SU(2)$ and $SU(3)\times SU(3)$. The effects of space-time symmetry were introduced only by requiring the Lagrangian to be a Lorentz scalar as well as chiral invariant. Hopkinson and Reya[4] considered nonlinear realizations of the Poincaré group itself. Since the Poincaré group is an exact symmetry rather than a spontaneously broken one, this approach was unsuccessful. The present paper will investigate the effects of space-time symmetry on nonlinear group realizations by requiring the dynamical-symmetry group to contain the Lorentz group as part of its physically exact subgroup. The nonlinear realization is required to become linear on this physical subgroup. Salam and Strathdee[5] treated the conformal group in this manner. Their results will be compared with the results of this study later. In this paper the methods of Coleman, Wess, and Zumino[6,7] for constructing nonlinear group realizations and invariant Lagrangians are applied to a simple 2-dimensional model.

The groups originally used by Coleman, Wess, and Zumino were compact. This compactness was used, however, only to prove the uniqueness of the nonlinear realization given by the canonical form developed by Coleman *et al.*[4] The use of this canonical form for noncompact groups still produces a valid nonlinear realization, but this particular nonlinear realization now no longer classifies all possible nonlinear realizations of the group. The problem of uniqueness will not be considered in this paper.

II. METHOD OF CONSTRUCTING NONLINEAR GROUP REALIZATIONS

Let G be a dynamical-symmetry group generated by the set of generators $\{H_n, A_l\}$ which satisfy the commutation relations $[A_j, A_k]\subset\{H_n\}$, $[A_j, H_k]\subset\{A_l\}$, and $[H_j, H_k]\subset\{H_n\}$. The set $\{H_n\}$ generates a subgroup H of G which one identifies with the unbroken physical symmetry of nature. Now make a coset decomposition of the group G with respect to the subgroup H so that any element g in G can be written as

$$g\in G \Rightarrow g=e^{i\xi\cdot A}e^{iu\cdot H}. \tag{2.1}$$

The action of any transformation g_0 in G on a coset representative again gives an element of the group G and so can be written

$$g_0e^{i\xi\cdot A}=e^{i\xi'\cdot A}e^{iu'\cdot H}, \tag{2.2}$$

where $\xi'=\xi'(\xi, g_0)$ and $u'=u'(\xi, g_0)$ are, in general, nonlinear functions of ξ and g_0. Nonlinear Goldstone fields are identified with the coset parameters ξ, and as fields the ξ are made to be functions of space-time. This is not due to any property of the dynamical-symmetry group, but is an added requirement so that one may meaningfully speak of derivatives of the fields and impose Lorentz invariance on the resulting Lagrangian. Let ψ be any other field in the theory which belongs to a known representation D_ψ of the subgroup H. Thus

$$e^{iu\cdot H}\in H \Rightarrow e^{iu\cdot H}: \psi \to D_\psi(e^{iu\cdot H})\psi\,, \tag{2.3}$$

where the colon means "acting on." Then a nonlinear realization of the group G is formed by ξ and ψ such that

$$g_0: \xi \to \xi'(\xi, g_0) \tag{2.4}$$

and

$$g_0: \psi \to D_\psi[e^{iu'(\xi, g_0)\cdot H}]\psi\,. \tag{2.5}$$

The nonlinearity in the transformation of ψ under g_0 is contained in the nonlinearity of $u'(\xi, g_0)$. The vacuum state is identified as the stability point of the subgroup H so that

$$h\in H \Rightarrow h: |0\rangle \to |0\rangle\,. \tag{2.6}$$

When the transformation g_0 on the coset representative $e^{i\xi\cdot A}$ is restricted to the subgroup H, the nonlinear transformation ξ' reduces to a linear one, $\xi'=D_\xi(g_0\in H)\xi$, where the representation of the subgroup D_ξ to which the ξ fields belong is determined by the structure of the group G and the subgroup H. Thus the physical properties of the Goldstone fields are determined by the group and cannot be introduced arbitrarily. The representation D_ξ can easily be calculated using the algebra of G. Let $e^{iu_0\cdot H}$ be an arbitrary transformation in the subgroup H:

$$\begin{aligned} e^{iu_0\cdot H}e^{i\xi\cdot A} &= e^{iu_0\cdot H}e^{i\xi\cdot A}e^{-iu_0\cdot H}e^{iu_0\cdot H} \\ &= e^{i\xi'\cdot A}e^{iu'\cdot H}\,. \end{aligned} \tag{2.7}$$

The evaluation of the three exponentials $e^{iu_0\cdot H}e^{i\xi\cdot A}e^{-iu_0\cdot H}$ in (2.7) involves only commutators of the form $[A_j, H_k]\subset\{A_l\}$, so that one may identify

$$e^{i\xi'\cdot A}=e^{iu_0\cdot H}e^{i\xi\cdot A}e^{-iu_0\cdot H}$$

and

$$u'=u_0\,.$$

Since the above expression for ξ' is true for any value of the parameter ξ, let ξ be small. Now let u_0 also be infinitesimal. Then upon expanding the exponentials one finds that

$$\begin{aligned} i\xi'\cdot A &= i\xi\cdot A+[\xi\cdot A, u_0\cdot H] \\ &= i\{[I_{ab}+iu_{0m}D_{\xi ab}(H_m)]\xi_b\}A_a\,. \end{aligned} \tag{2.8}$$

Therefore,

$$[\xi\cdot A, u_0\cdot H]=-\{[u_{0m}D_{\xi ab}(H_m)]\xi_b\}A_a\,. \tag{2.9}$$

One can thus calculate the representation D_ξ for the nonlinear fields by knowing only the commutation relations of the coset generators with those of the subgroup.

The nonlinear commutation relations of the Goldstone fields with the coset generators are determined by restricting the transformation $g_0\in G$ to the set of coset representatives $e^{i\alpha\cdot A}$. Then

$$e^{i\alpha\cdot A}e^{i\xi\cdot A}=e^{i\xi'(\xi,\alpha)\cdot A}e^{iu'(\xi,\alpha)\cdot H} \tag{2.10}$$

determines the transformation function ξ'. The commutation relations of the fields with the generators A_m are given in terms of this function:

$$[A_m, \xi_n]=i\,\frac{\partial\xi_n'}{\partial\alpha_m}(\xi,\alpha)\bigg|_{\alpha=0}\,. \tag{2.11}$$

For algebras with a simple structure the nonlinear commutation relations can be determined exactly by letting α become infinitesimal and by solving the following infinite series for $Q_n=(\partial\xi_n'/\partial\alpha_m)\alpha_m$:

$$\begin{aligned} i\alpha\cdot A+[i\alpha\cdot A, i\xi\cdot A]+\frac{1}{2!}[[i\alpha\cdot A, i\xi\cdot A], i\xi\cdot A]+\frac{1}{3!}[[[i\alpha\cdot A, i\xi\cdot A], i\xi\cdot A], i\xi\cdot A]+\cdots \\ =\frac{i\partial u_n'}{\partial\alpha_m}\alpha_m H_n+\left\{iQ\cdot A+\frac{1}{2!}[iQ\cdot A, i\xi\cdot A]+\frac{1}{3!}[[iQ\cdot A, i\xi\cdot A], i\xi\cdot A]+\cdots\right\}. \end{aligned} \tag{2.12}$$

If the algebra is not simple enough for the above equation to be solved explicitly, one can determine the transformation functions ξ' approximately by applying the Baker-Campbell-Hausdorff formula

$$\begin{aligned} e^Ce^D=\exp\{C+D+\tfrac{1}{2}[C,D]+\tfrac{1}{12}[[C,D],D] \\ -\tfrac{1}{12}[[C,D],C]+\cdots\} \end{aligned} \tag{2.13}$$

twice to evaluate the expression

$$e^{i\xi'\cdot A}=e^{i\alpha\cdot A}e^{i\xi\cdot A}e^{-iu'\cdot H}\,.$$

The resulting coefficients of the generators H determine a set of simultaneous equations for the u' in terms of ξ and α. The coefficients of the generators A determine ξ' as a function of ξ, α

and u'. Although tedious, this approximation can in principle be carried out to any order in ξ.

A Lagrangian invariant under the dynamical symmetry group G is constructed out of ξ and ψ by first constructing covariant derivatives $D_\mu \xi$, $D_\mu \psi$ which have transformation properties of the same form as the ψ field. Thus one wants

$$g_0\colon D_\mu\xi_a \to D_{ab}(e^{iu'\cdot H})D_\mu\xi_b$$

and similarly

$$g_0\colon D_\mu\psi_a \to D'_{ab}(e^{iu'\cdot H})D_\mu\psi_b\ .$$

Then by making the Lagrangian a function of only these quantities and explicitly invariant under the subgroup H, the Lagrangian is automatically invariant under the total group G. Thus $\mathcal{L} = f(D_\mu\xi, D_\mu\psi, \psi)$. The ξ fields do not transform in the above manner and thus can only appear in the Lagrangian as part of the covariant derivatives. The covariant derivatives $D_\mu\xi$ and $D_\mu\psi$ are calculated by evaluating the expression[7]

$$e^{-i\xi\cdot A}\partial_\mu e^{i\xi\cdot A} = i\partial_\mu\xi\cdot A + \frac{i^2}{2!}[\partial_\mu\xi\cdot A, \xi\cdot A] + \frac{i^3}{3!}[[\partial_\mu\xi\cdot A, \xi\cdot A], \xi\cdot A] + \cdots = i(p_\mu\cdot A + v_\mu\cdot H). \tag{2.14}$$

Then $D_\mu\xi = p_\mu$ and v_μ gives a correction to the ordinary derivative of ψ, $D_\mu\psi = \partial_\mu\psi + iv_\mu\cdot D_\psi(H)\psi$. The nonlinear fields ξ only appear in the Lagrangian along with their derivatives $\partial_\mu\xi$ and thus no mass term is possible. Also, since $D_\mu\xi$ and $D_\mu\psi$ involve nonlinear functions of ξ, terms such as $D_\mu\xi D_\mu\xi$ or $D_\mu\psi D_\mu\psi$ relate processes involving different numbers of the nonlinear fields ξ. In the phenomenological approach[8] the amplitudes for these processes are calculated by evaluating all tree diagrams, Feynman diagrams containing no internal loops, for a given interaction. Higher-order corrections are assumed to be already included in the physical coupling constants.

III. 2-DIMENSIONAL MODEL

Space-time symmetry is put into a simple 2-dimensional model to investigate its effect on the properties and couplings of the Goldstone fields. Let the dynamical-symmetry group be the 4-dimensional Poincaré group generated by the usual Poincaré generators $\{P_\mu, J_m, K_n\}$ ($\mu = 1, 2, 3, 4$; $m, n = 1, 2, 3$). This group is chosen merely for convenience since the properties of the Poincaré group are well known, and should not be interpreted as the physical Poincaré group. The exact physical symmetry in this model is generated by the set $\{J_3, K_3, P_3, P_0\}$, where P_3 and P_0 are momentum and energy operators, respectively, K_3 generates a Lorentz boost, and J_3 generates an internal rotation. Thus this example describes a space in which there is one space dimension and one time dimension, plus an internal charge. If the nonlinear formalism is straightforwardly applied to this model by making a coset decomposition of the 4-dimensional Poincaré group with respect to the above subgroup and then associating Goldstone fields with the six coset parameters, one finds that these Goldstone fields must belong to an unphysical, nilpotent representation of P_0 and P_3. Any operator involving transformations on space-time may be written as the sum of two parts, a "spin" part which acts on the functional form of the field and an "orbital" part which acts only on the argument of the field. The nonlinear formalism given in Sec. II only predicts the representation of the spin part of these operators. The orbital part is again introduced as an additional requirement on the fields in the same way as dealing with purely internal symmetries. For P_3 and P_0 to correspond to physical momentum and energy operators, the spin part of the operators must belong to the identity representation for any field. This means that the momentum and energy operators actually consist only of an orbital part. Since the dynamical-symmetry group now is interpreted as that group which described the internal symmetries and the spin parts of space-time symmetries, P_3 and P_0 must be deleted from the set of generators for this group. To insure closure of the group in this 2-dimensional model, P_1 and P_2 must also be deleted. The dynamical-symmetry group G used for this model is then generated by the spin parts of the usual 4-dimensional Lorentz group generators $\{J_m, K_n\}$ ($m, n = 1, 2, 3$). The subgroup H is generated only by J_3 and the spin part of K_3. Then momentum and energy operators may be introduced which behave physically for all fields.

A coset decomposition of the group G is made with respect to the subgroup H such that every element $g \in G$ is written as

$$g = \exp[i(-\theta_1 J_1 - \theta_2 J_2 + \eta_1 K_1 + \eta_2 K_2)] \times \exp[i(-\theta_3 J_3 + \eta_3 K_3)]\ . \tag{3.1}$$

The metric chosen for G is $(1, -1, -1, -1)$. The Goldstone fields are associated with the coset parameters θ_1, θ_2, η_1, η_2. The linear representation of the subgroup to which the Goldstone fields belong is determined by the commutation relation

$$[(-\theta_1 J_1 - \theta_2 J_2 + \eta_1 K_1 + \eta_2 K_2), (-\theta_3 J_3 + \eta_3 K_3)]\ .$$

This yields

$$D(J_3)=\begin{bmatrix}0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{bmatrix} \tag{3.2}$$

and

$$D(K_3)=\begin{bmatrix}0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & i & 0 & 0\\ -i & 0 & 0 & 0\end{bmatrix}. \tag{3.3}$$

Let the eigenvectors of J_3 correspond to the physical Goldstone fields. The eigenvectors have the form

$$\varphi_3^{\pm}=\frac{1}{\sqrt{2}}(\theta_2 \pm i\theta_1),$$
$$\varphi_0^{\pm}=\frac{1}{\sqrt{2}}(\eta_2 \pm i\eta_1), \tag{3.4}$$

where $\pm$ labels the charge eigenstates $\pm q$. The corresponding nonlinear generators are

$$F_3^{\pm}=\frac{1}{\sqrt{2}}(J_2 \mp iJ_1),$$
$$F_0^{\pm}=\frac{1}{\sqrt{2}}(K_2 \mp iK_1), \tag{3.5}$$

so that

$$\varphi_\nu^+ F_\nu^+ + \varphi_\nu^- F_\nu^- = (-\theta_1 J_1 - \theta_2 J_2 + \eta_1 K_1 + \eta_2 K_2),$$
$$\nu = 0, 3 .$$

Summations over the ν indices are implied using the metric (1, −1) in analogy with the 4-dimensional space-time metric (1, −1, −1, −1). The matrix representations of J_3 and K_3 in this basis are

$$D(J_3)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix} \tag{3.6}$$

and

$$D(K_3)=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}. \tag{3.7}$$

Under an infinitesimal Lorentz boost the charge eigenstates transform in the following way:

$$\varphi_3^{+\prime}=\varphi_3^+ + i\eta_3\varphi_0^+, \quad \varphi_3^{-\prime}=\varphi_3^- - i\eta_3\varphi_0^-, \tag{3.8}$$

$$\varphi_0^{+\prime}=\varphi_0^+ - i\eta_3\varphi_3^+, \quad \varphi_0^{-\prime}=\varphi_0^- + i\eta_3\varphi_3^-, \tag{3.9}$$

where η_3 is the boost parameter. The fields $\varphi_3^{\pm}$, $\varphi_0^{\pm}$ are not scalars under the Lorentz boost. If the 2-dimensional space of this model is interpreted as a slice out of 4-dimensional space, the pairs $(\varphi_3^+, \varphi_0^+)$ and $(\varphi_3^-, \varphi_0^-)=(\varphi_3^+, \varphi_0^+)^*$ transform like the third and zeroth components of a 4-vector.

It is convenient to investigate the nonlinear transformation properties of the Goldstone fields in terms of the linear combinations of these fields which are simultaneous eigenvectors of J_3 and K_3. These eigenvectors

$$\varphi_{-,\pm i}=\frac{1}{\sqrt{2}}(\varphi_3^- \pm i\varphi_0^-) \tag{3.10}$$

and

$$\varphi_{+,\pm i}=\frac{1}{\sqrt{2}}(\varphi_3^+ \mp i\varphi_0^+) \tag{3.11}$$

form charge pairs $\pm q$ under the operator J_3 and have eigenvalues $\pm i$ under the Lorentz boost K_3. These eigenvectors are actually the 1-dimensional irreducible representations of the 2-dimensional Lorentz group. These single eigenvectors are not individually interpreted as physical fields in order that a closer analogy may be made with real 4-dimensional Lorentz transformations. The linear combinations of the coset generators corresponding to the $\varphi_{a,b}$ fields are

$$F_{-,\pm i}=-\frac{1}{\sqrt{2}}(F_3^- \pm F_0^-) \tag{3.12}$$

and

$$F_{+,\pm i}=-\frac{1}{\sqrt{2}}(F_3^+ \mp iF_0^+). \tag{3.13}$$

In terms of these generators $F_{a,b}$ $(a=\pm,\ b=\pm i)$, the nonzero commutators expressing the Lie algebra of the group have the simple form

$$[J_3, F_{a,b}]=aF_{a,b}, \tag{3.14a}$$

$$[K_3, F_{a,b}]=bF_{a,b}, \tag{3.14b}$$

$$[F_{a,b}, F_{-a,-b}]=aJ_3-bK_3, \tag{3.14c}$$

so that the infinite series in expression (2.12) can be summed. The resulting equations are solved for

$$Q_{a,b}=\frac{\partial\varphi'_{a,b}}{\partial\varphi_{c,d}}\alpha_{c,d},$$

where $a, c=\pm$, $b, d=\pm i$, and summations over c and d are implied. Using Eq. (2.11) the commutation relations of the fields $\varphi_{a,b}$ with the generators $F_{c,d}$ are

$$[F_{c,d}, \varphi_{a,b}] = i \left. \frac{\partial \varphi'_{a,b}}{\partial \varphi_{c,d}} \right|_{\alpha=0}, \tag{3.15}$$

where $a, c = \pm$, $b, d = \pm i$. Carrying out the above algebraic procedure and expressing the resulting series in closed form gives the following nonzero commutation relations of the $\varphi_{a,b}$ fields with the coset generators:

$$[F_{a,b}, \varphi_{a,b}] = i \left\{ \frac{z}{2} \frac{d}{dz} \ln [j_0(z)] + 1 \right\} \tag{3.16}$$

and

$$[F_{-a,-b}, \varphi_{a,b}] = -i \frac{\varphi_{a,b}}{\varphi_{-a,-b}} \left\{ \frac{z}{2} \frac{d}{dz} \ln[j_0(z)] \right\} \tag{3.17}$$

for each $a = \pm$, $b = \pm i$. $j_0(z)$ is the usual zeroth-order spherical Bessel function with $z = 2 \times (\varphi_{a,b} \varphi_{-a,-b})^{1/2}$ with no summation implied. Using expressions (3.16) and (3.17) it is straightforward to verify that the Jacobi identity

$$[F_{a,b}, [F_{-a,-b}, \varphi_{a,b}]] - [F_{-a,-b}, [F_{a,b}, \varphi_{a,b}]] = [[F_{a,b}, F_{-a,-b}], \varphi_{a,b}] \tag{3.18}$$

is satisfied and thus a valid nonlinear realization of the group has been obtained. The commutation relations of the $\varphi_\nu^\pm$ fields with the $F_\nu^\pm$ generators are given in the terms of the above commutators. For example,

$$[F_3^+, \varphi_3^+] = -\tfrac{1}{2} \{ [F_{+,i}, \varphi_{+,i}] + [F_{+,-i}, \varphi_{+,-i}] \}. \tag{3.19}$$

They are nonlinear and in some cases also inhomogeneous.

The covariant derivatives of the $\varphi_{a,b}$ fields are calculated using Eq. (2.14) and have the form

$$D_\mu \varphi_{a,b} = \partial_\mu \varphi_{a,b} + \frac{1}{2\varphi_{-a,-b}} [1 - j_0(2(\varphi_{a,b}\varphi_{-a,-b})^{1/2})] \times (\varphi_{a,b} \partial_\mu \varphi_{-a,-b} - \varphi_{-a,-b} \partial_\mu \varphi_{a,b}), \tag{3.20}$$

where $a = \pm$ and $b = \pm i$. The covariant derivatives of the $\varphi_\nu^\pm$ fields may be expressed in terms of these in the following way:

$$D_\mu \varphi_3^+ = \frac{1}{\sqrt{2}} (D_\mu \varphi_{+,i} + D_\mu \varphi_{+,-i}), \tag{3.21a}$$

$$D_\mu \varphi_0^+ = \frac{-i}{\sqrt{2}} (D_\mu \varphi_{+,-i} - D_\mu \varphi_{+,i}), \tag{3.21b}$$

$$D_\mu \varphi_3^- = \frac{1}{\sqrt{2}} (D_\mu \varphi_{-,i} + D_\mu \varphi_{-,-i}), \tag{3.21c}$$

$$D_\mu \varphi_0^- = \frac{i}{\sqrt{2}} (D_\mu \varphi_{-,-i} - D_\mu \varphi_{-,i}). \tag{3.21d}$$

The couplings of the φ fields to other fields in the theory are determined by the functions v_μ in Eq. (2.14), where

$$v_{\mu J_3} = \frac{i}{2} \left\{ \frac{1}{\varphi_{+,-i}\varphi_{-,i}} \sin^2[(\varphi_{+,-i}\varphi_{-,i})^{1/2}] (\varphi_{+,-i} \partial_\mu \varphi_{-,i} - \varphi_{-,i} \partial_\mu \varphi_{+,-i}) + \frac{1}{\varphi_{+,i}\varphi_{-,-i}} \sin^2[(\varphi_{+,i}\varphi_{-,-i})^{1/2}] (\varphi_{+,i} \partial_\mu \varphi_{-,-i} - \varphi_{-,-i} \partial_\mu \varphi_{+,i}) \right\} \tag{3.22}$$

and

$$v_{\mu K_3} = \frac{1}{2} \left\{ \frac{\sin^2[(\varphi_{+,-i}\varphi_{-,i})^{1/2}]}{\varphi_{+,-i}\varphi_{-,i}} (\varphi_{+,-i} \partial_\mu \varphi_{-,i} - \varphi_{-,i} \partial_\mu \varphi_{+,-i}) - \frac{\sin^2[(\varphi_{+,i}\varphi_{-,-i})^{1/2}]}{\varphi_{+,i}\varphi_{-,-i}} (\varphi_{+,i} \partial_\mu \varphi_{-,-i} - \varphi_{-,-i} \partial_\mu \varphi_{+,i}) \right\}. \tag{3.23}$$

In order to demonstrate the use of the functions $v_{\mu J_3}$ and $v_{\mu K_3}$ in the construction of covariant derivatives, consider two types of additional fields ψ_1 and ψ_2 that one might want to include in the model. Let ψ_1 correspond to a particle like the pion which is a charge triplet and a pseudoscalar under Lorentz transformations. Then

$$D_{\psi_1}(J_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.24}$$

and

$$D_{\psi_1}(K_3) = 0. \tag{3.25}$$

The covariant derivatives $D_\mu \psi_1$ have the forms

$$D_\mu \psi_1^\pm = \partial_\mu \psi_1^\pm \pm \frac{1}{2} \left\{ \frac{\sin^2[(\varphi_{+,-i}\varphi_{-,i})^{1/2}]}{\varphi_{+,-i}\varphi_{-,i}} (\varphi_{+,-i} \partial_\mu \varphi_{-,i} - \varphi_{-,i} \partial_\mu \varphi_{+,-i}) + \frac{\sin^2[(\varphi_{+,i}\varphi_{-,-i})^{1/2}]}{\varphi_{+,i}\varphi_{-,-i}} (\varphi_{+,i} \partial_\mu \varphi_{-,-i} - \varphi_{-,-i} \partial_\mu \varphi_{+,i}) \right\} \psi_1^\pm \tag{3.26}$$

and

$$D_\mu \psi_1^0 = \partial_\mu \psi_1^0 . \tag{3.27}$$

Let ψ_2 correspond to a neutral, spinorlike particle. This would be a Λ^0-like particle in a model without strangeness. Then

$$D_{\psi_2}(J_3) = 0 \tag{3.28}$$

and

$$D_{\psi_2}(K_3) = \tfrac{1}{2}\gamma_3\gamma_4 = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} , \tag{3.29}$$

where γ_3, γ_4 are the 4×4 Dirac matrices. In this case the covariant derivative $D_\mu\psi_2$ has the form

$$D_\mu\psi_2 = \partial_\mu\psi_2 + \frac{i}{2}\left\{ \frac{\sin^2[(\varphi_{+,-i}\,\varphi_{-,i})^{1/2}]}{\varphi_{+,-i}\,\varphi_{-,i}} (\varphi_{+,-i}\,\partial_\mu\varphi_{-,i} - \varphi_{-,i}\,\partial_\mu\varphi_{+,-i}) - \frac{\sin^2[(\varphi_{+,i}\,\varphi_{-,-i})^{1/2}]}{\varphi_{+,i}\,\varphi_{-,-i}} (\varphi_{+,i}\,\partial_\mu\varphi_{-,-i} - \varphi_{-,-i}\,\partial_\mu\varphi_{+,i}) \right\} \gamma_3\gamma_4\psi_2 . \tag{3.30}$$

The Lagrangian is now constructed as a function of $D_\mu\varphi_\nu^\pm$, $D_\mu\psi_1$, $D_\mu\psi_2$, ψ_1, and ψ_2 which is explicitly invariant under the subgroup $U(1)\times L_2$ where U(1) is generated by J_3, and L_2 is the 2-dimensional Lorentz group. Although there is no spin operator in this simple 2-dimensional model, the fact that the Goldstone fields are not scalars under the Lorentz boost is suggestive of particles with nonzero spin in 4-dimensional space-time. If this 2-dimensional model is interpreted as a slice out of 4-dimensional space, the fields $\varphi_\nu^\pm$ ($\nu = 0, 3$) transform not as scalars (spin 0) but rather as components of fields with spin 1 or greater. For example, they could be interpreted as the third and zeroth components of a 4-vector under the Lorentz boost K_3; or they might be the $T_{\mu 3}$, $T_{\mu 0}$ components of a tensor, etc. To fix the spin, one would have to perform a similar analysis on a more "physical" group which includes the full Lorentz group as a subgroup. For the sake of argument, in the present simple model, the $\varphi_\nu^\pm$ fields will be assumed to be components of a 4-vector. The Lagrangian for these fields will then be constructed in analogy to that for a charged spin-1 field. Define $F_{\mu\nu}^\pm$ to be the antisymmetric combination and $S_{\mu\nu}^\pm$ to be the symmetric combination of the covariant derivatives $D_\mu\varphi_\nu^\pm$. Thus

$$F_{\mu\nu}^\pm = (D_\mu\varphi_\nu^\pm - D_\nu\varphi_\mu^\pm) \tag{3.31}$$

and

$$S_{\mu\nu}^\pm = (D_\mu\varphi_\nu^\pm + D_\nu\varphi_\mu^\pm). \tag{3.32}$$

The phenomenological Lagrangian then has the form

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{2}F_{\mu\nu}^+ F_{\mu\nu}^- - D_\mu\psi_1^+ D_\mu\psi_1^- - \tfrac{1}{2}D_\mu\psi_1^0 D_\mu\psi_1^0 - \bar\psi_2\gamma_\mu D_\mu\psi_2 \\ &-\tfrac{1}{2}m_1^2(2\psi_1^+\psi_1^- + \psi_1^0\psi_1^0) - m_2\bar\psi_2\psi_2 + \tfrac{1}{2}g_1 F_{\mu\nu}^+ F_{\mu\nu}^- \psi_1^0\psi_1^0 \\ &+\tfrac{1}{2}g_2 S_{\mu\nu}^+ S_{\mu\nu}^- \psi_1^0\psi_1^0 + \tfrac{1}{2}g_3 F_{\mu\nu}^+ F_{\mu\nu}^- \bar\psi_2\psi_2 + \cdots ,\end{aligned} \tag{3.33}$$

where summations over repeated indices $\mu, \nu = 0, 3$ are implied. The first four terms in the Lagrangian are kineticlike terms. The coefficients of these terms are determined by the requirement that the lowest-order term in each case reduces to the usual kinetic term for the free Lagrangian of φ_ν, ψ_1, and ψ_2, respectively. The remaining parts of the kineticlike terms represent interaction terms as shown in Fig. 1. For example, all the φ_ν self-interactions are contained in $\mathcal{L} = -\frac{1}{2}F_{\mu\nu}^+ F_{\mu\nu}^-$. Charge conservation and Lorentz invariance require that only terms involving even numbers of φ fields and two φ derivatives appear. The self-interaction terms thus represent vertices involving $2n$ φ fields, $n = 2, 3, \ldots$. The coefficient of each of these terms is not arbitrary, but is

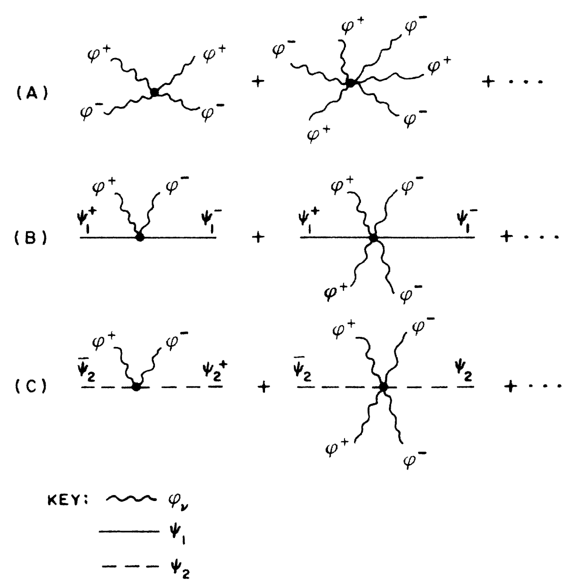

FIG. 1. Examples of interaction vertices contained in the kineticlike terms (A) $-\frac{1}{2}F_{\mu\nu}^+ F_{\mu\nu}^-$, (B) $-D_\mu\psi_1^+ D_\mu\psi_1^-$, and (C) $-\bar\psi_2\gamma_\mu D_\mu\psi_2$.

completely determined by the expansion coefficients of the covariant derivatives. The two terms $-\frac{1}{2}m_1{}^2(2\psi_1^+\psi_1^- + \psi_1^0\psi_1^0)$ and $-m_2\bar{\psi}_2\psi_2$ in the Lagrangian are mass terms for the additional ψ_1, ψ_2 fields. The remaining terms in the Lagrangian represent additional interaction terms. Here the coupling constants $g_1, g_2, g_3, \ldots$ are arbitrary, but once they are fixed the ratios between processes involving different numbers of the φ fields with ψ_1 and ψ_2 are determined.

If the 2-dimensional Lagrangian (3.33) is to be a model of 2-dimensional "reality," it is necessary that it be invariant under a physical Lorentz boost which contains both the orbital and internal parts, i.e., it must be explicitly invariant under the subgroup

$$\{P_0, P_3, J_3, K_{3\ \text{internal}} + K_{3\ \text{orbital}}\}\ . \qquad (3.34)$$

Pure spontaneous symmetry breaking, however, requires that the Lagrangian be explicitly invariant under

$$\{P_0, P_3, J_3, K_{3\ \text{internal}}, K_{3\ \text{orbital}}\}\ . \qquad (3.35)$$

To comply with the requirement of 2-dimensional physical reality the Lagrangian (3.33) has been constructed in accordance with (3.34). The terms in the Lagrangian are constructed to be scalars under the true physical 2-dimensional Lorentz group containing both the spin and orbital parts of the space-time operators. Then a term such as $\bar{\psi}_2\gamma_\mu D_\mu\psi_2$ which involves the coupling of the spin and orbital parts of these operators breaks the symmetry of the Lagrangian under the dynamical group G. This is the case, since by ignoring the orbital part, G treats the spin and orbital parts of the generators as though they were independent. Any coupling of these parts then breaks G. Thus the Lagrangian has two kinds of symmetry breaking. The dynamical symmetry G is broken spontaneously through the use of the nonlinear group realization methods of Coleman *et al.* to the subgroup H which contains the spin parts of the Lorentz group and the internal rotation. The invariance of the Lagrangian under the dynamical group G expected for a strictly spontaneously broken symmetry is, however, itself broken by the coupling of the spin and orbital parts of the space-time operators in the physical Lorentz group. If this is the only breaking of G which is not spontaneous, then the Goldstone fields φ are still required to enter the Lagrangian only in the form of their covariant derivatives and the covariant derivatives of the additional fields. The φ fields thus always appear in connection with a derivative $\partial_\nu\varphi$, so that no mass term for φ is possible.

In the phenomenological approach the amplitude for a given process is calculated by evaluating all tree diagrams for the process. Let a label the vertices governed by

$$\mathcal{L} = -\tfrac{1}{2}F^+_{\mu\nu}F^-_{\mu\nu}\ , \qquad (3.36)$$

let b label the vertices governed by

$$\mathcal{L} = (\tfrac{1}{2}g_1F^+_{\mu\nu}F^-_{\mu\nu} + \tfrac{1}{2}g_2S^+_{\mu\nu}S^-_{\mu\nu})\psi_1^0\psi_1^0\ , \qquad (3.37)$$

and let c label the vertices governed by

$$\mathcal{L} = (\tfrac{1}{2}g_4F^+_{\mu\nu}F^-_{\mu\nu} + \tfrac{1}{2}g_5S^+_{\mu\nu}S^-_{\mu\nu})\psi_1^0\ . \qquad (3.38)$$

Figure 2 shows the tree diagrams necessary for the calculation of the amplitudes for $\psi_1^0 \to 2\varphi$, $\psi_1^0 \to 4\varphi$, $\psi_1^0 \to \psi_1^0 + 2\varphi$, $\psi_1^0 \to \psi_1^0 + 4\varphi$, and $\psi_1^0 \to \psi_1^0 + 6\varphi$ in terms of these vertices. The first four processes determine the four arbitrary constants g_1, g_2, g_4, and g_5 appearing in the above Lagrangian terms. Then the amplitude for the process $\psi_1^0 \to \psi_1^0 + 6\varphi$ is determined.

To calculate the above diagrams, the field operators $\varphi_\nu^\pm$ must be appropriately quantized so that propagator functions may be determined. The Hamiltonian for the free part of the φ_ν Lagrangian, however, has the form

$$H = \int dx_3(\partial_0\varphi_3^+\partial_0\varphi_3^- - \partial_3\varphi_0^+\partial_3\varphi_0^-)\ , \qquad (3.39)$$

which is not positive-definite. An auxiliary condition must then be introduced. In analogy with 4-dimensional linear theories, one might try to impose the Lorentz condition on the $\varphi_\nu^\pm$ fields, so that $\partial_\mu\varphi_\mu^\pm = 0$. This condition requires that φ_3^a

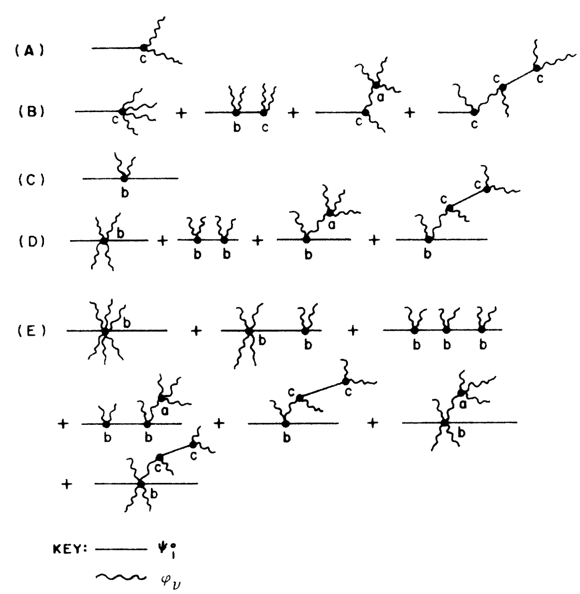

FIG. 2. Tree diagrams necessary for the calculation of the amplitudes (A) $\psi_1^0 \to 2\varphi$, (B) $\psi_1^0 \to 4\varphi$, (C) $\psi_1^0 \to \psi_1^0 + 2\varphi$, (D) $\psi_1^0 \to \psi_1^0 + 4\varphi$, and (E) $\psi_1^0 \to \psi_1^0 + 6\varphi$.

$=\pm\varphi_0^a$ ($a=\pm$). In terms of the eigenstates of the Lorentz boost K_3, the requirement is that $\varphi_{a,b}=\varphi_{-a,-b}$ ($a=\pm$, $b=\pm i$). This choice of an auxiliary condition is not, however, consistent with the commutation relations of the fields with the coset generators. These commutation relations (3.16) and (3.17) together require $[F_{a,b},\varphi_{a,b}]=\frac{1}{2}i$ for $\varphi_{a,b}=\varphi_{-a,-b}$, which is inconsistent with Eq. (3.16).

One possible cure for the inconsistency is to introduce additional symmetry breaking by giving a mass to the Goldstone fields. If one assumes the commutation relations (3.16) and (3.17) to remain valid, one can easily check that with massive Goldstone fields there is no inconsistency induced on the commutation relations by the imposition of the Lorentz condition. The propagator has the usual form for a massive spin-1 field:

$$\langle 0|T[\varphi_\mu^+(x)\varphi_\nu^-(y)]|0\rangle = \frac{-i}{(2\pi)^2}\int \frac{dp^2}{p^2+m^2-i\epsilon}\left(\delta_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}\right). \quad (3.40)$$

The tree diagrams are calculated using the usual methods of ordinary Lagrangian field theory. This is the approach used in chiral Lagrangian theories of the π meson.[9]

Another cure is to introduce gauge fields into the model. This approach is perhaps more satisfactory from the standpoint of the original model because it leaves the spontaneously broken symmetry intact. When gauge fields are introduced, the Goldstone fields are not treated as physical fields but as manifestations of a particular choice of gauge. They can be eliminated altogether via the Higgs mechanism.[10–12] Since this approach involves a significant alteration of the simple model presented, it will be the subject of a subsequent paper.

IV. CONCLUSION

In this paper 2-dimensional space-time symmetry has been introduced explicitly into a nonlinear group realization following the method of Coleman, Wess, and Zumino. In order to make a consistent physical interpretation of the resulting Goldstone fields, the group transformations are allowed to act only on the functional form of the field and thus predict only the spin part of the representation of the fields. The spin part of the physical momentum and energy operators is required to belong to the identity representation of these operators. The orbital parts of the space-time operators are introduced in the same way as for internal symmetries. The Lagrangian (3.33) is then constructed to be invariant under the physical Lorentz group. Because of the required coupling of the spin and orbital parts of the space-time operators in the physical Lorentz group, the nonlinear approach suffers from the same disease as other schemes that couple internal and space-time symmetry; viz., additional symmetry breaking in addition to the spontaneous breaking of the dynamical-symmetry group must be introduced to accommodate the orbital parts. The only essential difference between including the Lorentz group in the dynamical-symmetry group and simply requiring Lorentz invariance of the final Lagrangian is the natural appearance of Goldstone fields with nonzero spin. This is consistent with the results of Salam and Strathdee[5] with the conformal group. The Goldstone fields in that case formed a Lorentz 4-vector and a Lorentz scalar. The commutation relations of these fields with the generators of the dilation and special conformal transformations were linear and contained inhomogeneous terms. The commutation relations of the Goldstone fields with the coset generators in the present 2-dimensional model are also inhomogeneous and are truly nonlinear as expected for a nonlinear realization of the dynamical symmetry. In addition to the spin and masslessness of the Goldstone fields, these nonlinear transformations may make the construction of a suitable positive-definite free Hamiltonian for these fields difficult. This problem is partially overcome by introducing a small mass term for the Goldstone fields into the Lagrangian. The problem may be fully resolved by introducing gauge fields into the model. This will be discussed in a subsequent paper.

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Meson couplings from conserved vector current and partially conserved axial-vector current

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By requiring only that conserved vector current (CVC) and partially conserved axial-vector current (PCAC) be expressed by the field equations, a matrix formalism is developed for the nonlinear meson Lagrangian density, incorporating both conditions in any group representation. The pseudoscalar-meson "mass term" is given explicitly. The concept of the chiral covariant derivative is employed to treat a general system of vector, axial-vector, and pseudoscalar mesons in an elegant manner. In the context of invariant pseudoscalar-meson coupling constant f ($\approx 0.7 m_\pi^{-1}$?) and vector-meson coupling constant g (≈ 6) there follows immediately a relation between the (unrenormalized) axial-vector and vector-meson masses: $m_A^2 = m_V^2 + (g/2f)^2$.

I. INTRODUCTION

One of the major problems confronting attempts to describe the phenomena of elementary particles within a field-theoretical framework is the relationship of the internal quantum numbers to the dynamical properties of fields. The aspirations of mathematical esthetes to form a unified group structure providing "higher symmetries" appear to contain insufficient currency to conquer the towering difficulties involved. It would seem, therefore, that we should concentrate our efforts to understand the "internal" interactions of fields on the basis of their properties with respect to space-time with which we can deal effectively.

There are a number of beautiful and powerful theories employing dynamical subsidiary conditions or conservation laws in electrodynamics, gravitation,[1] and strong and weak interactions[2]; we may take as a relevant example the principle of conservation of vector current (CVC), leading to the so-called F-type coupling of the representative vector meson.[2] The purpose of the present paper is to point out that the power of at least one such dynamical condition, partial conservation of axial-vector current (PCAC),[3] has not heretofore been fully exploited. We show explicitly, in a nonlinear system of pseudoscalar mesons, with vector and axial-vector mesons, how PCAC in a form unified with CVC can completely determine the form of the couplings as a generalization of what is usually referred to as chiral dynamics. These results can be considered to be the (chiral) extension of the Yang-Mills theory[4]; they are model-independent, not only with respect to the form of the (unitary) pseudoscalar-meson functional, but also with respect to the representation of the "higher symmetry" (given that a second-rank tensorial, i.e., matrix, representation is valid). Further, a relation among the vector-meson mass, the axial-vector-meson mass, and the vector- and pseudoscalar-meson coupling constants follows directly.

In Sec. II we review the results of previous work[5-8] on nonlinear systems of pseudoscalar mesons, and give a general derivation of PCAC from the chiral dynamical form of the pseudoscalar meson Lagrangian in any representation [SU(2), SU(3), etc.]. Section III gives a brief resume of the formalism for vector mesons, with self-interactions, demonstrating how the supplementary condition follows from the field equations. The vector mesons are then added to the nonlinear pseudoscalar system with the development of the concept of covariant derivatives; it is shown how both CVC and (modified) PCAC are maintained. The axial-vector mesons are introduced into the combined system in Sec. IV; a broadening of the concept of the covariant derivative is evolved, along with a natural basis for the increase in the