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Regularization and renormalization of nonlinear pion-pion couplings*

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The fourth-order π-π scattering resulting from nonlinear pion-pion couplings is investigated to show how such couplings can be regularized by the method of auxiliary fields. Special care is required in the choice of coupling terms involving the auxiliary fields to ensure that all quartic divergences are canceled by the $\delta(0)$ terms in the regularized effective interaction. It is found that four counterterms are necessary for the elimination of divergences in the process under consideration. In the limiting case of massless pions, the π-π scattering result is free from infrared divergence, satisfies Adler's condition, and requires only two counterterms for renormalization.

I. INTRODUCTION

Nonlinear pion couplings have attracted widespread attention in recent years, and they appear promising in the light of experimental results. These couplings were first introduced[1] by postulating the so-called chiral symmetry, which however holds only for massless pions and therefore must be broken for real pions. Subsequently, it was shown[2] that the desired nonlinear couplings can be obtained simply by imposing the condition that the source function in the pion field equation be a complete divergence.

The scattering matrix elements obtained from the nonlinear pion couplings are not only highly complicated but also involve serious divergences. We shall show how the regularization and renormalization of such couplings can be carried out by investigating the fourth-order π-π scattering resulting from the pion-pion couplings. In view of the complex nature of the problem, we shall not discuss the regularization and renormalization for an arbitrary process up to any order. But it is hoped that the techniques described here will also be useful for other applications of nonlinear couplings.

For the evaluation of divergent integrals in quantum electrodynamics the method of auxiliary fields was formulated,[3] and it was ensured by using an indefinite metric with appropriate initial conditions that the auxiliary fields remain unobservable. Not only has this method been widely used in the past, but even in recent years it has been extensively applied to resolve anomalies in various applications.[4] The formulation of the method of auxiliary fields for nonlinear pion-pion couplings, which we shall discuss, is necessarily more complicated than that for the photon-electron coupling. But we shall find that considerable simplification can be achieved by making an important distinction between the divergences arising from the leaf diagrams[5] and those arising from other types of diagrams.

The π-π scattering due to nonlinear couplings has been investigated earlier by several authors using entirely different techniques. For instance, Bessis and Zinn-Justin[6] have used the nonlinear σ model based on the limit of the linear σ model as the σ mass tends to infinity, while Allen and Willey[7] have confined their work to the unregularized calculations of the scattering of massless pions. We shall, however, investigate the scattering for the realistic case of nonvanishing pion mass by directly using the nonlinear couplings and separating all divergences in an unambiguous manner by regularization.

The isovector pion field operator will be denoted by $\vec{\pi}$, and the pion mass by m. We shall denote the space-time coordinates as $x_\mu = (x_1, x_2, x_3, ix_0)$, and take $c = \hbar = 1$.

II. NONLINEAR PION-PION COUPLINGS WITH NONVANISHING PION MASS

We shall first express the nonlinear pion-pion couplings in a suitable form for practical applications. We impose the condition that the source function in the pion field equation be a complete divergence, so that

$$(\partial_\mu{}^2 - m^2)\vec{\pi} = \partial_\mu \vec{J}_{\mu 5} . \tag{2.1}$$

It then follows[2] that the Lagrangian density for the pion field is given in general by

$$L = -\tfrac{1}{2}(\partial_\mu \vec{\pi}\cdot\partial_\mu\vec{\pi} + m^2\vec{\pi}\cdot\vec{\pi}) + a(\vec{\pi}^2) + b(\vec{\pi}^2)\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi} + c(\vec{\pi}^2)(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 , \tag{2.2}$$

where $a(\vec{\pi}^2)$, $b(\vec{\pi}^2)$, and $c(\vec{\pi}^2)$ are expressible in terms of a single function $t(\vec{\pi}^2)$ as

$$a = \frac{m^2}{2}\int_0^{\vec{\pi}^2}\left[1 - \frac{t + 2t'\vec{\pi}^2}{f(1-4t^2\vec{\pi}^2)^{1/2}}\right] d(\vec{\pi}^2),$$

$$b = \tfrac{1}{2}(1 - t^2/f^2), \tag{2.3}$$

$$c = \frac{1}{2f^2\vec{\pi}^2}\left[t^2 - \frac{(t+2t'\vec{\pi}^2)^2}{1-4t^2\vec{\pi}^2}\right],$$

while $t(\vec{\pi}^2)$ can be expanded in powers of the coupling constant f as

$$t(\vec{\pi}^2) = f(1 + c_1 f^2\vec{\pi}^2 + c_2 f^4 \vec{\pi}^4 + \cdots), \tag{2.4}$$

$c_1, c_2, \ldots$ being arbitrary constants. Each term in the above Lagrangian density is an ordered product[8] of field operators, but for simplicity we have omitted the ordered-product notation.

Substitution of (2.4) into (2.3) leads to the expansions

$$a(\vec{\pi}^2) = -\tfrac{1}{4}(2+3c_1)f^2m^2\vec{\pi}^4 - \tfrac{1}{6}(6+10c_1+5c_2)f^4m^2\vec{\pi}^6 + O(f^6),$$

$$b(\vec{\pi}^2) = -c_1 f^2\vec{\pi}^2 - \tfrac{1}{2}(c_1{}^2 + 2c_2)f^4\vec{\pi}^4 + O(f^6),$$

$$c(\vec{\pi}^2) = -2(1+c_1)f^2 - 4(2+4c_1+c_1{}^2+c_2)f^4\vec{\pi}^2 + O(f^6),$$

and thus (2.2) becomes

$$L = -\tfrac{1}{2}(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi} + m^2\vec{\pi}\cdot\vec{\pi}) - \tfrac{1}{4}(2+3c_1)f^2m^2\vec{\pi}^4 - c_1 f^2\vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}) - 2(1+c_1)f^2(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2$$
$$-\tfrac{1}{6}(6+10c_1+5c_2)f^4m^2\vec{\pi}^6 - \tfrac{1}{2}(c_1{}^2+2c_2)f^4\vec{\pi}^4(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}) - 4(2+4c_1+c_1{}^2+c_2)f^4\vec{\pi}^2(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 + O(f^6). \tag{2.5}$$

Various models of nonlinear pion couplings can be obtained by assigning special values to $c_1, c_2, \ldots$. For instance, $c_1 = -\frac{2}{3}$ and $c_2 = \frac{2}{15}$ in Model A, $c_1 = 0$ and $c_2 = 0$ in Model B, while $c_1 = -1$ and $c_2 = 1$ in Model C, where Models A, B, and C have been described in Ref. 2.

We shall investigate the π-π scattering for an arbitrary model, and therefore we shall not assign special values to c_1 and c_2. However, in order to simplify the calculations, let us carry out a transformation of the pion field as

$$\vec{\pi} \to \vec{\pi}(1 + \lambda_1 f^2\vec{\pi}^2 + \lambda_2 f^4\vec{\pi}^4 + \cdots), \tag{2.6}$$

which transforms (2.5) into

$$L=-\tfrac{1}{2}(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+m^2\vec{\pi}^2)-\tfrac{1}{4}(2+3c_1+4\lambda_1)f^2m^2\vec{\pi}^4-(c_1+\lambda_1)f^2\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}-2(1+c_1+\lambda_1)f^2(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2$$
$$-\tfrac{1}{6}[6+10c_1+5c_2+6(2+3c_1)\lambda_1+3\lambda_1^2+6\lambda_2]f^4m^2\vec{\pi}^6-\tfrac{1}{2}(c_1^2+2c_2+8c_1\lambda_1+\lambda_1^2+2\lambda_2)f^4\vec{\pi}^4\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}$$
$$-4[2+4c_1+c_1^2+c_2+(4+5c_1)\lambda_1+\lambda_1^2+\lambda_2]f^4\vec{\pi}^2(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2+O(f^6)\,. \quad (2.7)$$

It is convenient to choose λ_1 and λ_2 such that the pion derivatives appear only in the form $\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}$ in L. This requires that

$$\lambda_1=-(1+c_1),\quad \lambda_2=1+3c_1+3c_1^2-c_2\,, \quad (2.8)$$

and (2.7) then reduces to

$$L=-\tfrac{1}{2}(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+m^2\vec{\pi}^2)$$
$$+\tfrac{1}{4}(2+c_1)f^2m^2\vec{\pi}^4+f^2\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}$$
$$-\tfrac{1}{6}(3+4c_1+3c_1^2-c_2)f^4m^2\vec{\pi}^6$$
$$-\tfrac{3}{2}f^4\vec{\pi}^4\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+O(f^6)\,. \quad (2.9)$$

III. EFFECTIVE INTERACTION

The interaction energy density in the interaction picture, resulting from the Lagrangian density (2.9), is given by

$$H_{\rm int}=-L_{\rm int}+2f^4\vec{\pi}^4\partial_0\vec{\pi}\cdot\partial_0\vec{\pi}+O(f^6)\,, \quad (3.1)$$

where

$$L_{\rm int}=\tfrac{1}{4}(2+c_1)f^2m^2\vec{\pi}^4+f^2\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}$$
$$-\tfrac{1}{6}(3+4c_1+3c_1^2-c_2)f^4m^2\vec{\pi}^6$$
$$-\tfrac{3}{2}f^4\vec{\pi}^4\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+O(f^6)\,, \quad (3.2)$$

and all field operators refer to the interaction picture. Moreover, the contractions required for the calculation of the scattering matrix elements are

$$\pi_i(x)^{\cdot}\pi_j(x')^{\cdot}=-i\delta_{ij}\Delta_F(x-x')\,,$$
$$\partial_\mu\pi_i(x)^{\cdot}\pi_j(x')^{\cdot}=-i\delta_{ij}\partial_\mu\Delta_F(x-x')\,, \quad (3.3)$$
$$\partial_\mu\pi_i(x)^{\cdot}\partial'_\nu\pi_j(x')^{\cdot}=-i\delta_{ij}\partial_\mu\partial'_\nu\Delta_F(x-x')$$
$$+i\delta_{ij}\delta_{\mu4}\delta_{\nu4}\delta(x-x')\,,$$

with

$$\Delta_F(x)=\lim_{\epsilon\to+0}\frac{1}{(2\pi)^4}\int dk\, e^{ik\cdot x}\frac{1}{k^2+m^2-i\epsilon}\,. \quad (3.4)$$

Usually the contributions from the noncovariant parts of the interaction energy density and the contractions are mutually canceled.[9] However, as explained in Appendix A, in the present case the effective interaction energy density takes the form

$$H_{\rm eff}=-L_{\rm int}-3if^2\vec{\pi}^2\delta(0)+\tfrac{9}{2}if^4\vec{\pi}^4\delta(0)$$
$$-3if^4\vec{\pi}^4\delta(0)+O(f^6) \quad (3.5)$$

with the effective contractions

$$\pi_i(x)^{\cdot}\pi_j(x')^{\cdot}=-i\delta_{ij}\Delta_F(x-x')\,,$$
$$\partial_\mu\pi_i(x)^{\cdot}\pi_j(x')^{\cdot}=-i\delta_{ij}\partial_\mu\Delta_F(x-x')\,, \quad (3.6)$$
$$\partial_\mu\pi_i(x)^{\cdot}\partial'_\nu\pi_j(x')^{\cdot}=-i\delta_{ij}\partial_\mu\partial'_\nu\Delta_F(x-x')\,.$$

The $\delta(0)$ terms in (3.5) are in agreement with the result of Gerstein *et al.*,[10] who have applied a more general but less conventional treatment to the nonlinear couplings of massless pions.

IV. REGULARIZATION

Regularization by the method of auxiliary fields[3] requires the choice of a Lagrangian density of the form

$$\overline{L}=-\tfrac{1}{2}(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+m^2\vec{\pi}\cdot\vec{\pi})$$
$$-\tfrac{1}{2}\sum_\alpha\eta^{(\alpha)}(\partial_\mu\vec{\pi}^{(\alpha)}\cdot\partial_\mu\vec{\pi}^{(\alpha)}+m^{(\alpha)2}\vec{\pi}^{(\alpha)}\cdot\vec{\pi}^{(\alpha)})+\overline{L}_{\rm int}\,, \quad (4.1)$$

where $\vec{\pi}^{(\alpha)}$ represents an auxiliary pion field with the mass $m^{(\alpha)}$, and α takes the values $1,2,\ldots,n$. The number of required auxiliary fields depends on the degree of the highest divergence in the integrals to be evaluated, and $\eta^{(\alpha)}$ is equal to 1 or -1 according to whether the field is normal or abnormal. For the present purpose, it is sufficient to take $n=3$ such that $\eta^{(\alpha)}=-1$ for $\alpha=1$ or 3, and $\eta^{(\alpha)}=1$ for $\alpha=2$, which implies that

$$\eta^{(\alpha)2}=1,\quad \sum_\alpha\eta^{(\alpha)}=-1\,. \quad (4.2)$$

The coupling terms in (4.1) are to be obtained by introducing the auxiliary fields into the coupling terms of (2.9) in a suitable way, and it is convenient to define

$$\vec{\phi}=\sum_\alpha\vec{\pi}^{(\alpha)}\,. \quad (4.3)$$

Before we discuss the regularization of the pion-pion coupling terms, let us examine the π-π scattering according to the unregularized effective interaction (3.5). The second-order scattering matrix element for this process is free from divergence. We therefore consider the fourth-order scattering for which the interaction diagrams are shown in Fig. 1. We have not represented the contributions of the $\delta(0)$ terms by means of separate diagrams, because it can be shown that the $\delta(0)$ terms simply cancel the quartic divergences in the contributions of the diagrams in Fig. 1. This figure contains the *self-energy* diagrams (a)–(d), the *two-pion-exchange* (TPE) diagrams (e)–(g), and the *single-vertex* diagram (h), and it should be noted that the self-energy and single-vertex diagrams are *leaf* diagrams.[5]

The aim of regularization is to express all divergent integrals in an unambiguous form, so that divergences can be isolated in a covariant way and eliminated by renormalization. But the divergences in the leaf diagrams are already in an unambiguous form, and they can be completely absorbed within renormalization constants. Therefore, it is necessary only to regularize the contributions of the TPE diagrams. This purpose can be achieved in the following manner: In the f^2 coupling terms of the unregularized Lagrangian density (2.9) we replace $\vec{\pi}$ by $\vec{\pi}+\vec{\phi}/\sqrt{2}$ and drop all terms involving the products of two or more auxiliary fields, which gives

$$\tfrac{1}{4}(2+c_1)f^2m^2\vec{\pi}^4+f^2\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}$$
$$\rightarrow\tfrac{1}{4}(2+c_1)f^2m^2(\vec{\pi}^4+2\sqrt{2}\,\vec{\pi}^2\vec{\pi}\cdot\vec{\phi})+f^2(\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+\sqrt{2}\,\vec{\pi}\cdot\vec{\phi}\,\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+\sqrt{2}\,\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\phi}).$$

These coupling terms indeed lead to the regularization of the TPE contributions, but they also lead to a modification of the $\delta(0)$ terms in the effective interaction, which results in noncancellation of the quartic divergences. We therefore further introduce all possible f^4 coupling terms involving the auxiliary fields that can contribute to the fourth-order π-π scattering, and they are given by

$$a_1f^4m^2\vec{\pi}^4\vec{\phi}^2+a_2f^4\vec{\pi}^4\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi}+a_3f^4\vec{\pi}^2\vec{\phi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi},$$

where the constants a_1, a_2, and a_3 will be chosen later in such a way that the appropriate $\delta(0)$ terms are restored in the effective interaction and the TPE contributions remain regularized. Thus, by regularizing the coupling terms in (2.9) as described above, we arrive at the regularized Lagrangian density

$$\overline{L}=-\tfrac{1}{2}(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+m^2\vec{\pi}\cdot\vec{\pi})-\tfrac{1}{2}\sum_\alpha\eta^{(\alpha)}(\partial_\mu\vec{\pi}^{(\alpha)}\cdot\partial_\mu\vec{\pi}^{(\alpha)}+m^{(\alpha)2}\vec{\pi}^{(\alpha)}\cdot\vec{\pi}^{(\alpha)})+\tfrac{1}{4}(2+c_1)f^2m^2(\vec{\pi}^4+2\sqrt{2}\,\vec{\pi}^2\vec{\pi}\cdot\vec{\phi})$$
$$+f^2(\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+\sqrt{2}\,\vec{\pi}\cdot\vec{\phi}\,\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+\sqrt{2}\,\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\phi})-\tfrac{1}{6}(3+4c_1+3c_1^2-c_2)f^4m^2\vec{\pi}^6-\tfrac{3}{2}f^4\vec{\pi}^4\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}$$
$$+a_1f^4m^2\vec{\pi}^4\vec{\phi}^2+a_2f^4\vec{\pi}^4\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi}+a_3f^4\vec{\pi}^2\vec{\phi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+O(f^6). \qquad (4.4)$$

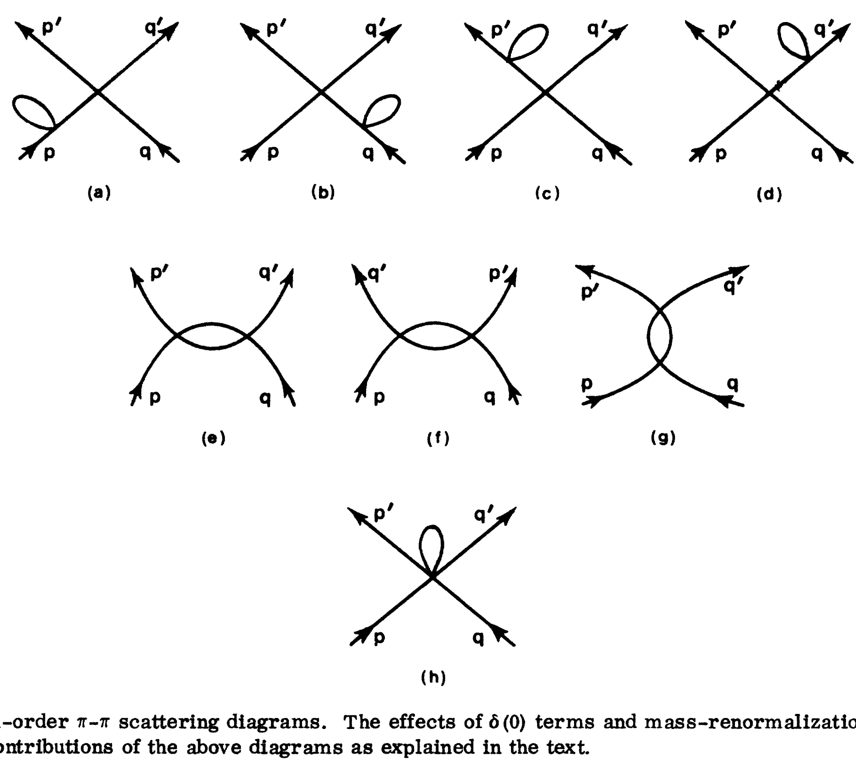

FIG. 1. Fourth-order π-π scattering diagrams. The effects of $\delta(0)$ terms and mass-renormalization terms have been included in the contributions of the above diagrams as explained in the text.

The Lagrangian density (4.4) leads to the interaction energy density in the interaction picture

$$\bar{H}_{\rm int} = -\bar{L}_{\rm int} + f^4\vec{\pi}^4\partial_0\vec{\pi}\cdot\partial_0\vec{\pi} + 2\sqrt{2}\, f^4\vec{\pi}^4\partial_0\vec{\pi}\cdot\partial_0\vec{\phi} + f^4\vec{\pi}^4\partial_0\vec{\phi}\cdot\partial_0\vec{\phi} + 4\sqrt{2} f^4\vec{\pi}^2\vec{\pi}\cdot\vec{\phi}\partial_0\vec{\pi}\cdot\partial_0\vec{\pi} + 4f^4\vec{\pi}^2\vec{\pi}\cdot\vec{\phi}\partial_0\vec{\pi}\cdot\partial_0\vec{\phi} + 4f^4\vec{\pi}\cdot\vec{\phi}\vec{\pi}\cdot\vec{\phi}\partial_0\vec{\pi}\cdot\partial_0\vec{\pi} + O(f^6), \tag{4.5}$$

where

$$\bar{L}_{\rm int} = \tfrac{1}{4}(2+c_1)f^2m^2(\vec{\pi}^4 + 2\sqrt{2}\,\vec{\pi}^2\vec{\pi}\cdot\vec{\phi}) + f^2(\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi} + \sqrt{2}\,\vec{\pi}\cdot\vec{\phi}\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi} + \sqrt{2}\,\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\phi}) - \tfrac{1}{6}(3+4c_1+3c_1^2-c_2)f^4m^2\vec{\pi}^6 - \tfrac{3}{2}f^4\vec{\pi}^4\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi} + a_1f^4m^2\vec{\pi}^4\vec{\phi}^2 + a_2f^4\vec{\pi}^4\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi} + a_3f^4\vec{\pi}^2\vec{\phi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi} + O(f^6). \tag{4.6}$$

By following the same argument as described in Appendix A for the unregularized case, it is found that (4.5) is equivalent to the effective interaction energy density

$$\bar{H}_{\rm eff} = -\bar{L}_{\rm int} - 3if^2\vec{\pi}^2\delta(0) - 3\sqrt{2}\,if^2\vec{\pi}\cdot\vec{\phi}\,\delta(0) + \tfrac{9}{2}if^4\vec{\pi}^4\delta(0) + 3a_2\,if^4\vec{\pi}^4\delta(0) - 3a_3\,if^4\vec{\pi}^2\vec{\phi}^2\delta(0) + O(f^6) \tag{4.7}$$

with the effective contractions given by (3.6) and

$$\phi_i(x)^{\cdot}\phi_j(x')^{\cdot} = -i\delta_{ij}\hat{\Delta}_F(x-x'),$$
$$\partial_\mu\phi_i(x)^{\cdot}\phi_j(x')^{\cdot} = -i\delta_{ij}\partial_\mu\hat{\Delta}_F(x-x'), \tag{4.8}$$
$$\partial_\mu\phi_i(x)^{\cdot}\partial'_\nu\phi_j(x')^{\cdot} = -i\delta_{ij}\partial_\mu\partial'_\nu\hat{\Delta}_F(x-x'),$$

where

$$\hat{\Delta}_F(x) = \sum_\alpha \eta^{(\alpha)}\Delta_F^{(\alpha)}(x), \tag{4.9}$$

and $\Delta_F^{(\alpha)}(x)$ can be obtained from (3.4) by replacing m^2 by $m^{(\alpha)2}$. It is reasonable to require that if we drop all terms involving the auxiliary fields in the regularized effective interaction (4.7), it should reduce to the unregularized result (3.5). But an examination of the $\delta(0)$ terms shows that this condition is fulfilled only if $a_2=-1$. It can also be verified that the π-π scattering contribution resulting from the terms with the coefficients a_1, a_2, and a_3 in (4.7) is regularizable only if $a_1=a_2$ and $a_3=0$. We shall, therefore, set

$$a_1=a_2=-1, \quad a_3=0. \tag{4.10}$$

V. SECOND-ORDER π-π SCATTERING

Let us consider the scattering of two pions whose momentum four-vectors are p and q in the initial state and p' and q' in the final state, so that

$$p+q=p'+q',$$
$$p^2=q^2=p'^2=q'^2=-m^2, \tag{5.1}$$
$$p\cdot p'=q\cdot q', \quad p\cdot q'=q\cdot p', \quad p\cdot q=p'\cdot q'.$$

It will be useful to note the result for the second-order scattering process. The contribution of the scattering operator for this process can be obtained from

$$S(f^2) = if^2\int dx\,[\tfrac{1}{4}(2+c_1)m^2\vec{\pi}^4 + \vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}] \tag{5.2}$$

by substituting

$$\vec{\pi}=\vec{\pi}^+(\vec{p})e^{ip\cdot x}, \quad \vec{\pi}=\vec{\pi}^+(\vec{q})e^{iq\cdot x},$$
$$\vec{\pi}=\vec{\pi}^-(\vec{p}')e^{-ip'\cdot x}, \quad \vec{\pi}=\vec{\pi}^-(\vec{q}')e^{-iq'\cdot x} \tag{5.3}$$

for the four pion-field operators in every possible way. Thus,

$$S(f^2) = i(2\pi)^4\delta(p+q-p'-q')f^2 \times[2(2+c_1)m^2(T_A+T_B+T_C) + 8(p\cdot p'\,T_A + p\cdot q'\,T_B - p\cdot q\,T_C)], \tag{5.4}$$

where

$$T_A = \vec{\pi}^-(\vec{q}')\cdot\vec{\pi}^+(\vec{q})\,\vec{\pi}^-(\vec{p}')\cdot\vec{\pi}^+(\vec{p}),$$
$$T_B = \vec{\pi}^-(\vec{p}')\cdot\vec{\pi}^+(\vec{q})\,\vec{\pi}^-(\vec{q}')\cdot\vec{\pi}^+(\vec{p}), \tag{5.5}$$
$$T_C = \vec{\pi}^-(\vec{q}')\cdot\vec{\pi}^-(\vec{p}')\,\vec{\pi}^+(\vec{q})\cdot\vec{\pi}^+(\vec{p}).$$

VI. FOURTH-ORDER π-π SCATTERING

As explained in Sec. IV, the fourth-order π-π scattering diagrams shown in Fig. 1 can be divided into three categories. We shall consider the contribution arising from each category separately.

A. Self-energy diagrams

The contribution of the scattering operator for the diagrams (a)–(d) is expressible as

$$S_{\rm SE} = -i(2\pi)^4\delta(p+q-p'-q')f^2\left[\frac{\Sigma(p)}{p^2+m^2}+\frac{\Sigma(q)}{q^2+m^2}+\frac{\Sigma(p')}{p'^2+m^2}+\frac{\Sigma(q')}{q'^2+m^2}\right]$$
$$\times[2(2+c_1)m^2(T_A+T_B+T_C)+8(p\cdot p'\,T_A+p\cdot q'\,T_B-p\cdot q\,T_C)] \quad (6.1)$$

with

$$\Sigma(p) = \frac{if^2}{(2\pi)^4}\int dk\left[\frac{5(2+c_1)m^2}{k^2+m^2}+\frac{6(k^2+p^2)}{k^2+m^2}-6\right], \quad (6.2)$$

where $\Sigma(p)$ represents the contribution of the self-energy part, and the last term within the square brackets in (6.2) arises from the term $-3if^2\vec{\pi}^2\delta(0)$ in (4.7). It is also possible to write (6.2) as

$$\Sigma(p) = -i(2-5c_1)f^2m^2\Delta_F(0)+6if^2\Delta_F(0)(p^2+m^2), \quad (6.3)$$

where $\Delta_F(0)$ is the Lorentz-invariant divergent integral

$$\Delta_F(0) = \frac{1}{(2\pi)^4}\int dk\,\frac{1}{k^2+m^2}. \quad (6.4)$$

Further, a renormalization of the pion mass in the Lagrangian density (4.4) introduces the counter-terms

$$\delta\mathcal{L}_1 = \tfrac{1}{2}\delta(m^2)\vec{\pi}^2-\tfrac{1}{4}(2+c_1)f^2\delta(m^2)\vec{\pi}^4+\cdots, \quad (6.5)$$

which lead to the additional fourth-order scattering contribution

$$\delta S_{\rm SE} = i(2\pi)^4\delta(p+q-p'-q')f^2\left\{\left[\frac{\delta(m^2)}{p^2+m^2}+\frac{\delta(m^2)}{q^2+m^2}+\frac{\delta(m^2)}{p'^2+m^2}+\frac{\delta(m^2)}{q'^2+m^2}\right]\right.$$
$$\times[2(2+c_1)m^2(T_A+T_B+T_C)+8(p\cdot p'\,T_A+p\cdot q'\,T_B-p\cdot q\,T_C)]$$
$$\left.-2(2+c_1)\,\delta(m^2)(T_A+T_B+T_C)\right\}. \quad (6.6)$$

After adding (6.1) and (6.6), substituting (6.3), and taking

$$\delta(m^2) = -i(2-5c_1)f^2m^2\Delta_F(0), \quad (6.7)$$

we can put the resulting contribution in the form

$$S_{\rm SE}+\delta S_{\rm SE} = i(2\pi)^4\delta(p+q-p'-q')$$
$$\times\{A_1[2(2+c_1)m^2(T_A+T_B+T_C)+8(p\cdot p'\,T_A+p\cdot q'\,T_B-p\cdot q\,T_C)]+8A_2(T_A+T_B+T_C)\}, \quad (6.8)$$

where

$$A_1 = -12if^4\Delta_F(0),$$
$$A_2 = \tfrac{1}{4}i(2+c_1)(2-5c_1)f^4m^2\Delta_F(0). \quad (6.9)$$

B. Single-vertex diagram

We associate with diagram (h) the contribution arising from the terms[11]

$$\tfrac{1}{6}(3+4c_1+3c_1^2-c_2)f^4m^2\vec{\pi}^6 + \tfrac{3}{2}f^4\vec{\pi}^4\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+\tfrac{9}{2}if^4\vec{\pi}^4\delta(0)$$

in the effective interaction energy density (4.7), and thus for the single-vertex diagram

$$S_{\rm SV} = i(2\pi)^4\delta(p+q-p'-q')$$
$$\times\{B_1[2(2+c_1)m^2(T_A+T_B+T_C)$$
$$+8(p\cdot p'T_A+p\cdot q'T_B-p\cdot q\,T_C)]$$
$$+8B_2(T_A+T_B+T_C)\}, \quad (6.10)$$

where

$$B_1 = 15if^4\Delta_F(0),$$
$$B_2 = -\tfrac{1}{4}i(6-41c_1-42c_1^2+14c_2)f^4m^2\Delta_F(0). \quad (6.11)$$

C. Two-pion-exchange diagrams

We shall now consider the TPE diagrams (e), (f), and (g), and it would be convenient here to include also the contribution of the fourth-order terms

$$f^4m^2\vec{\pi}^4\vec{\phi}^2+f^4\vec{\pi}^4\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi}-3if^4\vec{\pi}^4\delta(0)$$

appearing in the effective interaction (4.7) with the condition (4.10), since these terms were not taken into account in the treatment of the single-vertex diagram (h). By carrying out the appropriate contractions in the scattering operator, it is then found that

$$S_{\text{TPE}}=f^4\int dx\int dx'\{[\tfrac{7}{4}c_1^2m^4\vec{\pi}(x)\cdot\vec{\pi}(x)\vec{\pi}(x')\cdot\vec{\pi}(x')+c_1^2m^4\vec{\pi}(x)\cdot\vec{\pi}(x')\vec{\pi}(x)\cdot\vec{\pi}(x')$$
$$+3c_1m^2\partial_\mu\vec{\pi}(x)\cdot\partial_\mu\vec{\pi}(x)\vec{\pi}(x')\cdot\vec{\pi}(x')-4c_1m^2\partial_\mu\vec{\pi}(x)\cdot\vec{\pi}(x')\partial_\mu\vec{\pi}(x)\cdot\vec{\pi}(x')$$
$$-\partial_\mu\vec{\pi}(x)\cdot\partial_\mu\vec{\pi}(x)\partial'_\nu\vec{\pi}(x')\cdot\partial'_\nu\vec{\pi}(x')+4\partial_\mu\vec{\pi}(x)\cdot\partial'_\nu\vec{\pi}(x')\partial_\mu\vec{\pi}(x)\cdot\partial'_\nu\vec{\pi}(x')]\,\overline{\Delta}_F(x-x')\,\Delta_F(x'-x)$$
$$-[10c_1m^2\vec{\pi}(x)\cdot\vec{\pi}(x)\vec{\pi}(x')\cdot\partial'_\nu\vec{\pi}(x')+4\partial_\mu\vec{\pi}(x)\cdot\partial_\mu\vec{\pi}(x)\vec{\pi}(x')\cdot\partial'_\nu\vec{\pi}(x')]\,\overline{\Delta}_F(x-x')\,\partial'_\nu\Delta_F(x'-x)$$
$$+[12\vec{\pi}(x)\cdot\partial_\mu\vec{\pi}(x)\vec{\pi}(x')\cdot\partial'_\nu\vec{\pi}(x')+8\vec{\pi}(x)\cdot\vec{\pi}(x')\partial_\mu\vec{\pi}(x)\cdot\partial'_\nu\vec{\pi}(x')$$
$$-8\partial_\mu\vec{\pi}(x)\cdot\vec{\pi}(x')\vec{\pi}(x)\cdot\partial'_\nu\vec{\pi}(x')]\,\overline{\Delta}_F(x-x')\,\partial_\mu\partial'_\nu\Delta_F(x'-x)\}$$
$$+f^4\int dx\,[(6+5c_1)m^2\vec{\pi}(x)\cdot\vec{\pi}(x)\vec{\pi}(x)\cdot\vec{\pi}(x)+8\vec{\pi}(x)\cdot\vec{\pi}(x)\partial_\mu\vec{\pi}(x)\cdot\partial_\mu\vec{\pi}(x)]\,\overline{\Delta}_F(0),\tag{6.12}$$

where

$$\overline{\Delta}_F(x)=\Delta_F(x)+\hat{\Delta}_F(x),\tag{6.13}$$

and it should be observed that $\Delta_F(x)$, $\hat{\Delta}_F(x)$, and $\overline{\Delta}_F(x)$ are even functions of x. The integrands in (6.12) have been simplified by expressing them in terms of Δ_F and $\overline{\Delta}_F$, transferring all derivatives from $\overline{\Delta}_F$ by dropping four-divergences, and applying the relations

$$(\partial_\mu{}^2-m^2)\vec{\pi}=0,$$
$$(\partial_\mu{}^2-m^2)\,\Delta_F(x-x')=-\delta(x-x').\tag{6.14}$$

The total contribution of the TPE diagrams can be obtained by substituting (5.3) into (6.12) for the four pion-field operators in every possible way. Thus, after simplification with the help of (5.1),

$$S_{\text{TPE}}=f^4\,\delta(p+q-p'-q')$$
$$\times\int dk\{14c_1^2m^4(T_Ax_A+T_Bx_B+T_Cx_C)+4c_1^2m^4[(T_B+T_C)x_A+(T_C+T_A)x_B+(T_A+T_B)x_C]$$
$$+24c_1m^2(p\cdot p'T_Ax_A+p\cdot q'T_Bx_B-p\cdot q\,T_Cx_C)$$
$$-16c_1m^2[p\cdot p'(T_B+T_C)x_A+p\cdot q'(T_C+T_A)x_B-p\cdot q(T_A+T_B)x_C]$$
$$-8[(p\cdot p')^2T_Ax_A+(p\cdot q')^2T_Bx_B+(p\cdot q)^2T_Cx_C]$$
$$+16[(p\cdot p')^2(T_B+T_C)x_A+(p\cdot q')^2(T_C+T_A)x_B+(p\cdot q)^2(T_A+T_B)x_C]$$
$$+40c_1m^2k_\mu[(p'_\mu-p_\mu)T_Ax_A+(q'_\mu-p_\mu)T_Bx_B+(p_\mu+q_\mu)T_Cx_C]$$
$$+16k_\mu[(p'_\mu-p_\mu)p\cdot p'T_Ax_A+(q'_\mu-p_\mu)p\cdot q'T_Bx_B-(p_\mu+q_\mu)p\cdot q\,T_Cx_C]$$
$$+24k_\mu k_\nu[(p'_\mu-p_\mu)(p'_\nu-p_\nu)T_Ax_A+(q'_\mu-p_\mu)(q'_\nu-p_\nu)T_Bx_B+(p_\mu+q_\mu)(p_\nu+q_\nu)T_Cx_C]$$
$$+16k_\mu k_\nu[(p'_\mu+p_\mu)(q'_\nu+q_\nu)(T_B-T_C)x_A-(p'_\mu+q_\mu)(q'_\nu+p_\nu)(T_C-T_A)x_B$$
$$+(p'_\mu-q'_\mu)(p_\nu-q_\nu)(T_A-T_B)x_C]$$
$$+8(6+5c_1)m^2(T_A+T_B+T_C)\,\overline{\Delta}_F(k)+64(p\cdot p'T_A+p\cdot q'T_B-p\cdot q\,T_C)\,\overline{\Delta}_F(k)\},\tag{6.15}$$

where

$$x_A=\Delta_F(k)\,\overline{\Delta}_F(k+p'-p),$$
$$x_B=\Delta_F(k)\,\overline{\Delta}_F(k+q'-p),\tag{6.16}$$
$$x_C=\Delta_F(k)\,\overline{\Delta}_F(k+p+q),$$

and

$$\Delta_F(k)=\frac{1}{k^2+m^2},$$
$$\overline{\Delta}_F(k)=\frac{1}{k^2+m^2}+\sum_\alpha\eta^{(\alpha)}\frac{1}{k^2+m^{(\alpha)2}}.\tag{6.17}$$

Note that, for finite values of k, (6.15) reduces to the unregularized result for S_{TPE} as the auxiliary masses tend to infinity, so that the effect of regularization is confined to infinitely large values of k.

The complete TPE contribution can be obtained with the help of the results given in Appendix B, where the integrals appearing in (6.15) have been fully evaluated. However, we shall confine ourselves to an examination of the divergences, and therefore, by retaining only the divergent terms in Eqs. (B3), (B5), (B6), and (B7), we set

$$\begin{aligned}
&\int dk\, \Delta_F(k)\, \overline{\Delta}_F(k+l) = \pi^2 i \ln(\xi^2/m^2)\,,\\
&\int dk\, k_\mu \Delta_F(k)\, \overline{\Delta}_F(k+l) = -\tfrac{1}{2}\pi^2\, i\, l_\mu \ln(\xi^2/m^2)\,,\\
&\int dk\, k_\mu k_\nu \Delta_F(k)\, \overline{\Delta}_F(k+l)\\
&\qquad = \tfrac{1}{2}\pi^2\, i\, \delta_{\mu\nu}[\xi^2 \ln 2 - (\tfrac{1}{6} l^2 + m^2)\ln(\xi^2/m^2)]\\
&\qquad\quad + \tfrac{1}{3}\pi^2\, i\, l_\mu l_\nu \ln(\xi^2/m^2)\,,\\
&\int dk\, \overline{\Delta}_F(k) = \pi^2 i[2\xi^2 \ln 2 - m^2 \ln(\xi^2/m^2)]\,,
\end{aligned} \tag{6.18}$$

where ξ ultimately tends to infinity.

With the use of (6.18), together with the relations (5.1) and

$$\begin{aligned}
2p\cdot q' p\cdot q &= (p\cdot q')^2 + (p\cdot q)^2 - (p\cdot p')^2 - 2m^2 p\cdot p' - m^4\,,\\
2p\cdot q\, p\cdot p' &= (p\cdot q)^2 + (p\cdot p')^2 - (p\cdot q')^2 - 2m^2 p\cdot q' - m^4\,,\\
-2p\cdot p' p\cdot q' &= (p\cdot p')^2 + (p\cdot q')^2 - (p\cdot q)^2 + 2m^2 p\cdot q - m^4\,,
\end{aligned} \tag{6.19}$$

the divergent part of the TPE contribution (6.15) can be expressed as

$$\begin{aligned}
S_{\mathrm{TPE}} &= i(2\pi)^4\, \delta(p+q-p'-q')\\
&\times \{C_1[2(2+c_1)m^2(T_A+T_B+T_C) + 8(p\cdot p' T_A + p\cdot q' T_B - p\cdot q\, T_C)]\\
&\quad + 8C_2(T_A+T_B+T_C) + 8C_3[(p\cdot p')^2 + (p\cdot q')^2 + (p\cdot q)^2](T_A+T_B+T_C)\}\,,
\end{aligned} \tag{6.20}$$

where

$$\begin{aligned}
C_1 &= (f^4/16\pi^2)[19\xi^2 \ln 2 - (3-10c_1)m^2 \ln(\xi^2/m^2)]\,,\\
C_2 &= (f^4 m^2/64\pi^2)[3(2+7c_1)\,\xi^2 \ln 2 - (2+9c_1 - c_1{}^2)m^2 \ln(\xi^2/m^2)]\,,\\
C_3 &= (f^4/6\pi^2)\ln(\xi^2/m^2)\,.
\end{aligned} \tag{6.21}$$

VII. RENORMALIZATION

It follows from the results of the preceding section that all divergences in the fourth-order π-π scattering can be removed in the following way:

In addition to the mass-renormalization counterterms given by (6.5), let us also introduce in the Lagrangian density the counterterms

$$\delta\overline{L}_2 = -\delta(f^2)[\tfrac{1}{4}m^2(2+c_1)\vec{\pi}^4 + \vec{\pi}^2 \partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}] - \lambda\vec{\pi}^4 - \kappa[(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi})(\partial_\nu\vec{\pi}\cdot\partial_\nu\vec{\pi}) + 2(\partial_\mu\vec{\pi}\cdot\partial_\nu\vec{\pi})(\partial_\mu\vec{\pi}\cdot\partial_\nu\vec{\pi})]\,. \tag{7.1}$$

The contribution to π-π scattering resulting from the above coupling terms is

$$\begin{aligned}
\delta S(f^4) = -i(2\pi)^4\, \delta(p+q-p'-q')\{&\delta(f^2)[2(2+c_1)m^2(T_A+T_B+T_C) + 8(p\cdot p' T_A + p\cdot q' T_B - p\cdot q\, T_C)]\\
&+ 8\lambda(T_A+T_B+T_C) + 8\kappa[(p\cdot p')^2 + (p\cdot q')^2 + (p\cdot q)^2](T_A+T_B+T_C)\}\,,
\end{aligned} \tag{7.2}$$

which will cancel all divergences appearing in (6.8), (6.10), and (6.20) provided that

$$\begin{aligned}
\delta(f^2) &= A_1 + B_1 + C_1\,,\\
\lambda &= A_2 + B_2 + C_2\,,\\
\kappa &= C_3\,.
\end{aligned} \tag{7.3}$$

We find that the divergences in fourth-order π-π scattering can be removed by renormalization with the use of four counterterms. Two of these counterterms represent the conventional mass and coupling-constant renormalizations, while the third counterterm is reminiscent of the well-known linear pseudoscalar pion-nucleon coupling. The fourth counterterm, which involves four derivatives and a logarithmic divergence, has a less familiar form, but the extra divergence that necessitates the introduction of this term has also been observed by earlier authors.[6,7] It should also be noted that our results do not single out any specific model of nonlinear pion-pion couplings as being superior to the others.

VIII. SCATTERING OF MASSLESS PIONS

Although we do not regard a theory of massless pions as realistic, we shall add some remarks on the fourth-order π-π scattering process with $m=0$.

The total fourth-order π-π scattering contribution consists of (6.8), (6.10), and (6.15), which correspond to the self-energy, single-vertex, and TPE diagrams, respectively. The contributions (6.8) and (6.10) are evidently free from infrared divergence for $m=0$. By letting $m\to 0$ and using Eq. (B8), we also find that (6.15) is free from infrared divergence. Further, if we set not only $m=0$ but also $p=0$ without requiring q, p', and q' to vanish, then it can be verified that the scattering contributions (6.8), (6.10), and (6.15) completely vanish, which is in agreement with Adler's condition.[12]

Finally, it should be noted that, for $m=0$, the π-π scattering result no longer depends on the parameters c_1 and c_2, and thus it becomes model independent. Moreover, the renormalization constants $\delta(m^2)$ and λ, given by (6.7) and (7.3), vanish for $m=0$, and consequently only two counterterms survive in the renormalization procedure.

APPENDIX A: DERIVATION OF $\delta(0)$ TERMS

Since we are interested in the complications due to the presence of the field derivatives in the interaction, we shall drop nonderivative terms in the interaction energy density (3.1), which then reduces to

$$H_{\rm int}=-f^2\vec{\pi}^2\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+\tfrac{3}{2}f^4\vec{\pi}^4\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+2f^4\vec{\pi}^4\partial_0\vec{\pi}\cdot\partial_0\vec{\pi}+O(f^6), \tag{A1}$$

where noncovariant terms appear in fourth and higher orders. Our aim is to determine what modification should be carried out in (A1) to compensate for the neglect of the noncovariant term in the contraction

$$\partial_\mu\pi_i(x)^\bullet\,\partial'_\nu\pi_j(x')^\bullet=-i\delta_{ij}\partial_\mu\partial'_\nu\Delta_F(x-x')+i\delta_{ij}\delta_{\mu4}\delta_{\nu4}\delta(x-x') \tag{A2}$$

of the field derivatives.

Let us first consider the second-order contribution of the scattering operator due to the second-order interaction term in (A1). When the two field derivatives are contracted, the noncovariant part of the contraction (A2) gives rise to the contribution

$$-3f^2\int dx\,\vec{\pi}^2\,\delta(0),$$

which is equivalent to adding the term

$$-3if^2\vec{\pi}^2\,\delta(0) \tag{A3}$$

to the interaction energy density.

Similarly, let us consider the fourth-order scattering contribution due to the fourth-order covariant interaction term in (A1). If we carry out a contraction of the two field derivatives, the noncovariant part of the contraction gives rise to the contribution

$$\tfrac{9}{2}f^4\int dx\,\vec{\pi}^4\,\delta(0),$$

which is equivalent to an interaction energy density term

$$\tfrac{9}{2}if^4\vec{\pi}^4\,\delta(0). \tag{A4}$$

We now consider the fourth-order scattering contribution due to the second-order interaction term. It is easy to see that the inclusion of the $\delta(0)$ term (A3) in the interaction will reproduce the fourth-order contribution arising from the noncovariant part of the contraction when two field derivatives belonging to the same vertex are contracted. Therefore, we shall ignore such contractions, and examine only those contributions where field derivatives belonging to different vertices are contracted. For this purpose, the following two cases must be considered separately:

(a) When only one contraction between field derivatives belonging to different vertices is carried out, the contribution arising from the noncovariant part of the contraction is

$$2if^4\int dx\,\vec{\pi}^4\,\partial_0\vec{\pi}\cdot\partial_0\vec{\pi},$$

which is exactly canceled by the contribution due to the fourth-order noncovariant interaction term in (A1) without any contraction.

(b) When two contractions between field derivatives belonging to different vertices are carried out, the contribution involving the noncovariant parts of the contractions is

$$6f^4\int dx\,\vec{\pi}^4\{[\partial_0\partial'_0\Delta_F(x-x')]_{x'=x}+\tfrac{1}{2}\delta(0)\}.$$

On the other hand, the contribution due to the fourth-order noncovariant interaction term with the contraction of the two $\partial_0\vec{\pi}$'s is

$$-6f^4\int dx\,\vec{\pi}^4\{[\partial_0\partial'_0\Delta_F(x-x')]_{x'=x}+\delta(0)\}.$$

These two contributions do not cancel each other, but together yield

$$-3f^4\int dx\,\vec{\pi}^4\,\delta(0),$$

which is equivalent to the introduction of

$$-3if^4\bar{\pi}^4\delta(0) \tag{A5}$$

in the interaction energy density.

It follows from the above arguments that the noncovariant term in the contraction (A2) not only cancels the contributions due to the noncovariant part of the interaction (A1), but also generates, according to (A3), (A4), and (A5), the $\delta(0)$ interaction terms

$$-3if^2\bar{\pi}^2\delta(0)+\tfrac{9}{2}if^4\bar{\pi}^4\delta(0)-3if^4\bar{\pi}^4\delta(0)+O(f^6)\,. \tag{A6}$$

APPENDIX B: EVALUATION OF DIVERGENT INTEGRALS

All the divergent integrals in the TPE contribution (6.15) can be made convergent over the k space by means of the procedure given in Ref. 3, for with an appropriate choice of the auxiliary fields it is possible to convert

$$\overline{\Delta}_F(k)=\frac{1}{k^2+m^2}+\sum_\alpha \eta^{(\alpha)}\frac{1}{k^2+m^{(\alpha)2}}$$

into

$$\overline{\Delta}_F(k)=2\int_0^{\xi^2}dz_1\int_0^{\xi^2}dz_2\,\frac{1}{(k^2+m^2+z_1+z_2)^3}\,, \tag{B1}$$

where $\xi^2\to\infty$.

Let us consider the integral

$$I=\int dk\,\frac{1}{k^2+m^2}\,\overline{\Delta}_F(k+l)\,, \tag{B2}$$

where l is an arbitrary four-vector. By using (B1), and combining the denominators with the help of the identity

$$\frac{1}{ab^3}=3\int_0^1 du\,\frac{u^2}{[a+(b-a)u]^4}\,,$$

(B2) can be expressed as

$$I=6\int dk\int_0^{\xi^2}dz_1\int_0^{\xi^2}dz_2\int_0^1 du\,\frac{u^2}{[k^2+2uk\cdot l+ul^2+m^2+u(z_1+z_2)]^4}\,.$$

The integration over the k space can be carried out easily after shifting the origin as $k_\mu\to k_\mu-ul_\mu$, and thus

$$I=\pi^2 i\int_0^{\xi^2}dz_1\int_0^{\xi^2}dz_2\int_0^1 du\,\frac{u^2}{[u(1-u)l^2+m^2+u(z_1+z_2)]^2}\,.$$

Then, integrating over z_1 and z_2, and letting $\xi\to\infty$, we obtain

$$I=\pi^2 i[\ln(\xi^2/2m^2)+1-F(m^2,l^2)]\,, \tag{B3}$$

where

$$F(m^2,l^2)=\left(\frac{4m^2+l^2}{l^2}\right)^{1/2}\ln\frac{[(4m^2+l^2)/l^2]^{1/2}+1}{[(4m^2+l^2)/l^2]^{1/2}-1}\,. \tag{B4}$$

Similarly,

$$I_\mu=\int dk\,\frac{k_\mu}{k^2+m^2}\,\overline{\Delta}_F(k+l)$$

$$=-\tfrac{1}{2}\pi^2 i l_\mu[\ln(\xi^2/2m^2)+\tfrac{3}{2}-F(m^2,l^2)]\,, \tag{B5}$$

$$I_{\mu\nu}=\int dk\,\frac{k_\mu k_\nu}{k^2+m^2}\,\overline{\Delta}_F(k+l)$$

$$=\tfrac{1}{2}\pi^2 i\delta_{\mu\nu}[\xi^2\ln 2-(m^2+\tfrac{1}{6}l^2)\ln(\xi^2/2m^2)-\tfrac{4}{3}m^2-\tfrac{11}{36}l^2+\tfrac{1}{6}(4m^2+l^2)F(m^2,l^2)]$$

$$+\tfrac{1}{3}\pi^2 i l_\mu l_\nu[\ln(\xi^2/2m^2)+\tfrac{11}{6}+2m^2/l^2-(1+m^2/l^2)F(m^2,l^2)]\,, \tag{B6}$$

and

$$\bar{I} = \int dk\, \bar{\Delta}_F(k)$$

$$= \pi^2 i[2\xi^2 \ln 2 - m^2 \ln(\xi^2/2m^2) - m^2]. \tag{B7}$$

It is interesting to note that when $m \to 0$, (B3), (B5), (B6), and (B7) reduce to

$$\begin{aligned} I &= \pi^2 i[\ln(\xi^2/2l^2)+1], \\ I_\mu &= -\tfrac{1}{2}\pi^2 i l_\mu[\ln(\xi^2/2l^2)+\tfrac{3}{2}], \\ I_{\mu\nu} &= \tfrac{1}{2}\pi^2 i\delta_{\mu\nu}[\xi^2\ln 2 - \tfrac{1}{6}l^2\ln(\xi^2/2l^2) - \tfrac{11}{36}l^2] + \tfrac{1}{3}\pi^2 i l_\mu l_\nu[\ln(\xi^2/2l^2)+\tfrac{11}{6}], \\ \bar{I} &= 2\pi^2 i\xi^2 \ln 2, \end{aligned} \tag{B8}$$

which shows that they are free from infrared divergence.

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