Electron-positron annihilation and hadronic sources*

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The canonical leading light-cone contribution $\partial_\mu\Delta_+(x)p_\nu f(x\cdot p)+\cdots$ to the electroproduction $[\gamma(q)+h(p)\to$ anything; γ=photon, h=hadron] matrix element $\langle h(p)|J_\mu(x)J_\nu(0)|h(p)\rangle$ is such that $f(\lambda)$, by virtue of the $J_\mu J_\nu$ operator-product expansion (OPE), is analytic for small λ. The corresponding contribution $\partial_\mu\Delta_+(x)p_\nu\bar{f}(x\cdot p)+\cdots$ to the annihilation $[e^+e^-\to\gamma(q)\to h(p)$ + anything] matrix element $\langle 0|R(J_\mu(x)\hat{S}(p))R(J_\nu(x)\hat{S}^\dagger(-p))|0\rangle$, where $\hat{S}(p)$ is the Fourier transform of a source operator $S(y)$ for $h(p)$, is such that $\bar{f}(\lambda)$ can be singular for $\lambda\to 0$; $\bar{f}(\lambda)\sim\lambda^{-\sigma}$. We show how the multiple OPE's among the operators involved can determine the degree σ of this short-distance singularity provided long-distance effects are not important. We refer to σ as the "slant" of the matrix element and we explicitly calculate σ in terms of the minimal dimension d of the source $S(y)$. In the canonical case, we find $\sigma=d-\frac{1}{2}$ if h is a pseudoscalar particle and $\sigma=d-1$ if h is a spinor particle. These singularities imply that the scaling functions behave like ω^σ for $\omega\equiv q^2/2q\cdot p\to\infty$ and that the multiplicities behave like $(\sqrt{q^2})^{\sigma-3}$ for $q^2\to\infty$. These results provide handles on the heretofore elusive source dimensions d. For example, if logarithmic or greater multiplicities are observed ($\sigma\geq 3$) along with canonical scaling, it can be concluded that d cannot have its canonical *elementary* value ($d=\frac{5}{2}$ for spinors, $d=3$ for scalars) but rather $d\geq\frac{7}{2}$ (spinors) or $d\geq 4$ (scalars). These results can be readily generalized to noncanonical cases. For example, they lead to simple explanations of known results in renormalized perturbation theory (ϕ^3 theory, ϕ^4 theory, pseudoscalar-meson theory, quantum electrodynamics). Since λ (and ω) is dimensionless, the dimensional analysis involved in our treatment is on a different footing from that which determines the usual short-distance and light-cone singularities of current products. The slants are nevertheless simple functions of the field and source dimensions. Phenomenologically, the operatorial nature of our approach makes it easily extendable to the treatment of other inclusive lepton-hadron processes with one or more particles observed in the final state.

I. INTRODUCTION

The development of operator-product expansions (OPE's)[1-3] represents a significant advance in quantum field theory, enabling definite predictions to be made in many areas of particle physics: current algebra,[2] semihadronic scattering processes,[3] broken symmetry,[4] vector-meson dominance,[5] radiative corrections.[6,7] OPE's are in general relevant in any physical process involving current operators at large virtual mass, and so it is natural to apply them to inclusive processes involving currents when one or more hadrons are detected in the final state. Typical of processes of this type, which are of great current interest, are e^+e^- annihilation (via single-photon exchange, illustrated in Fig. 1),

$$\gamma(q)\to h(p)+\text{anything}, \tag{1.1}$$

and one-particle inclusive electroproduction,

$$\gamma(q)+h(p)\to h'(p')+\text{anything}, \tag{1.2}$$

where $h'(p')$ represents the detected final hadron. Indeed, many investigations of such processes have been made both within[8-12] and outside[13-16] the OPE framework.

One crucial difference exists between these processes and other cases of successful applications of the ideas of OPE's. In all the previous cases[7] it is sufficient to consider the operatorial nature of the product of *two* current operators at short distance (SD) or on the light cone (LC), because the important quantities can always be expressed as matrix elements of products of two currents. For illustration consider a ϕ^4 theory with scalar photons and hadrons. The electroproduction process

$$\gamma(q)+h(p)\to\text{anything} \tag{1.3}$$

is completely specified by the matrix element

$$W(\kappa,\nu)=\int dx\, e^{iq\cdot x}\langle p|[j(x),j(0)]|p\rangle\,, \tag{1.4}$$

where the variables are $\kappa\equiv q^2$, $\nu\equiv q\cdot p$, and $j(x)=:\phi^2(x):$. In the Bjorken limit $\nu\to\infty$ with $\omega\equiv-\kappa/2\nu$ fixed the process is described in terms of the LC expansion[17]

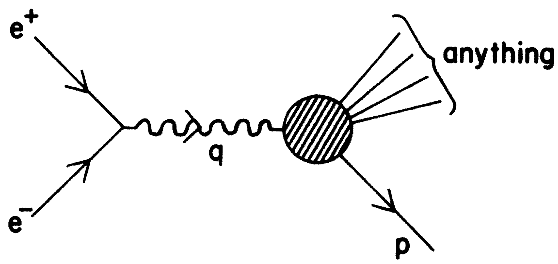

FIG. 1. Electron-positron annihilation via single-photon exchange into one hadron of momentum p plus "anything" X.

$$j(x)j(0)\to\Delta_+(x):\varphi(x)\phi(0): \\ =\Delta_+(x)\sum_n x^{\alpha_1}\cdots x^{\alpha_n}\mathcal{O}^{(n)}_{\alpha_1\cdots\alpha_n}(0), \quad (1.5)$$

with $\dim\phi=1$, $\dim j=2$, $\dim\mathcal{O}^{(n)}=n+2$, so that level $\mathcal{O}^{(n)}\equiv\dim\mathcal{O}^{(n)}-\text{spin}\mathcal{O}^{(n)}=2$.[18] Then $W(\kappa,\nu)$ scales in the Bjorken[19] limit,

$$\nu W(\kappa,\nu)\to F(\omega),\quad 0\leq\omega<1 \quad (1.6)$$

where

$$F(\omega)=\pi\int d\lambda\, e^{-i\omega\lambda}f(\lambda), \quad (1.7)$$

$$f(\lambda)\equiv\langle p|:\phi(x)\phi(0):|p\rangle|_{x^2=0}, \quad (1.8)$$

and $\lambda\equiv x\cdot p$, $\omega\equiv-\kappa/2\nu$. The amplitude in, for example, the annihilation process (1.1), while an appropriate discontinuity of $\langle p|T[j(x)j(0)]|p\rangle$, is, however, *not* directly expressible in any similar way. Instead,[8,11]

$$\overline{W}(\kappa,\nu)=\int dx\, e^{iq\cdot x}\langle 0|\mathcal{A}(x,0;p)|0\rangle, \quad (1.9)$$

with

$$\mathcal{A}(x,0;p)=\int dy\,dz\, e^{ip\cdot(y-z)}R(j(x)S(y))R(j(0)S^\dagger(z)), \quad (1.10)$$

where R denotes the retarded commutator, S is a *source* operator for h, and $\nu\equiv q\cdot p$. Process (1.2) is then described by the same operator $\mathcal{A}$ between $|p\rangle$. It has been convincingly argued[12,13,20] that the LC singularities of (1.10) are the same as that of the LC expansion (1.5), so that

$$\nu\overline{W}(\kappa,\nu)\to\overline{F}(\omega),\quad 1\leq\omega\leq\infty \quad (1.11)$$

where $\omega\equiv+\kappa/2\nu$ for the annihilation process,

$$\overline{F}(\omega)=\pi\int d\lambda\, e^{+i\omega\lambda}\overline{f}(\lambda), \quad (1.12)$$

$$\overline{f}(\lambda)\equiv\langle 0|\mathcal{B}(x,0;p)|0\rangle|_{x^2=0}, \quad (1.13)$$

$$\mathcal{B}(x,0;p)\equiv\int dy\,dz\, e^{ip\cdot(y-z)}R(\phi(x)S(y))R(\phi(0)S^\dagger(z)), \quad (1.14)$$

and $\mathcal{B}(x,0;p)$ is regular as $x^2\to0$. Interesting quantities like multiplicities in such processes turn out to be controlled by the behavior of $\overline{F}(\omega)$ as $\omega\to\infty$, or equivalently by the behavior of $\overline{f}(\lambda)$ as $\lambda\to0$. In this paper we shall investigate possible singularities in $\overline{f}(\lambda)$ as $\lambda\to0$ with the aid of OPE's.[21]

Model-independent information on such singularities has previously been lacking. This is in contrast with electroproduction where the coefficient function $f(x\cdot p)$ is presumably analytic[22] for small $x\cdot p$ and exhibits Regge behavior for large $x\cdot p$. Our purpose here is to attempt to supply this model-independent information.

We will show in this paper that, as long as infrared effects are not dominant, the *degree of the singularity is actually determined by the dimensions of the currents and hadronic sources h(p).* This means that in scale-invariant theories or limits, *both* the nature of the scaling [e.g., the power a in $\overline{W}(q^2,\nu)\to\nu^a\overline{F}(\omega)$] and the asymptotic behavior of the structure functions [e.g., the power b in $\overline{F}(\omega)\sim\omega^b$ for $\omega\to\infty$] are determined by dimensional analysis, even though ω is dimensionless. These two limits are quite distinct: The first is determined by the leading LC singularity (more precisely, by the smallest-level[18] fields in the current-current OPE) and is independent of the source dimension, whereas the second involves all nonleading LC singularities and depends on the (minimal) source dimension. Nevertheless, we shall see that the second limit is determined by the largest value of a new quantity which we shall call the "slant" and which we calculate as an explicit function of the current and source dimensions and the minimal levels. Our results thus make possible the experimental determination of the heretofore elusive dimensions of hadronic sources. They also lead to model-independent determinations of inclusive quantities such as multiplicities. The operator nature of our analysis makes it readily applicable to any inclusive process.

To make progress in this direction, we have to face the problem of the nature of the expansion of *quadrilocal* operators like (1.10) or (1.14). A simple example illustrates the subtlety involved. If one has the complete SD expansion of the operator product $\phi(x_1)\phi(x_2)\cdots\phi(x_n)$, it is possible to obtain equal-time commutators of any combination of ϕ and composite operators formed from ϕ's. Conversely it is possible to infer some properties of these operator products from the known commutation relations of the various operators.[23] The following relations hold in a model considered in Ref. 23:

$$[\dot\phi(x),\phi(0)]\,\delta(x^0)=-i\delta^4(x), \quad (1.15)$$

$$[j(x), \dot{\phi}(0)]\,\delta(x^0) = 0, \tag{1.16}$$

$$[J(x), \dot{\phi}(0)]\,\delta(x^0) \propto i\delta^4(x) j(0), \tag{1.17}$$

where $j(x), J(x)$ are composite operators formed by an appropriate limiting process from $\phi\phi$ and $\phi\phi\phi$, respectively. It is easy to see[23] that the OPE $j(x_1)\phi(x_2)\phi(x_3)$ must contain a term

$$j(x_1)\phi(x_2)\phi(x_3) \to \cdots + \frac{\ln(x_1 - x_2)^2}{(x_1 - x_3)\cdot(x_2 - x_3)}\, j\!\left(\frac{x_1 + x_2 + x_3}{3}\right) + \cdots \tag{1.18}$$

in order to satisfy (1.15)–(1.17). Thus there can be singularities in the variables $(x_i + x_j)^2$ which become felt only as $x_i \cdot x_j \to 0$ in addition to those at $x_i^2 = x_j^2 = 0$. The point is that while in products of two operators, SD singularities imply LC singularities, there *can* be singularities in products of more than two operators at SD which do not manifest themselves on the LC.

Motivated by this, we shall attempt to deduce the SD singularities of the above type in (1.10) or (1.14) from the knowledge of OPE's involving two operators. The Fourier transform would convert these singularities into singularities in λ in (1.14), even as $\mathcal{B}$ is nonsingular on the LC. In these considerations we shall continue to be guided by the lessons learned from OPE's of two operators: dimensional analysis, and that nature is well described by the LC expansions deduced canonically from the quark-gluon model[7] provided composite operators are consistently defined,[23,24] although the formalism is clearly still applicable outside the canonical framework.

We show that, in the absence of infrared-divergence effects, the degree of the λ singularity is given in terms of the quantity which we call the *slant,* and which can be computed explicitly in any given field theory, provided one is given as input the scale dimension of the hadronic source $S(y)$. In the scalar theory,

$$\bar{f}(\lambda) \underset{\lambda \to 0}{\sim} (\text{const})\,(\lambda)^{-\Sigma}, \tag{1.19}$$

$$\Sigma = D + \Delta - 3, \tag{1.20}$$

where $D = \dim S$, $\Delta = \dim \phi$ (= 1 canonically). We shall refer to singularities occurring only at SD and not on the LC as *slant singularities*. We see that such singularities are expected on the basis of dimensional analysis alone.[25]

In view of the role played in our analysis of hadronic sources and their dimensions, some remarks on their significance are in order. We recall that a local (scalar) field $\Phi(y)$ is a good interpolating field for the (scalar) hadron H if[26]

$$\langle 0|\Phi(y)|H(p)\rangle \neq 0, \tag{1.21}$$

where $|H(p)\rangle$ is an (in or out) state consisting of a single H particle of momentum p and mass $m = \sqrt{p^2}$. Then we can choose

$$\langle 0|\Phi(y)|H(p)\rangle = e^{ip\cdot y}. \tag{1.22}$$

An S-matrix element involving H as an external particle is then the residue of the pole at $p^2 = m^2$ of the Green's function

$$G = \int dy\, e^{ip\cdot y} \cdots \langle 0|T[\Phi(y)\cdots]|0\rangle. \tag{1.23}$$

In terms of the source operator

$$S(y) = (\Box + m^2)\Phi(y) \tag{1.24}$$

for H, the S-matrix element has the form

$$S = \int dy\, e^{ip\cdot y} \cdots \langle 0|T[S(y)\cdots]|0\rangle + (\text{equal-time commutators}). \tag{1.25}$$

There is, however, an enormous freedom in choosing $\Phi(y)$ and $S(y)$—any field satisfying (1.22) can be used in (1.25) without changing S.[26] An obvious example is

$$\Phi'(y) = \frac{1}{m^2}\Box\Phi(y). \tag{1.26}$$

These different fields give, of course, different off-shell extrapolations of (1.23) and (1.25).

Suppose now that the theory is (asymptotically) scale-invariant so that dimensions can be assigned to the local fields.[27] Given a good interpolating field ϕ of some dimension, it is easy to construct (infinitely many) other good interpolating fields of higher dimension, as, for example, in (1.26), and so no physical significance can be attached to source dimensions. There is, however, no method for arbitrarily *decreasing* the dimension of a local field. It therefore makes sense to speak of the interpolating field of *minimal dimension* among the class of good interpolating fields. (It is irrelevant to us if there is more than one such field.) Such fields always exist in perturbation theory and solvable models—they are the fields occurring in the usual Lagrangians. We shall assume the existence of minimal-dimensional interpolating fields and sources.[28] These minimal dimensions *are* physically significant. Physical consequences of values of these dimensions for the pseudoscalar octet have been discussed at length by Wilson.[2,29] Our result (1.19) is an explicit illustration of this significance. The amplitude (1.9) is on-shell with regard to H and is therefore independent of the source S used, but its behavior is, nevertheless, dependent upon the *minimum* possible source dimension D.

In spite of their importance, the values of these dimensions have been particularly elusive. Partial conservation of axial-vector current (PCAC) tells us that the dimension Δ of the pion field satisfies $1 \leqslant \Delta < 4$.[2] ($\Delta = 1$ in the σ model and $\Delta = 3$ in the gluon model.) The nucleon field dimension is even more elusive. The presumed bound-state nature of these hadrons contributes to the elusiveness. A major consequence of our analysis is the handle it puts on these dimensions. Inclusive semihadronic processes such as e^+e^- annihilation are seen to be excellent probes of minimal hadronic dimensions.

It must be stressed that our use of sources should not be confused with the dubious application[30] of LC expansions with sources to exclusive processes such as electromagnetic form factors or photoproduction amplitudes. Such applications, for example, suggest the form-factor behavior

$$G(\kappa) \equiv \int dx\, e^{iq\cdot x} \langle p | T[j(x)S(0)] | 0 \rangle \underset{\kappa\to\infty}{\sim} C\kappa^{D/2-2},$$

where $D = \dim S$. Using the field Φ, one, however, finds

$$G(\kappa, p^2) \equiv \int dx\, e^{iq\cdot x} \langle p | T[j(x)\Phi(0)] | 0 \rangle \sim \kappa^{D/2-3},$$

which has no pole in p^2. One finds in fact that $C = 0$ if the OPE for jS is obtained from that of $j\Phi$. Infinitely many nonleading LC singularities are in fact needed to build up the pole.[31]

The following sections are organized as follows. Notations and conventions are established in Sec. II with a review of relevant aspects of deep-inelastic scattering and annihilation. The single-particle spectrum in deep-inelastic annihilation is treated in Sec. III for the canonical scalar case. The SD behavior (1.19) is derived from the SDOPE's involved in (1.14) (including all nonleading terms) and discussed in detail in Sec. III A. The specific integrations involved are performed in the Appendix. Consequences of this result are discussed in Sec. III B, where a comparison with perturbation theory is made. These results are generalized to vector currents in Sec. III C.

The (physically interesting) canonical spinor case in which the currents are constructed out of spinor fields is treated in Sec. IV. The sources are either (pseudo) scalar (dimension D) or spinor (dimension d). In Sec. IV A the appropriate formalism is given and in Sec. IV B the SD singularities are derived (i.e., the slants are determined). The physical consequences are discussed in Sec. IV C. The asymptotic behavior of $\overline{F}_2(\omega)$ is deduced and compared with perturbation theory. The interesting implications for the multiplicities of the produced particles are also discussed. It is found that *if logarithmic or greater multiplicity is observed (along with canonical scaling and the absence of dominant infrared contributions), one must have* $d \geqslant \frac{7}{2}$ *and* $D \geqslant 4$ *so that canonical elementary sources* ($d = \frac{5}{2}$, $D = 3$) *would be ruled out.*

Generalizations to noncanonical theories are considered in Sec. V. Section VI contains general discussions, conclusions, and suggests extensions to other processes.

II. REVIEW OF DEEP-INELASTIC SCATTERING AND ANNIHILATION

To establish notation and conventions, and for subsequent reference, we review in this section the space-time approach to deep-inelastic scattering and annihilation. More details can be found in the original papers[3,20] and the recent extensive reviews.[7]

A. Canonical formalism

We begin with the canonical scalar ϕ^4 theory with scalar photons $\gamma(q)$ coupled to the scalar current $j = :\phi^2:$ and a scalar target $H(p)$ of mass $p^2 = M^2$. The amplitude

$$W(\kappa, \nu) = \int dx\, e^{iq\cdot x} \langle H(p) | [j(x), j(0)] | H(p) \rangle^{\text{conn}} \tag{2.1}$$

for the inclusive process $\gamma(q) + H(p) \to$ anything is the discontinuity in ν of the amplitude

$$M(\kappa, \nu) = -i \int dx\, e^{iq\cdot x} \langle H(p) | R(j(x)j(0)) | H(p) \rangle^{\text{conn}} \tag{2.2}$$

for the forward exclusive reaction $\gamma(q) + H(p) \to \gamma(q) + H(p)$ in the physical region $\kappa < 0$, $\nu > 0$.[32] The variables are $\kappa = q^2$, $\nu = q\cdot p$. W can also be expressed in terms of the operator (1.10) as[11]

$$W(\kappa, \nu) = \int dx\, e^{iq\cdot x} \langle 0 | \mathcal{A}(x, 0; -p) | 0 \rangle . \tag{2.3}$$

The canonical LCOPE (1.5) gives the scaling behavior (1.6)–(1.8) in the Bjorken limit $\nu \to \infty$ with $\omega = -\kappa/2\nu$ fixed in the physical region $0 \leqslant \omega \leqslant 1$. The scaling function $F(\omega)$ can also be expressed in terms of the operator (1.14) as

$$F(\omega) = \int d\lambda\, e^{-i\lambda\omega} \langle 0 | \mathcal{B}(x, 0; -p) | 0 \rangle^{\text{conn}}_{x^2=0}, \quad 0 \leqslant \omega \leqslant 1 . \tag{2.4}$$

The representation (1.7) is more informative than (2.4) in that it expresses $F(\omega)$ or the Fourier transform of a function $f(\lambda)$ which, by virtue of the LCOPE, is analytic, at least for sufficiently small λ.

With further plausible and, in the physical (vector photon) case, experimentally suggested,[33] assumptions, much more can be said about $F(\omega)$. For $\omega \to 0$, the Regge behavior[7]

$$F(\omega) \underset{\omega \to 0}{\sim} \omega^{1-\alpha} , \tag{2.5}$$

where α is the leading Regge-pole intercept at $t=0$, is expected to occur as a consequence of the Regge behavior $W(\kappa,\nu) \sim \beta(\kappa)\nu^{\alpha}$ of W in the Regge limit $\nu \to \infty$ with κ fixed, and for $\omega \to 1$, the threshold behavior[34]

$$F(\omega) \underset{\omega \to 1}{\sim} (1-\omega)^{p+1} , \tag{2.6}$$

where p specifies the decrease of the transition form factor $\langle H(p)|j(0)|H^*(p')\rangle \sim (p\cdot p')^{-p}$, is expected to occur as a consequence of the resonance dominance of W near threshold according to duality. Thus much of the structure of $F(\omega)$ is understandable on the basis of field-theoretic results like LCOPE's and phenomenological principles like Regge behavior and duality. In x space, Eqs. (1.7) and (2.5) give

$$f(\lambda) \underset{\lambda \to \infty}{\sim} \lambda^{\alpha} \tag{2.7}$$

and this, together with the analyticity of $f(\lambda)$ for small λ, specifies $f(\lambda)$ in the asymptotic domains of interest.

In the physically more interesting case with vector photons coupled to the conserved vector current $j_\mu = i : \phi^\dagger \overleftrightarrow{\partial}_\mu \phi :$, (2.1) becomes

$$W_{\mu\nu} = \int dx\, e^{iqx} \langle H(p)|[j_\mu(x), j_\mu(0)]|H(p)\rangle \tag{2.8}$$

$$= \left(p_\mu - \nu\frac{q_\mu}{\kappa}\right)\left(p_\nu - \nu\frac{q_\nu}{\kappa}\right) W_2(\kappa,\nu) + \left(\frac{q_\mu q_\nu}{\kappa} - g_{\mu\nu}\right) W_1(\kappa,\nu) . \tag{2.9}$$

The canonical LCOPE becomes

$$j^\mu(x) j^\nu(y) \to -\Delta_+(x-y)\overleftrightarrow{\partial}{}^\mu_x \overleftrightarrow{\partial}{}^\nu_y [\phi^\dagger(x)\phi(y) + \phi^\dagger(y)\phi(x)] , \tag{2.10}$$

and the consequent scaling laws are

$$\nu W_2(\kappa,\nu) \to F_2(\omega) , \quad W_1(\kappa,\nu) \to F_1(\omega) . \tag{2.11}$$

Because of the explicit structure of (2.10), one finds further that $F_1(\omega) \equiv 0$. This result, in disagreement with experiment,[33] is a consequence of the charged scalar constitution of j_μ. Equation (2.10) also gives

$$F_2(\omega) = -4\omega^2 F(\omega) , \tag{2.12}$$

with $F(\omega)$ given by (2.3) and (1.14).

We come finally to the most physically relevant case in which the conserved vector currents J_μ are constructed out of spinor fields ψ with electric charge matrix Q: $J_\mu = : \overline{\psi}\gamma_\mu Q \psi :$. The target can either be a scalar particle $H(p)$, with source $S(y)$, or a spinor particle $h(p)$, with source $s(y)$, although in the latter case a spin average will always be understood. The inclusive amplitude has the form (2.9) and satisfies the scaling laws (2.11). The canonical LCOPE is [3,7,35]

$$J_\mu(x) J_\nu(0) \to i g_{\mu\nu\alpha\beta} \partial^\alpha \Delta_+(x) \tfrac{1}{2} : [\overline{\psi}(x)\gamma^\beta Q^2 \psi(0) - \overline{\psi}(0)\gamma^\beta Q^2 \psi(x)] : , \tag{2.13}$$

with

$$g_{\mu\nu\alpha\beta} = g_{\mu\alpha} g_{\nu\beta} + g_{\nu\alpha} g_{\mu\beta} - g_{\mu\nu} g_{\alpha\beta} , \tag{2.14}$$

which implies the relation

$$2\omega F_1(\omega) = F_2(\omega) , \tag{2.15}$$

in good agreement with experiment.[33]

B. Generalizations

The only known field theories in which the purely canonical formalism outlined above is valid are free-field theories. It is therefore important to consider generalizations of this formalism. It might be hoped that a suitable generalization is provided by a formal use of scale-invariant, canonical field equations [e.g., $(\Box + M^2)\phi = g : \phi^3 :$ in ϕ^4 theory or $(i\partial\!\!\!/ - m)\psi = g B\!\!\!/ \psi$ in the vector-gluon model], and canonical equal-time field commutation relations. The LCOPE's so generated do have the forms (1.5), (2.10), and (2.13). This approach is unfortunately inconsistent—the assumptions are mutually inconsistent so that the currents cannot be simple Wick products and the bilocals in (1.5), (2.10), and (2.13) must have further LC singularities unless the theories are free ($g=0$). The only known generalization which is not inconsistent in this sense and which gives the scaling laws (2.11) are R-invariant[36] theories with reducible[37] scale invariance.[23] In the scalar (ϕ^4) version of such theories, the scalar current is given by

$$j(x) = \lim_{\xi \to 0} \frac{\phi(x+\xi)\phi(x) - \Delta_+(\xi)}{a + b\ln\xi^2} , \tag{2.16}$$

and the LCOPE has the form

$$j(x)j(0) \to \Delta_+(x) \sum_n x^{\alpha_1} \cdots x^{\alpha_n} \mathcal{O}_{\alpha_1 \cdots \alpha_n}(0) \equiv \Delta_+(x) j(x;0) . \tag{2.17}$$

The bilocal $j(x;0)$ is not given by a simple Wick product as in (1.5), but rather by[24]

$$j(x;0) = \lim_{\xi \to 0} \lim_{x^2 \to 0} \frac{\phi(x+\xi)\phi(0)}{\ln(x+\xi)^2} . \tag{2.18}$$

Another form of the bilocal is[24]

$$j(x;0) = \int d a \sigma(a) : \phi(ax)\phi(0): . \tag{2.19}$$

In this theory, the scaling function $F(\omega)$ is the Fourier transform of $j(x;0)$ and is therefore a simple generalization of (1.7). Similar remarks apply to the gluon version of this class of theories.

Further departures from the naive canonical framework involve greater singularities and consequently violate the scaling laws (2.11). It is nevertheless worthwhile to consider such departures both because we can then compare our general analysis with known results in perturbation theory and because the scaling laws might break down at higher energies. The simplest departure is renormalized perturbation theory itself, which has the great advantages of explicitness and known consistency. In finite orders of renormalized theories, the above canonical results are violated by logarithmic factors, $(\ln x^2)^k$ in position space and $(\ln\kappa)^k$ in momentum space, with the power κ increasing as the order of perturbation theory increases. This represents a breakdown of asymptotic scale invariance. The breakdown is, however, weak and the general results we obtain assuming canonical dimensions will be valid in these theories apart from similar logarithmic factors. In the superrenormalizable $\lambda\phi^3$ theory, the scaling laws (2.11) are satisfied in each order even though, because of the dimensional coupling constant λ, the theory is not scale-invariant. What happens is that, because of infrared effects, the effective coupling constant becomes the dimensionless quantity λ^2/M^2, at least for the ladder diagrams.

A further departure from the canonical framework is provided by scale-invariant theories with anomalous (i.e., noncanonical) dimensions. In such theories, the scaling laws (2.11) and canonical LCOPE's can be violated by powers of κ and x^2, respectively. It is even possible that the levels of the fields in the LCOPE's are not constant, so that expansions look like

$$j(x)j(0) \to \Delta_+(x) \sum_n (x^2)^{-a_n} x^{\alpha_1} \cdots x^{\alpha_n} \mathcal{O}^{(n)}_{\alpha_1 \cdots \alpha_n}(0) , \tag{2.20}$$

and bilocals cannot be defined. Examples of theories with anomalous dimensions are Gell-Mann–Low eigenvalue theories,[38] conformally invariant perturbation theories,[39] and theories of critical indices in statistical mechanics.[40] The application of our results to such theories will be discussed in Sec. V.

C. Annihilation

The simplest inclusive semileptonic process is electron-positron annihilation into hadrons with no hadrons observed: $\gamma(q) \to$ anything. The total cross section is[7,41]

$$\sigma(\kappa) = -\frac{16}{3}\frac{\pi^2\alpha^2}{\kappa^2}\int dx\, e^{iqx}\langle 0| J_\mu(x) J^\mu(0)|0\rangle , \tag{2.21}$$

with α the fine-structure constant. The behavior for $\kappa \to \infty$ is given by the c-number piece of the $J_\mu(x)J_\nu(0)$ (LC or SD—they are the same here) OPE. The canonical result is

$$J_\mu(x)J_\nu(0) \to A(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\frac{1}{(x^2 - i\epsilon x_0)^2} I + \text{operators} , \tag{2.22}$$

where I is the unit operator, and the constant A is model-dependent, and is not of interest to us. Equation (2.22) implies the scaling behavior

$$\sigma(\kappa) \to (-32\pi^5\alpha^2 A)\frac{1}{\kappa} . \tag{2.23}$$

Since J_μ and I must have canonical dimensions, (2.23) follows from scale invariance alone. Logarithmic and/or power deviations can occur in the non-scale-invariant theories mentioned in subsection B. Present experiments are consistent with (2.23), but, because of low energies and large errors, are not compelling.

III. CANONICAL SCALAR CASE

The one-particle spectrum in deep-inelastic annihilation [process (1.1)] will be considered in this section under the simplifying assumptions that only scalar fields are involved and only canonical LC and SD singularities appear. The resulting simplifications are notational only, and will be removed in Secs. IV and V. In subsections A and B, we take the photon to be scalar for further simplicity. The simple extension to vector photons is given in subsection C.

A. Short-distance singularities

The formalism relevant for the canonical scalar case is given in Eqs. (1.9)–(1.14). Our purpose here is to determine the c-number SD singularities of

$$\mathcal{B}(x, 0; p) = \int dy\, dz\, e^{ip\cdot(y-z)} R(\phi(x)S(y)) R(\phi(0)S^\dagger(z)). \tag{3.1}$$

These will give us the singularities of[42]

$$\bar{f}(x\cdot p, x^2) \equiv \langle 0|\mathcal{B}(x, 0; p)|0\rangle , \tag{3.2}$$

which, for $x^2 = 0$, will give us the singularities of (1.13) for $\lambda \to 0$.

To determine the small-λ behavior of $\bar{f}(\lambda)$, we use the SDOPE

$$R(\phi(x)S(y)) \to \sum_l (x-y)_R^{d_l - D - \Delta} \mathcal{O}^{(l)}\left(\frac{x+y}{2}\right). \tag{3.3}$$

Here we have written

$$\Delta = \dim\phi, \quad D = \dim S, \quad d_l = \dim \mathcal{O}^{(l)}, \tag{3.4}$$

and R indicates the retarded $i\epsilon$ prescription. For notational simplicity, we have not indicated the Lorentz index structure in (3.3) (e.g., $x^{-2}\mathcal{O}$ may be $x^{-4}x^\alpha x^\beta \mathcal{O}_{\alpha\beta}$). All nonleading terms are included in (3.3), but terms involving mass factors have been omitted since they will be seen to be irrelevant. Similarly,

$$R(\phi(0)S^\dagger(z)) \to \sum_m (-z)^{d_m - D - \Delta} \mathcal{O}^{\dagger(m)}(\tfrac{1}{2}z), \tag{3.5}$$

and

$$\mathcal{O}^{(l)}\left(\frac{x+y}{2}\right)\mathcal{O}^{\dagger(m)}(\tfrac{1}{2}z) \to \sum_n \left(\frac{x+y-z}{2}\right)^{d_{lmn} - d_l - d_m} \times \mathcal{P}^{(lmn)}(0), \tag{3.6}$$

where $d_{lmn} = \dim \mathcal{P}^{(lmn)}$. We shall substitute (3.3), (3.5), (3.6), into (3.1) to determine the singularities in (3.2). This procedure is certainly incorrect if only a finite number of terms were kept in (3.3) or (3.5) since less singular terms are accompanied by higher-dimensional operators which yield more singular contributions to (3.6).[43] We assume, however, as is true in perturbation theory, that the procedure is correct if all terms are kept.

For the vacuum expectation value (3.2), only the (dimension zero) unit operators $\mathcal{P}^{(lmn)} = c_{lmn} I$ in (3.6) contribute and we obtain

$$\bar{f}(x\cdot p, x^2) \to \int dy\, dz\, e^{ip\cdot(y-z)} \sum_{lmn} (x-y)_R^{d_l - D - \Delta}(-z)_R^{d_m - D - \Delta}\left(\frac{x+y-z}{2}\right)_W^{-d_l - d_m} c_{lmn}, \tag{3.7}$$

where W indicates the Wightman $i\epsilon$ prescription. The terms in (3.7) are written somewhat symbolically in that (contracted) Lorentz indices have not been indicated (e.g., $(x-y)^2 z^2$ could be $[(x-y)\cdot z]^2$). A sum over all possible such contractions with arbitrary coefficients consistent with the indicated dimensions should be understood in (3.7).

The integrations in (3.7) are explicitly performed in the Appendix. The result is

$$\bar{f}(x\cdot p, x^2) \to \lambda^{-\Sigma}(a + b\ln x^2 + c\ln\lambda) + O(x^2) + O(1/x^2), \tag{3.8}$$

where

$$\Sigma = D + \Delta - 3. \tag{3.9}$$

We refer to Σ as the (vacuum, or level zero) slant of (3.1). We have assumed in (3.8) that $\Sigma \geq 0$.[44] Here a, b, and c are constants contributed to by each term in (3.7) with $c_{lmn} \neq 0$ for each arrangement of the Lorentz contractions. Any of these constants may vanish but we assume, as is true in perturbation theory, that they are not infinite. If $a = b = c = 0$, then the next-leading term, with slant $\Sigma - 1$, must be kept. The $\ln x^2$ term in (3.8) is signaling the inconsistency of the naive canonical framework mentioned in Sec. II B. It would not occur if the current were properly defined, as, for example, in (2.16). The limit in (2.18) or the line integral in (2.19) explicitly removes this term. We will put $b = 0$, reserving a more careful treatment for Sec. V. For simplicity we also put $c = 0$ since the $\ln\lambda$ factor does not affect our results.

The $O(1/x^2)$ terms in (3.8) occur if $\Sigma > 0$ and have the form $(x^2)^{-\Sigma+N}$, $N = 0, 1, 2, \ldots, \Sigma - 1$. The scaling assumptions (1.11) require that the coefficients of the light-cone singularities vanish. Since the space-time limit of interest is $x^2 \to 0$ first and then $\lambda \to 0$, the $O(x^2)$ terms in (3.8) do not contribute. To avoid possible confusion, however, we note that their associated slants may be greater than (3.9). The $O(x^2)$ term, for example, has an associated slant of $\Sigma + 2$. This is important if one is interested in the differentiation $\Box_x \bar{f}(x\cdot p)$. It does *not* follow from (3.8) that $\Box_x \bar{f}(x\cdot p) \to \lambda^{-\Sigma-2}$. The $O(x^2)$ contribution to (3.8) also gives a term $\lambda^{-\Sigma-2}$ to $\Box_x \bar{f}$ [since $\Box_x(\lambda^{-\Sigma-2}x^2) \sim 8\lambda^{-\Sigma-2} + \cdots$] and these terms cancel, leaving $\lambda^{-\Sigma-1}$.

With these understandings, we obtain from (3.8) the important result

$$\bar{f}(\lambda) \underset{\lambda\to 0}{\sim} a\lambda^{-\Sigma}. \tag{3.10}$$

We have thus determined the maximum possible SD singularity. It is specified by the combination (3.9) of current and source dimensions which constitute the slant. It is important to emphasize that, because of the possibility that $a = 0$, Σ only gives the maximal possible singularity. This is well illustrated by considering the use in the above analysis of a source $S'(y)$ of dimension D' greater than D. We would then obtain $\bar{f} \to a'\lambda^{-\Sigma'}$ with $\Sigma' = D' + \Delta - 3 > \Sigma$. Since we are free to use any source for H [i.e., $\bar{f}(\lambda)$, being on-shell with regard to H, must be independent of the chosen source], consistency requires that $a' = 0$. *The first possibly nonvanishing coefficient a occurs for the source* *of minimal dimension D.* A specific example is provided by the source $S'(y)=\Box_y S(y)$ of dimension $D'=D+2$. Since the OPE's (3.3)–(3.6) commute with differentiations,[1] the effect of using S' rather than S is to replace (3.7) by the same expression with $\Box_y\Box_z$ before the summation sign. Integration by parts reveals that $\bar{f}$ is changed by the factor $(p')^2(p')^2=1$; i.e., $\bar{f}$ is unchanged (as it must be). Thus we again obtain (3.10) (as we must), and the maximum *possible* leading term $a'\lambda^{-\Sigma'}$ is not present because $a'=0$. There is no such reason for the coefficient a associated with the source S of minimal dimension D to vanish and, in fact, in perturbation theory it does *not* vanish.

If we worked with hadronic fields $\Phi(y)$ rather than sources $S(y)=(\Box+1)\Phi(y)$, then we would only be interested in slant singularities associated with double poles $(p^2-1)^{-2}$. The use of all the nonleading terms in the SD expansions enables such poles to occur. [This is quite distinct from the dubious use (cf. Sec. I) of leading LC singularities of source-current products to determine on-shell quantities like electromagnetic form factors or photoproduction amplitudes. A finite number of singularities cannot give rise to the necessary pole in the field-current amplitude.] If such poles did not occur with slant Σ,[45] then a would vanish in (3.10). Again, Σ is the maximal possible slant.

We should remark that we have obtained Eq. (3.10) by taking the limits in (3.1) in a definite order. Taking the limits in other orders does not increase the slant.

The singularity (3.10) results from the short-distance singularities in (3.3), (3.5), (3.6) although our inclusion of all terms makes the multiple expansion in (3.7) valid away from the short-distance region. The slant singularity would still be the same even if the region of integration in (3.7) were restricted to only some compact neighborhood of the origin. This would mean that we need only assume a finite radius of convergence in the short-distance expansions (3.5) and (3.6). For sufficiently small values of the exponents in (3.7), however, the integral becomes infrared (large distance) singular. Such infrared contributions can lead to larger slants than (3.10). We must ignore such contributions since they come from regions of integration in (3.1) where (3.3), (3.5), (3.6) are not necessarily valid. Infrared effects are never present canonically and may lead to violations of the scaling laws. The presence of such effects would invalidate our conclusions. They are, however, present in perturbation theory and our methods are easily generalized to include their effect. Discussion will be given in a forthcoming publication.[45]

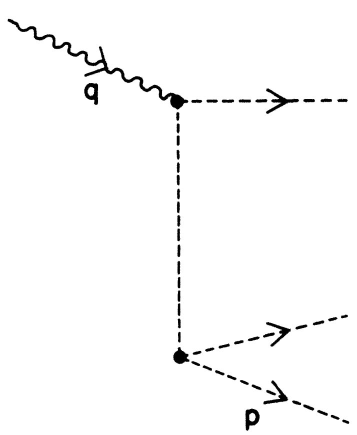

FIG. 2. Lowest-nontrivial-order contribution to $e^+e^- \to h(p)$ + anything in φ^3 theory with scalar photons.

B. Consequences

Using (3.10) in (1.12) gives the desired asymptotic behavior

$$\bar{F}(\omega) \underset{\omega\to\infty}{\sim} \text{const}\times\omega^{\Sigma-1}, \quad \Sigma=D+\Delta-3. \quad (3.11)$$

This is the promised result that this behavior is determined by dimensional analysis.

Our result (3.11) can be nicely illustrated in (scalar) perturbation theories. The simplest example is massive $\lambda\phi^3$ theory. This theory is not (asymptotically) scale-invariant since $\dim\lambda=1$, but for the ladder diagrams infrared effects render the effective coupling constant to be the dimensionless quantity λ/m, where m is the mass of the ϕ particle. Our results are therefore applicable. The source is $S=:\phi^2:$, so that

$$D=2,\ \Delta=1,\ \Sigma=0\ (\phi^3 \text{ theory}) \quad (3.12)$$

and (3.11) becomes

$$\bar{F}(\omega) \underset{\omega\to\infty}{\sim} \text{const}\times\omega^{-1}\ (\phi^3 \text{ theory}). \quad (3.13)$$

The lowest-order contribution to $\bar{F}(\omega)$, corresponding to the Feynman diagram of Fig. 2, has been evaluated in Ref. 14 and the result is in exact agreement with (3.13). Also, the asymptotic behavior of $\bar{F}(\omega)$ in the ladder model has been evaluated in Ref. 67, and the result is again (3.13).

We consider next ϕ^4 theory. This theory is formally scale-invariant but the perturbation expansion breaks the invariance by logarithmic factors. Our results are therefore only valid up to such factors. The source is $S=:\phi^3:$ so that

$$D=3,\ \Delta=1,\ \Sigma=+1\ (\phi^4 \text{ theory}) \quad (3.14)$$

and (3.11) becomes

$$\bar{F}(\omega) \underset{\omega\to\infty}{\sim} \text{const}\ (\phi^4 \text{ theory}). \quad (3.15)$$

The lowest-order contribution to $\bar{F}(\omega)$, corresponding to the Feynman diagram of Fig. 3, has been evaluated in Ref. 14 and the result agrees with (3.15).

C. Vector currents

The generalization of the above analysis to the case of conserved vector currents $j_\mu = i:\phi^\dagger \overleftrightarrow{\partial}_\mu \phi:$ is straightforward. The amplitude $\bar{W}_{\mu\nu}$ can be decomposed as in (2.9) into two scalar amplitudes $\bar{W}_{1,2}(\kappa,\nu)$ which yield two scaling functions $\bar{F}_{1,2}(\omega)$ as in (2.11). The relevant LCOPE is given in Eq. (2.10). As in electroproduction, we find $\bar{F}_1(\omega)\equiv 0$ and

$$\bar{F}_2(\omega) = -4\omega^2 \bar{F}(\omega), \tag{3.16}$$

with $\bar{F}(\omega)$ given by (1.12)–(1.14). Thus

$$\bar{F}_2(\omega) \underset{\omega\to\infty}{\sim} \text{const}\times\omega^{\Sigma+1} \tag{3.17}$$

IV. CANONICAL SPINOR CASE

In this section, we continue to work within the context of the canonical framework but we use vector currents constructed out of charged spinor fields, e.g., quarks. Such a construction is indicated by the electroproduction experiments.[33] We allow the observed final hadron to be either (pseudo) scalar (e.g., a pion) or spinor (e.g., a nucleon).

A. Formalism

The electric current is now $J_\mu = :\bar{\psi}\gamma_\mu Q\psi:$ and the final hadron is either a spinor particle $h(p)$ with source $s_k(y)$ (see Ref. 46) or a scalar particle $H(p)$ with source $S(y)$. An average over the spin of h will always be taken. The dimensions will be written[47]

$$\delta = \dim\psi, \quad d = \dim s, \quad D = \dim S. \tag{4.1}$$

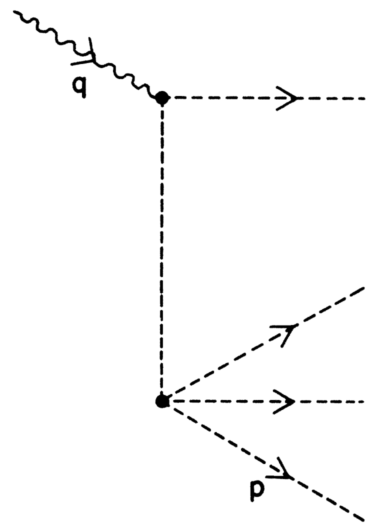

FIG. 3. Lowest-nontrivial-order contribution to $e^+e^- \to h(p)$ + anything in ϕ^4 theory with scalar photons.

The amplitude is decomposed as usual:

$$\bar{W}_{\mu\nu} = \left(p_\mu - \frac{\nu q_\mu}{\kappa}\right)\left(p_\nu - \frac{\nu q_\nu}{\kappa}\right)\bar{W}_2(\kappa,\nu) + \left(\frac{q_\mu q_\nu}{\kappa} - g_{\mu\nu}\right)\bar{W}_1(\kappa,\nu), \tag{4.2}$$

and the scaling laws are

$$\nu\bar{W}_2(\kappa,\nu)\to\bar{F}_2(\omega), \quad \bar{W}_1(\kappa,\nu)\to\bar{F}_1(\omega), \quad 1\le\omega\le\infty \tag{4.3}$$

with

$$2\omega\bar{F}_1(\omega) = \bar{F}_2(\omega). \tag{4.4}$$

For the spinor source, we have the representation

$$\bar{W}_{\mu\nu} = \int dx\, e^{iq\cdot x}\langle 0|\mathfrak{A}_{\mu\nu}(x,0;p)|0\rangle, \tag{4.5}$$

where[48]

$$\mathfrak{A}_{\mu\nu}(x,0;p) = \int dy\,dz\, e^{ip\cdot(y-z)} R(J_\mu(x)s_k(y))R(J_\nu(0)\bar{s}_l(z))(\not{p}-1)_{kl}. \tag{4.6}$$

Using the LCOPE (2.13) in the scaling limit gives rise to the operator

$$\mathfrak{B}_\mu(x,0;p') = \int dy\,dz\, e^{ip\cdot(y-z)}[R(\bar{\psi}_i(x)s_k(y))R(\psi_j(0)\bar{s}_l(z)) - (x\leftrightarrow 0)](\gamma_\mu Q^2)_{ij}(\not{p}-1)_{kl}. \tag{4.7}$$

We define

$$\langle 0|\mathfrak{B}_\mu(x,0;p)|0\rangle_{x^2=0} = p_\mu\bar{f}(x\cdot p) + x_\mu\bar{g}(x\cdot p) \tag{4.8}$$

and

$$\bar{F}_T(\omega) = \pi\int d\lambda\, e^{i\omega\lambda}\bar{f}(\lambda), \quad 1\le\omega\le\infty; \tag{4.9}$$

the scaling function of interest is given by[49]

$$\bar{F}_2(\omega) = \omega\bar{F}_T(\omega). \tag{4.10}$$

For the scalar source, everything is the same with $\bar{s}(z)(\not{p}-1)s(y)$ replaced by $S^\dagger(z)S(y)$. We shall denote the double-helicity-flip scaling function in this case by $\bar{\mathfrak{F}}_2(\omega)$.

B. Short-distance singularities

To find the behavior of $\bar{f}(\lambda)$ for $\lambda \to 0$, we must determine the slant singularities associated with the operator (4.7). To accomplish this, we need only repeat the procedure used in Sec. III A for the scalar case, being careful with the spinor indices. With the spinor source, the relevant OPE's are

$$R(\bar{\psi}_i(x)s_k(y)) \to \sum_l (x-y)_R{}^{d_l-d-\delta}\mathcal{O}_{ik}^{(l)}\left(\frac{x+y}{2}\right), \tag{4.11}$$

$$R(\psi_j(0)\bar{s}_l(z)) \to \sum_m (-z)^{d_m-d-\delta}\bar{\mathcal{O}}_{jl}^{(m)}(\tfrac{1}{2}z), \tag{4.12}$$

and

$$\mathcal{O}_{ik}^{(l)}\left(\frac{x+y}{2}\right)\bar{\mathcal{O}}_{jl}^{(m)}(\tfrac{1}{2}z) \to \sum_n \left(\frac{x+y-z}{2}\right)^{d_{lmn}-d_l-d_m}\mathcal{P}_{iklj}^{(lmn)}(0). \tag{4.13}$$

As in Sec. III A, these expressions are somewhat symbolic in that vector and spinor contractions have not been explicitly indicated [e.g., $z^2\mathcal{O}_{jl}$ may be $(z\cdot\gamma_{jl})z^\alpha\mathcal{O}_\alpha$].

For (4.8), only the unit operator $\mathcal{P}_{iklj}^{(lmn)} = C_{iklj}^{lmn}I$ of zero dimension contributes. When (4.11)–(4.13) are substituted into (4.7) and the spinor contractions are made, the largest slant is seen to come from terms of the form

$$\partial_\mu p\cdot\partial(x+y-z)^{-d_l-d_m+2}\,. \tag{4.14}$$

Using the scalar case result (3.10), we obtain a contribution to (4.8) of the form

$$\partial_\mu p\cdot\partial(\lambda)^{3-d-\delta+1} \sim p_\mu(\lambda)^{2-d-\delta}\,. \tag{4.15}$$

Thus

$$\bar{f}(\lambda) \underset{\lambda\to 0}{\sim} a\lambda^{-\sigma}, \tag{4.16}$$

where

$$\sigma = d+\delta-2 \tag{4.17}$$

is the vacuum slant of (4.7). No rearrangement of the vector or spinor contractions can give rise to a greater singularity than (4.16). We note, however, (see below) that if infrared effects are important, a larger singularity can be present.

All of the discussion in Sec. III A concerning the scalar case result (3.10) applies to (4.16) as well. We repeat only the important remark that (4.16) gives the maximum *possible* SD singularity and this occurs for the source s of minimum dimension d. Even for the source of minimum dimension the constant a may vanish, although it does not vanish in perturbation theory.[50]

For the scalar source S, the largest slant in $\mathcal{B}_\mu$ comes from terms of the form

$$\partial_\mu(x+y-z)^{-d_l-d_m+1}\,. \tag{4.18}$$

These contribute singularities of the form

$$\partial_\mu(\lambda)^{3-D-\delta+1/2} \sim p_\mu(\lambda)^{5/2-D-\delta} \tag{4.19}$$

to $\langle 0|\mathcal{B}_\mu|0\rangle_{x^2=0}$. Thus

$$\bar{f}(\lambda) \underset{\lambda\to 0}{\sim} A\lambda^{-\Sigma}, \tag{4.20}$$

where

$$\Sigma = D+\delta-\tfrac{5}{2} \tag{4.21}$$

is the (vacuum) slant of $\mathcal{B}_\mu$ for a scalar source of dimension D.

C. Consequences

The asymptotic behavior of the scaling function $\bar{F}_2(\omega)$ is determined by using (4.16) in (4.9) and (4.10). We obtain

$$\bar{F}_2(\omega) \underset{\omega\to\infty}{\sim} \text{const}\times\omega^\sigma, \quad \sigma = d+\delta-2\,. \tag{4.22}$$

Thus the maximum allowed rate of growth is given by the slant. This rate is seen to increase with the source dimension.

Our result (4.22) can be verified in perturbation theory. Scale invariance in massive quantum electrodynamics (QED) ($\mathcal{H}_I = e:\bar{\psi}\not{B}\psi:$) and pseudoscalar-meson theory (PS) ($\mathcal{H}_I = g:\bar{\psi}\gamma_5\psi\phi:$) is only logarithmically broken in finite order of perturbation theory and so our result is applicable.[51] The spinor sources are

$$s = :\not{B}\psi: \quad \text{(QED)} \tag{4.23}$$

and

$$s = :\gamma_5\psi\phi: \quad \text{(PS)} \tag{4.24}$$

so that

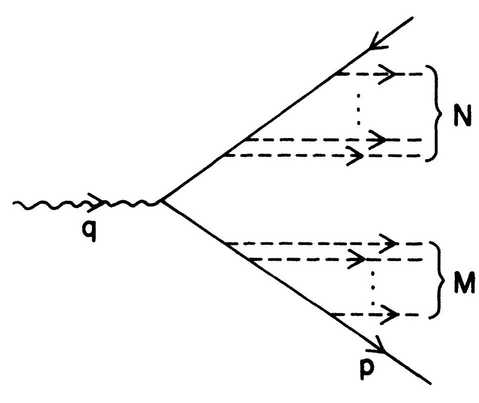

FIG. 4. Class of diagrams contributing to $e^+e^- \to h(p)$ + anything in quantum electrodynamics or pseudoscalar theory. Dashed lines may be vector or pseudoscalar. Solid lines are spinor.

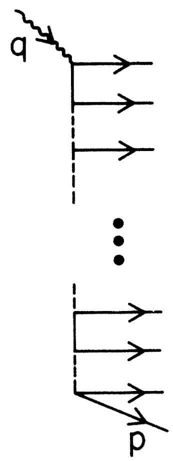

FIG. 5. Diffractive contribution to $e^+e^- \to h(p)$ + anything in quantum electrodynamics or pseudoscalar theory. Dashed lines may be vector or pseudoscalar. Solid lines are spinor.

$$d=\tfrac{5}{2},\ \delta=\tfrac{3}{2},\ \sigma=2 \quad \text{(QED, PS)} \tag{4.25}$$

and (4.22) becomes

$$\overline{F}_2(\omega) \underset{\omega\to\infty}{\sim} \text{const}\times\omega^2 \quad \text{(QED, PS)}. \tag{4.26}$$

The asymptotic behavior of the diagrams of Fig. 4 have been explicitly evaluated in both QED and PS theory.[15,16] The result is

$$\nu \overline{W}_2^{(N,M)}(\kappa,\nu) \to (\ln\kappa)^{N+M}\overline{F}_2^{(N,M)}(\omega), \tag{4.27}$$

with

$$\overline{F}_2^{(N,M)}(\omega) \to \text{const}\times(\ln\omega)^{M-1}\omega^2. \tag{4.28}$$

Thus, apart from the expected logarithmic factor, (4.28) is in agreement with our result (4.26). The great amount of work involved in deriving (4.28) should be compared with the relative ease with which we derived (4.26).[52]

In each order of perturbation theory, the (diffractive) diagrams of Fig. 5 are as important as the (rainbow) diagrams of Fig. 4. The asymptotic behavior of these diagrams has also been explicitly evaluated in both QED and PS theory.[16] For PS theory, the results are again in agreement with (4.26). For QED, however, the dominant asymptotic behavior $[\overline{F}_2(\omega)\sim\omega^3]$ is an (noncanonical) infrared effect and so is not given by (4.26). This behavior actually follows from the generalization of our analysis to include infrared effects, as will be discussed in a subsequent paper.[53]

One of the most important consequences of our result (4.22) is that it provides a heretofore lacking (model-independent) handle on the (minimal) source dimension d if long-distance effects are not dominant. Thus, given canonical scaling ($\delta=\tfrac{3}{2}$), from an experimental observation of the form $\overline{F}_2(\omega)\to\omega^b$, one can immediately conclude that the minimal source dimension d satisfies $d \geq b+\tfrac{1}{2}$. This is the most reliable method we know of for learning about d. From a theoretical point of view, (4.22) can be used to test field-theoretic models. Such a model must be consistent with this bound.

One might worry that nonleading contributions to $\nu\overline{W}_2$ might make it difficult to experimentally determine b. To investigate this, we include the first nonleading LC contribution in the scalar current-current expansion:

$$j(x)j(0) \to \Delta_+(x) : \phi(x)\phi(0): + (\text{const})m^2 \ln x^2 : \phi(x)\phi(0): + \text{const}\times(\ln x^2) : j(x)j(0):. \tag{4.29}$$

For large ν and ω, we obtain

$$W(\kappa,\nu) \to \text{const}\times\nu^{-1}\omega^{D+\Delta-4} + \text{const}\times\nu^{-2}m^2\omega^{D+\Delta-4} + \text{const}\times\nu^{-2}\omega^{D+2\Delta-5}. \tag{4.30}$$

With $\Delta=1$, this becomes

$$W(\kappa,\nu) \to \text{const}\times\nu^{-1}\omega^{D-3} + \text{const}\times\nu^{-2}m^2\omega^{D-3} + \text{const}\times\nu^{-2}\omega^{D-3}. \tag{4.31}$$

The nonleading contributions are seen to fall faster with ν and not more slowly with ω. This argument can be extended to general nonleading singularities. Any such singularity is accompanied by operators of proportionately higher dimension and these effects precisely cancel. We conclude that nonleading LC contributions are never more important than the leading LC contributions.

It similarly follows from (4.20) that the asymptotic behavior of the scaling function for the case of the scalar source is

$$\overline{\mathfrak{F}}_2(\omega) \underset{\omega\to\infty}{\sim} \text{const}\times\omega^{\Sigma}, \quad \Sigma = D+\delta-\tfrac{5}{2}. \tag{4.32}$$

This result provides a handle on the minimal dimension of the produced scalar hadron (e.g., the pion). Thus, with canonical scaling ($\delta=\tfrac{3}{2}$), an observation $\overline{\mathfrak{F}}_2(\omega)\to\omega^B$ implies that the minimal source dimension D satisfies $D \geq B+1$.

The bound (4.22) is easily converted into a bound on the (scaling contribution to the) multiplicity of hadron h in electron-positron annihilation. This is significant because of the greater ease with which multiplicities can be measured compared to asymptotic behaviors such as (4.22). Assuming that the total annihilation cross section scales [$\sigma(\kappa)\to\text{const}\,\kappa^{-1}$. See Sec. IIC], the spinor particle multiplicity $N_h(\kappa)$ satisfies

$$N_h(\kappa) \sim \int_1^{(1/2)\sqrt{\kappa}} \frac{d\omega}{\omega^4}\overline{F}_2(\omega) \sim \begin{cases} (\sqrt{\kappa})^{d+\delta-5}, & d+\delta-5>0 \\ \ln\kappa, & d+\delta-5=0 \\ \text{const}, & d+\delta-5<0. \end{cases} \tag{4.33}$$

Thus the maximal possible multiplicity is determined by our dimensional analysis. Note that, according to (4.31) (and its generalizations), the contributions of nonleading LC singularities to the multiplicity are always dominated by the leading LC contribution (4.33). Note also that, because energy-momentum conservation requires that $N(\kappa) < \sqrt{\kappa}$, if d is such that $d+\delta-5>1$, the constant a in (4.16) must vanish.

The result (4.33) is most interesting because of the relative smallness of the power $d+\delta-5$. The point is that it is not possible to obtain the expected logarithmic (or greater) multiplicity in the canonical field ($\delta=\frac{3}{2}$) canonical elementary source ($d=\frac{3}{2}+1=\frac{5}{2}$) case where $d+\delta-5=-1$. In the canonical field ($\delta=\frac{3}{2}$) case, $d=\frac{7}{2}$ is needed to get the expected logarithmic multiplicity, and so *if logarithmic or greater multiplicity is observed (along with canonical scaling), it can be concluded that $d \geq \frac{7}{2}$ so that a canonical elementary source ($d=\frac{5}{2}$) would be ruled out and the existence of bound states of higher dimension would be established.* In other words, all that is needed to establish the composite[54] (i.e., nonelementary) nature of the observed (spinor) hadrons is to observe logarithmic or greater multiplicity together with canonical scaling. More generally, an observed multiplicity $(\sqrt{\kappa})^M$ with canonical scaling implies that the minimal source dimension satisfied $d \geq M+\frac{7}{2}$.

The above connection between source dimension and multiplicity is perhaps somewhat surprising. One might intuitively expect that it is easiest to produce low-dimensional particles. A physical way of understanding our result is the following. Given that strongly bound states of high dimensions exist, at a given energy κ, these states will be most likely produced since it requires much energy to break apart the bound state into its constituents.

The corresponding results for the scalar source S multiplicity are

$$N_H(\kappa) \sim (\sqrt{\kappa}\,)^{D+\delta-11/2}. \tag{4.34}$$

The implications of this result with regard to produced scalar or pseudoscalar particles are strictly analogous to the spinor case. Here $D \geq 4$ is needed in order to get logarithmic or greater multiplicity in the canonical scaling case, whereas the canonical elementary value is $D=3$.

V. GENERALIZATIONS

It is not difficult to extend our analysis to the generalizations of the canonical framework mentioned in Sec. II B. We will only explicitly consider scalar currents and hadrons, the further generalizations to include spin being obvious. The first class of generalizations we will consider are those theories, such as reducibly scale-invariant R-invariant theories,[23,24] in which the LCOPE has the form (2.17) with canonical singularities but with the bilocal $j(x;0)$ of the form (2.18) or (2.19).[55] In such theories, the canonical scaling law (1.11) remains valid but the scaling function is determined according to (1.12) and

$$\bar{f}(x\cdot p)=\langle 0|\mathcal{C}(x,0;p)|0\rangle|_{x^2=0} \tag{5.1}$$

by the operator

$$\mathcal{C}(x,0;p)=\lim_{\xi\to 0}\lim_{x^2\to 0}\frac{\mathcal{B}(x+\xi,0;p)}{\ln(x+\xi)^2}, \tag{5.2}$$

or

$$\mathcal{C}(x,0;p)=\int da\,\sigma(a)\mathcal{B}(ax,0;p), \tag{5.3}$$

where $\mathcal{B}$ is still given by (1.14). Our result (3.10) is still obtained, but now the extra LC singularities which occur in (3.8) will be automatically disposed of. This is important since it is nice to know that (3.10) and its implications are valid in theories which do not suffer from the inconsistencies[23] of the naive canonical framework. It is also nice to be able to avoid the embarrassment of having to arbitrarily lop off the extra singularities in (3.8). In a properly formulated theory, the correct current and bilocal definitions, like (2.16) and (2.18) or (2.19), enable one to derive the scaling law (1.11) and the asymptotic behavior (3.11) in a purely mathematical fashion.

We consider next theories in which the bilocals have the previous forms (1.5), (2.18), or (2.19), but in which there are anomalous dimensions. The LCOPE becomes

$$j(x)j(0)\to \text{const}\times(x^2)^{-1-r}j(x;0), \tag{5.4}$$

and the scaling law becomes

$$\nu^{1-r}\overline{W}(\kappa,\nu)\to\overline{F}(\omega), \tag{5.5}$$

with $\overline{F}(\omega)$ having the previous definitions and properties. The result (3.10) is again obtained, but now Δ, as well as D, is *a priori* unknown.

For use below, we note here an alternate derivation of our previous results in the canonical case. Instead of working with the operator $\mathcal{B}$ [Eq. (1.14)], which depends on the fields ϕ and sources S, we could have worked directly with the operator $\mathcal{A}$ [Eq. (1.10)], which depends on the currents j and sources S. We could follow the same procedure employed for $\mathcal{B}$ in Sec. III and the Appendix, except that we would seek the slant associated with the $1/x^2$ term in $\langle 0|\mathcal{A}|0\rangle$. This term turns out to have a slant which is one less than the slant of the constant (in x^2) term for a given integrand, but this is made up for by the fact that the dimension of j is twice the dimension of ϕ. That is, the previous approach,

$$\langle \mathcal{A} \rangle \sim \frac{1}{x^2}\langle \mathcal{B} \rangle \sim \frac{1}{x^2}\lambda^{-(D+\Delta-3)} = \frac{1}{x^2}\lambda^{2-D}, \qquad (5.6)$$

and the present approach,

$$\langle \mathcal{A} \rangle \sim \frac{1}{x^2}\lambda^{-(D+2\Delta-3)+1} = \frac{1}{x^2}\lambda^{2-D}, \qquad (5.7)$$

give the same scaling law (1.11) (a consequence of the $1/x^2$ LC singularity) and the same asymptotic behavior (3.11) (a consequence of the λ^{2-D} slant singularity). Even at the purely canonical level, the second approach is useful since it eliminates the need to know that $\langle \mathcal{A} \rangle \to \Delta_+(x)\langle \mathcal{B} \rangle$ follows from (1.5).

The real advantage of the second approach is that it can be used when the bilocal does not have the form (1.5), (2.18), or (2.19), or even when the bilocal does not exist, as in (2.20). One need only insert the OPE's for $R(j, s)$, $R(j, s^+)$, etc., into $\langle \mathcal{A} \rangle$ and evaluate the integrals as we have previously done. As long as the theory is (asymptotically) scale-invariant, well-defined LC and slant singularities will be obtained.

VI. DISCUSSION

The main result of this paper has been the derivation (assuming nondominant infrared effects) of the short-distance behaviors

$$\bar{f}_h(\lambda) \underset{\lambda\to 0}{\sim} a\lambda^{-\sigma}, \quad \sigma = d + \delta - 2 \qquad (6.1a)$$

$$\bar{f}_H(\lambda) \underset{\lambda\to 0}{\sim} A\lambda^{-\Sigma}, \quad \Sigma = D + \delta - \tfrac{5}{2} \qquad (6.1b)$$

of the coefficients of the leading LC singularities in $\mathcal{A}_{\mu\nu}$ [Eq. (4.6)] which contribute to $\overline{W}_{\mu\nu}$ [Eq. (4.5)], the amplitude for e^+e^- annihilation into hadron h (spinor particle with source s of minimal dimension d) or H (scalar particle with source S of minimal dimension D) plus anything, when the electromagnetic current is given by $J_\mu = :\bar{\psi}\gamma_\mu Q\psi:$ (properly defined) with $\delta = \dim\psi$. In terms of operators,

$$\mathcal{A}_{\mu\nu}(x, 0; p) \underset{x^2\to 0}{\longrightarrow} \partial_\mu\Delta_+(x)p_\nu\bar{f}(\lambda)I + \cdots \qquad (6.2)$$

or

$$\mathcal{B}_\mu(x, 0; p) \underset{x^2\to 0}{\longrightarrow} p_\mu\bar{f}(\lambda)I + \cdots . \qquad (6.3)$$

The slants σ and Σ thus specify the maximum possible SD singularity associated with the leading LC singularity.

Both the degree +1 of the leading LC singularity $[\Delta_+(x) \sim (x^2)^{-1}]$ and the degree σ of the leading SD singularity are determined (in asymptotically scale-invariant theories with no dominating infrared effects) by dimensional analysis. The nature of these dimensional analyses are, however, remarkably distinct. Infinitely many terms in the OPE's of the operators in $\mathcal{A}_{\mu\nu}$ contribute to (6.1). The reason that so simple a result as (6.1) is nevertheless obtained is that each of the terms in (3.7) (for each possible contraction of the Lorentz indices) contributes in the same way, independently of the intermediate dimensions d_l or l_m. The precise result (6.1) could not, unlike the case for the LC singularity in (1.5), be guessed *a priori* by matching dimensions—the function $\bar{f}(\lambda)$ and the variable $\lambda = x\cdot p$ are dimensionless. An explicit evaluation, as performed in the Appendix, of the relevant integrals is necessary to obtain (6.1).

The contrast with the behavior of the coefficient $f(\lambda)$ of the leading LC contribution to electroproduction is especially to be noted. The LCOPE (1.5) implies that $f(\lambda)$ is analytic (for small λ), independently of the source (i.e., independently of the matrix element). The behaviors (6.1) are, in the contrary, singular and significantly dependent on the nature of the source. This source dependence means that more information is needed in the annihilation case, but the rewards are commensurate: Annihilation can be used to probe the (minimal) source dimensions. Also, the small λ behavior is much more useful in the annihilation case in that it leads to directly observable results (since $\omega\to\infty$ is in the physical region for annihilation).

The behavior of the transverse scaling function

$$\bar{F}_T(\omega) = \omega^{-1}\bar{F}_2(\omega) = \pi\int d\lambda\, e^{i\lambda\omega}\bar{f}(\lambda) \qquad (6.4)$$

for large ω is directly determined by (6.1):

$$\bar{F}_2(\omega) \underset{\omega\to\infty}{\sim} \text{const}\times\omega^\sigma \quad (\text{or } \omega^\Sigma \text{ for } H). \qquad (6.5)$$

So both the scaling limit of $\overline{W}_{\mu\nu}$ *and* the asymptotic behavior of the scaling function are determined by dimensional analysis. Again the differences in these dimensional analyses should be stressed: $\nu\overline{W}_2(\kappa, \nu)$ is (in the engineering sense) dimensionless while κ and ν have dimensions, whereas both $\bar{F}_2(\omega)$ and ω are dimensionless. The principal importance of (6.5) is the handle it provides on the source dimensions. It will be of much interest to compare (6.5) with other determinations of the source dimensions, e.g., with the nonleading terms in the total annihilation cross section.[29]

The results (6.5) in turn determine the asymptotic behavior of the multiplicity of h (or H) particles in annihilation:

$$N_h(\kappa) \to \text{const} \times (\sqrt{\kappa})^{d+\delta-5}, \tag{6.6a}$$

$$N_H(\kappa) \to \text{const} \times (\sqrt{\kappa})^{D+\delta-11/2}. \tag{6.6b}$$

These results provide an even more accessible handle on the source dimensions. If canonical scaling ($\delta = \frac{3}{2}$) remains valid and if the expected logarithmic or greater multiplicity is observed, then (6.6) imply that the sources in nature cannot have the canonical elementary dimensions ($d = \frac{5}{2}$, $D = 3$) but rather $d \geq \frac{7}{2}$, $D \geq 4$ (always assuming nondominant infrared effects).

The multiplicities $N(\kappa)$ will be soon measured at large κ and then (6.6) can be used to learn about the source dimensions. Until then, all that is known about the dimension D of the pion source is the PCAC restriction $3 \leq D < 6$, and essentially nothing is known about the dimension d of the nucleon source.[56] The information supplied by (6.6) should therefore be most welcome. The expected logarithmic or greater multiplicity will imply the also expected (but as yet unproved) composite nature of the observed hadrons. The constituent quark model is, in this connection, suggested. There, $s \sim \not{\partial} : \psi\psi\psi :$ and $S \sim \Box : \bar{\psi}\psi :$, so that, in the free-field limit, one has the large dimensions $d = \frac{11}{2}$ and $D = 5$. In this case the free-field dimensional assignments should not be taken seriously, however, since the strong binding should appreciably change the dimensions and since the canonical value $\dim\psi = \frac{3}{2}$ should only be correct for current quarks and not for constituent quarks.[57,58]

The results (6.5) [and, for the scalar case, (3.11)] provide simple explanations of previous results in perturbation theory. For example, they show why the asymptotic behaviors of the scaling function in the ϕ^3 and ϕ^4 theories are a power different and why those in PS theory and QED are similar.[51] The only perturbation theory which scales (in four dimensions) is the superrenormalizable ϕ^3 theory. There our results imply that, as long as $\dim\phi$ is unity, only constant scaling multiplicity is possible no matter how many diagrams one sums. More generally, if a sum of diagrams in any renormalizable theory scales, they can only lead to constant multiplicity for the canonical elementary particles. In order to obtain both scaling and nonconstant multiplicity in perturbation theory, suitably many graphs must be summed to obtain bound states of high dimensions or at least noncanonical elementary particles. It should be noted here that the logarithmic multiplicity obtained in some perturbation theory calculations[15] is a consequence of scaling violations and not of compositeness.

Our results are also applicable to conformally invariant quantum field theories,[39] whether or not they lead to Bjorken scaling. Consider such a theory in which either Bjorken scaling does obtain or at least is only violated by small corrections to the canonical dimensions of the fields in the current-current OPE. Then results like (6.5) and (6.6) will be valid, with d (or D) the anomalous dimension of the sources corresponding to the basic fields in the theory. Since the scalar (Δ_c) and spinor (δ_c) field dimensions in these theories must satisfy the conditions

$$\Delta_c > 1, \quad \delta_c > \tfrac{3}{2}, \tag{6.7}$$

the multiplicities can be power-behaved. Deviations from Bjorken scaling will increase the powers.[59] That such power behavior *is* obtained for multiplicities in these theories has been argued by Polyakov.[60]

Phenomenologically, the results (6.5) put annihilation on the same footing as electroproduction in that the behaviors of the scaling functions in all the asymptotic limits can now be predicted with some reliability. Equation (6.5) is the annihilation counterpart of the Regge behavior (2.5), and even the electroproduction threshold behavior (2.6) is also expected to occur for annihilation.[61]

The small-ω behavior in electroproduction and the large-ω behavior in annihilation may be related by more than analogy. An unexpected relation was noticed by Gribov and Lipatov[16] in their study of perturbation theory, linking the behavior of $\bar{F}_2(\omega)$ and $\bar{F}_2(1/\omega)$,

$$\bar{F}_2(-\omega) = -\omega^3 F_2(1/\omega). \tag{6.8}$$

This has been called a reciprocity relation. Equation (6.8) holds in perturbation theory if the current constituents coincide with the physical particle, but the connection between the behavior of $F_2(\omega)$ as $\omega \to 0$ (Regge limit) and that of $\bar{F}_2(\omega)$ as $\omega \to \infty$ given by (6.8) may go deeper than that. Pomeron dominance, known to give logarithmic multiplicities in the purely hadronic case,[62] when coupled with the reciprocity relation, would predict logarithmic multiplicities in e^+e^- annihilation, where the Regge limit is not relevant. More generally, the Regge behavior

$$F_2(\omega) \underset{\omega \to 0}{\sim} \beta\omega^{1-\alpha}, \tag{6.9}$$

combined with the reciprocity relation (6.8) (for large ω) gives

$$\bar{F}_2(\omega) \underset{\omega \to \infty}{\sim} \beta\omega^{\alpha+2}, \tag{6.10}$$

$$N(\kappa) \sim (\sqrt{\kappa})^{\alpha-1}. \tag{6.11}$$

Comparison with (6.5) then gives

$$\alpha+2=\sigma=d+\delta-2 \quad (\text{or} =\Sigma=D+\delta-\tfrac{5}{2} \text{ for } H), \tag{6.12}$$

thus providing a striking connection between Regge intercepts and minimal dimensions.[63] In a canonical framework this becomes

$$\alpha=d-\tfrac{5}{2}=D-3 \tag{6.13}$$

and could explain why the Pomeron intercept is at (or near) unity.

Our method, being of an operator nature, is of course applicable to other processes of current interest, in particular, to the one-particle spectrum in electroproduction (1.2). There, the dependence of the multiparticle amplitude on several variables complicates the procedure: for example, one needs the limit $x\cdot p'\to 0$, $x\cdot p$ fixed, which is not just the $x^\mu\to 0$ limit. Details of our treatment of this process are given in a subsequent publication.[64]

It should be kept in mind that, in order to derive the results (6.1) on which our conclusions are based, we had to make assumptions which are considerably stronger than those necessary for deriving the nature of ordinary SD and LC singularities. These assumptions are, however, correct in perturbation theory and our conclusions are readily verifiable experimentally. Taking the optimistic attitude that the assumptions are correct in nature, our results constitute a large extension of the region of applicability of OPE's in particle physics. Space-time techniques are now applicable to rather detailed studies of the final states in inclusive current-hadron processes. The fundamental questions concerning the dynamical origin of scaling and the internal structure of the hadrons unfortunately remain as intractable as ever.

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APPENDIX: THE SLANT INTEGRAL

Consider the basic slant integral

$$I_1=\int dy\,dz\,e^{ip\cdot(y-z)}(x-y)_R^{-2a}(x+y-z)_W^{-2b}(z)_R^{-2c}, \tag{A1}$$

where $z_W^{-2c}\equiv(z^2)_W^{-c}=(z^2-i\epsilon z_0)^{-c}$, etc. Let $\xi\equiv y+z$, $\eta=y-z$, and we have

$$I_1\sim\int d\eta\,d\xi\,e^{ip\cdot\eta}\frac{1}{(2x-\xi-\eta)^{2a}}\frac{1}{(x+\eta)^{2b}}\frac{1}{(\xi-\eta)^{2c}}. \tag{A2}$$

Successively let $\xi\equiv\xi'+\eta$, $\eta'\equiv x-\eta$, and then

$$I_1\sim e^{ip\cdot x}\int d\eta\,e^{-ip\cdot\eta'}\int d\xi'\frac{1}{(2\eta'-\xi')^{2a}}\frac{1}{(2x-\eta')^{2b}}\times\frac{1}{(\xi')^{2c}}. \tag{A3}$$

We can now rename the integration variables to get

$$I_1\sim e^{ip\cdot x}\int dz\,e^{-ip\cdot z}\int dy\frac{1}{(2z-y)^{2a}}\frac{1}{(2x-z)^{2b}}\frac{1}{y^{2c}}. \tag{A4}$$

I_1 is analytic in a, b, and c. Since the singularity depends only on $a+b+c$, we can perform the integral for certain values of a, b, and c, and analytically continue to ascertain its value for the same value of $a+b+c$. Thus let

$$I_2=\int dz\,dy\,e^{ip\cdot z}\theta(2z^0-y^0)\delta((2z-y)^2)\times\frac{1}{[(2x-z)^2]^n}\theta(y^0)\delta(y^2), \tag{A5}$$

so that

$$a+b+c=n+2. \tag{A6}$$

The dy integral can be evaluated trivially:

$$\begin{aligned}J&=\int dy\,\theta(2z^0-y^0)\delta((2z-y)^2)\theta(y^0)\delta(y^2)\\&=\tfrac{1}{4}\pi\int_0^\infty d|\vec{y}|\,|\vec{y}|\int_{-1}^{+1}d\alpha\,\theta(2z^0-|\vec{y}|)\\&\qquad\times\delta(z^2-z^0|\vec{y}|+|\vec{z}|\,|\vec{y}|\alpha).\end{aligned} \tag{A7}$$

Demanding that $-1\le\alpha\le+1$ in the δ function gives

$$J=\tfrac{1}{4}\pi\int_{z^0-|\vec{z}|}^{z^0+|\vec{z}|}d|\vec{y}|\,|\vec{y}|\,\theta(z^2)\frac{\theta(z^0)}{|\vec{z}|\,|\vec{y}|}=\tfrac{1}{2}\pi\theta(z^0)\theta(z^2). \tag{A8}$$

Thus, by renaming $2x\to x$, we get

$$I_2=\tfrac{1}{2}\pi\int dz\,e^{ip\cdot z}\theta(z^0)\theta(z^2)\frac{1}{[(x-z)^2]^n}. \tag{A9}$$

I_2 is most conveniently evaluated in terms of LC variables, with

$$\begin{aligned}&a=x\cdot p+[(x\cdot p)^2-x^2p^2]^{1/2},\quad b=x\cdot p-[(x\cdot p)^2-x^2p^2]^{1/2},\\&I_2=\tfrac{1}{2}\pi^2\int_0^\infty d\alpha\int_0^\infty d\beta\int_0^{4\alpha\beta}dy_\perp{}^2\,e^{ip^0(\alpha+\beta)}\theta(\alpha+\beta)\frac{1}{[4(ab-a\beta-\alpha b+\alpha\beta)-y_\perp{}^2]^n}\\&\quad=\frac{\pi^2}{8(n-1)}\int d\alpha\,d\beta\,e^{ip^0(\alpha+\beta)}\left\{\frac{1}{[b(a-\alpha)-a\beta]^{n-1}}-\frac{1}{[(a-\alpha)(b-\beta)]^{n-1}}\right\}.\end{aligned} \tag{A10}$$

The Fourier transforms can be performed with the aid of the formulas [65]

$$\int_0^\infty \frac{e^{-\mu x}dx}{(x+\beta)^n} = \frac{1}{(n-1)!}\sum_{k=1}^{n-1}(k-1)!\,(-\mu)^{n-k-1}\beta^{-k} - \frac{(-\mu)^{n-1}}{(n-1)!}e^{\beta\mu}\,\mathrm{Ei}(-\beta\mu)\,, \tag{A11a}$$

$$\int_0^\infty \frac{e^{-\mu x}dx}{x+\beta} = -e^{\beta\mu}\,\mathrm{Ei}(-\mu\beta)\,, \tag{A11b}$$

and

$$\int_0^\infty dx\, e^{ikx}\,\mathrm{Ei}(\lambda x+\mu) = \frac{i}{k}[\mathrm{Ei}(\mu) - e^{-ik\mu/\lambda}\,\mathrm{Ei}(ik\mu/\lambda+\mu)]. \tag{A11c}$$

The result is

$$\begin{aligned}
I_2 = \frac{\pi}{2}\,\frac{\pi}{4(n-1)(n-2)!}\Bigg(&(-1)^n\Bigg\{\sum_{j=1}^{n-3}(ip^0)^{n-j-1}(-1)^j(j-1)!\frac{1}{b-a}\left[\left(\frac{1}{b}\right)^j\left(\frac{1}{a}\right)^{n-2}-\left(\frac{1}{a}\right)^j\left(\frac{1}{b}\right)^{n-2}\right]\\
&+\frac{1}{a-b}(ip^0)^{n-2}\left[\left(\frac{1}{a}\right)^{n-2}\mathrm{Ei}(-ip^0b)e^{ip^0b}-\left(\frac{1}{b}\right)^{n-2}\mathrm{Ei}(-ip^0a)e^{ip^0a}\right]\Bigg\}\\
&-\frac{1}{(n-2)!}\left[\sum_{k=1}^{n-2}(k-1)!\,(ip^0)^{n-k-2}(-1)^k\left(\frac{1}{a}\right)^k-(ip^0)^{n-2}e^{ip^0a}\,\mathrm{Ei}(-ip^0a)\right]\\
&\times\left[\sum_{k=1}^{n-2}(k-1)!\,(ip^0)^{n-k-2}(-1)^k\left(\frac{1}{b}\right)^k-(ip^0)^{n-2}e^{ip^0b}\,\mathrm{Ei}(-ip^0b)\right]\Bigg)\\
=\frac{\pi^2}{8(n-1)(n-2)!}&(p^2)^{n-2}\\
\times\Bigg\{&(-1)^n\Bigg[\sum_{j=1}^{n-3}i^{n+j-1}(j-1)!\frac{-1}{2\lambda}\left(\frac{1}{[\lambda-(\lambda^2-x^2p^2)^{1/2}]^j[\lambda+(\lambda^2-x^2p^2)^{1/2}]^{n-2}}-\frac{1}{[\lambda+(\lambda^2-x^2p^2)^{1/2}]^j[\lambda-(\lambda^2-x^2p^2)^{1/2}]^{n-2}}\right)\\
&+i^{n-2}\frac{1}{2\lambda}\left(\frac{1}{[\lambda+(\lambda^2-x^2p^2)^{1/2}]^{n-2}}\exp\{i[\lambda-(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda-(\lambda^2-x^2p^2)^{1/2}])\right.\\
&\left.-\frac{1}{[\lambda-(\lambda^2-x^2p^2)^{1/2}]^{n-2}}\exp\{i[\lambda+(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda+(\lambda^2-x^2p^2)^{1/2}])\right)\Bigg]\\
&-\frac{1}{(n-2)!}\left(\sum_{k=1}^{n-2}(k-1)!\,i^{n+k-2}\frac{1}{[\lambda+(\lambda^2-x^2p^2)^{1/2}]^k}-i^{n-2}\exp\{i[\lambda+(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda+(\lambda^2-x^2p^2)^{1/2}])\right)\\
&\times\left(\sum_{k=1}^{n-2}(k-1)!\,i^{n+k-2}\frac{1}{[\lambda-(\lambda^2-x^2p^2)^{1/2}]^k}-i^{n-2}\exp\{i[\lambda-(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda-(\lambda^2-x^2p^2)^{1/2}])\right)\Bigg\}\,.
\end{aligned} \tag{A12a}$$

Equation (A12a) holds for general $n \geqslant 4$. The results for $n = 1$, 2, and 3 are

$$\begin{aligned}
\int dy\, e^{ipy}\theta(y^0)\theta(y^2)\frac{1}{(x-y)^2} \sim -\frac{1}{p^2}\Bigg(&-\frac{[\lambda-(\lambda^2-x^2p^2)^{1/2}]}{(\lambda^2-x^2p^2)^{1/2}}\exp\{i[\lambda-(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda-(\lambda^2-x^2p^2)^{1/2}])\\
&+\frac{[\lambda+(\lambda^2-x^2p^2)^{1/2}]}{2(\lambda^2-x^2p^2)^{1/2}}\exp\{i[\lambda+(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda+(\lambda^2-x^2p^2)^{1/2}])\Bigg),
\end{aligned} \tag{A12b}$$

$$\begin{aligned}
\int dy\, e^{ipy}\theta(y^0)\theta(y^2)\frac{1}{[(x-y)^2]^2} \sim \tfrac{1}{8}\pi\Bigg\{&\frac{i}{2(\lambda^2-x^2p^2)^{1/2}}\Big[\exp\{i[\lambda-(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda-(\lambda^2-x^2p^2)^{1/2}])\\
&-\exp\{i[\lambda+(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda+(\lambda^2-x^2p^2)^{1/2}])\Big]\\
&-e^{2i\lambda}\,\mathrm{Ei}(-i[\lambda+(\lambda^2-x^2p^2)^{1/2}])\,\mathrm{Ei}(-i[\lambda-(\lambda^2-x^2p^2)^{1/2}])\Bigg\},
\end{aligned} \tag{A12c}$$

$$\int dy\, e^{ipy}\theta(y^0)\theta(y^2)\frac{1}{[(x-y)^2]^3}$$

$$\sim -\tfrac{1}{32}\pi p^2\Big(-\frac{1}{2[\lambda+(\lambda^2-x^2p^2)^{1/2}](\lambda^2-x^2p^2)^{1/2}}\exp\{i[\lambda-(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda-(\lambda^2-x^2p^2)^{1/2}])$$

$$+\frac{1}{2[\lambda-(\lambda^2-x^2p^2)^{1/2}](\lambda^2-x^2p^2)^{1/2}}\exp\{i[\lambda+(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda+(\lambda^2-x^2p^2)^{1/2}])+\frac{1}{x^2p^2}$$

$$+\frac{i}{[\lambda+(\lambda^2-x^2p^2)^{1/2}]}\exp\{i[\lambda-(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda-(\lambda^2-x^2p^2)^{1/2}])$$

$$+\frac{i}{[\lambda-(\lambda^2-x^2p^2)^{1/2}]}\exp\{i[\lambda+(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-i[\lambda+(\lambda^2-x^2p^2)^{1/2}])$$

$$-e^{i2\lambda}\,\mathrm{Ei}(-i[\lambda+(\lambda^2-x^2p^2)^{1/2}])\,\mathrm{Ei}(-i[\lambda-(\lambda^2-x^2p^2)^{1/2}])\Big)\,. \tag{A12d}$$

Another set of relevant integrals is

$$\int dy\, e^{ipy}\theta(y^0)\delta(y^2)\frac{1}{(x-y)^2}\sim\frac{\pi i}{2(\lambda^2-x^2p^2)^{1/2}}\Big(-\exp\{\tfrac{1}{2}i[\lambda+(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-\tfrac{1}{2}i[\lambda+(\lambda^2-x^2p^2)^{1/2}])$$

$$+\exp\{\tfrac{1}{2}i[\lambda-(\lambda^2-x^2p^2)^{1/2}\}\,\mathrm{Ei}(-\tfrac{1}{2}i[\lambda-(\lambda^2-x^2p^2)^{1/2}])\Big)\,, \tag{A12e}$$

$$\int dy\, e^{ipy}\theta(y^0)\delta(y^2)\frac{1}{[(x-y)^2]^2}$$

$$\sim\frac{\pi p^2}{4(\lambda^2-x^2p^2)^{1/2}}\left\{\frac{1}{\lambda+(\lambda^2-x^2p^2)^{1/2}}\exp\{\tfrac{1}{2}i[\lambda-(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-\tfrac{1}{2}i[\lambda-(\lambda^2-x^2p^2)^{1/2}])\right.$$

$$\left.-\frac{1}{\lambda-(\lambda^2-x^2p^2)^{1/2}}\exp\{\tfrac{1}{2}i[\lambda+(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-\tfrac{1}{2}i[\lambda+(\lambda^2-x^2p^2)^{1/2}])\right\}, \tag{A12f}$$

and

$$\int dy\, e^{ipy}\theta(y^0)\delta(y^2)\frac{1}{[(x-y)^2]^n}$$

$$\sim\frac{\pi(-1)^{n-1}}{4(n-1)!}\frac{1}{(\lambda^2-x^2p^2)^{1/2}}(p^2)^{n-1}\left\{\sum_{k=1}^{n-2}\left[(k-1)!(\tfrac{1}{2}i)^{n-k-2}\left(\frac{1}{\lambda+(\lambda^2-x^2p^2)^{1/2}}\right)^{n-k-1}(-1)^k\left(\frac{1}{x^2}\right)^k\right.\right.$$

$$\left.-(k-1)!(\tfrac{1}{2}i)^{n-k-2}\left(\frac{1}{\lambda-(\lambda^2-x^2p^2)^{1/2}}\right)^{n-k-1}(-1)^k\left(\frac{1}{x^2}\right)^k\right]$$

$$-(\tfrac{1}{2}i)^{n-2}\left(\frac{1}{\lambda+(\lambda^2-x^2p^2)^{1/2}}\right)^{n-1}\exp\{\tfrac{1}{2}i[\lambda-(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-\tfrac{1}{2}i[\lambda-(\lambda^2-x^2p^2)^{1/2}])$$

$$-(\tfrac{1}{2}i)^{n-2}\left(\frac{1}{\lambda-(\lambda^2-x^2p^2)^{1/2}}\right)^{n-1}$$

$$\left.\times\exp\{\tfrac{1}{2}i[\lambda+(\lambda^2-x^2p^2)^{1/2}]\}\,\mathrm{Ei}(-\tfrac{1}{2}i[\lambda+(\lambda^2-x^2p^2)^{1/2}])\right\}, \tag{A12g}$$

(A12g) being valid for $n\geqslant 3$.

For the purpose of investigating the effect of numerators, it is also useful to have the integral

$$\int dy\, e^{ipy} \frac{(y^2)^m \theta(y^0)\theta(y^2)}{[(x-y)^2]^n}$$
$$\sim (p^2)^{n-m-2} \Bigg(-\pi \sum_{k=1}^{m-1} \frac{m!(n-k-2)!}{(m-k)!(n-1)!} \frac{(-1)^{m-k}}{4^{n-m-1}}$$
$$\times \Bigg(\frac{1}{(n-k-2)!} \sum_{j=1}^{n-k-2} \frac{(j-1)!(n-k-j-2)!}{(n-m-j-2)!} i^{n-k+j-2} \frac{1}{[\lambda + (\lambda^2 - x^2p^2)^{1/2}]^j}$$
$$- \frac{(n-k-1)i^{n-k-1}}{(n-m-1)!} \exp\{i[\lambda + (\lambda^2 - x^2p^2)^{1/2}]\} \operatorname{Ei}(-i[\lambda + (\lambda^2 - x^2p^2)^{1/2}]) \Bigg)$$
$$\times \Bigg(\frac{1}{(n-k-2)!} \sum_{j=1}^{n-k-2} \frac{(j-1)!(n-k-j-2)!}{(n-m-j-2)!} i^{n-k+j-2} \frac{1}{[\lambda - (\lambda^2 - x^2p^2)^{1/2}]^j}$$
$$- \frac{(n-k-1)i^{n-k-1}}{(n-m-1)!} \exp\{i[\lambda - (\lambda^2 - x^2p^2)^{1/2}]\} \operatorname{Ei}(-i[\lambda - (\lambda^2 - x^2p^2)^{1/2}]) \Bigg)$$
$$+ \pi \frac{m!(n-m-1)!}{(n-1)!} \frac{1}{(n-m-1)!}$$
$$\times \Bigg\{ (-1)^{n-m} \Bigg[\sum_{j=1}^{n-m-3} i^{l+j-1}(j-1)! \frac{-1}{2\lambda} \Bigg(\frac{1}{[\lambda - (\lambda^2 - x^2p^2)^{1/2}]^j [\lambda + (\lambda^2 - x^2p^2)^{1/2}]^{n-m-2}}$$
$$- \frac{1}{[\lambda + (\lambda^2 - x^2p^2)^{1/2}]^j [\lambda - (\lambda^2 - x^2p^2)^{1/2}]^{n-m-2}} \Bigg)$$
$$+ \frac{1}{2\lambda} i^{n-m-2} \Bigg(\frac{1}{[\lambda + (\lambda^2 - x^2p^2)^{1/2}]^{n-m-2}} \exp\{i[\lambda - (\lambda^2 - x^2p^2)^{1/2}]\} \operatorname{Ei}(-i[\lambda - (\lambda^2 - x^2p^2)^{1/2}])$$
$$- \frac{1}{[\lambda - (\lambda^2 - x^2p^2)^{1/2}]^{n-m-2}} \exp\{i[\lambda + (\lambda^2 - x^2p^2)^{1/2}]\} \operatorname{Ei}(-i[\lambda + (\lambda^2 - x^2p^2)^{1/2}]) \Bigg) \Bigg]$$
$$- \frac{1}{(n-m-2)!} \Bigg(\sum_{k=1}^{n-m-2} (k-1)!\, i^{n-m+k-2} \frac{1}{[\lambda + (\lambda^2 - x^2p^2)^{1/2}]^k}$$
$$- i^{n-m-2} \exp\{i[\lambda + (\lambda^2 - x^2p^2)^{1/2}]\} \operatorname{Ei}(-i[\lambda + (\lambda^2 - x^2p^2)^{1/2}]) \Bigg)$$
$$\times \Bigg(\sum_{k=1}^{n-2} (k-1)!\, i^{m-m+k-2} \frac{1}{[\lambda - (\lambda^2 - x^2p^2)^{1/2}]^k}$$
$$- i^{n-m-2} \exp\{i[\lambda - (\lambda^2 - x^2p^2)^{1/2}]\} \operatorname{Ei}(-i[\lambda - (\lambda^2 - x^2p^2)^{1/2}]) \Bigg) \Bigg\} \Bigg). \quad \text{(A12h)}$$

The variable $a = \lambda + (\lambda^2 - x^2p^2)^{1/2} \to 2(x \cdot p)$ as $x^2 \to 0$, $\lambda \to 0$, so that I_2 indeed possesses slant singularities. It is also singular as $x^2 \to 0$, but we shall only be interested in the piece which diverges at most like $\ln(x^2p^2)$. Then the slant singularity is

$$I_2 \underset{\lambda\to 0}{\sim} \frac{\pi}{2} \frac{\pi(-1)^n i^{n-2}}{(n-1)!2^{n+1}} \frac{1}{\lambda^{n-1}} (p^2)^{n-2} \times (-3\ln 2 + \ln x^2p^2 - 2\ln\lambda). \quad \text{(A13)}$$

The singularities are always the result of divergence of the integral near the origin, and would *not* be changed even if the integration is cut off at a finite distance. This is the basis of our optimistic attitude toward "infrared" effects manifested in some perturbation-theoretic examples. Written in terms of a, b, and c, this means

$$I_1(a,b,c) \underset{\lambda\to 0}{\sim} \text{const} \times \lambda^{-(a+b+c-3)}, \quad \text{(A14)}$$

or that the slant is given by

$$\sigma = a + b + c - 3. \quad \text{(A15)}$$

The variable $b = \lambda - (\lambda^2 - x^2p^2)^{1/2} \sim x^2p^2/2\lambda + \cdots$ as $x^2 \to 0$, $\lambda \to 0$. If the term containing $\ln(x^2p^2)$ has the form $\ln(x^2p^2)\lambda^{-\sigma}$, then the $1/x^2$ term will have the form $(1/x^2)\lambda^{-\sigma+1}$, and so forth. There are also terms like $x^2\lambda^{-\sigma-2}$ arising from the expansion of the exponentials. Applying $\Box_x$ to (A1) increases the slant only by *one*, because of these various cancellations.

We now proceed to consider the effect of inserting various inner products. We are interested in three kinds of inner products: (1) linear combinations of $x\cdot y$, $y\cdot z$, ... etc., (2) $y\cdot p$, $z\cdot p$, (3) $y\cdot k$, $z\cdot k$ where k is any external vector and $x\cdot k$ may or may not vanish.

Consider first inserting a factor of $y\cdot k$ in the integrand (A5). We take

$$\vec{y}\cdot\vec{z}=|\vec{z}|\,|\vec{y}|\cos\theta,$$
$$\vec{y}=|\vec{y}|\,(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta),\qquad \text{(A16)}$$
$$\vec{y}\cdot\vec{k}=|\vec{y}|\,|\vec{k}|\,(\sin\beta\sin\theta\cos\phi+\cos\beta\cos\theta),$$

where

$$\cos\beta=\frac{\vec{z}\cdot\vec{k}}{|\vec{z}|\,|\vec{k}|},\quad \cos\theta=\frac{z^0|\vec{y}|-z^2}{|\vec{z}|\,|\vec{y}|}. \qquad \text{(A17)}$$

Then we find

$$\int dy\,\theta(2z^0-y^0)\delta((2z-y)^2)(y\cdot k)\theta(y^0)\delta(y^2)=\tfrac{1}{2}\pi(z\cdot k)\theta(z^0)\theta(z^2). \qquad \text{(A18)}$$

Thus we can concentrate on finding the slant of

$$I_3=\int dz\,dy\,e^{ip\cdot z}(2z^0-y^0)\delta((2z-y)^2)\times\frac{z\cdot k}{[(2x-z)^2]^n}\theta(y^0)\delta(y^2). \qquad \text{(A19)}$$

We choose $p=(p^0,\vec{0})$, and parametrize

$$\vec{z}\cdot\vec{x}=|\vec{z}|\,|\vec{x}|\cos\theta,$$
$$\vec{z}=|\vec{z}|\,(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta), \qquad \text{(A20)}$$
$$\vec{z}\cdot\vec{k}=|\vec{z}|\,|\vec{k}|\,(\sin\beta\sin\theta\cos\phi+\cos\beta\cos\theta),$$

where

$$\cos\beta=\frac{\vec{x}\cdot\vec{k}}{|\vec{x}|\,|\vec{k}|}. \qquad \text{(A21)}$$

Then,

$$I_3\sim\int_0^\infty dz^0\,e^{ip^0z^0}\int_0^{z^0}d|\vec{z}|\,|\vec{z}|^2\int_{-1}^{+1}d(\cos\theta)\,\frac{(z^0k^0-|\vec{z}|\,|\vec{k}|\cos\theta\cos\beta)}{(4x^2+z^2-4x^0z^0+4|\vec{x}|\,|\vec{z}|\cos\theta)^n}. \qquad \text{(A22)}$$

We can ignore any factor of x^0 or $|\vec{x}|$ relative to terms without them, and the leading term in I_3 will be

$$I_3\sim\frac{1}{|\vec{x}|^2}\int_0^{z_0}d|\vec{z}|\,|\vec{z}|\,\frac{z^2|\vec{z}|\,|\vec{k}|\cos\beta}{(4x^2+z^2-4x^0z^0-4|\vec{x}|\,|\vec{z}|)^{n-1}}+\text{nonleading terms},$$
$$\sim\frac{|\vec{k}|}{|\vec{x}|^2}\int_0^{z_0}d|\vec{z}|\,|\vec{z}|\,\frac{|\vec{z}|\cos\beta}{(4x^2+z^2-4x^0z^0-4|\vec{x}|\,|\vec{z}|)^{n-2}}+\text{nonleading terms}. \qquad \text{(A23)}$$

By contrast,

$$I_2\sim\frac{1}{|\vec{x}|}\int_0^{z_0}d|\vec{z}|\,|\vec{z}|\,\frac{1}{(4x^2+z^2-4x^0z^0-4|\vec{x}|\,|\vec{z}|)^{n-1}}+\text{nonleading terms}. \qquad \text{(A24)}$$

The integral has a slant identical to I_2. If $x\cdot k\to 0$ also, then I_3 would have the same slant as I_2. If $x\cdot k\neq 0$, then in this frame $|\vec{k}|\sim x\cdot k/|\vec{x}|$, and I_3 with the extra $|k|$ factor would have its slant increased by one. To see this succinctly, note that

$$I_3=-ik_\mu\frac{\partial}{\partial p_\mu}I_2, \qquad \text{(A25)}$$

where I_2 given in (A12a) is a function with singularities as $x\cdot p\pm[(x\cdot p)^2-x^2p^2]^{1/2}\to 0$. Clearly

$$k_\mu\frac{\partial}{\partial p_\mu}f(x\cdot p+[(x\cdot p)^2-x^2p^2]^{1/2})=\left(x\cdot k+\frac{(x\cdot k)(x\cdot p)-x^2k\cdot p}{[(x\cdot p)^2-x^2p^2]^{1/2}}\right)f'(x\cdot p+[(x\cdot p)^2-x^2p^2]^{1/2}), \qquad \text{(A26)}$$

and the above statement follows since f' in this case is one power more singular than f.

In particular we can put $k=p$, and obtain the result that $z\cdot p$ or $y\cdot p$ in the numerator does not change the slant.

For $x\cdot y$, $y\cdot z$ etc., we notice that by (A18) we need only consider $x\cdot z$ and z^2. The insertion of z^2 in the numerator of (A24) decreases the slant by one relative to I_2. The effect of $x\cdot z$ is obtained from (A23) by replacing k with x: It decreases the slant by one.

We conclude by listing our results:

(1) $x \cdot y$, $y \cdot z, \ldots$, etc. decreases the slant by one;

(2) $y \cdot p$, $z \cdot p$ does not change the slant;

(3) $y \cdot k$, $z \cdot k$ does not change the slant for $x \cdot k = 0$;

(4) $y \cdot k$, $z \cdot k$ increases the slant by one if $x \cdot k \neq 0$.

The independence of the result (3.10) of the way contractions occur in the integrand in (3.7) is an immediate consequence of the result (1) listed above. For example, the integrands

$$(x^2)^{-a}(y^2)^{-b}(z^2)^{-c},$$

$$(x^2)^{-a-1}(y^2)^{-b-1}(z^2)^{-c}(x \cdot y)^2,$$

$$(x^2)^{-a-1}(y^2)^{-b-1}(z^2)^{-c-1}(x \cdot y)(x \cdot z)(y \cdot z),$$

etc.,

all give the same slant.[66]

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[29]K. G. Wilson, in Proceedings of the 1970 Midwest Conference on Theoretical Physics, 1970 [SLAC Report No. SLAC-PUB-737 (unpublished)]. In particular, it is shown here that the dimension Δ_π of the pion field determines corrections to the scaling behavior of the e^+e^- total cross section.

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[31]LC techniques can still be used to obtain information about the exclusive processes if used in connection with mass dispersion relations and compositeness, as in Refs. 4 and 5. A nice critique is given in Ref. 30. Although the correctness of the methods was previously the object of some debate, it is by now generally accepted. See, for example, M. Ciafaloni and P. Menotti, Nuovo Cimento Lett. 6, 545 (1973); Nucl. Phys. B54, 483 (1973); J. Sucher and C. H. Woo, Phys. Rev. D 7, 3372 (1973). See also J. Jersak, H. Leutwyler, and J. Stern, Nucl. Phys. B57, 413 (1973).

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[36]An R transformation adds a constant to each elementary field in the theory.

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[42]We put $p^2 = 1$ in the remainder of the paper.

[43]The mass terms are unimportant because they correspond to less singular terms which are *not* accompanied by higher-dimensional operators.

[44]Note that in the free-field case Σ is negative, and $\bar{f}(\lambda)$ is analytic.

[45]It should be noted that we are only free to assume the irrelevance of infrared contributions when we use OPE's with sources $S(y)$. If, as described above, we instead use OPE's with (smaller dimensional) fields $\Phi(y)$, such contributions can no longer be ignored. Indeed, the occurrence of the necessary poles $[(p^2-1)^{-1}]$ is a manifestly infrared effect.

[46]k is a spinor index $k = 1, \ldots, 4$.

[47]Recall that the dimension of the spinor source is the dimension of the field plus one.

[48]We use the summation convention for spinor indices.

[49]F_T is the transverse structure function. The longitudinal structure function vanished because of (4.4).

[50]See the discussion in Sec. IV C.

[51]Except in cases where infrared effects are important.

[52]The sum of these diagrams does not give rise to a bilocal but rather to an LC expansion of the form (2.20). See N. Christ, B. Hasslacher, and A. Mueller, Phys. Rev. D 6, 3543 (1972). The generalization of our methods to include such situations is discussed in Sec. V.

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[54]Or at least their noncanonical nature.

[55]This form occurs in lowest nontrivial order to φ^4 theory. See Ref. 18, Eq. (2.33).

[56]There have been many proposed models for baryon currents, but none has yet received any experimental support.

[57]The distinction between these two types of quarks has been particularly stressed by M. Gell-Mann and H. Fritzsch. See M. Gell-Mann, in *Elementary Particle Physics*, proceedings of the XI Schladming Conference (Acta Phys. Austriaca Suppl. IX), edited by P. Urban (Springer, New York, 1972), p. 733. This distinction suggests that, e.g., the pseudoscalar pion source $:\bar{\psi}\gamma_5\psi:$ may not be the source of minimal dimension.

[58]In the gauge-invariant spinor theories of H. P. Dürr and N. J. Winter [Nuovo Cimento 70A, 467 (1970)], the spinor potential ψ has dimension $\frac{1}{2}$, but the physical baryon field of minimal dimension might be ψ, or $\psi\psi\psi$ of dimension $\frac{3}{2}$, or $\Box\psi$ of dimension $\frac{5}{2}$. Our result can clearly distinguish between these possibilities.

[59]The total annihilation cross section $\sigma(q^2)$ will, of course, always scale in these theories.

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