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Generalized eikonal functions in potential theory

H. M. Fried*
Physics Department, Brown University, Providence, Rhode Island 02912

Yukap Hahn
Physics Department, University of Connecticut, Storrs, Connecticut 06268
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A potential-theory formalism is described within which exponentialized corrections to simple Glauber theory may be developed. An alternate formalism with direct generalization to relativistic field theory is developed for potential scattering at all angles, and some numerical work is presented to confirm and justify the approximations employed.

I. INTRODUCTION

One of the perennial problems of theoretical physics is the derivation of potential-theory scattering amplitudes better than (or at least different from) those of a previously published approximation. Is it possible to improve the Glauber[1] eikonal amplitude in some unique way? Is the Saxon-Schiff[2] formulation for scattering at all angles the best of the simple approximations? Do these amplitudes have generalizations to relativistic potential theory? To field theory?

Recently, there have been a sequence of calculations defining corrections[3] to simple Glauber theory, along with some field-theoretic computations[4] of high-energy, large-momentum-transfer proton-proton scattering, representing a relativistic generalization of the Schiff wide-angle potential-theory approximation.[5] These calculations are all connected, in the sense that they may be reached starting from the same, fundamental forms for the scattering amplitude. It is the purpose of these remarks to exhibit this underlying formalism, and to demonstrate one facet of it by producing an alternative to the Saxon-Schiff formalism, one with immediate generalization to relativistic quantum field theory. When applied to nonrelativistic potential theory one finds a simple approximation not much better and probably no worse than those given by other approximation schemes. The lack of numerical comparisons of different approximation schemes in the truly wide-angle region is to be deplored.

It must be emphasized that the techniques employed here are really not new; rather, their use in these contexts is most natural and straightforward, but does not seem to have appeared in the potential-theory literature. An important exception is the recent paper by Harrington,[6] with results analogous to our Eq. (18) of Sec. II. The soft/hard expansion was introduced in an excellent paper by Mahanthappa.[7] Exact propagator representations are well known and have frequently been used in field theory.[8]

The plan of this paper is as follows. Section II contains the basic formalism, defining corrections to Glauber theory and generating an alternate formalism for scattering at all angles. A simplified derivation of formulas for the latter case is presented in Sec. III. Section IV contains some numerical computations carried out for a Gaussian potential, while a final section is devoted to a brief discussion. Our main result is contained in (29) and (30) [or (38) and (43)], which exhibits the result of both the small-angle formulation[4] and the large-angle theory.[5]

II. FORMALISM

A. Fundamental forms

Perhaps the simplest approach begins by writing down the definition of an amplitude appropriate to the scattering of a nonrelativistic particle by a time-independent potential $V(r)$,

$$f(\vec{p}_f, \vec{p}_i) = -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)\int d^3z\, e^{-i\vec{p}_f\cdot\vec{z}}\, V(\vec{z})\,\psi_{(+)}(\vec{z}) , \tag{1}$$

where $\psi_{(+)}(\vec{z})$ satisfies the formal integral equation

$$\psi_{(+)} = \psi_i + (E - H_0 + i\epsilon)^{-1} V\psi_{(+)} , \tag{2}$$

with $\psi_i(r) = e^{i\vec{p}_i\cdot\vec{r}}$, $H_0 = -(\hbar^2/2m)\nabla^2$, $|\vec{p}_f| = |\vec{p}_i| \equiv p$, and the normalization chosen so that $\sigma_{\text{tot}} = (4\pi/p)\text{Im}f(\vec{p}_i, \vec{p}_i)$, $d\sigma/d\Omega = |f|^2 = (p^2/\pi)(d\sigma/dt)$. It is not difficult to manipulate (1) into an equivalent form, most suitable for subsequent eikonal purposes,

$$f = -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)(\psi_f, (E - H_0)G^V_{(+)}(E - H_0)\psi_i) . \tag{3}$$

where $G^V_{(+)} \equiv (E - H_0 - V + i\epsilon)^{-1}$ is the exact, potential-dependent propagator of the problem.

Another and somewhat more convenient approach begins by writing down an expression for the potential-theory scattering amplitude obtained directly from the reduction formulas[9] of relativistic S-matrix theory; with $S/\langle 0|S|0\rangle = 1 + iT$, one has

$$\langle p'|T|p\rangle = (2\pi)^{-3}(2E2E')^{-1/2} \times \int d^4x\, e^{ip\cdot x} \int d^4y\, e^{-ip'\cdot y} K_x K_y \times \overline{\Delta}_c(y, x|U), \qquad (4)$$

where p, p' are the four-dimensional equivalents of $\vec{p}_i, \vec{p}_f$, $K_x \equiv m^2 - \partial_x{}^2$, and $\overline{\Delta}_c[U]$ represents the exact, relativistic, causal Green's function for a particle propagating in an external field $U(x) \equiv (2m/\hbar^2)V(x)$,

$$\overline{\Delta}_c(y, x|U) = \langle y|(m^2 - \partial^2 + U - i\epsilon)^{-1}|x\rangle . \qquad (5)$$

The consequence of the independence of V on t has not yet been extracted from (4), but one may note the similarity in structure of (3) and (4); in both cases one is asked to integrate appropriate plane waves over an exact Green's function, inserting operators which would, by themselves, vanish on the energy shell ($E - H_0 \to 0$) or on the mass shell ($K_x \to m^2 + p^2 \to 0$). Since (4) is manifestly relativistic, it is somewhat simpler to begin by studying an exact representation of $\overline{\Delta}_c[U]$, and hence of (4), one which easily leads into subsequent and more properly relativistic considerations. However, all steps may be performed in an identical way for $G_{(+)}^V$, and the nonrelativistic amplitude of (3). For the usual situation in which $V(x) = V(\vec{r})$, energy is conserved, and the relation between the $f(\vec{p}_f, \vec{p}_i)$ of (1) and the (nonrelativistic approximation to) $\langle p'|T|p\rangle$ of (4) is given by

$$\langle p'|T|p\rangle = (2\pi)^{-2}(2E2E')^{-1/2}\,\delta(E - E')4\pi f(\vec{p}_f, \vec{p}_i). \qquad (6)$$

B. An exact propagator representation

What is a useful representation for $\overline{\Delta}_c[U]$? The simplest and therefore most frequently performed operation, generating the Born series, is obtained by the expansion of $\overline{\Delta}_c[U]$ in powers of U. A far more useful form may be obtained by first writing an exact representation for the partial Fourier transform,

$$\overline{\Delta}_c(y, x|U) = \sum_p \langle y|p\rangle\langle p|(m^2 - \partial^2 + U - i\epsilon)^{-1}|x\rangle ,$$

i.e.,

$$\overline{\Delta}_c(y, x|U) = (2\pi)^{-4} \times \int d^4p\, e^{ip\cdot y}\langle p|(m^2 - \partial^2 + U - i\epsilon)^{-1}|x\rangle . \qquad (7)$$

It is convenient to exponentiate this operator,

$$\langle p|(m^2 - \partial^2 + U - i\epsilon)^{-1}|x\rangle = i\int_0^\infty d\xi \langle p|\exp[-i\xi(m^2 - \partial^2 + U)]|x\rangle = i\int_0^\infty d\xi\, e^{-i\xi(m^2+p^2)}\,\mathfrak{F}(\xi; p; x),$$

where

$$\mathfrak{F}(\xi; p; x) \equiv \langle p|e^{-i\xi\partial^2} e^{i\xi(\partial^2 - U)}|x\rangle , \qquad (8)$$

and construct a differential equation for $\mathfrak{F}$,

$$\frac{\partial \mathfrak{F}(\xi; p; x)}{\partial \xi} = -i\langle p|e^{-i\xi\partial^2} U e^{i\xi(\partial^2 - U)}|x\rangle = -i\langle p|e^{-i\xi\partial^2} U e^{i\xi\partial^2} e^{-i\xi\partial^2} e^{i\xi(\partial^2 - U)}|x\rangle .$$

Inserting

$$1 = \sum_k |k\rangle\langle k| = \sum_k |p+k\rangle\langle p+k| ,$$

this becomes

$$\frac{\partial \mathfrak{F}(\xi; p; x)}{\partial \xi} = -\int d^4k \langle p|e^{-i\xi\partial^2} U e^{i\xi\partial^2}|p+k\rangle \times \mathfrak{F}(\xi; p+k; x).$$

Finally, since $\langle p|U|p+k\rangle = \tilde{U}(-k)$ [if $\langle x|U|y\rangle = \delta^4(x-y)U(x)$], and setting $\mathfrak{F}(\xi; p; x) \equiv e^{-ip\cdot x} \times f(\xi; p; x)$ so that $f(0; p; x) = 1$, we obtain the relation

$$\frac{\partial f(\xi; p; x)}{\partial \xi} = -i\int d^4k\, \tilde{U}(-k)e^{-ik\cdot x} e^{-i\xi(k^2 + 2k\cdot p)} \times f(\xi; p+k; x). \qquad (9)$$

A somewhat neater equation may be made out of this by noting that $f(\xi; p+k; x) \equiv e^{k\cdot\partial/\partial p} f(\xi; p; x)$, and using the Baker-Hausdorff lemma[10] to replace $e^{-i\xi(k^2+2k\cdot p)} e^{k\cdot\partial/\partial p}$ by $e^{-ik\cdot(2\xi p + i\partial/\partial p)}$ and so obtain

$$\frac{\partial f(\xi; p; x)}{\partial \xi} = -i\int d^4k\, \tilde{U}(-k)e^{-ik\cdot(x + 2\xi p + i\partial/\partial p)} f(\xi; p; x) = -i\,U\left(x + 2\xi p + i\frac{\partial}{\partial p}\right) f(\xi; p; x). \qquad (10)$$

Equation (10) has the formal solution

$$f(\xi; p; x) = \left(\exp\left[-i\int_0^\xi d\xi' U\left(x + 2\xi' p + i\frac{\partial}{\partial p}\right)\right]\right)_+ , \qquad (11)$$

where $(\ )_+$ denotes a "time ordering" of the ξ' parameters in the expansion of the exponential of (11). Alternatively, f is the solution to the integral equation

$$f(\xi;p;x)=1-i\int_0^{\xi} d\xi' U\left(x+2\xi' p+i\frac{\partial}{\partial p}\right)f(\xi';p;x). \tag{12}$$

Finally, in terms of this function, one has the exact representation

$$\overline{\Delta}_c(y,x|U)=i(2\pi)^{-4}\int d^4p\, e^{ip\cdot(y-x)} \times\int_0^{\infty} d\xi\, e^{-i\xi(m^2+p^2)}f(\xi;p;x). \tag{13}$$

C. An exact amplitude representation

It will be most convenient to multiply U by a parameter λ, where $0\leqslant\lambda\leqslant 1$, and so consider $\overline{\Delta}_c[\lambda U]$. One observes that

$$\lambda\frac{\partial}{\partial\lambda}\overline{\Delta}_c[\lambda U]=\int d^4z\, U(z)\frac{\delta}{\delta U(z)}\overline{\Delta}_c[\lambda U], \tag{14}$$

a property which obviously holds for any sum of polynomial functionals. Equivalently, since such polynomials may be obtained by appropriate functional differentiation with respect to a source $j(z)$, it is sufficient to think of $\overline{\Delta}_c[\lambda U]$ in the form $\exp[\lambda\int d^4u\, j(u)U(u)]$, in which case (14) follows immediately.

Note also that

$$\overline{\Delta}_c(y,x|\lambda U)|_{\lambda=0}=\Delta_c(y-x) =(2\pi)^{-4}\int d^4k\, e^{ik\cdot(y-x)} \times(k^2+m^2-i\epsilon)^{-1},$$

and therefore

$$K_x\Delta_c(y-x)=\delta^{(4)}(y-x),$$

$$K_xK_y\Delta_c(y-x)=(2\pi)^{-4}\int d^4k\, e^{ik\cdot(y-x)}(k^2+m^2),$$

and

$$\int d^4x\, e^{ip\cdot x}K_xK_y\Delta_c(y-x)=e^{ip\cdot y}(p^2+m^2)=0,$$

when p is on the mass shell. Hence, in (4), one may replace $\overline{\Delta}_c(y,x|\lambda U)$ by

$$\begin{aligned}\overline{\Delta}_c(y,x|\lambda U)-\Delta_c(y-x)&=\int_0^1 d\lambda\frac{\partial}{\partial\lambda}\overline{\Delta}_c(y,x|\lambda U)\\ &=\int_0^1\frac{d\lambda}{\lambda}\int d^4z\, U(z)\frac{\delta}{\delta U(z)}\overline{\Delta}_c(y,x|\lambda U)\\ &=-\int_0^1 d\lambda\int d^4z\,\overline{\Delta}_c(y,z|\lambda U)U(z)\overline{\Delta}_c(z,x|\lambda U),\end{aligned} \tag{15}$$

where the last line follows from the easily proven property $[\delta/\delta U(z)]\overline{\Delta}_c(y,x|U)=-\overline{\Delta}_c(y,z|U)\overline{\Delta}_c(z,x|U)$. Thus one may write

$$\langle p'|T|p\rangle=-(2\pi)^{-3}(2E2E')^{-1/2}\int_0^1 d\lambda\int d^4z\left[\int d^4y\, e^{-ip'\cdot y}K_y\overline{\Delta}_c(y,z|\lambda U)\right]U(z)\left[\int d^4x\, e^{ip\cdot x}K_x\overline{\Delta}_c(z,x|\lambda U)\right]. \tag{16}$$

Each of the bracketed expressions of (16) has a lovely, mass-shell simplification, which follows directly from (13) and the symmetry property $\overline{\Delta}_c(x,y)|U)=\overline{\Delta}_c(y,x|U)$,

$$\int d^4y\, e^{-ip'\cdot y}K_y\overline{\Delta}_c(y,z|\lambda U)|_{p'^2+m^2=0}=e^{-ip'\cdot z}f(\infty;p';z),$$

$$\int d^4x\, e^{ip\cdot x}K_x\overline{\Delta}_c(x,z|\lambda U)|_{p^2+m^2=0}=e^{ip\cdot z}f(\infty;-p;z).$$

Hence, (16) simplifies to

$$\langle p'|T|p\rangle=-(2\pi)^{-3}(2E2E')^{-1/2}\int_0^1 d\lambda\int d^4z\, e^{iq\cdot z}f(\infty;p';z)\,U(z)f(\infty;-p;z), \tag{17}$$

where $q=p-p'$ and $t=-|t|=-q^2$. Finally, if U depends on $\vec{r}$ only, an energy-conserving δ function may be extracted; with the normalization of (6), one finds a representation for the exact scattering amplitude,

$$f(\vec{p}',\vec{p}) = -\frac{1}{4\pi}\int_0^1 d\lambda \int d^3z\, e^{i\vec{q}\cdot\vec{z}}\left(\exp\left[-i\lambda\int_0^\infty d\xi\, U\left(z+2\xi p'+i\frac{\partial}{\partial p'}\right)\right]\right)_+ U(\vec{z})$$
$$\times\left(\exp\left[-i\lambda\int_0^\infty d\xi\, U\left(z-2\xi p-i\frac{\partial}{\partial p}\right)\right]\right)_+ . \tag{18}$$

Note that microscopic reversibility (time reversal) is satisfied, $f(\vec{p}',\vec{p})=f(-\vec{p},-\vec{p}')$, and will so continue under any approximation which treats each bracket of (18) in like manner.

D. Eikonal approximations

The simplest eikonal approximation may practically be read off from (18). Physically, one supposes that p and p' are almost the same, that $\vec{q}^2/\vec{p}^2 \ll 1$. This implies that any variations $\delta\vec{p}$, $\delta\vec{p}'$ are small, and hence one expects to be able to drop the $\partial/\partial\vec{p}$, $\partial/\partial\vec{p}'$ operators of (18). When this is done, the complicated ordered exponentials become ordinary exponentials, and one has

$$f \simeq f_{\text{eik}}(\vec{q},E) = -\frac{1}{4\pi}\int_0^1 d\lambda\int d^3z\, e^{i\vec{q}\cdot\vec{z}}\, U(z)\exp\left\{-i\lambda\int_0^\infty d\xi\left[U(z+2\xi p')+U(z-2\xi p)\right]\right\}. \tag{19}$$

Equation (19) is almost in the form of a conventional eikonal amplitude, but before completing the argument it is worthwhile to see how this approximation may be phrased in terms of the basic Eq. (9). One thinks of all $|\vec{k}|$ components as small ("soft") compared to $|\vec{p}|$, so that (9), with $U\to\lambda U$, is replaced by the approximate relation

$$\frac{\partial f(\xi;\vec{p};\vec{x})}{\partial\xi} = -i\lambda\int d^3k\,\tilde{U}(-k)\, e^{-\vec{k}\cdot\vec{x}-2i\xi\vec{k}\cdot\vec{p}} \times f(\xi;\vec{p};\vec{x}), \tag{20}$$

which, in effect, leads to a boson version of the Bloch-Nordsieck approximation scheme.[8] *A posteriori*, we shall see that this is a reasonable approximation in the forward direction, $\vec{q}^2 \ll \vec{p}^2$.

If one neglects all q/p dependence in (19), the only $\vec{q}\neq 0$ dependence will remain in the phase $e^{i\vec{q}\cdot\vec{z}}$. In a coordinate system where $\vec{p}=\hat{e}_3 p$, at high energy and small angles $q_3 \sim O(q_\perp^2/p)\sim 0$, so that this phase becomes $e^{i\vec{q}_\perp\cdot\vec{z}_\perp}$, where $\vec{z}_\perp(x,y) \equiv \vec{b}$, the impact parameter. The ξ integral of (19) is trivial, producing a factor

$$\exp\left[-\frac{i\lambda}{2p}\int\frac{d^2k}{(2\pi)^2}\,\tilde{U}(k_\perp)\, e^{i\vec{k}\cdot\vec{b}}\right],$$

while the $\int_{-\infty}^{+\infty} dz_3$ may be performed directly, to yield the form of a perfect differential for the λ integration. One immediately finds

$$f_{\text{eik}}(\vec{q},E) = \frac{ip}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\left[1-e^{i\chi(\vec{b},p)}\right], \tag{21}$$

where

$$i\chi = -\frac{i}{2p}\int\frac{d^2k}{(2\pi)^2}\,\tilde{U}(k_\perp)\, e^{i\vec{k}\cdot\vec{b}}.$$

With $U=(2m/\hbar^2)V$, one recovers the familiar eikonal amplitude. The basic, self-consistent justification of the method is that, in the forward direction, $\vec{q}_\perp^{\,2} \ll \vec{p}^2$, $\int d^2b$ emphasizes large b, $b\sim q_\perp^{-1}$, and therefore only small $k_\perp \sim b^{-1}\sim q_\perp \ll p$ can enter. For large p and real V, $i\chi \sim -iO(p^{-1})$, and the effect is suppressed, so that $f\sim f_{\text{Born}}$. For high-energy-particle purposes, one may imagine that V is both absorptive and energy-dependent, effects required by experiment and which, happily, follow from numerous field-theoretic calculations[8]; then f can be vastly different from its first few Born approximations.

E. Corrections and refinements

What sort of corrections may be made to this simplest eikonal amplitude? Computations using simple potentials have shown that f_{eik} is accurate at forward angles, and is not particularly good at larger angles, as expected. There is no unique way of developing corrections to f_{eik}; rather, there are a variety of methods which may be employed, either to extend the range of q^2 for which the approximation is accurate, or to refashion the entire method and obtain new expressions valid in other (large) q^2 regions. Recent work[3] of Wallace and Baker deal with the first approach, while the older work[5,2] of Schiff, and of Saxon and Schiff, consider the second. For specific problems of potential theory, a major consideration is ease of computation (i.e., minimizing computer time), a feature not built into the analysis of this section.

Corrections may be defined by returning to the exact Eq. (9), and writing the ansatz $f(\xi;p;x) = \exp[-i\int_0^\xi d\xi'\,\psi(\xi';p;x)]$. Substitution into (9) then yields

$$\psi(\xi;p;x)=\int d^4k\,\tilde{U}(-k)\,e^{-ik\cdot x}e^{-i\xi(k^2+2k\cdot p)}\exp\left\{-i\int_0^\xi d\xi'[\psi(\xi';p+k;x)-\psi(\xi';p;x)]\right\}\ , \tag{22}$$

an impossible integral equation. There is no unique method of defining approximations to ψ, and different methods generate different approximations. For example, one natural approach is to rewrite (22) as

$$\psi_\lambda(\xi;p;x)=\int d^4k\,\tilde{U}(-k)\,e^{-i\xi(\lambda k^2+2k\cdot p)}\exp\left\{-i\int_0^\xi d\xi'[\psi_\lambda(\xi';p+\lambda k;x)-\psi_\lambda(\xi';p;x)]\right\}, \tag{23}$$

and expand ψ_λ in powers of λ, $\psi_\lambda(\xi;p;x)=\sum_{j=0}^\infty\lambda^j\psi_{(j)}(\xi;p;x)$. The most glaring lack of uniqueness is the present expansion of the k^2 exponent [this corresponds to different boson Bloch-Nordsieck (BN) expansions[8] in field theory]. From (23) one easily computes $\psi_{(0)}, \psi_{(1)}$, etc.; the simplest corrections to the previous estimate of (11) generate

$$f(\infty;p;x)\simeq\exp\left\{-i\int_0^\infty d\xi[U(z+2\xi p)+i\xi\partial_z^{\,2}U(z+2\xi p)]-2i\int_0^\infty d\xi\int_0^\xi \xi'\,d\xi'[\partial_z^\mu U(z+2\xi p)][\partial_z^\mu U(z+2\xi' p)]\right\}. \tag{24}$$

When inserted into (18), for a time-independent potential, one finds already-exponentiated corrections to the simple theory, which presumably extend the quality of the approximation to larger values of q/p.

F. Scattering at all angles

Systematic approximation schemes (e.g., that of Blankenbecler and Sugar[3]) may be invented to describe scattering at both small and large momentum transfer. The method employed here first saw service in wide-angle field-theoretic calculations (in the hadronic bremsstrahlung models[8] of Gaisser, Fried, Raman, Moreno, and Kirby), and represents a particular application of the decomposition introduced by Mahanthappa.[7] The basic idea is to split V into two parts, $V=V_S+V_H$, which are treated differently throughout. If one sets

$$\tilde{U}(\vec{k})\equiv\tilde{U}(\vec{k})\,e^{-\beta\vec{k}^2}+\tilde{U}(\vec{k})(1-e^{-\beta\vec{k}^2}), \tag{25}$$

where $\tilde{U}_S(\vec{k})\equiv e^{-\beta\vec{k}^2}\tilde{U}(\vec{k})$, $\tilde{U}_H(\vec{k})\equiv(1-e^{-\beta\vec{k}^2})\tilde{U}$, and β is a constant to be specified, then effectively $\tilde{U}_S$ contains only $\vec{k}^2$ values $\lesssim\beta^{-1}$ in any subsequent integration; and conversely for $\tilde{U}_H$.

One then expands the fundamental propagator in powers of U_H,

$$(m^2-\partial^2+U_S+U_H)^{-1}=(m^2-\partial^2+U_S)^{-1}-(m^2-\partial^2+U_S)^{-1}\times U_H(m^2-\partial^2+U_S)^{-1}+\cdots,$$

so that the expression (4) for $\langle p'|T|p\rangle$ becomes a sum of terms corresponding to the expansion in powers of U_H. For any $\overline{\Delta}_c[U_S]$, there exists the equation

$$\frac{\partial f(\xi;p;x)}{\partial\xi}=-i\int d^3k\,\tilde{U}(-\vec{k})\,e^{-\beta\vec{k}^2}e^{-i\vec{k}\cdot\vec{x}-i\xi(\vec{k}^2+2\vec{k}\cdot\vec{p})}\times f(\xi;\vec{p}+\vec{k};x), \tag{26}$$

and for a specific $|\vec{p}|=|\vec{p}'|\equiv p$, one may now choose β so that $|\vec{k}|<\beta^{-1/2}$ means $|\vec{k}|\ll p$, i.e., $\beta^{-1}\ll p^2$ or $\beta p^2\gg1$. This guarantees that the k entering the integrand are always negligible compared to p, and hence the BN approximation previously used is correct:

$$\overline{\Delta}_c[U_S]\to\overline{\Delta}_{c,\mathrm{BN}}^{(\beta)}[U_S],$$

and

$$f\to f_{\mathrm{eik}}^{(\beta)}(\vec{p},E)+\cdots.$$

Note that if q is restricted to very small values, the β dependence of $f_{\mathrm{eik}}^{(\beta)}$ is irrelevant, since only corresponding small k can enter; but away from the forward direction, the β dependence of $f_{\mathrm{eik}}^{(\beta)}$ should be retained.

The first correction to f_{eik} may be obtained by expanding

$$\overline{\Delta}_c(y,x|U)\simeq\overline{\Delta}_c(y,x|U_S)-\int d^4z\,\overline{\Delta}_c(y,z|U_S)\,U_H(z)\,\overline{\Delta}_c(z,x|U_S)+\cdots. \tag{27}$$

Performing all previous computations for the first right-hand-side term of (27) generates $f_{\mathrm{eik}}^{(\beta)}$, for $\beta\vec{p}^2\gg1$. The correction term linear in V_H, when substituted into (4), yields

$$\langle p'|T_{2H}|p\rangle = -(2\pi)^{-3}(4EE')^{-1/2}\int d^4z\left[\int d^4y\, e^{-ip'\cdot y}K_y\bar{\Delta}_c(y,z|U_S)\right]U_H(z)\left[\int d^4x\, e^{ip\cdot x}K_x\bar{\Delta}_c(z,x|U_S)\right], \quad (28)$$

which has almost the same form as the T_{eik} of (16), except for the absence of all λ dependence. This corresponds to an important difference in counting the interactions of the scattering particle with the potential, for the single interaction with V_H is counted separately from the multiple V_S interactions. But all technical manipulations leading up to (19) may be applied to (28), and without any restrictions on q/p we can write

$$f(\vec{q},E) = f_{0H}^{(\beta)}(\vec{q},E) + f_{1H}^{(\beta)}(\vec{q},E) + \cdots,$$

where nH denotes the number of V_H powers involved, and the superscript β is a reminder that $\beta \neq 0$. The entire amplitude must be independent of β; but one has the expectation that the different pieces, specifically $f_{0H}^{(\beta)}$ and $f_{1H}^{(\beta)}$, provide quality approximations to the scattering amplitude in the different regions,

$$f_{0H}^{(\beta)}(\vec{q},E) = -\frac{1}{4\pi}\int_0^1 d\lambda\int d^3z\, e^{i\vec{q}\cdot\vec{z}}U_S(\vec{z})\exp\left\{-i\lambda\int_0^\infty d\xi[U_S(z+2\xi p') + U_S(z-2\xi p)]\right\} \quad (29)$$

and

$$f_{1H}^{(\beta)}(\vec{q},E) = -\frac{1}{4\pi}\int d^3z\, e^{i\vec{q}\cdot\vec{z}}U_H(\vec{z})\exp\left\{-i\int_0^\infty d\xi[U_S(z+2\xi p') + U_S(z-2\xi p)]\right\}. \quad (30)$$

Except for the β dependence, (29) is just (19), while (30) is analogous to Schiff's wide-range result. One can argue that for large q^2, (30) is equivalent to Schiff's result, for if at sufficiently high energies, a small β may be chosen to satisfy $\beta > 1/p^2$, then β may be set equal to zero in the V_S factors, which generates Schiff's expression except for the replacement of his V by V_H. However, at large q, only small z may enter; and one may expect to be able to neglect the z dependence inside V_S. Then, one simply calculates the Fourier transform of $V_H(z)$, and obtains $\tilde{U}_H(q) = (1 - e^{-\beta q^2})\,\tilde{U}(q)$. But if $\beta p^2 > 1$, and $q_{\max}{}^2 = 4p^2$, then $\beta q^2 > 1$, so that $\tilde{U}_H(q) \simeq \tilde{U}(q)$. In this way, one expects $f_{1H}^{(\beta)}$ to reproduce Schiff's wide-angle form. In fact, qualitative arguments may be given to suggest that $|f_{1H}^{(\beta)}| \gg |f_{0H}^{(\beta)}|$ for large q^2, while the reverse is true for small angles. This will be explicitly shown, for a Gaussian potential, in Sec. IV. We will also note that (29) + (30) is nearly β-independent.

III. A SIMPLER FORMALISM

The derivation of Eqs. (28) and (30) may be carried out in a somewhat more direct manner. Using standard forms[9] of nonrelativistic potential theory ($\hbar = m = c = 1$),

$$f(\vec{p}_i,\vec{p}_f) = -\frac{1}{2\pi}\int d^3r\,\Phi_f^* V(\vec{r})\Psi_i^{(+)}, \quad (31)$$

where

$$\Phi_i = e^{(i\vec{p}_i\cdot\vec{r})},\quad \Phi_f = e^{(i\vec{p}_f\cdot\vec{r})},$$

$$\Psi_i^{(+)} = \Phi_i + G_{(+)}V\Phi_i, \quad (32)$$

and

$$G_{(+)} = (E + i\epsilon - T - V)^{-1},$$

$$T = -\tfrac{1}{2}\nabla^2 \quad (\hbar = m = 1),$$

with

$$p_i = p_f = (2E)^{1/2} \equiv p.$$

The differential cross section is $\sigma(\theta,p) = |f|^2$. We treat the soft component of V to provide the necessary distortions, while V_H is to be treated approximately by the first-order perturbative theory. Using the two-potential formula,[9] we obtain

$$f = -\frac{1}{2\pi}\int d^3r\,\Phi_f^* V_S\Psi_{Si}^{(+)} - \frac{1}{2\pi}\int d^3r\,\Psi_{Sf}^{(-)}V_H\Psi_i^{(+)}$$

$$\equiv f_S + \sum_H f_H, \quad (33)$$

where

$$(T + V_S - E)\Psi_{Si}^{(+)} = 0. \quad (34)$$

First consider f_S, which involves only V_S. We can solve (34) for Ψ_{Si} and evaluate the integral for f_S in several different approximations. Since we are here mainly interested in the large-angle behavior of f, we could follow the procedure of Lévy and Sucher[4] and Harrington[6] and obtain

$$f_S \simeq -\frac{1}{2\pi}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V_S(\vec{r})(e^{i\chi_S} - 1)(i\chi_S)^{-1}$$

$$= -\frac{1}{2\pi}\int_0^1 d\lambda\int d^3r e^{i\vec{q}\cdot\vec{r}}\, e^{i\lambda\chi_S}V_S(\vec{r}), \quad (35)$$

where

$$\chi_S = -\int_0^\infty d\xi [V_S(\vec{r} - \xi\vec{p}_i) + V_S(\vec{r} + \xi\vec{p}_f)]$$
$$\equiv \chi_{Si} + \chi_{Sf} \tag{36}$$

and

$$\vec{q} \equiv \vec{p}_i - \vec{p}_f . \tag{37}$$

Alternatively, we may set[11] either

$$f_S \simeq -\frac{1}{2\pi}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V_S(\vec{r})\, e^{i\chi_{Si}}, \tag{38a}$$

$$f_S \simeq -\frac{1}{2\pi}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V_S(\vec{r}) e^{i\chi_S/2}, \tag{38b}$$

or

$$f_S \simeq -\frac{1}{2\pi}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V_S(\vec{r}) e^{i\chi_{Sa}}, \tag{38c}$$

where

$\chi_{Sa} = \chi_{Si}$,

with $\vec{p}_i$ replaced by $\vec{p}_a$,

$$\vec{p}_a \equiv p\,\frac{\vec{p}_i + \vec{p}_f}{|\vec{p}_i + \vec{p}_f|} . \tag{39}$$

The form (38c) is the usual Glauber amplitude in which V is replaced by V_S. It was shown earlier[11] that all the forms listed above, (35) and (38), give cross sections of comparable accuracy at large values of q.

Incidentally, note that, e.g.,

$$\chi_{Si} = -\int_0^\infty d\xi\, V_S(\vec{r} - \xi\vec{p}_i)$$
$$= -\frac{1}{p}\int_{-\infty}^{z_i} dz'\, V_S(\vec{r}'),$$
$$\vec{r}' \equiv \vec{b} + \vec{z}', \quad \hat{z}_i \times \hat{p}_i = 0 \tag{40a}$$

and

$$\chi_{S_f} = -\int_0^\infty d\xi\, V_S(\vec{r} + \xi\vec{p}_f)$$
$$= -\frac{1}{p}\int_{z_f}^{\infty} dz'\, V_S(\vec{r}'),$$
$$\vec{r}' \equiv \vec{b} + \vec{z}', \quad \hat{z}_f \times \hat{p}_f = 0 . \tag{40b}$$

The second part of f in (33) is much more difficult to evaluate, because both $\Psi_{Sf}^{(-)}$ and $\Psi_i^{(+)}$ appear. We assume again that $\Psi_i^{(+)}$ is dominated by $\Psi_{Si}^{(+)}$, and write

$$\sum_H f_H \sim f_H = -\frac{1}{2\pi}\int d^3r\, \Psi_{Sf}^{(-)}\, V_H\, \Psi_{Si}^{(+)}, \tag{41}$$

with

$$\Psi_{Si}^{(+)} \simeq e^{i\chi_{Si}} e^{i\vec{p}_i\cdot\vec{r}} \quad (\approx \Psi_i^{(+)}),$$
$$\Psi_{Sf}^{(-)} \approx e^{i\chi_{Sf}} e^{-i\vec{p}_f\cdot\vec{r}} . \tag{42}$$

That is,

$$f_H \approx -\frac{1}{2\pi}\int d^3r\, e^{i\chi_S}\, V_H(\vec{r}), \tag{43}$$

which is of first order in V_H, but the wave functions are distorted initially and finally by the soft component V_S. This is then very similar to Schiff's large-angle formula,[5] except for the fact that V_S and V_H are now playing different roles in f_H. (In his derivations of the large-angle formula, Schiff distinguishes hard and soft components.) Note that the eikonal phase factor $e^{i\chi_S}$ in (43) does not contain a factor $\frac{1}{2}$ as compared with (38b). In fact, for small χ_S, (38b) agrees with (35) to second order in χ_S. For details of this difference, we refer to Ref. 12.

The form f_S and f_H given above may be simplified further for the purpose of numerical evaluation. Noting that the complexity of these amplitudes arises mainly from the fact that the initial and final momentum directions are different for $\vec{q} \neq 0$, we may set the average momentum vector $\vec{p}_a$ for $\vec{p}_i$ and $\vec{p}_f$. Then, the amplitude f becomes

$$f \to f_a \approx -\frac{1}{2\pi}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}[V_S(\vec{r})\, e^{i\chi_{Sa}} + V_H(\vec{r})\, e^{i\chi_{Sa}}]$$
$$= -\int_0^\infty b\,db\, J_0(qb)$$
$$\times \int_{-\infty}^{\infty} dz[V_S(\vec{r})\, e^{i\chi_{Sa}} + V_H(\vec{r})\, e^{i\chi_{Sa}}]$$
$$= -ip\int_0^\infty b\,db\, J_0(qb)\, (e^{i\chi_{Sa}} - 1 + i\chi_{Ha}\, e^{i\chi_{Sa}})$$
$$\equiv f_{aS} + f_{aH}, \tag{44}$$

where

$$\chi_{Sa}(b) = -\frac{1}{p}\int_{-\infty}^{+\infty} V_S\, dz,$$
$$\chi_{Ha}(b) = -\frac{1}{p}\int_{-\infty}^{+\infty} V_H\, dz . \tag{45}$$

The form (44) involves a double integration as both χ_{Sa} and χ_{Ha} are functions of the impact parameter b.

IV. CALCULATIONS

A very extensive study of the various approximation procedures used in high-energy potential scattering has been carried out previously.[11] For ready comparison of the formalism presented in Secs. II and III, we adopt the same form of the po-

TABLE I. The amplitudes and the differential cross sections $|f|^2$ are calculated for the choice $\beta = 10/p^2 = 2.5$. For each entry in the table, we have listed the value for f_S, f_H, $f = f_S + f_H$, and the exact amplitude f_{EX}, and the Glauber amplitude f_G ($p = 2.0$).

q	Ref	Imf	$\|f\|^2$	
0.0	5.99	0.52	36.1	(S)
	0.33	0.50	0.4	(H)
	6.31	1.02	40.9	$(S+H)$
	6.31	1.32	41.5	(EX)
	6.21	1.31	40.3	(G)
0.4	3.17	0.37	10.2	
	1.51	0.49	2.5	
	4.68	0.86	22.6	
	4.63	1.11	22.7	
	4.59	1.10	22.2	
0.8	0.32	0.14	0.12	
	1.44	0.35	2.19	
	1.76	0.49	3.33	
	1.60	0.63	3.16	
	1.69	0.63	3.26	
1.2	0.078	0.035	0.007	
	0.072	0.128	0.022	
	0.150	0.163	0.049	
	0.065	0.198	0.043	
	0.116	0.218	0.061	

TABLE II. Same as Table I, except that $\beta = 1/p^2 = \frac{1}{4}$ and $p = 2.0$.

q	Ref	Imf	$\|f\|^2$	
0.0	6.26	1.14	40.4	(S)
	−0.04	0.14	0.0	(H)
	6.22	1.28	40.3	$(S+H)$
	6.31	1.32	41.5	(EX)
	6.21	1.31	40.3	(G)
0.4	4.45	0.91	20.7	
	0.15	0.13	0.0	
	4.61	1.04	22.3	
	4.63	1.11	22.7	
	4.59	1.10	22.2	
0.8	1.46	.52	2.40	
	0.25	0.09	0.07	
	1.71	0.61	3.28	
	1.66	0.63	3.16	
	1.69	0.63	3.26	
1.2	0.074	0.218	0.053	
	0.042	0.038	0.003	
	0.115	0.256	0.079	
	0.065	0.198	0.043	
	0.116	0.218	0.061	

TABLE III. A two-dimensional approximation to $f_a = f_{aS} + f_{aH}$. The parameter $\beta = 10/p^2$, $p = 2.0$. For each entry in the table, we have listed the values for f_{aS}, f_{aH}, f_a, f_{EX}, and f.

q	Ref	Imf	$\|f\|^2$	
0.0	6.36	0.59	40.8	(S)
	−0.05	0.46	0.2	(H)
	6.31	1.05	40.9	$(S+H)$
	6.31	1.32	41.5	(EX)
	6.31	1.02	40.9	(G)
0.4	3.18	0.41	10.3	
	1.49	0.48	2.5	
	4.67	0.90	22.6	
	4.63	1.11	22.7	
	4.68	0.86	22.6	
0.8	0.35	0.14	0.14	
	1.40	0.37	2.11	
	1.75	0.51	3.33	
	1.66	0.63	3.16	
	1.76	0.49	3.33	
1.2	−0.003	0.022	0.001	
	0.152	0.126	0.039	
	0.149	0.148	0.044	
	0.065	0.198	0.043	
	0.150	0.163	0.049	

TABLE IV. Same as Table III, except that $\beta = 1/p^2 = 0.25$.

q	Ref	Imf	$\|f\|^2$	
0.0	6.26	1.17	40.5	(S)
	−0.04	0.13	0.0	(H)
	6.22	1.30	40.3	$(S+H)$
	6.31	1.32	41.5	(EX)
	6.22	1.28	40.3	(G)
0.4	4.44	0.97	20.6	
	0.15	0.13	0.0	
	4.59	1.09	22.3	
	4.63	1.11	22.7	
	4.61	1.04	22.3	
0.8	1.45	0.53	2.39	
	0.24	0.10	0.07	
	1.69	0.63	3.26	
	1.66	0.63	3.16	
	1.71	0.61	3.28	
1.2	0.075	0.177	0.037	
	0.042	0.040	0.003	
	0.117	0.217	0.061	
	0.065	0.198	0.043	
	0.115	0.256	0.079	

tential

$$V(r)=\tfrac{1}{2}g\,e^{-Ar^2}(1+\rho r^2), \tag{46}$$

with

$$g=-0.4,\quad \rho=0.3,\quad A=0.2, \tag{47}$$

and choose

$$p=2.0,$$

all in the units $m=\hbar=c=1$. This potential produces diffraction maxima at $q=0$ and $q\approx 1.5$ and the first minimum at $q\approx 1.3$. The evaluation of V_S and V_H is straightforward, and we obtain, with $d\equiv 1+4A\beta$,

$$V_S(r)=\frac{g}{2}\,\frac{e^{-Ar^2/d}}{d^{3/2}}\left[1+\frac{3\rho}{2A}-\frac{\rho}{A}\left(\frac{\frac{3}{2}+6A\beta}{d^2}\right)+\frac{\rho r^2}{d^2}\right], \tag{48}$$

and $V_H(r)\equiv V(r)-V_S(r)$. The explicit shapes of V, V_S, and V_H are shown in Fig. 1 for the choices

$$\beta=10p^{-2} \text{ and } \beta=p^{-2}. \tag{49}$$

The evaluation of f_S and f_H given by (35) and (43) requires multiple integrations involving four variables, one integration for the evaluation of χ_S, and three integrations over d^3r. To simplify the calculation, we chose the z axis to be parallel to $\vec{p}_a$ in a symmetric way. Then

$$\chi_S=-\int_0^\infty d\xi[V_S(R_i)+V_S(R_f)], \tag{50}$$

where

$$R_i^2=(\vec{r}-\xi\vec{p}_i)^2=r^2+\xi^2p^2+2\xi px\sin(\tfrac{1}{2}\theta)-2\xi pz\cos(\tfrac{1}{2}\theta),$$

$$R_f^2=(\vec{r}+\xi\vec{p}_f)^2=r^2+\xi^2p^2+2\xi px\sin(\tfrac{1}{2}\theta)+2\xi pz\cos(\tfrac{1}{2}\theta),$$

with

$$r^2=x^2+y^2+z^2,\quad q=2p\sin(\tfrac{1}{2}\theta),$$

Here, the $dy\,dz$ integrations are for $0\leqslant y, z<\infty$, while the dx integration is still over the range $-\infty<x<\infty$. The result of this calculation is presented in Tables I and II.

Because of the many (4) variables involved in the integrations for f_S and f_H, we were not able to perform a more accurate evaluation of f. This problem grows severe as q is increased, due to the oscillatory behavior of the integrand and subsequent cancellations. Thus, our result beyond $q\gtrsim 1.2$ is not reliable.

We have also evaluated the approximate amplitudes f_{aS} and f_{aH} of (44), which involve only a double integral each. The result is given in Tables III and IV. Although f_a is extremely simple to evaluate, the result compares well with f_S and f_H of Tables I and II. In fact, f_a is as simple as the Glauber amplitude. We expect, however, that the original form for f, with (38) and (43), should be more accurate at larger angles than the form (44).

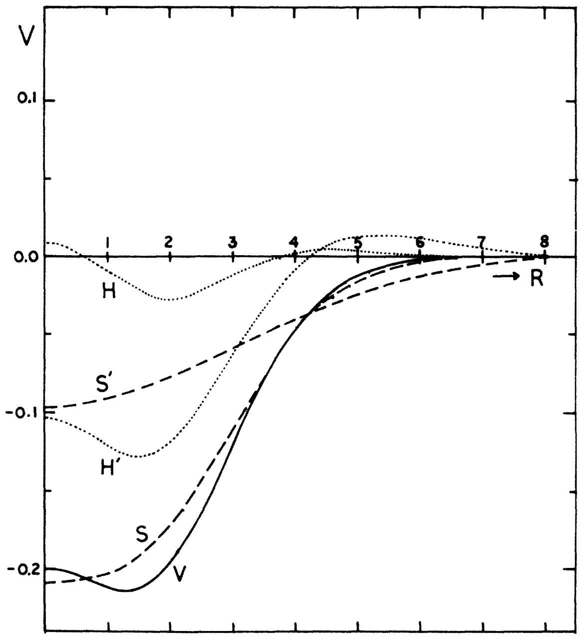

FIG. 1. V_H and V_S for different values of β. The curves S and H correspond to the potentials with $\beta=1/p^2$, while S' and H' are for $\beta=10/p^2$, where $p=2.0$ in natural units. The potential $V=V_S+V_H$ is also shown.

V. DISCUSSION

The result given in Table I clearly shows that the V_S contribution indeed dominates the amplitude in the low-q region ($q\leqslant 0.4$), while the effect of V_H is large for $q\geqslant 0.8$, all for the choice $\beta=10/p^2$. This feature shows up especially nicely at $q\approx 0.0$ and $q\approx 0.8$, but less distinctly in other regions where contributions from both terms in f are appreciable. Therefore, the dominance of V_S and V_H in certain regions of q holds only up to a moderate value of q, beyond which more careful analyses are required.

The second striking feature of our results is that the two-potential formula and the approximations introduced to derive f_S and f_H are such that the resulting total amplitude $f\cong f_S+f_H$ is insensitive to the choice of the parameter β. This can be seen by comparing the results in Tables I and II. This feature is especially useful in view of the fact that there is no reliable *a priori* criterion one can use to determine the value for β. The results in the Tables also suggest that the derivation of f_S and f_H is fairly reliable.

Although the evaluation of $|f|^2$ in the forms (35) and (43) is rather time-consuming, the resulting cross sections in the region $(q/p) \leq 0.6$ are not much of an improvement over the simple Glauber amplitude. This fact only confirms the extreme sensitivity of the large-angle behavior of the cross sections to the accuracy of the approximations made. Owing to numerical difficulty, we were not able to study the region beyond $(q/p) \geq 0.6$.

On the other hand, $|f_a|^2$ of Tables III and IV indicate that both f_S and f_H can be simplified further without scattering accuracy too much. In fact, f_a has the same general structure as f_S+f_H, and yet is as easy to evaluate as the Glauber form. This may not be the case for larger q.

Finally, we emphasize that the approximate separation $f \simeq f_S+f_H$, while apparently not appreciably better than the amplitude of other approximations, has the virtue of possessing an immediate generalization[8] to relativistic particle scattering in a field-theory context. For potential theory, different approximations to (23) generate different corrections to the Glauber amplitude.

One of the noteworthy features of the analysis of this paper is the natural resolution of the well-known mismatch between the large-angle amplitude derived by Schiff[5] and the small-angle formula of Lévy and Sucher.[4] As discussed in the last paper of Ref. 4, the difficulty originally arose from the assumptions one makes in the counting procedure[12] when the perturbation series is summed. Depending on whether a single hard collision among the multiple-scattering processes is assumed to make a distinct contribution to the amplitude, we obtain two different eikonal amplitudes of the form f_S or f_H, but *not* both, with of course V_S and V_H replaced by V. We have shown that both these physical pictures are essentially correct if the interactions responsible for the soft and hard collisions are carefully defined.

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